THE NONABELIAN HODGE CORRESPONDENCE

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Abstract. We review the basics of the Hodge structures and variations of Hodge structures, including a proof of Griffiths transversality. We then explain the nonabelian Hodge correspondence, which is a correspondence between *Higgs bundles* and representations of the fundamental group of a compact Kähler manifold.

1. The Hodge decomposition and Hodge structures

Let us first recall the Hodge decomposition. For a d-dimensional smooth projective variety $X/\mathbb{C}$, there is a canonical decomposition of the (algebraic) de Rham cohomology

$$H^n_{\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X).$$

There are several perspectives on the decomposition:

- (complex-analytic) letting $X^{\mathrm{an}} := X(\mathbb{C})$ viewed as a compact Kähler variety, by GAGA (for more details, see [Gro66]), the algebraic and analytic de Rham cohomology coincides: $H^n_{\mathrm{dR}}(X,\mathbb{C}) = H^n_{\mathrm{dR}}(X^{\mathrm{an}},\mathbb{C})$. Then by the theory of harmonic forms,

$$H^n_{\mathrm{dR}}(X^{\mathrm{an}},\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X^{\mathrm{an}}), \tag{1.1}$$

  where $H^{p,q}(X^{\mathrm{an}})$ is the subgroup of cohomology classes which are represented by (p,q)-forms, i.e., they can locally be written as finite sums of the form

$$f dz_1 \wedge \cdots \wedge dz_p \wedge \overline{dw_1} \wedge \cdots \wedge \overline{dw_q}.$$

- (algebraic) recall that the algebraic de Rham cohomology is the hypercohomology of the complex of sheaves $\Omega^\bullet = [\mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d_X]$. It has a Hodge filtration,

$$F^p\Omega^\bullet = [0 \to \cdots 0 \to \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \xrightarrow{d} \cdots]. \tag{1.2}$$

  The spectral sequence attached to this filtration is the *Hodge to de Rham spectral sequence*

$$E_1^{p,q} = H^q(X,\Omega^p_X) \implies H^{p+q}_{\mathrm{dR}}(X/\mathbb{C}), \tag{1.3}$$

  which degenerates at the E_1-page, which gives a decomposition

$$H^n_{\mathrm{dR}}(X,\mathbb{C}) = \bigoplus_{p+q=n} H^q(X,\Omega^p_X).$$

Note that since in the complex-analytic setting, the analytic de Rham complex

$$\Omega^\bullet_{X^{\mathrm{an}}} = [\mathcal{O}_{X^{\mathrm{an}}} \xrightarrow{d} \Omega_{X^{\mathrm{an}}} \xrightarrow{d} \cdots]$$

is quasi-isomorphic to the constant sheaf $\underline{\mathbb{C}}$ sitting in degree zero, the de Rham cohomology is isomorphic to the singular cohomology (the de Rham theorem):

$$H^n_{\mathrm{dR}}(X,\mathbb{C}) \simeq H^n_{\mathrm{sing}}(X,\mathbb{C}).$$

Thus we in fact have an abelian group $H^n = H^n_{\text{sing}}(X, \mathbb{Z})$ whose base change $H^n_{\mathbb{C}} := H^n \otimes_{\mathbb{Z}} \mathbb{C}$ decomposes as a direct sum as in (1.1). Our goal is to provide an *abstract framework* to work with such decompositions.

Definition 1.1. A *(pure) Hodge structure of weight* n is an abelian group H and a direct sum decomposition known as a *Hodge decomposition*

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}$$

into complex subspaces $H^{p,q}$ such that $\overline{H^{p,q}} = H^{q,p}$.

However, the following equivalent definition will behave better in families:

Definition 1.2. A *(pure) Hodge structure of weight* n is an abelian group H and a finite[1] decreasing filtration known as a *Hodge filtration*

$$\cdots \subset F^{p+1}H_{\mathbb{C}} \subset F^p H_{\mathbb{C}} \subset F^{p-1}H_{\mathbb{C}} \subset \cdots,$$

such that for any p,

$$H_{\mathbb{C}} = F^p H_{\mathbb{C}} \oplus \overline{F^{n-p+1} H_{\mathbb{C}}}.$$

