Sensitivity to Cumulative Perturbations for a Class of Piecewise Constant Hybrid Systems

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***Abstract*—We consider a class of continuous-time hybrid dynamical systems that correspond to subgradient flows of a piecewise linear and convex potential function with finitely many pieces, and which include the fluid-level dynamics of the Max-Weight scheduling policy as a special case. We study the effect of an external disturbance/perturbation on the state trajectory, and establish that the magnitude of this effect can be bounded by a constant multiple of the *integral* of the perturbation.**

I. Introduction

We consider a class of continuous-time, non-expansive, hybrid systems that are subject to an external disturbance/perturbation, and develop a bound on the effect of the perturbation on the state trajectory, in terms of the **integral** of the perturbation.

In order to appreciate the issues that arise, and the usefulness of such a result, let us consider a discrete-time system of the form $x(t+1) = f\big(x(t)\big)$, $t = 0, 1, \dots$, and its perturbed counterpart

$$\widetilde{x}(t+1) = f\big(\widetilde{x}(t)\big) + u(t), \qquad t = 0, 1, \dots \tag{1}$$

Here, $x(t)$ and $u(t)$ take values in $\mathbb{R}^n$ and we assume that the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is non-expansive, in the sense that

$$\|f(x) - f(y)\| \le \|x - y\|, \qquad \forall\, x, y \in \mathbb{R}^n,$$

for a given norm $\|\cdot\|$. A straightforward induction yields a bound on the distance of the perturbed trajectory from the original one: if $\widetilde{x}(0) = x(0)$, then

$$\big\|\widetilde{x}(t) - x(t)\big\| \le \sum_{\tau=0}^{t-1} \big\|u(t)\big\|. \tag{2}$$

However, our goal is to derive stronger bounds, of the form

$$\big\|\widetilde{x}(t) - x(t)\big\| \le C \max_{k<t} \Big\| \sum_{\tau=0}^{k} u(\tau) \Big\|, \tag{3}$$

for some constant $C > 0$ independent of $u(\cdot)$. Note that a bound which is *linear* in the cumulative perturbation is clearly the tightest possible, even for the trivial system $f(x) = x$.

A bound of the form (3) is not valid in general, even for non-expansive systems, or gradient fields of convex functions [1]. Nevertheless, we show that such a bound is valid for continuous-time hybrid systems driven by a piecewise constant drift, determined by the subdifferential of a piecewise linear and convex function with finitely many pieces. Within this class of systems, the dynamics are automatically non-expansive with respect to the Euclidean norm. Furthermore, this class is fairly broad, in the sense that it actually contains a seemingly larger class of non-expansive *finite-partition hybrid systems*[1] [2]. Finite-partition systems often arise in the context of systems that are controlled through the selection of a particular action at each time among the elements of a finite set. They have attracted broad interest, due to numerous applications to communication networks [3], [4], processing systems [5], manufacturing systems, and inventory management [6], [7], etc.

A prominent example to which our results apply are the fluid-level dynamics of the celebrated Max-Weight policy for real-time job scheduling [3]. This policy is used for scheduling in queueing systems: at each time, it chooses a service vector (from a finite set) that maximizes a weighted sum of the current queue lengths (see Fig. 1 for a simple example). This policy and its properties, e.g., *stability* [8], [9], [10], [11] and *state space collapse* [12], [13], have been studied extensively over the past three decades.

When the Max-Weight policy is applied to a discrete-time stochastic setting, the perturbation $u(\cdot)$ in (1) is the sample path of a stochastic process, and captures the fluctuations in job arrivals. Under usual probabilistic assumptions, $\sum_{\tau=0}^{t-1} \|u(\tau)\|$ grows at the rate of t, whereas $\max_{k<t} \big\| \sum_{\tau=0}^{k} u(\tau) \big\|$ only grows as (roughly) $\sqrt{t}$, with high probability. This fact, in combination with the main result of this paper, leads to tighter than earlier available probabilistic bounds on the fluctuations of the Max-Weight trajectories from their deterministic (fluid) counterparts, and opens the way for new results [14], such as strengthening the state-space collapse results in [13]. More specifically, in [14], we study in detail the discrete-time Max-Weight dynamics: we use the results of this paper to prove a bound similar to (3), and also address a state-space collapse conjecture posed in [13]. Furthermore, our approach also enables us to settle another open problem

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[1] In a finite-partition hybrid system, the domain is partitioned into a finite number of regions, and system trajectories have a constant drift in the interior of each region.

that was posed in [15], on delay-stability in the presence of heavy-tailed traffic, as will be reported in a forthcoming paper.

