MILNOR K-THEORY AND THE BLOCH-GABBER-KATO THEOREM

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ABSTRACT. An exposition on the proof of Bloch-Gabber-Kato theorem, relating Milnor K-theory and the module of differentials, mostly following [GS17].

CONTENTS

1. SHORT REVIEW OF MILNOR K-THEORY

Milnor K-theory plays a central role in number theory:

Definition 1 (Milnor K-theory [Mil70]). For a field F, define the graded ring

$$K_*^M(F) := \mathbb{Z} \oplus F^\times \oplus (F^\times)^{\otimes 2} \oplus \cdots / \langle a \otimes (1-a) : a \neq 0, 1 \in F^\times \rangle.$$

The relations $a \otimes (1-a) = 0$ are called the *Steinberg relations*. Let $K_j^M(F)$ be the j-th graded piece of the ring. The element $a_1 \otimes \cdots \otimes a_j \in K_*^M(F)$ is denoted as $\{a_1, \ldots, a_j\}$.

We will mainly be interested in Milnor K-theory when F has positive characteristic.

Example 1.1. $K_0^M(F) = \mathbb{Z}, K_1^M(F) = F^\times$, since there are no relations in degrees 0 and 1.

Example 1.2 ([Mil70, Example 1.5]). If $F = \mathbb{F}_q$ is a finite field, then $K_\bullet^M(F) = \mathbb{Z} \oplus F^\times$, i.e., $K_2^M(F) = 0$. Indeed, $F^\times$ is cyclic of order $q-1$, so it has $\varphi(q-1) \geq \frac{q-1}{2}$ multiplicative generators, where φ is Euler's totient function. Thus, by the pigeonhole principal the two subsets of the size $q-2$ set $F \setminus \{0,1\}$

$$\{g : g \in F^\times \text{ is a generator}\} \text{ and } \{1-g : g \in F^\times \text{ is a generator}\}$$

intersect, and hence $g_1 = 1 - g_2$ for some generators $g_1, g_2 \in F^\times$. But then $g_1 \otimes g_2 = 1$ by the Steinberg relation, so $K_2^M(F) = 0$.

Milnor K-theory has some functoriality properties:

Proposition-Definition 1 ([Mil70, Lemma 2.1],[GS17, Prop 7.1.4])**.** Let F be a field with a discrete valuation v with residue class field k. There exists a unique homomorphism $\partial = \partial_v$, called the *boundary homomorphism* from K_nF to $K_{n-1}k$ such that

$$\partial_v(\{\pi, u_2, \ldots, u_n\}) = \{\overline{u}_2, \ldots, \overline{u}_n\}$$

for every prime element π, i.e., $\mathrm{ord}_v\pi = 1$ and for all units $u_2, \ldots, u_n$.

Moreover, once a prime element π is fixed, there is a unique homomorphism $s_\pi^M \colon K_n^M(F) \to K_n^M(k)$, called *specialization*, with the property

$$s_\pi^M(\{\pi^{i_1}u_1, \ldots, \pi^{i_n}u_n\}) = \{\overline{u}_1, \ldots, \overline{u}_n\},$$

for all integers $i_1, \ldots, i_n$ and units $u_1, \ldots, u_n$.

Remark 1. In particular, if $u_1, \ldots, u_n$ are units, then $\partial_v(\{u_1, \ldots, u_n\}) = 1$, since

$$\partial_v(\{\pi, u_2, \ldots, u_n\}) = \partial_v(\{\pi u_1, u_2, \ldots, u_n\}),$$

where both π and πu_1 are prime.

The residue map allows for the following proposition, which assists in many computations of Milnor K-theory:

Proposition 1.3 ([Mil70, Thm 2.3])**.** *There is a split exact sequence*

$$0 \to K_n^M(F) \to K_n^M F(t) \xrightarrow{\bigoplus \partial_\pi} \bigoplus K_{n-1}^M F[t]/\pi \to 0,$$

where the direct sum runs over all monic irreducible polynomials $\pi \in F[t]$.

Example 1.4. If $F = \mathbb{F}_q(t)$ is the field of rational functions over the finite field $\mathbb{F}_q$, then

$$K_n^M(F) = \begin{cases} \mathbb{Z} & n = 0 \\ F^\times & n = 1 \\ \bigoplus_{\pi \in \mathbb{F}_q[t]} (\mathbb{F}_q[t]/\pi)^\times & n = 2 \\ 0 & n > 2, \end{cases}$$

where $\pi \in \mathbb{F}_q[t]$ runs through the monic irreducible polynomials. Indeed, by Proposition 1.3 there are exact sequences

$$0 \to K_2(\mathbb{F}_q) = 0 \to K_2(F) \to \bigoplus_{\pi \in \mathbb{F}_q[t]} K_1(\mathbb{F}_q[t]/\pi) = \bigoplus_{\pi \in \mathbb{F}_q[t]} (\mathbb{F}_q[t]/\pi)^\times \to 0$$

and

$$0 \to K_3(\mathbb{F}_q) = 0 \to K_3(F) \to \bigoplus_{\pi \in \mathbb{F}_q[t]} K_2(\mathbb{F}_q[t]/\pi) = 0 \to 0.$$