Remark 1.3. Given a Hodge decomposition, we obtain a Hodge filtration by

$$F^p H_{\mathbb{C}} := \bigoplus_{i \geq p} H^{i,n-i}.$$

Conversely, given a Hodge filtration, we obtain a Hodge decomposition by

$$H^{p,q} := F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}}. \tag{1.4}$$

Remark 1.4. Hodge filtrations behave better in families because it directly comes from the spectral sequence (1.3);

$$F^p H_{\mathbb{C}} := \bigoplus_{i \geq p} H^{n-i}(X, \Omega^i_X)$$

is simply the image of the hypercohomology of the Hodge filtration $F^p\Omega^\bullet_{X^{\text{an}}}$ from (1.2) in $H^n_{\text{dR}}(X, \mathbb{C})$.

There is a natural notion of a morphism of Hodge structures:

Definition 1.5. Let (H, F) and (H', F) be two Hodge structures of weight n. Then a homomorphism $f\colon H \to H'$ is a *homomorphism of Hodge structures* when

$$f(F^p H_{\mathbb{C}}) \subset F^p H'_{\mathbb{C}}.$$

Remark 1.6. By (1.4) a homomorphism of Hodge structures automatically also satisfies $f(H^{p,q}) \subset H^{p,q}$. However, for complex Hodge structures (i.e., without choosing an integral form H of $H_{\mathbb{C}}$) this need not hold.

1.7. **Polarization of Hodge structures.** The cohomology of compact Kähler manifolds carry an additional piece of structure, which is abstractly the following:

Definition 1.8. A *polarization* on a Hodge structure (H, F) of weight n is a bilinear form S on $H_{\mathbb{C}}$ such that:

- if n is even S is symmetric, and if n is odd S is anti-symmetric;
- when $p + p' \neq n$,
$$S(H^{p,n-p}, H^{p',n-p'}) = 0,$$
- if $v \in H^{p,n-p}$ is nonzero,
$$\sqrt{-1}^{\,n-2p} S(v, \overline{v}) > 0.$$

[1] i.e., such that for $p \gg 0$, $F^p H_{\mathbb{C}} = 0$ and $F^{-p} H_{\mathbb{C}} = H_{\mathbb{C}}$.

A polarization on the cohomology of compact Kähler mainfolds is provided by the Hodge-Riemann form:

Theorem 1.9. *Let (X,ω) be a d-dimensional Kähler manifold. The* Hodge-Riemann form *is the bilinear form*

$$S(\alpha,\beta) := (-1)^{n(n-1)/2}\int_X \alpha\wedge\beta\wedge\omega^{d-n}$$

for $[\alpha],[\beta]\in H^n_{\mathrm{dR}}(X,\mathbb{C})$. Then S provides a polarization of the Hodge structure on $H^n_{\mathrm{dR}}(X,\mathbb{Z})$.

Remark 1.10. Given a complex variation of Hodge structures, we may not be able to recover $H^{p,q}$ from F^pH; we cannot use (1.4) since $\overline{F^qH}$ does not make sense. But given a polarization, we may let

$$H^{p,q} = F^pH \cap (F^{p+1}H)^{\perp} = \bigoplus_{i\geq p} H^{i,n-i} \cap \bigoplus_{i<p+1} H^{i,n-i}.$$

2. Variations of Hodge structures

Given a smooth morphism $f\colon X\to Y$ of smooth proper varieties over $\mathbb{C}$, the pushforward

$$R^nf_*\underline{\mathbb{Z}}_{X^{\mathrm{an}}}$$

is a local system on Y^{an}, whose stalk at $x\in X(\mathbb{C})$ is the cohomology $H^n(f^{-1}(x)^{\mathrm{an}},\mathbb{Z})$, each of which carries a Hodge structure of weight n by our discussion above. Thus, we want a notion of Hodge structures for local systems on varieties. Let us work with local systems over $\mathbb{C}$ from now on:

Definition 2.1. A *variation of Hodge structures* on a complex manifold X is a $\mathbb{C}$-local system L on X or by Riemann-Hilbert, a vector bundle $V = L\otimes_{\mathbb{C}}\mathcal{O}_X$ with a flat connection $\nabla\colon V\to V\otimes_{\mathcal{O}_X}\Omega^1_X$, together with a decreasing Hodge filtration $F^\bullet V$ on V, such that:

- the filtration induces a Hodge structure of weight n on each stalk of V; and
- *Griffiths transversality* is satisfied, i.e.,

$$\nabla(F^pV)\subset F^{p-1}V\otimes\Omega^1_X.$$

The variation of Hodge structures is furthermore *polarized* if there is a morphism of local systems

$$S\colon L\otimes L\to\underline{\mathbb{C}}$$

which give a polarization of Hodge structures at each stalk.

