11. Semialgebraic Lifting

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- Primal SDP relaxation
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Primal and Dual Formulations So Far

Positivity of one polynomial: does there exist x such that $f(x) < 0$?

- Dual SDP relaxation: f is SOS
- Primal SDP relaxation: lifting

Semialgebraic feasibility: does there exist x such that $f_i(x) \geq 0$ and $h_j(x) = 0$ for all i, j

- Positivstellensatz is exact dual. Finite degree condition is an SDP: does there exist s_i, r_{ij}, t_i such that s_i, r_{ij} is SOS and

$$-1 = s_0 + \sum_i s_i f_i + \sum_{i,j} r_{ij} f_i f_j + \cdots + \sum_i t_i h_i$$

- Questions: what is the dual? It should give a *convex relaxation* of the primal feasible set

Valid Inequalities for the Primal

Does there exist $x \in \mathbb{R}^n$ such that

$$f_i(x) \geq 0 \qquad \text{for all } i = 1, \ldots, m$$

We can add a *parametrized family* of valid inequalities of the form

$$f_i(x)(a_{00} + a_{10}x + a_{01}y + a_{11}xy + \ldots)^2 \geq 0$$
$$(a_{00} + a_{10}x + a_{01}y + a_{11}xy + \ldots)^2 \geq 0$$

- Any vector a of coefficients defines a valid inequality
- The multipliers are *squares*; i.e., extreme rays of the SOS cone

  The Lagrange duality construction forms linear combinations of these, resulting in a dual with *SOS multipliers*

Lifting

We can represent these multipliers as

$$a^T z = \begin{bmatrix} a_{00} & a_{10} & a_{01} & a_{11} & \dots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ \vdots \end{bmatrix}$$

so an equivalent feasibility problem is: does there exist x such that

$$\begin{aligned} f_i(x)(a^T z)^2 &\geq 0 \qquad \text{for all } a, i \\ (a^T z)^2 &\geq 0 \qquad \text{for all } a \end{aligned}$$

now lift; let $Y = zz^T$, then we have

$$(a^T z)^2 = a^T Y a$$

Lifted Problem

The lifted problem is: does there exist $x \in \mathbb{R}^n$ such that

$$\begin{aligned} a^T\big(f_i(x)Y\big)a &\geq 0 \qquad \text{for all } a, i \\ a^TYa &\geq 0 \qquad \text{for all } a \\ Y &= zz^T \end{aligned}$$

Since Y defines a quadratic form, we have equivalently

$$\begin{aligned} f_i(x)Y &\succeq 0 \qquad \text{for all } i \\ Y &\succeq 0 \\ Y &= zz^T \end{aligned}$$

Example

suppose $f(x) = x^2 + 3x + 1$; does there exist x such that $f(x) < 0$?

Apply the lifting

$$Y = \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T = \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix}$$

Then

$$f(x)Y = \begin{bmatrix} x^2 + 3x + 1 & x^3 + 3x^2 + x & x^4 + 3x^3 + x^2 \\ & x^4 + 3x^3 + x^2 & x^5 + 3x^4 + x^3 \\ & & x^6 + 3x^5 + x^4 \end{bmatrix}$$

$$= \begin{bmatrix} Y_{13} + 3Y_{12} + Y_{11} & Y_{23} + 3Y_{13} + Y_{12} & Y_{33} + 3Y_{23} + Y_{13} \\ \iddots & \iddots & * \\ \iddots & * & * \end{bmatrix}$$

Primal SDP Relaxation

Relaxing the constraint $Y = zz^T$, we have the SDP

$$
\begin{aligned}
Y &\text{ is Hankel} \\
Y_{11} &= 1 \\
Z &= \begin{bmatrix} Y_{13} + 3Y_{12} + Y_{11} & Y_{23} + 3Y_{13} + Y_{12} \\ Y_{23} + 3Y_{13} + Y_{12} & Y_{33} + 3Y_{23} + Y_{13} \end{bmatrix} \\
Z &\succeq 0 \\
Y &\succeq 0
\end{aligned}
$$

