12. Further Applications

- Domain of attraction of Lyapunov functions
- Matrix copositivity
- Geometric theorem proving
- Deciding quantum entanglement

Domain of attraction for Lyapunov functions

For a *given* Lyapunov function, want to estimate the domain of attraction.

We can compute the largest sublevel set that is invariant, i.e., the optimization problem:

$$\gamma_0 := \inf_{x \in \mathbb{R}^n} V(x) \qquad \text{subject to} \begin{cases} \dot{V}(x) = 0 \\ x \neq 0 \end{cases}$$

The invariant subset is given by the connected component of the Lyapunov function sublevel set $\mathcal{S} := \{x \,|\, V(x) < \gamma_0\}$ that includes the origin.

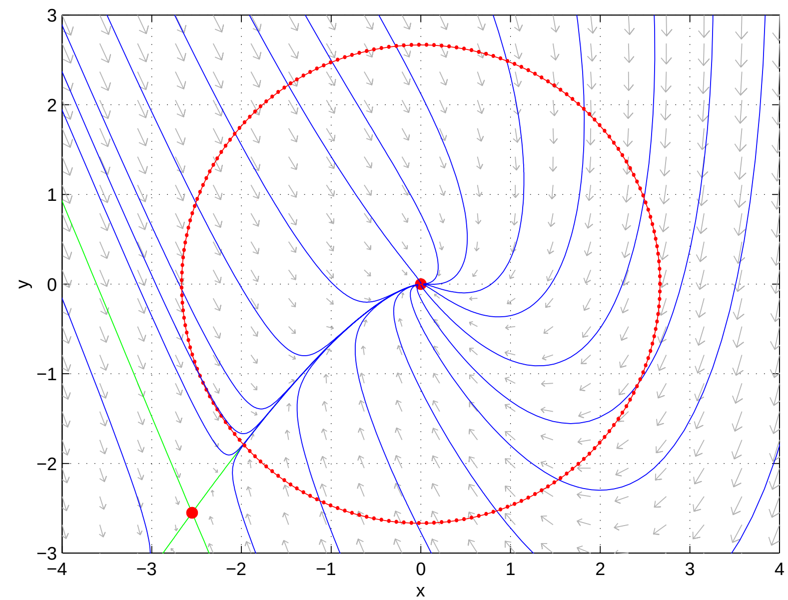
Using the SOS machinery, we easily obtain lower bounds on γ_0, which immediately provide estimates for the attracting region.

Example: Domain of attraction

Consider the system:

$$\dot{x} = -x + y$$
$$\dot{y} = 0.1x - 2y - x^2 - 0.1x^3$$

and Lyapunov function $V(x,y) := x^2 + y^2$.
The system has three fixed points.



We can consider the condition:

$$(V(x,y) - \gamma)(x^2 + y^2) + (p_1 + p_2 x + p_3 y + p_4 xy) \cdot \dot{V}(x,y) \quad \text{is a sum of squares.}$$

Clearly, $V(x,y) \geq \gamma$ holds for every (x,y) with $\dot{V} = 0$.
For this example, the obtained value of γ is the best possible.

Matrix copositivity

- A matrix $M \in \mathbb{S}^n$ is *copositive* if

$$x^T M x \geq 0 \quad \forall x \in \mathbb{R}^n, \, x_i \geq 0.$$

- Quadratic form is nonnegative on the nonnegative orthant.
- The set of copositive matrices is a convex closed cone, but...
- Checking copositivity is coNP-complete (Murty & Kabadi 1992).
- There exist necessary and sufficient conditions, usually in terms of principal minors. But, exponential time in the worst case (to be expected).