Then, as desired, we have:

Proposition 2.2. *Given a smooth proper morphism $f\colon X\to Y$ of smooth varieties over $\mathbb{C}$, the pushforward $R^nf_*\underline{\mathbb{C}}_{X^{\mathrm{an}}}$ is a variation of Hodge structures on Y.*

Proof (from [PS08, Corollary 10.31]). The Gauss-Manin connection on $R^nf_*\underline{\mathbb{C}}_{X^{\mathrm{an}}}$ can be described as follows: $\Omega^\bullet_{X^{\mathrm{an}}}$ has a *Koszul filtration*

$$\mathrm{Koz}^q\Omega^\bullet_{X^{\mathrm{an}}} = f^*\Omega^q_{Y^{\mathrm{an}}}\wedge\Omega^{\bullet-q}_{X^{\mathrm{an}}},$$

whose graded pieces are

$$\mathrm{gr}^q_{\mathrm{Koz}}\,\Omega^\bullet_{X^{\mathrm{an}}}\simeq f^*\Omega^q_{Y^{\mathrm{an}}}\otimes_{\mathcal{O}_{X^{\mathrm{an}}}}\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}[-q].$$

Now there is a short exact sequence of complexes

$$0\to f^*\Omega^1_{Y^{\mathrm{an}}}\otimes\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}[-1]\to\mathrm{Koz}^0/\mathrm{Koz}^2\to\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}\to 0, \tag{2.1}$$

and the Gauss-Manin connection is the connecting homomorphism by obtaining Rf_*:

$$\delta\colon R^nf_*\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}\to R^{n+1}f_*(f^*\Omega_{Y^{\mathrm{an}}}\otimes\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}[-1])\simeq\Omega_{Y^{\mathrm{an}}}\otimes R^nf_*\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}.$$

Upon passing to the *filtration bête* (stupid filtration) $\sigma^{\geq p}$ in (2.1), we have a short exact sequence

$$0 \to f^*\Omega^1_{Y^{\mathrm{an}}} \otimes \sigma^{\geq p-1}\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}}[-1] \to \sigma^{\geq p}(\mathrm{Koz}_0/\mathrm{Koz}_2) \to \sigma^{\geq p}\Omega^\bullet_{X^{\mathrm{an}}/Y^{\mathrm{an}}} \to 0,$$

which shows δ maps F^p into $F^{p-1} \otimes \Omega^1_{Y^{\mathrm{an}}}$. □

If variations of Hodge structures are generalizations of Hodge filtrations, the following is the generalization of Hodge decompositions:

Definition 2.3. A holomorphic vector bundle V on a complex manifold X is a *system of Hodge bundles* if there is a decomposition

$$V = \bigoplus_{p+q=n} V^{p,q},$$

together with homomorphisms $\theta\colon V^{p,q} \to V^{p-1,q+1} \otimes \Omega^1_X$ such that $\theta \wedge \theta = 0$.

Example 2.4. When $f\colon X \to Y$ is a smooth proper morphism of smooth varieties, we simply have $E^{p,q} = R^q f_* \Omega^p_{X/Y}$ and θ_y is the cup product with the Kodaira-Spencer deformation class

$$\eta_y \in \mathrm{Hom}\left(T_{Y,y}, H^1(X_y, T_{X_y})\right) \simeq H^1(X_y, T_{X_y}) \otimes \Omega^1_{Y,y}.$$

This is in fact an example of a Higgs bundle, which is the key player in the following story about nonabelian Hodge theory:

Definition 2.5. A *Higgs bundle* on a smooth variety $X/\mathbb{C}$ is one of the following equivalent pieces of data:

- a vector bundle V and a one-form $\theta\colon V \to V \otimes \Omega^1_X$ such that $\theta \wedge \theta = 0$;
- a vector bundle V on X with an action of $\mathrm{Sym}^\bullet TX$; or
- a coherent sheaf $\mathcal{V}$ on $T^*X = \mathbf{Spec}_X(\mathrm{Sym}^\bullet TX)$ such that $V = f_*\mathcal{V}$ is a vector bundle on X.