Milnor K-theory is closely related to cycles, and is "motivic" in nature:

Proposition 1.5 ([NS89, Thm 4.9])**.** *For any field F, there is a natural isomorphism $\mathrm{CH}^j(K, j) \xrightarrow{\sim} K_j^M(F)$.*

When F has characteristic p, the ring $K_\bullet^M(F)$ is also closely related to differentials on F.

1.1. **The norm map.** Let F'/F be a finite extension. Then [GS17, Section 7.3] constructs a family of maps $N_{F'/F}\colon K_n^M(F') \to K_n^M(F)$ such that:

(1) the map $N_{F'/F}\colon K_0^M(F') \to K_0^M(F)$ is multiplication by $[F' : F]$;
(2) the map $N_{F'/F}\colon K_1^M(F') \to K_1^M(F)$ is multiplication by the field norm $N_{F'/F}(a) = \det_F(F' \xrightarrow{m_a} F')$;
(3) for $\alpha \in K_n^M(F)$ and $\beta \in K_m^K(F')$,
$$N_{F'/F}(\{\alpha_{F'}, \beta\}) = \{\alpha, N_{F'/F}(\beta)\},$$
where α_F is the image of α under the natural homomorphism $K_\bullet^M(F) \to K_\bullet^M(F')$; and
(4) for a tower of field extensions $F''/F'/F$,
$$N_{F''/F} = N_{F'/F} \circ N_{F''/F}.$$

2. Background on differentials in characteristic p

For any F-vector space V, let ${}^pV$ denote an alternative F-vector space structure on V, given by $a \cdot w := a^p w$ for any $a \in F$ and $w \in V$.

Consider the module $\Omega_F^n := \Omega_{F/\mathbb{Z}}^n$ of absolute differentials over F. There is a chain complex
$$\Omega_F^\bullet := [\Omega_F^0 \xrightarrow{d^0} \Omega_F^1 \xrightarrow{d^1} \Omega_F^2 \xrightarrow{d^2} \cdots],$$
and let $B_F^n := \operatorname{im}(d^{n-1})$ be the n-cocycles, and let $Z_F^n := \ker(d^n)$ be the n-coboundaries. We may define the cohomology $H^n(\Omega_F^\bullet) := Z_F^n/B_F^n$.

Proposition 2.1 (Cartier, [GS17, Thm 9.4.3])**.** *There is an isomorphism*
$$\gamma\colon \Omega_F^n \to {}^pH^n(\Omega_F^\bullet),$$
defined by $\gamma(da_1 \wedge \cdots \wedge da_n) := a_1^{p-1}da_1 \wedge \cdots \wedge a_n^{p-1}da_n$*, where* $a_i \in F$*.*

Let
$$\nu(n)_F := \ker(\gamma - \mathrm{id}\colon \Omega_F^n \to \Omega_F^n/B_F^n).$$

Example 2.2. If $F = \mathbb{F}_p[x]/f(x)$ is a finite field, where $f(x) \in \mathbb{F}_p[x]$ is irreducible, we have
$$\Omega_F^1 \cong F dx/f'(x)dx \cong \mathbb{F}_p[x]/(f(x), f'(x)) = 0.$$
In particular $\nu(n)_F = 0$ for $n \geq 1$.

Example 2.3. If $F = \mathbb{F}_q(t)$ is the field of rational functions on the finite field $\mathbb{F}_q$, we have $\Omega_F^1 = F dt$, hence $\Omega_F^n = 0$ for $n \geq 2$. Thus, $\nu(n)_F = 0$ for $n \geq 2$. For $n = 1$,
$$\begin{aligned}\nu(1)_F &= \{f(t)dt \in Fdt : f(t)^p t^{p-1}dt \equiv f(t)dt \pmod{B}\} \\ &= \{f(t)dt \in Fdt : f(t)^p t^{p-1} - f(t) = g'(t), \text{ for some } g \in F\}.\end{aligned}$$

We now define
$$\begin{aligned} d\log\colon (F^\times)^{\otimes n} &\to \nu(n)_F \\ a_1 \otimes \cdots \otimes a_n &\mapsto a_1^{-1}da_1 \wedge \cdots \wedge a_n^{-1}da_n.\end{aligned}$$
Here $d\log$ a priori only maps into Ω_F^n, but its image lies in $\nu(n)_F$ since
$$\begin{aligned}\gamma(a_1^{-1}da_1 \wedge \cdots \wedge a_n^{-1}da_n) &= a_1^{-p}a_1^{p-1}da_1 \wedge \cdots \wedge a_n^{-p}a_n^{p-1}da_n \\ &= a_1^{-1}da_1 \wedge \cdots \wedge a_n^{-1}da_n.\end{aligned}$$