Copositivity applications

- Very important in quadratic programming.
- Characterization of local solutions.
- Valid inequalities for linearly constrained problems.
- Minimization of quadratic forms over polyhedra
- Consider the optimization problem

$$f^* = \text{minimize } x^T Q x \qquad \text{subject to } \begin{cases} Ax \geq 0 \\ x^T x = 1 \end{cases}$$

If $Q \succeq A^T C A + \gamma I$ with C copositive, then $f^* \geq \gamma$, since

$$x^T Q x \geq (Ax)^T C (Ax) + \gamma x^T x \geq \gamma.$$

We want *computable* sufficient conditions for copositivity.

More copositivity

We could use the P-satz, but we present first a different approach.

To check copositivity of M, consider the fourth order form:

$$P(\mathbf{z}) := \mathbf{z}^T M \mathbf{z} = \sum_{i,j} m_{ij} z_i^2 z_j^2, \qquad \mathbf{z} = [z_1^2, z_2^2, \ldots, z_n^2]^T.$$

M is copositive if and only if the form $P(\mathbf{z})$ is nonnegative.

Hard, but can check if $P(\mathbf{z})$ is a SOS form.

Equivalent to a well-known sufficient condition: if

$$M = P + N, \qquad P \succeq 0, \quad N \geq 0.$$

then M is copositive.

Necessary and sufficient for $n \leq 4$, counterexamples exist for $n \geq 5$.

Stronger SDP conditions

Consider the family of $2(r+1)$-forms given by

$$P_r(\mathbf{z}) = \left(\sum_{i=1}^{n} z_i^2\right)^r P_0(\mathbf{z}).$$

If P_i is a sum of squares, then P_{i+1} is also a sum of squares. For $r = 1$, we have the following sufficient condition:

Thm: Consider the system of LMIs given by:

$$\begin{aligned} M - \Lambda^i &\geq 0, & i &= 1, \ldots, n \\ \Lambda^i_{ii} &= 0, & i &= 1, \ldots, n \\ \Lambda^i_{jj} + \Lambda^j_{ji} + \Lambda^j_{ij} &= 0, & i &\neq j \\ \Lambda^i_{jk} + \Lambda^j_{ki} + \Lambda^k_{ij} &\geq 0, & i &\neq j \neq k \end{aligned}$$

If feasible, then M is copositive.

Copositivity: P-satz interpretation

These LMIs also have a simple P-satz interpretation, via the homogeneous identity:

$$(\sum_i x_i)(x^T M x) = \sum_i (x^T S_i x) x_i + \sum_{ijk} s_{ijk} x_i x_j x_k$$

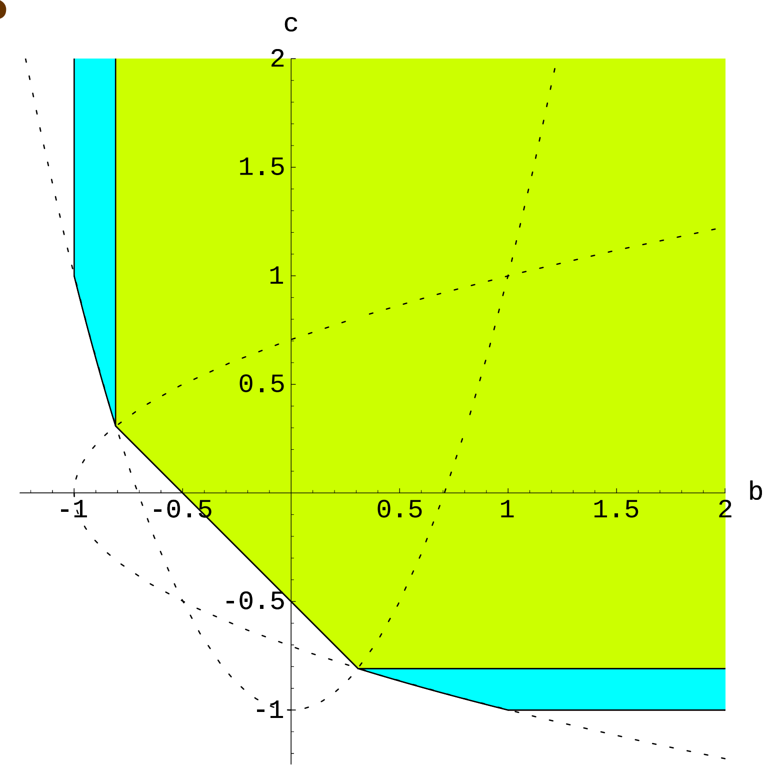
where the S_i are PSD quadratic forms, and the s_{ijk} are nonnegative scalars. A P-satz certificate for nonegativity over $x_i \geq 0$.