3. Simpson's correspondence

Let X be a compact Kähler manifold. Then Hodge decomposition gives an isomorphism

$$\mathrm{Hom}(\pi_1(X), \mathbb{C}) \simeq H^1(X, \mathbb{C}) \simeq H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X), \tag{3.1}$$

so homomorphisms $\pi_1(X) \to \mathbb{C}$ can be thought of as pairs of a cohomology class $e \in H^1(X, \mathcal{O}_X)$ and a holomorphic 1-form $\xi \in H^0(X, \Omega^1_X)$. The nonabelian Hodge correspondence is analogous: n-dimensional representations of $\pi_1(X)$

$$\mathrm{Hom}(\pi_1(X), \mathrm{GL}_n(\mathbb{C})) \simeq H^1(\pi_1(X), \mathrm{GL}_n(\mathbb{C}))$$

are (roughly) classified by a cohomology class in $H^1(X, \mathrm{GL}_n(\mathcal{O}_X))$, i.e., a n-dimensional vector bundle V on X, with a endomorphism-valued one-form θ—this is exactly the data of a Higgs bundle.

For a more precise statement, we restrict our attention to stable Higgs bundles:

Definition 3.1. A Higgs bundle V on X is *stable* (resp., *semi-stable*) if for any subsheaf U preserved by θ and a hyperplace section H,

$$\frac{\chi(U(nH))}{\mathrm{rk}(U)} < \frac{\chi(V(nH))}{\mathrm{rk}(V)} \text{ (resp., } \frac{\chi(U(nH))}{\mathrm{rk}(U)} \leq \frac{\chi(V(nH))}{\mathrm{rk}(V)})$$

for $n \gg 0$.

Remark 3.2. In general stable vector bundles behave better in families, so they are better when considering moduli spaces. For example, the vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbf{P}^1$ is not stable since $\mathcal{O}(1)$ is a subsheaf with higher slope. They are problematic since there is a family of exact sequences

$$0 \to \mathcal{O}(-1) \to E_t \to \mathcal{O}(1) \to 0$$

where E_t corresponds to the extension class $t \in H^1(\mathbf{P}^1, \mathcal{O}(-2)) \simeq \mathbb{C}$. Generically $E_t \simeq \mathcal{O}^{\oplus 2}$ has trivial Chern classes, but at $t = 0$ the vector bundle $E_0 = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ has $c_2 = -1$.

Example 3.3. When V is an irreducible polarized complex variation of Hodge structures, the associated system of Hodge bundles $V = \bigoplus_{p+q=n} V^{p,q}$ is stable.

Now, the nonabelian Hodge theorem states:

Theorem 3.4 ([Sim91, Theorem 1])**.** *There are bijections:*

$$\{\textit{stable Higgs bundle with trivial Chern classes}\} \simeq \{\textit{irreducible representations of } \pi_1(X)\}$$
$$\{\textit{semistable Higgs bundle with trivial Chern classes}\} \simeq \{\textit{representations of } \pi_1(X)\}.$$

Remark 3.5. Formally, the theorem looks very similar to the Riemann-Hilbert correspondence: representations of $\pi_1(X)$ (or equivalently, $\mathbb{C}$-local systems L on X) correspond to a vector bundle $V = L \otimes \mathcal{O}_X$ with a flat section $\nabla\colon V \to V \otimes \Omega^1_X$. *However*, note that for Higgs bundles $\theta\colon V \to V \otimes \Omega^1_X$ is required to be $\mathcal{O}_X$-linear, i.e.,

$$\theta(fv) = f\theta(v),$$

while for connections we require

$$\nabla(fv) = f\nabla(v) + v \otimes df.$$

Alternatively, Riemann-Hilbert is a correspondence between representations of $\pi_1(X)$ with D_X-modules, while nonabelian Hodge theory predicts a correspondence with modules of

$$\operatorname{gr} D_X \simeq \operatorname{Sym}(TX).$$

In the construction of the nonabelian Hodge correspondence, one starts with the vector bundle $V = L \otimes \mathcal{O}_X$, and keep the topological vector bundle, but modify the complex structure on it.