This gives a ring homomorphism

$$\mathrm{Tens}_{\mathbb{Z}}(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times)^{\otimes 2} \oplus \cdots \to \nu(\bullet)_F.$$

In fact, $d\log$ factors through Milnor K-theory:

Lemma 2.4 ([GS17, Lem 9.5.1])**.** *The map $d\log$ factors through the quotient $(F^\times)^{\otimes n} \to K_n^M(F)/p$. It thus defines a graded ring homomorphism $\psi_F\colon K_\bullet^M(F)/p \to \nu(\bullet)_F$, sending $\{a_1, \dots, a_n\}$ to $a_1^{-1}da_1 \wedge \cdots \wedge a_n^{-1}da_n$ for every $a_1, \dots, a_n \in F^\times$.*

Proof. We have $p\nu(n)_F \subset p\Omega_F^n = 0$, since $pda = d(pa) = 0$. Thus it suffices to check the Steinberg relations. For $a \neq 0, 1$ we have

$$\begin{aligned} d\log(a \otimes (1-a)) &= d\log(a) \wedge d\log(1-a) \\ &= \frac{1}{a(1-a)} da \wedge d(1-a) \\ &= -\frac{1}{a(1-a)} da \wedge da \\ &= 0. \end{aligned}$$

□

2.1. **Statement of the Bloch-Gabber-Kato theorem.** The following theorem relates the a priori very distinct objects K_n^M and $\nu(n)$:

Theorem 2.5 (Bloch-Gabber-Kato theorem [GS17, Thm 9.5.2])**.** *Let F be a field of characteristic $p > 0$. The following is an isomorphism:*

$$\psi_F\colon K_\bullet^M(F)/p \to \nu(\bullet)_F.$$

Example 2.6. When F is a finite field, this is clear by Examples 1.2 and 2.2. More generally, for perfect fields $F = F^p$ clearly $K_n^M(F) = 0$ and $\nu(n)_F = 0$ for any $n \geq 1$.

Example 2.7. Let $F = \mathbb{F}_q(t)$. Then Example 2.3 gives a description of $\nu(1)_F$, and Theorem 2.5 claims an isomorphism, on the 1-st graded piece,

$$\begin{aligned} \psi_F\colon F^\times/(F^\times)^p &\to \{f(t) \in F : f(t)^p t^{p-1} - f(t) = g'(t) \text{ for some } g \in F\}. \\ u &\mapsto \frac{u'}{u}. \end{aligned}$$

Injectivity is clear, since $u'/u = 0$ implies $u' = 0$, so $u \in F^p = \mathbb{F}_q(t^p)$.

When $n = 1$, Theorem 2.5 is:

Theorem 2.8 (Jacobson and Cartier, [GS17, Thm 9.2.2])**.** *For every field F of characteristic $p > 0$, the sequence*

$$1 \to F^\times \xrightarrow{p} F^\times \xrightarrow{d\log} \Omega_F \xrightarrow{\gamma - 1} {}^p\Omega_F^1/{}^pB_F^1$$

is exact.

In fact, Jacobson and Cartier's theorem is a key ingredient in the proof of Theorem 2.5. We present a proof in Section 4

As a first step to prove Theorem 2.5, we have the following functoriality property:

Lemma 2.9 ([GS17, Lem 9.5.4]). *Given a finite separable extension F'/F, the diagram*

$$\begin{array}{ccc} K^M_\bullet(F')/p & \xrightarrow{\psi_{F'}} & \nu(\bullet)_{F'} \\ {\scriptstyle N_{F'/F}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\ K^M_\bullet(F)/p & \xrightarrow{\psi_F} & \nu(\bullet)_F \end{array}$$

commutes. Here, the homomorphism $\mathrm{tr}\colon \Omega^n_{F'} \to \Omega^n_F$ *is given by the composition*

$$\Omega^n_{F'} \cong F' \otimes_F \Omega^n_F \xrightarrow{\mathrm{tr}\otimes 1} F \otimes_F \Omega^n_F = \Omega^n_F.$$

Remark 2. The lemma allows us to reduce Theorem 2.5 to finitely-generated extensions $F/\mathbb{F}_p$, since any field F can be written as a colimit of finitely-generated extensions.

3. A digression—the motivic story

Since Theorem 2.5 describes $K^M_\bullet(F)/p$ when F has characteristic p, we also mention the story of Milnor K-theory away from the characteristic of p, i.e., $K^M_\bullet(F)/\ell$ where $\ell \neq 0 \in F$.