As is apparent from our discussion of the Max-Weight policy, one may be ultimately interested in a discrete-time system, as opposed to the continuous-time systems considered in this paper. However, we found it more natural to start with the development of the core concepts and results within the more elegant continuous-time framework in this paper, and then translate them back to the discrete-time framework. For instance, [1] shows that if a continuous-time system admits a bound of the form (3), then its discrete-time counterpart obeys a similar bound.

Regarding related literature, we are not aware of any work that resembles the main result of this paper. Some seemingly related research threads deal with *input-to-state stability*[2] [16], [17], [18], [19], [20], integral input-to-state stability [21], incremental input-to-state stability [22], incrementally integral input-to-state stability [23], and robust input-to-state stability [24]. However, we note that integral input-to-state stability [21] and incrementally integral input-to-state stability [23] are concerned only with generalizations of the weak bound in (2). Furthermore, incremental input-to-state stability [22] involves generalizations of a sensitivity bound in terms of $b = \max_{k<t} \big\|u(k)\big\|$. A bound of the form Cb, for some constant C that does not depend on t, would typically be stronger than ours; however, such a bound does not hold in our setting, even for the simplest system where $f(x) = x$.

It is worth pointing out that for systems with additive disturbances, $x(t+1) = f\big(x(t)\big) + u(t)$, input-to-state stability and the bound (3) do not imply one another[3].

Another key difference is that input-to-state stability results usually rely on Lyapunov-type arguments [25]. However, Lyapunov functions seem to be inadequate for our purposes. This is because our bounds (as can be seen in the proof given in Section IV) rely in a delicate manner on the relative orientation of the two trajectories $x(\cdot)$ and $\widetilde{x}(\cdot)$, in conjunction with the local "landscape" of the potential function. Furthermore, as shown in [1], the desired sensitivity bound (3) fails to carry over if the number of constant-drift regions is not finite. This means that a Lyapunov-based argument would have to make essential use of our finiteness assumption, something for which we are not aware of having any precedents in the literature.

The rest of this paper is organized as follows. In the next section we discuss some preliminaries and our notational conventions. In Section III we state our main theorem. In Section IV we provide the core of the proof, while relegating some of the details to the Appendix. Finally, in Section V we discuss possible extensions, open problems and challenges, and directions for future research.

[2]A discrete time dynamical system $x(t+1) = f\big(x(t), u(t)\big)$ with state $x(\cdot)$ and external disturbance (or control) $u(\cdot)$ is said to be input-to-state stable if there exists a continuous and strictly increasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(0) = 0$ and a function $\beta : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, strictly increasing in the first argument and decreasing in the second argument with $\beta(0, \cdot) = 0$ and $\lim_{t\to\infty} \beta(s, t) = 0$, for all $s \geq 0$; such that $\|x(t)\| \leq \beta\big(\|x(0)\|, t\big) + \max_{k\leq t} \gamma\big(\|u(k)\|\big)$, for all trajectories $x(\cdot)$, all disturbances $u(\cdot)$, and all times t [16].

[3]For example, the discrete-time system $x(t+1) = x(t)$ satisfies (3), but is not input-to-state stable. Conversely, the two-dimensional and two-region discrete-time system with $f(x, y) = \big(x/2, y/2\big)$ for $x \geq 0$ and $f(x, y) = \big(x/4, y/4\big)$ for $x < 0$, is input-to-state stable but (3) fails to hold. This is because, for a trajectory initialized at $\big(x(0), y(0)\big) = (0, 4)$, a small perturbation of the initial condition $\big(\widetilde{x}(0), \widetilde{y}(0)\big) = (-\epsilon, 4)$, will result in a distance larger than 1 at the next time step.