Example 3.6. Let us focus our attention on $\mathbb{G}_a$-Higgs bundles, i.e., two-dimensional unipotent Higgs bundles. Then the nonabelian Hodge correspondence predicts a bijection between:

- semistable rank two Higgs bundles V sitting in a short exact sequence
$$0 \to \mathcal{O} \to V \to \mathcal{O} \to 0,$$
which as a vector bundle is classified by a class in $H^1(X, \mathcal{O}_X)$ and the Higgs field $\theta\colon V \to V \otimes \Omega^1_X$ must descend to $\theta\colon \mathcal{O}_X \to \Omega^1_X$, i.e., $\theta \in H^0(X, \Omega^1_X)$ where the condition $\theta \wedge \theta = 0$ is automatic; and
- representations of $\pi_1(X)$ sitting in a short exact sequence
$$0 \to \mathbb{C}_{\mathrm{triv}} \to \pi_1(X) \to \mathbb{C}_{\mathrm{triv}} \to 0,$$
i.e., classified by $\operatorname{Hom}(\pi_1(X), \mathbb{C})$.

This correspondence is exactly the one described in (3.1).

Example 3.7. For line bundles, the nonabelian Hodge correspondence predicts a bijection between:

- line bundles L with trivial Chern class, classified by a class in
$$\operatorname{Pic}^0(X) := \ker\left(H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})\right) \simeq H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}),$$
with a Higgs field $\theta\colon L \to L \otimes \Omega^1_X$. By twisting away the line bundle, we simply have $\theta \in H^0(X, \Omega^1_X)$; and

- homomorphisms $\pi_1(X) \to \mathbb{C}^\times$, which are classified by

$$\mathrm{Hom}(\pi_1(X), \mathbb{C}^\times) \simeq H^1(X, \mathbb{C})/H^1(X, \mathbb{Z}).$$

This is given by the isomorphism

$$H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z}) \times H^0(X, \Omega^1_X) \simeq H^1(X, \mathbb{C})/H^1(X, \mathbb{Z})$$

by quotienting the usual isomorphism $H^1(X, \mathbb{C}) \simeq H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)$ by a lattice $H^1(X, \mathbb{Z})$. Note that in particular the isomorphism does not preserve the complex structure.

We outline several special important cases of Theorem 3.4. When $\theta = 0$, the trivial Higgs field, we have:

Theorem 3.8 (Hitchin-Kobayashi, Narasimhan-Seshadri)**.** *There are bijections*

$$\{\textit{stable vector bundle with trivial Chern classes}\} \simeq \{\textit{unitary irreducible representations of } \pi_1(X)\}$$
$$\{\textit{semistable vector bundle with trivial Chern classes}\} \simeq \{\textit{unitary representations of } \pi_1(X)\},$$

which associates to a unitary local system L on X the vector bundle $L \otimes \mathcal{O}_X$.

Remark 3.9. A key point of the theorem is that given a unitary local system L on X, the vector bundle $L \otimes \mathcal{O}_X$ is stable. When X is a curve, this follows from the following argument. Suppose $E \subset L \otimes \mathcal{O}_X$ is a positive-degree sub-bundle. By taking wedge powers we may assume E is a line bundle. Then by taking finite étale covers $f\colon X' \to X$, the cohomology group $\Gamma(X', f^*E)$ grows arbitrarily, but by Proposition 3.10

$$\Gamma(X', f^*(L \otimes \mathcal{O}_X)) \simeq \Gamma(X', f^*L), \tag{3.2}$$

which has bounded dimension, a contradiction.

Proposition 3.10 ([NS64])**.** *Let X be a compact Riemann surface and let L be a local system with unitary monodromy. Then*

$$\Gamma(X, L) \simeq \Gamma(X, L \otimes \mathcal{O}_X).$$

Proof. Choose a point $x \in X$. A section of L is simply a $\pi_1(X, x)$-equivariant function $f\colon \widetilde{X} \to L_x$, where $\widetilde{X} \to X$ is the universal cover. Then L_x carries a $\pi_1(X, x)$-equivariant hermitian form $\|\cdot\|$. Now $\|f\|$ will be a subharmonic function on X, which must be constant since X is compact. Thus f is constant. □

Remark 3.11. For Proposition 3.10, unitarity is important. Indeed, if $E/\mathbb{C}$ is an elliptic curve, it has a Tate uniformization $E \simeq \mathbb{C}^\times/\langle q^{\mathbb{Z}}\rangle$, and it has a line bundle L_q with monodromy q. Then the holomorphic function z on $\mathbb{C}^\times$ descends to a nonconstant section in $H^0(E, L_q)$.