3.1. **The norm residue theorem.** Let F be a field and $\ell > 0$ be an integer such that $\ell \in F^\times$. The short exact sequence

$$1 \to \mu_\ell \to \overline{F}^\times \xrightarrow{\ell} \overline{F}^\times \to 1$$

where $\overline{F}$ is the separable closure of F gives rise to the isomorphism $F^\times/\ell \to H^1_{\text{ét}}(F, \mu_\ell)$. By cup products, we have a homomorphism $\partial\colon (F^\times)^{\otimes q}/\ell \to H^q_{\text{ét}}(F, \mu_\ell^{\otimes q})$. Moreover, since $\partial(x \otimes (1-x)) = 0$ for any $x \neq 0, 1$, we obtain a homomorphism $K^M_q(F)/\ell \to H^q_{\text{ét}}(F, \mu_\ell^{\otimes q})$.

Theorem 3.1 (Norm residue theorem, Voevodsky). *For any field F and an integer $\ell \in F$ invertible, $\partial\colon K^M_q(F)/\ell \to H^q_{\acute{e}t}(F, \mu_\ell^{\otimes q})$ defined above is an isomorphism.*

In fact, given a smooth scheme X/F, there is an object $\mathbb{Z}(j)^{\text{ét}}_X \in D(X_{\text{ét}})$ which interpolates between these two objects, i.e., with the properties:

- $\mathbb{Z}(j)^{\text{ét}}_X/\ell^r \cong \mu_\ell^{\otimes r}$ when $1/\ell \in \mathcal{O}_X$; and
- $\mathbb{Z}(j)^{\text{ét}}_X/p^r \cong W_r\Omega^j_{\log}[-j]$ when $p = 0 \in \mathcal{O}_X$.

4. The proof of Jacobson and Cartier's theorem

We will follow the proof in [GS17, §9.3], due to Katz.

Definition 2. A *connection* on a finite dimensional F-vector space V is a homomorphism $\nabla\colon V \to \Omega^1_F \otimes_F V$ such that

$$\nabla(av) = a\nabla(v) + da \otimes v$$

for all $a \in F$ and $v \in V$.

Example 4.1. The main tool here will be to study, for a differential form $\omega \in \Omega^1_F$, the map $\nabla_\omega\colon F \to \Omega^1_F$ defined by

$$\nabla_\omega(a) := da + a\omega.$$

Indeed, for any $a, b \in F$,

$$\begin{aligned}\nabla_\omega(ab) &= d(ab) + ab\omega \\ &= a(db + b\omega) + da \cdot b \\ &= a\nabla_\omega(b) + da \cdot b.\end{aligned}$$

Notably, a 1-form $\omega \in \Omega^1_F$ is logarithmic (i.e., in the image of $d\log\colon F^\times \to \Omega^1_F$) if and only if $\nabla_\omega(a) = da + a\omega = 0$ for some $a \in F^\times$.

A connection ∇ gives rise to a F-linear map $\nabla_*\colon \mathrm{Der}_{F^p}(F) \to \mathrm{End}_{F^p}(V)$ sending a derivation V to the composition

$$V \xrightarrow{\nabla} \Omega^1_F \otimes_F V \xrightarrow{D\otimes \mathrm{id}} F \otimes_F V \cong V.$$

where the F-derivation $D\colon F \to F$ is identified with a homomorphism $\Omega^1_F \to F$ via the universal property.

Remark 3. Although ∇_* is F-linear, the element $\nabla_* D$ is only F^p-linear in general. For $a \in F$ and $v \in V$,

$$\begin{aligned}\nabla_* D(av) &= D \otimes \mathrm{id}(a\nabla(v) + da \otimes v) \\ &= aD \otimes \mathrm{id}(\nabla(v)) + D(a)v \\ &= a\nabla_* D(v) + D(a)v.\end{aligned}$$

Recall that $D(da) = D(a)$ since we abuse notation by calling both the differential $K \to K$ and the homomorphism $\Omega^1_K \to K$ as D. Here if $a = b^p$ for some $b \in F$ then $D(a) = pb^{p-1}D(b) = 0$, showing linearity.

Example 4.2. For the connection ∇_ω,

$$\nabla_{\omega *} D(a) = D(da + a\omega) = D(a) + aD(\omega).$$

Recall that the F-vector space $\mathrm{End}_{F^p}(V)$ has two natural operations:

- the Lie bracket $[\phi, \psi] := \phi \circ \psi - \psi \circ \varphi$; and
- the p-th iterate $\phi^{\circ p}$.