- We have relaxed the valid inequality $f(x)Y \succeq 0$ to positivity of its principal 2×2 submatrix
- We can include as many monomials z as we like

SDP Dual

The SDP dual is: does there exist α, λ, P such that

$$\begin{bmatrix} -\alpha & 0 & -\lambda \\ 0 & 2\lambda & 0 \\ -\lambda & 0 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2P_{11} & 3P_{11} + 2P_{12} & P_{11} + 6P_{12} + P_{22} \\ & 0 & 2P_{12} + 3P_{22} \\ & & P_{22} \end{bmatrix} \succeq 0$$

$$P \succeq 0$$

$$\alpha > 0$$

To interpret this, multiply left and right by z^T and z, giving

$$-\alpha - (x^2 + 3x + 1)(P_{11} + 2P_{12}x + P_{22}x^2) \text{ is SOS}$$

$$(P_{11} + 2P_{12}x + P_{22}x^2) \text{ is SOS}$$

that is

$$\boxed{-\alpha = s_0 + s_1 f}$$

Positivstellensatz and Duality

We have the Positivstellensatz refutation

$$-\alpha = s_0 + \sum_i s_i f_i$$

- *Dual SDP relaxation:* express the SOS constraints as SDP constraints
- *Primal SDP relaxation:* relax the *lifting*

$$\begin{aligned} f_i(x) Y &\succeq 0 \quad \text{for all } i \\ Y &\succeq 0 \\ Y_{11} &= 1 \\ Y &= \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix}^T \end{aligned}$$

Convex Relaxation of Semialgebraic Sets

Given a semialgebraic set, we have the lifting

$$\begin{aligned} f_i(x)Y &\succeq 0 \\ Y &\succeq 0 \\ Y_{11} &= 1 \\ Y &= \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix} \begin{bmatrix} 1 \\ x \\ \vdots \end{bmatrix}^T \end{aligned}$$

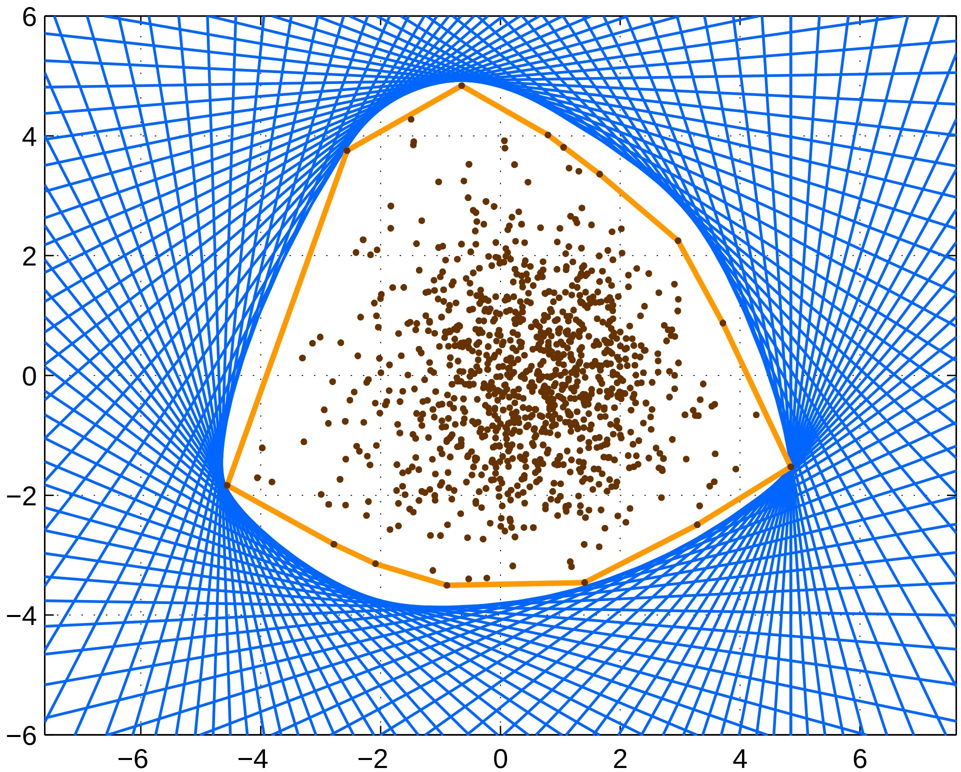
- Projecting the feasible set onto the space spanned by x gives a convex relaxation of the original semialgebraic set
- We don't need to compute the projection explicitly
- To tighten the relaxation, include more monomials in Y – equivalently, increase the degree of the multipliers in the refutation