In the special case of Example 3.3 (so the local system on X additionally has a Hodge structure, and the Higgs bundle comes from a system of Hodge bundles), we have:

Theorem 3.12. *There is a bijection between:*

- *irreducible polarized complex variations of Hodge structures; and*
- *stable systems of Hodge bundles with vanishing Chern classes.*

4. Interpretations in terms of moduli spaces

Our goal now is to re-write Theorem 3.4 in the language of (coarse) moduli spaces. We let M_{Rep} be the quotient stack

$$M_{\mathrm{Rep}} = \mathrm{Hom}(\pi_1(X), \mathrm{GL}_n)/\mathrm{GL}_n,$$

where GL_n acts on $\mathrm{Hom}(\pi_1(X), \mathrm{GL}_n)$ by conjugation, which is well-behaved since $\pi_1(X)$ is a finitely generated group. The moduli space of Higgs bundles is more subtle.

Recall that we may view a Higgs bundle (V, θ) as a coherent sheaf $\mathcal{V}$ on $T^*X = \mathbf{Spec}_X(\mathrm{Sym}^\bullet TX)$. A point $(x, \xi) \in T^*X$ is in the support of $\mathcal{V}$ if and only if the linear function $T_xX \to \mathbb{C}$ is an eigenvalue of the action of θ_x on E_x. Now $\mathcal{V}$ will be finite and flat over $\mathcal{O}_X$, so in particular $\mathrm{supp}(\mathcal{V})$ will be $d = \dim(X)$-dimensional.

Definition 4.1. For a projective variety Z with ample line bundle $\mathcal{O}_Z(1)$, a coherent sheaf $\mathcal{E}$ on Z has *pure dimension* d if $\dim(\mathrm{supp}(\mathcal{F})) = d$ for every subsheaf $\mathcal{F} \subset \mathcal{E}$.

Now we use the following general fact:

Proposition 4.2. *Let Z be a projective variety with ample line bundle $\mathcal{O}_Z(1)$. Then there is a projective variety $M_Z(P)$ which is a coarse moduli space for semistable coherent sheaves of pure dimension d and Hilbert polynomial P, up to Jordan equivalence.*[2]

For us let Z be the projective closure of T^*X, i.e.,

$$Z = \mathbf{Proj}_X(\mathrm{Sym}^\bullet TX \otimes \mathbb{C}[t]).$$

Then there is a ample $\mathcal{O}_Z(1)$ whose restriction to T^*X is the pullback of $\mathcal{O}_X(1)$. Then, the Higgs bundles on X are coherent sheaves $\mathcal{E}$ of pure dimension d on Z whose support is contained in T^*X, and the vanishing Chern classes means $\mathcal{E}$ has the appropriate Hilbert polynomial P. Thus the moduli space M_{Higgs} of Higgs bundles with trivial Chern classes on X is an open subset of the moduli space $M_Z(P)$.

Now, we have the following version of Theorem 3.4:

Theorem 4.3. *For any smooth projective variety $X/\mathbb{C}$ there is a real-analytic isomorphism of coarse moduli spaces*

$$M_{\mathrm{Higgs}} \simeq M_{\mathrm{Rep}}.$$

The formalism of the moduli spaces allows us to prove the following:

Corollary 4.1. *Any representation of $\pi_1(X)$ may be deformed to a representation underlying a polarized complex variation of Hodge structure. In particular, a rigid representation (i.e., an isolated point in the moduli space) must already underly a complex variation.*

Proof. There is an action of $\mathbb{C}^\times$ on M_{Higgs}:

$$(E, \theta) \mapsto (E, t\theta),$$

and the points which underly a system of Hodge bundles may be characterized as the fixed points. Now although M_{Higgs} is not compact, $E_0 := \lim_{t\to 0} tE$ exists, and is a $\mathbb{C}^\times$-fixed point. The points of M_{Higgs} which underly a system of Hodge bundles exactly correspond to representations of $\pi_1(X)$ underlying a complex variation of Hodge structures by Theorem 3.12. □

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