The subspace $\mathrm{Der}(F)$ is stable under both of these operations. Indeed, for any derivation $D \in \mathrm{Der}(F)$,

$$D^{\circ p}(ab) = \sum_{i=0}^{p} \binom{p}{i} D^{\circ i} a D^{\circ (p-i)} b = D^{\circ p} ab + aD^{\circ p} b.$$

Thus, it is a natural condition for a connection to require that the map ∇_* respect these operations on $\mathrm{Der}(F)$ and $\mathrm{End}(V)$:

Definition 3. The connection ∇ is *flat* if

$$\nabla_*[D_1, D_2] = [\nabla_* D_1, \nabla_* D_2]$$

for all $D_1, D_2 \in \mathrm{Der}(K)$ and ∇ is a *p-connection* if $\nabla_*(D^{\circ p}) = (\nabla_* D)^{\circ p}$ for all $D \in \mathrm{Der}(K)$.

Remark 4. This definition is completely analogous to flat connections in differential geometry. A connection of a vector bundle $E \to M$ is a $\mathbb{R}$-linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the product rule, and it is *flat* if the curvature

$$F_\nabla(X, Y)(s) := \nabla_X \nabla_Y - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

disappears everywhere.

We have:

Theorem 4.3 ([GS17, Thm 9.3.3])**.** *The following are equivalent for a differential form $\omega \in \Omega^1_F$:*

(1) ω is logarithmic
(2) $\omega \in \nu(1)_F$
(3) The connection ∇_ω is a flat p-connection.

Proof. That (1) implies (2) is clear.

We will check (2) implies (3). Denoting L_a for multiplication by a, we have

$$\begin{aligned}[\nabla_{\omega*}D_1, \nabla_{\omega*}D_2] &= [D_1D_2] + [D_1L_{D_2\omega}] - [D_2L_{D_1\omega}] - [L_{D_1\omega}L_{D_2\omega}]\\ &= [D_1D_2] + L_{D_1D_2\omega} - L_{D_2D_1\omega}\\ &= \nabla_{\omega*}[D_1D_2] - L_{[D_1D_2]\omega} + L_{D_1D_2\omega} - L_{D_2D_1\omega},\end{aligned}$$

where the second equality is since $D \in \mathrm{Der}(F)$ if and only if $[D, L_a] = L_{Da}$ for all $a \in F$. Here, I claim that for all derivations $D_1, D_2 \in \mathrm{Der}(F)$ and all $\omega \in \Omega^1_F$,

$$(D_1 \wedge D_2)(d\omega) = D_1D_2\omega - D_2D_1\omega - [D_1D_2]\omega. \tag{4.1}$$

Given this claim, $L_{[D_1D_2]\omega} + L_{D_1D_2\omega} - L_{D_2D_1\omega} = -L_{(D_1\wedge D_2)(d\omega)} = 0$ since $d\omega = 0$, so we are done.

Now, to check (4.1) it suffice to look at $\omega = adb$ with $a, b \in F$, in which case

$$\begin{aligned}(D_1 \wedge D_2)(d\omega) &= D_1(a)D_2(b) - D_2(a)D_1(b)\\ &= D_1D_2(adb) - D_2D_1(adb) - [D_1D_2](adb)\\ &= D_1D_2\omega - D_2D_1\omega - [D_1D_2]\omega.\end{aligned}$$

Proving ∇_ω is a p-connection is similar.

Finally, we check (3) implies (1). It suffices to prove this for F/F^p a finite extension. We use the following lemma:

Lemma 4.4 ([GS17, Thm 9.3.6])**.** *Let F/E be a finite extension with $F^p \subset E$, and let V be a K-vector space equipped with a flat p-connection ∇. Then setting $V^\nabla := \{v \in V : \nabla(v) = 0\}$, the natural map*

$$F \otimes_E V^\nabla \to V$$

is an isomorphism.

Applying this theorem to $V = F$ we obtain a nonzero vector $v \in F$ such that $\nabla_\omega(v) = 0$, so that $\omega = -d\log(v)$. □

5. Surjectivity of the differential symbol

The goal of this section is to prove:

Proposition 5.1 (Surjectivity of ψ_F [GS17, Thm 9.6.1])**.** *Let F be finitely-generated over $\mathbb{F}_p$. The group $\nu(n)_F$ is additively generated by the elements of the form $a_1^{-1}da_1 \wedge \cdots \wedge a_n^{-1}da_n$.*

We first prove the following lemma:

Lemma 5.2 ([GS17, Prop 9.6.3])**.** *Let F/F^p be a purely inseparable extension of degree p. Then for any F-linear map $g\colon F \to F^p$, there exists a finite extension E/F^p of degree prime to p such that the induced map $g'\colon EF \to E$ satisfies: there exists $c \in EF^\times$ such that $g(c^i) = 0$ for $1 \le i \le p-1$.*