The Cut Polytope

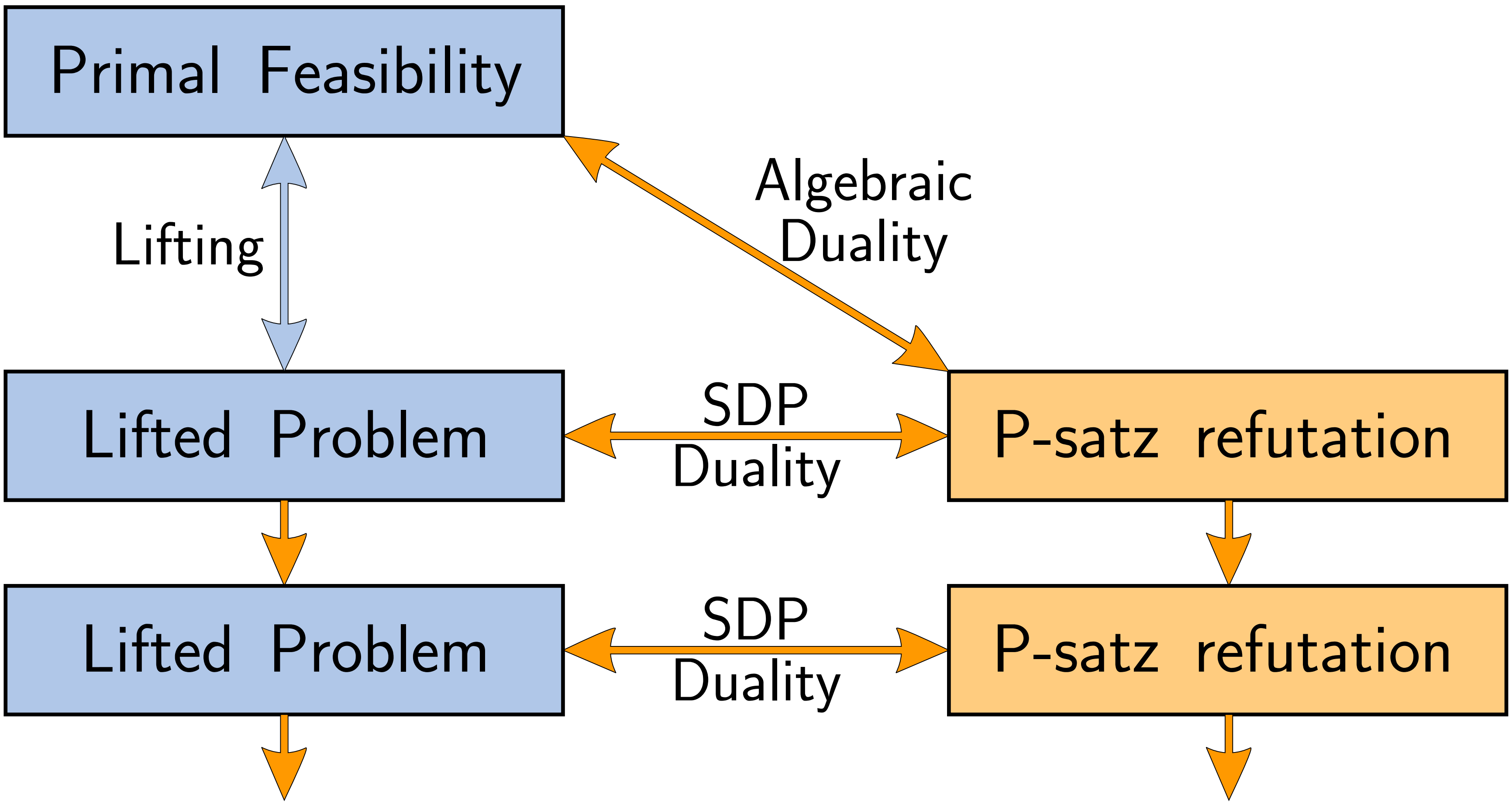
The feasible set of the MAXCUT problem is