Similar interpretations for the other relaxations $(r > 1)$.

Example: cyclic copositive matrices

Consider 5×5 matrices of the form

$$\begin{bmatrix} 1 & b & c & c & b \\ b & 1 & b & c & c \\ c & b & 1 & b & c \\ c & c & b & 1 & b \\ b & c & c & b & 1 \end{bmatrix}$$



What conditions should b, c satisfy for the matrix to be copositive?

What about the relaxations? How powerful are they?

The inner region is the P+N relaxation ($r = 0$).
The outer region corresponds to the case $r = 1$, and coincides *exactly* with the region of copositivity.

Example: Structured Singular Value

- Structured singular value μ and related problems: provides better upper bounds.
- μ is a measure of robustness: how big can a structured perturbation be, without losing stability.
- A standard semidefinite relaxation: the μ upper bound.
  - Morton and Doyle's counterexample with four scalar blocks.
  - Exact value: approx. 0.8723
  - Standard μ upper bound: 1
  - New bound: 0.895

Geometric Inequalities

Ono's inequality: For an *acute* triangle,

$$(4K)^6 \geq 27 \cdot (a^2+b^2-c^2)^2 \cdot (b^2+c^2-a^2)^2 \cdot (c^2+a^2-b^2)^2$$

where K and a, b, c are the area and lengths of the edges.
The inequality is true if:

$$\left.\begin{array}{rcl} t_1 := a^2+b^2-c^2 & \geq & 0 \\ t_2 := b^2+c^2-a^2 & \geq & 0 \\ t_3 := c^2+a^2-b^2 & \geq & 0 \end{array}\right\} \Rightarrow (4K)^6 \geq 27 \cdot t_1^2 \cdot t_2^2 \cdot t_3^2$$

A simple proof: define

$$s(x,y,z) = (x^4+x^2y^2-2y^4-2x^2z^2+y^2z^2+z^4)^2 + 15 \cdot (x-z)^2(x+z)^2(z^2+x^2-y^2)^2.$$

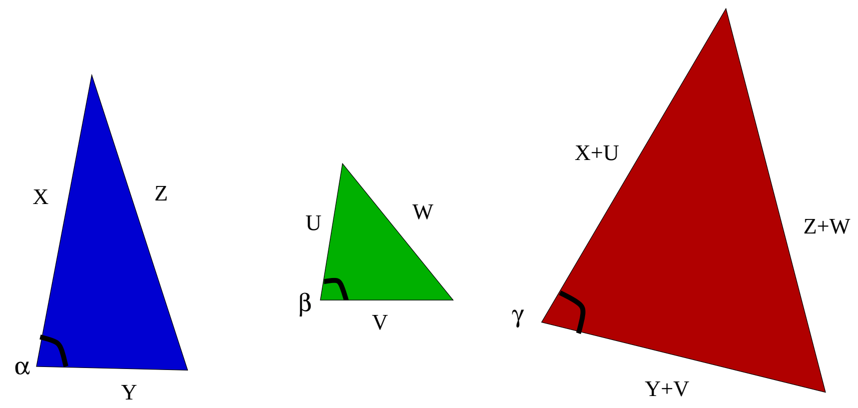
We have then

$$(4K)^6 - 27 \cdot t_1^2 \cdot t_2^2 \cdot t_3^2 = s(a,b,c) \cdot t_1 \cdot t_2 + s(c,a,b) \cdot t_1 \cdot t_3 + s(b,c,a) \cdot t_2 \cdot t_3$$

therefore *proving* the inequality.