Proof. The F^p-subspaces $\ker(g) \subset F$ and $dF \subset \Omega^1_{F/F^p}$ both have codimension 1 (by Proposition 2.1), so there is a F-isomorphism $\phi\colon F \to \Omega^1_{F/F^p}$ taking $\ker(g)$ to dF. Let $\omega = a(db/b) \in \Omega^1_{F/F^p} \setminus dF$, with $a, b \in F^\times$. Now $\Omega^1_{F/F^p}/dF$ is a one-dimensional F^p-vector space, so there exists a $\rho \in (F^p)^\times$ such that

$$a^p \frac{db}{b} \in \rho a \frac{db}{b} + dF.$$

Let $E = F^p(u)$ with $u^{p-1} = \rho$. Now, in $\Omega^1_{FE/E}$,

$$(u^{-1}a)^p \frac{db}{b} \in u^{-1} a \frac{db}{b} + d(FE),$$

i.e., $u^{-1}\omega \in \nu(1)_F$. Thus by Theorem 2.8 there is a $y \in F^\times$ with $u^{-1}\omega = dy/y$. Now, the following are equivalent:

- $g(x) = 0$;
- $xa \in \ker(g)$;
- $xdy/y \in dF$ for all $x \in F$

Moreover, the elements $y^i dy$ for $0 \le i \le p-2$ span dF. Thus $c = y$ works. □

Now, consider $F^p \subset E \subset F$ and suppose F/E has degree p^r, with a p-basis $\{b_1, \dots, b_r\}$, so $d\log b_i$ forms a K-basis for $\Omega^1_{F/E}$. Let $\begin{bmatrix} r \\ n \end{bmatrix}$ denote the set of strictly increasing functions from $\{1, \dots, n\}$ to $\{1, \dots, r\}$, and for each $s \in \begin{bmatrix} r \\ n \end{bmatrix}$ set

$$\omega_s := d\log b_{s(1)} \wedge \cdots \wedge d\log b_{s(n)},$$

which forms a F-basis for $\Omega^1_{F/E}$. Now we may define a filtration on $\Omega^n_{F/E}$ by setting

$$\Omega^n_{F/E,<s} := F\{\omega_s : s \in \begin{bmatrix} r \\ n \end{bmatrix}\}$$
$$B^n_{F/E,<s} := d\Omega^{n-1}_{F/E,<s}.$$

Considering the lexicographic ordering on $\begin{bmatrix} r \\ n \end{bmatrix}$, we have $s < s'$ then $\Omega^n_{F/E,<s} \subset \Omega^n_{F/E,s'}$.

Now, we have a "filtered version" of Proposition 5.1:

Proposition 5.3 ([GS17, Prop 9.6.5])**.** *Let F/E be a finite extension of degree p^r, as above. Fix $s \in \begin{bmatrix} r \\ n \end{bmatrix}$ and assume $a \in F$ satisfies*

$$(\gamma - 1)(a\omega_s) = (a^p - a)\omega_s \in \Omega^n_{F/E,<s} + B^n_{F/E}.$$

(recall that $B^n_{F/E}$ is only a E-vector space.) Then, for some finite extension F'/F of degree coprime to p,

$$a\omega_s \in \Omega^n_{F'/E,<s} + \operatorname{Im}(d\log).$$

Proof. The proof is quite technical. □

Given Proposition 5.3 we can prove Proposition 5.1:

Proof of Proposition 5.1. F/F^p is a finite extension. Assume $d\log$ is not surjective. Then since

$$\Omega^n_F = \sum_{s \in \begin{bmatrix} r \\ n \end{bmatrix}} \Omega^n_{F,\le s},$$

we may pick a minimal $s = s(F)$ such that there exists a $\omega \in \nu(n)_F$ not in the image of $d\log$ such that $\omega = \omega' + \eta$ with $\omega' \in \Omega^n_{F,\leq s}$ and $\eta \in B^n_F$. Now $\Omega^n_{F,\leq s} = \Omega^n_{F,<s} + F\omega_s$ so $\omega' = a\omega_s + \omega''$ for some $a \in F$ and $\omega'' \in \Omega^n_{F,<s}$.

Now $a\omega_s = \omega - \eta - \omega''$, so

$$(\gamma - 1)(a\omega_s) = (\gamma - 1)\omega - (\gamma - 1)\eta - (\gamma - 1)\omega'' \in \Omega^n_{F,<s} + B^n_F,$$

since $(\gamma - 1)\eta, (\gamma - 1)\omega \in B^n_F$ and $(\gamma - 1)\omega'' \in \Omega^n_{F,<s}$.

By Proposition 5.3 there is a finite extension F'/F of degree coprime to p such that $a\omega_s \in \Omega^n_{F'} + \mathrm{Im}(d\log)$.