$$C = \{ \, X \in \mathbb{S}^n \mid X = vv^T, \; v \in \{-1, 1\}^n \, \}$$

A simple SDP relaxation gives the outer approximation to its convex hull
Here $n = 11$; the set has affine dimension 55; a projection is shown below



A General Scheme



Distinguished Representations

We have a basic semialgebraic S

$$S = \left\{ x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \right\}$$

Which polynomials are non-negative on S?

- Every polynomial in $\mathbf{cone}\{g_1, \ldots, g_m\}$ is non-negative on S
- But are there others? Recall radicality of ideals.

The Positivstellensatz gives an exact test, since $f(x) \geq 0$ for all $x \in S$ iff

$$\left\{ x \in \mathbb{R}^n \mid f(x) < 0, g_i(x) \geq 0 \right\} \text{ is empty}$$

Distinguished Representations

If S is *compact*, then Schmüdgen showed

$$f(x) > 0 \text{ for all } x \in S \qquad \Longrightarrow \qquad f \in \mathbf{cone}\{g_1, \ldots, g_m\}$$

- More explicitly, this means

$$f = s_0 + \sum_i s_i g_i + \sum_{i,j} r_{ij} g_i g_j + \cdots$$

  for some SOS polynomials $s_i, r_{ij}, \ldots$

- Also notice

$$f(x) \geq 0 \text{ for all } x \in S \qquad \Longleftarrow \qquad f \in \mathbf{cone}\{g_1, \ldots, g_m\}$$

Certificate of Positivity

The Positivstellensatz implies $f(x) \geq 0$ on S if and only if

$$sf = 1 + s_0 + \sum_i s_i g_i + \sum_{i,j} r_{ij} g_i g_j + \cdots$$

- Schmüdgen's distinguished representation implies that, to prove *strict* positivity, one can assume the multiplier s is a nonnegative constant
- i.e., one can *prove* positivity using *fewer axioms*. Consequently
  - proofs may become longer
  - need assumptions on S
- So we can *fix* the multiplier s, without *theoretical* loss, but this may require *higher degree* certificates
- Theoretical justification for optimization of polynomials over compact domains; e.g., Lyapunov stability in a basin of attraction

Reducing the Axiom Set

If there is a *single* polynomial g_k such that

$$\left\{ x \in \mathbb{R}^n \mid g_k(x) \geq 0 \right\} \text{ is compact}$$

then Putinar's result holds:

$$f(x) > 0 \text{ for all } x \in S \quad \Longrightarrow \quad f = s_0 + \sum_i s_i g_i \text{ for some SOS } s_i$$

- Stronger assumptions about S mean we can reduce axiom set further; we don't need to take products

Handelman Representations

Suppose that S is defined by *linear inequalities*

$$S = \left\{ x \in \mathbb{R}^n \mid b - Ax \geq 0 \right\}$$

and S is compact, with nonempty interior.

Then, if $f(x) > 0$, we have for $W \subset \mathbb{N}^m$

$$f = \sum_{\alpha \in W} c_\alpha \prod_{i=1}^{m} (b_i - a_i^T x)^{\alpha_i} \quad \text{for some } c_\alpha > 0$$

- No SOS polynomials, just constants c_α. Hence solvable using LP
- But proofs may be extremely long

Distinguished Representations

	Products	No products
SOS coefficients	Schmüdgen *compactness*	Putinar *compactness++*
Scalar coefficients	Handelman *compactness* *linear inequalities*	Lagrange *convexity* *constraint qualifications*

- *Strong duality* results
- Positivstellensatz requires no assumptions
- Tradeoffs between computation, assumptions, and proof lengths