Geometric Inequalities (II)

- A geometric inequality arising from circle packings (R. Peretz):



$$\alpha\cdot(X+Y-Z)+\beta\cdot(U+V-W) \leq \gamma\cdot((X+U)+(Y+V)-(Z+W))$$

- Not easy to prove. *Not* semialgebraic, in the standard form.
- The inequality holds if certain polynomial expression is nonnegative.
- Using SOS/SDP, we will obtain a very concise proof.

Geometric inequalities: reduction to a polynomial

It can be shown that the theorem is true if:

$$
\begin{aligned}
L(a,b,c,d) \;=\; & a^2b^2\,(a-b)^2 + (a-b)^2\,c^3d^3 + a^2d^2\,(1-ab)\left(1+ab-2\,b^2\right) - \\
& -adbc\left(2-4\,ab+ba^3+ab^3\right) + b^2c^2\,(1-ab)\left(1+ab-2\,a^2\right) + \\
& +\left(c^2b\,(1-ab)\left(2\,a-b-ab^2\right) - cd\left(a^2+b^2+2\,a^3b^3-4\,a^2b^2\right)\right. \\
& \left. +d^2a\,(1-ab)\left(2\,b-a-a^2b\right)\right) cd
\end{aligned}
$$

is nonnegative in $[0,1]^4$.

The statement of the theorem is invariant under interchange of the two triangles.

This translates into *symmetries* of the polynomial: we can simultaneously interchange a, b and c, d.

Can use symmetry reduction to simplify the problem, and achieve faster computation times.

Geometric inequalities: solution

We solve the symmetry-reduced SDPs, and obtain:

$$\begin{aligned} L(a,b,c,d) &= L_1 + L_2 + L_3 \\ L_1 &= (c+d)(-a^2b + ab^2 - ad + bc - bcd + adc - ab^2c + a^2bd)^2 \\ L_2 &= (1-c)(1-d)(ab-1)^2(ad-bc)^2 \\ L_3 &= (1-c)(1-d)(a-b)^2(ab-cd)^2. \end{aligned}$$

From this, stronger conclusions on the sign of L can be derived. Not only it is nonnegative on the open unit hypercube $(0,1)^4$, but the same property holds on the much larger region $\mathbb{R} \times \mathbb{R} \times \{c+d \geq 0, (1-c)(1-d) \geq 0\}$.

An *verifiable* certificate for nonnegativity.

As a consequence, the original geometric inequality is now proved.

Entanglement and Quantum Mechanics

- Entanglement is a behavior of quantum states, which cannot be explained classically.
- Responsible for many of the non-intuitive properties, and computational power of quantum devices.

A bipartite mixed quantum state ρ is *separable* (not *entangled*) if

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \qquad \sum p_i = 1$$

for some ψ_i, ϕ_i.

Given ρ, how to *decide* and *certify* if it is entangled?

Deciding entanglement

The set of separable states is convex by definition.

We can certify entanglement by using *entanglement witnesses*, linear functionals that are nonnegative in all separable states.

$$\forall \rho_{\mathsf{sep}} \ \langle Z, \rho_{\mathsf{sep}} \rangle \geq 0, \qquad \langle Z, \rho \rangle < 0.$$

The first condition is computationally difficult, since it reduces to nonnegativity of a bihermitian form:

$$\langle Z, xx^* \otimes yy^* \rangle = \sum_{ijkl} Z_{ij,kl} x_i x_j^* y_k y_l^*$$

We can now apply the SOS hierarchies.

The first level corresponds to a well-known criterion (PPT).

The other levels are stronger, can detect *many* entangled states.