The argument shows that $s(F') < s(F)$. Thus, eventually $s(F)$ will not exist, and there exists some extension $\widetilde{F}/F$ of degree coprime to p for which $\psi^n_{F'} \colon K^M_n(F')/p \to \nu(n)_{F'}$ is surjective. Now, in the diagram:

$$\begin{array}{ccc}
K^M_n(F)/p & \xrightarrow{\psi_F} & \nu(n)_F \\
\downarrow & & \downarrow \\
K^M_n(F')/p & \xrightarrow{\psi_{F'}} & \nu(n)_{F'} \\
{\scriptstyle N_{F'/F}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\
K^M_n(F)/p & \xrightarrow{\psi_F} & \nu(n)_F
\end{array}$$

The vertical composition $\nu(n)_F \to \nu(n)_{F'} \xrightarrow{\mathrm{tr}} \nu(n)_F$ is $[F' : F]$, which is an isomorphism since $\nu(n)$ is p-torsion. Thus, tr is surjective, and since $\psi_{F'}$ is surjective, the homomorphism ψ_F must be surjective as well. □

6. Injectivity of the differential symbol

We hope to prove:

Theorem 6.1 (Injectivity of ψ^n_F [GS17, Thm 9.7.1])**.** *For all finitely generated extensions $F/\mathbb{F}_p$, the differential symbol $\psi^n_F \colon K^M_n(F)/p \to \nu(n)_F$ is injective.*

The first step is to use Proposition 1.3 to allow for induction on the transcendence degree:

Lemma 6.2 ([GS17, Prop 9.7.2])**.** *Assume that ψ^n_F and ψ^{n-1}_E are injective, for any finite extension E/F. Then so is $\psi^n_{F(t)}$.*

Proof. There is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^M_n(F)/p & \longrightarrow & K^M_n(F(t))/p & \longrightarrow & \bigoplus_{P \in (\mathbb{A}^1_F)_0} K^M_{n-1}(\kappa(P))/p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi^n_F} & & \downarrow{\scriptstyle \psi^n_{F(t)}} & & \downarrow{\scriptstyle \bigoplus i_P} & & \\
0 & \xrightarrow{\varphi_f} & \Omega^n_{F[t]} & \longrightarrow & \Omega^n_{F(t)} & \longrightarrow & \bigoplus_{P \in (\mathbb{A}^1_F)_0} \Omega^n_{F(t)}/\Omega^n_{F[t]_P}. & &
\end{array}$$

Here, i_P is the composite of $\psi^{n-1}_{\kappa(P)}$ with the map $j_P \colon \Omega^{n-1}_{\kappa(P)} \to \Omega^n_{F(t)}/\Omega^n_{F[t]_P}$ given by

$$j_P(x_0 dx_1 \wedge \cdots \wedge dx_{n-1}) = \widetilde{x}_0 d\widetilde{x}_1 \wedge \cdots \wedge d\widetilde{x}_{n-1} \wedge \pi_P^{-1} d\pi_P$$

where $P = (\pi_P) \subset F[t]$ and the $\widetilde{x}_i$ are arbitrary lifts of $x_i \in \kappa(P)$ to $F[t]_P$.

Thus, it suffices to check the injectivity of i_P, which reduces to the injectivity of j_P.

The module $\Omega^1_{F[t]_P}$ is a free $F[t]_P$-module on a basis consisting of $d\pi_P$ and some other elements da_i. Thus, $\Omega^n_{F[t]_P}$ has a basis consisting of n-fold exterior products of these forms. Hence the relation $\widetilde{\omega}\wedge\pi_P^{-1}d\pi_P = 0$ can hold only for a lift $\widetilde{\omega} \in \Omega^n_{F[t]_P}$ of $\omega \in \Omega^n_{\kappa(P)}$ if $\widetilde{\omega}$ is a linear combination of basis elements involving $d\pi_P$. But then the image of ω in $\Omega^{n-1}_{\kappa(P)}$ is 0, as desired. □

To prove Theorem 6.1, we proceed by induction on n, the case $n = 0$ being obvious. Let d be the transcendence degree of $F/\mathbb{F}_p$. Then there exists a scheme-theoretic point of codimension 1 (i.e., a divisor) on the affine space $\mathbb{A}^{d+1}_{\mathbb{F}_p}$ whose local ring R has residue field isomorphic to F. Let us define:

Definition 4. Let $\widetilde{F}$ be a fractional field of R, and M its maximal ideal. Let $K^M_n(R)/p$ be the kernel of the residue map $\partial_M \colon K_n(\widetilde{F})/p \to K^M_{n-1}(F)/p$. The analogous construction on the differential side is

$$\nu(n)_R := \ker(\Omega^n_R \xrightarrow{\gamma-1} \Omega^n_R/B^n_R).$$

Now the differential symbol ψ^n_F restricts to a homomorphism $\psi^n_R \colon K^M_n(R)/p \to \nu(n)_R$.

Denote by $K^M_n(R,M)/p$ the kernel of the specialization map $s^M_R \colon K^M_n(R) \to K^M_n(F)$, which is independent of the choice of the prime element, and by $\nu(n)_{R,M}$ the kernel of the reduction map $\rho_R \colon \nu(n)_R \to \nu(n)_F$. Then ψ^n_R restricts further to a map $\psi^n_{R,M} \colon K^M_n(R,M)/p \to \nu(n)_{R,M}$.

Lemma 6.3. *With notations as above, assume that the differential symbol*

$$\psi^n_{R,M} \colon K_n(R,M)/p \to \nu(n)_{R,M}$$

is surjective. Then the symbol ψ^n_F is injective.

Proof. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K^M_n(R,M)/p & \longrightarrow & K^M_n(R)/p & \longrightarrow & K^M_n(F)/p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi^n_{R,M}} & & \downarrow{\scriptstyle \psi^n_R} & & \downarrow{\scriptstyle \psi^n_F} & & \\
0 & \longrightarrow & \nu(n)_{R,M} & \longrightarrow & \nu(n)_R & \longrightarrow & \nu(n)_F. & &
\end{array}$$

□

Thus to prove Theorem 6.1 it suffices to prove the surjectivity of $\psi^n_{R,M}$.

Definition 5. If R is a semi-local Dedekind ring with field of fractions $\widetilde{F}$ and maximal ideals $M_1, \ldots, M_r$, denote its Jacobson radical by $I := M_1 \cap \cdots \cap M_r$. By the Chinese remainder theorem $R/I \cong R/M_1 \times \cdots \times R/M_r$, a direct product of fields. Therefore, we may define

$$K^M_n(R/I) := K^M_n(R/M_1) \oplus \cdots \oplus K^M_n(R/M_r).$$

Let $K^M_n(R)/p \subset K^M_n(\widetilde{F})/p$ be the kernel of $\oplus\partial_{M_i} \colon K^M_n(\widetilde{F})/p \to K^M_n(R/I)/p$.

The group $K^M_n(R,I)/p$ is the kernel of

$$\oplus s^{M_i}_R \colon K^M_n(R)/p \to \bigoplus K^M_n(R/M_i)/p = K^M_n(R/I)/p.$$

As in the case for local rings, ψ_F^n restricts to a homomorphism $\psi_R^n\colon K_n^M(R)/p \to \nu(n)_R$, and the following diagram commutes:

$$\begin{array}{ccc} K_n^M(R)/p & \xrightarrow{\oplus s_R^{M_i}} & K_n^M(R/I)/p = \bigoplus K_n^M(R/M_i)/p \\ \Big\downarrow{\scriptstyle \psi_R^n} & & \Big\downarrow{\scriptstyle \oplus \psi_{R/M_i}^n} \\ \nu(n)_R & \longrightarrow & \nu(n)_{R/I} = \bigoplus \nu(n)_{R/M_i}. \end{array}$$

Thus, it restricts to a homomorphism $\psi_{R,I}^n\colon K_n(R,I)/p \to \nu(n)_{R,I}$, where $\nu(n)_{R,I}$ is the kernel of the bottom map $\nu(n)_R \to \nu(n)_{R/I}$.

Thus, the statement to be proven is:

Proposition 6.4 ([GS17, Prop 9.7.6])**.** *Let k be a perfect field of characteristic $p > 0$ and R a semi-local Dedekind domain which is obtained as a localization of a finitely-generated k-algebra. Then the differential symbol*

$$\psi_{R,I}^n\colon K_n^M(R,I) \to \nu(n)_{R,I}$$

is surjective.

The proof follows a similar strategy as the proof of Proposition 5.1, using the integral version of Theorem 2.8 to prove the injectivity of ψ_F^n:

Corollary 6.5 ([GS17, Lemma 9.7.9])**.** *Let $R \supset T \supset R^p$ be an extension of semi-local Dedekind rings which arise as localizations of finitely generated algebras over a perfect field k of characteristic $p > 0$. Assume that the arising extension F/F_0 of fraction field is finite. Then the sequence*

$$1 \to R^\times/T^\times \xrightarrow{d\log} \Omega^1_{R/T} \xrightarrow{\gamma_R - 1} \Omega^1_{R/T}/B^1_{R/T}$$

is exact.

References

[GS17] Philippe Gille and Tamás Szamuely, *Central simple algebras and Galois cohomology*, 2nd revised and updated edition ed., Camb. Stud. Adv. Math., vol. 165, Cambridge: Cambridge University Press, 2017 (English).

[Mil70] John W. Milnor, *Algebraic K-theory and quadratic forms. With an appendix by J. Tate*, Invent. Math. **9** (1970), 318–344 (English).

[NS89] Yu. P. Nesterenko and A. A. Suslin, *Homology of the general linear group over a local ring and the Milnor K- theory*, Izv. Akad. Nauk SSSR, Ser. Mat. **53** (1989), no. 1, 121–146 (Russian).