II. Preliminaries

A. Notation

We denote by $\mathbb{R}_+$ the set of non-negative real numbers. For a column vector $v \in \mathbb{R}^n$, we denote its transpose and Euclidean norm by v^T and $\big\|v\big\|$, respectively. For any set $S \subseteq \mathbb{R}^n$, $\mathrm{span}(S)$ stands for the span of the vectors in S. Furthermore, if p is a point in $\mathbb{R}^n$, then $p + S$ stands for the set $\big\{p + x \,\big|\, x \in S\big\}$, and $d\big(p\,,\, S\big)$ for the Euclidean distance between p and S, with the convention that $d\big(p\,,\, S\big) = \infty$ if S is empty. Similarly, we let $d(p, \{x\}) = \big\|p - x\big\|$ for $p, x \in \mathbb{R}^n$. We finally let $A \backslash B = A \bigcap B^c$, for any two sets A and B, where B^c is the complement of B.

B. Perturbed Dynamical Systems

As in [26], we identify a dynamical system with a set-valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ and the associated differential inclusion $\dot{x}(t) \in F(x(t))$. We start with a formal definition, which allows for the presence of perturbations.

Definition 1 (Perturbed Trajectories)**.** Consider a dynamical system $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, and let $U : \mathbb{R} \to \mathbb{R}^n$ be a right-continuous function, which we refer to as the *perturbation*. Suppose that there exist measurable and integrable functions $\widetilde{x}(\cdot)$ and $\zeta(\cdot)$ of time that satisfy

$$\begin{aligned} &\widetilde{x}(t) = \int_0^t \zeta(\tau)\, d\tau \,+\, U(t), \qquad \forall\, t \geq 0, \\ &\zeta(t) \in F\big(\widetilde{x}(t)\big), \qquad \forall\, t \geq 0. \end{aligned} \tag{4}$$

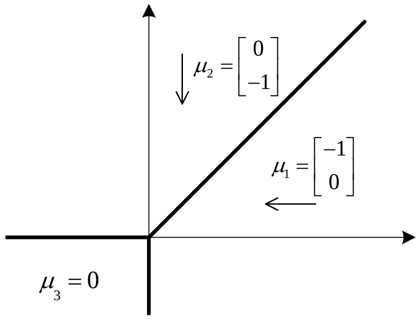
We then call U the *perturbation*. Any such $\widetilde{x}$ and ζ is called a *perturbed trajectory* and a *perturbed drift*, respectively. In the special case where U is identically zero, we also refer to $\widetilde{x}$ as an *unperturbed trajectory*.

Note that a perturbed trajectory is automatically right-continuous. In the absence of the perturbation $U(\cdot)$, Eq. (4) becomes the differential inclusion $\dot{x} \in F(x(t))$ (almost everywhere). When perturbations are present, U is often absolutely continuous, of the form $\int_0^T u(\tau)\, d\tau$, for some measurable function $u(\cdot)$. In this case, we are essentially dealing with the differential inclusion $\dot{\widetilde{x}}(t) \in F(\widetilde{x}(t)) + u(t)$. However, the integral formulation in Definition 1 is more useful because it also applies to cases where U is not absolutely continuous, e.g., if U is a sample path of a Wiener or a jump process.

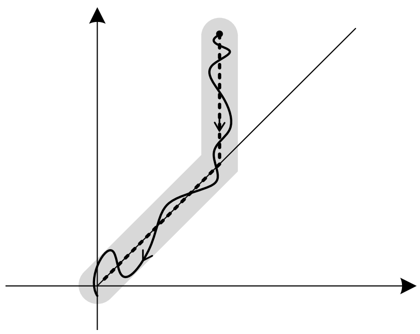
C. Classes of Systems

We now introduce some classes of systems of interest. A dynamical system F is called *non-expansive* if for any pair of unperturbed trajectories $x(\cdot)$ and $y(\cdot)$, and if $0 \leq t_1 \leq t_2$, then

$$\big\|x(t_2) - y(t_2)\big\| \leq \big\|x(t_1) - y(t_1)\big\|. \tag{5}$$



(a)



(b)

Fig. 1. Consider a simple discrete-time network of two parallel queues with no arrivals and a unit-rate server. The Max-Weight policy always serves a longest queue. Part (a) illustrates the corresponding continuous-time (the so called fluid-level) dynamics of this system. The state vector (x_1, x_2) describes the workload at each queue. (To avoid dealing with differential inclusions involving boundary constraints, we extend the naturally nonnegative state of the system to all of $\mathbb{R}^2$.) We have three regions indicated in the figure. The set $F(x)$ is a singleton in the interior of each region and it is the convex hull of multiple vectors on the boundaries of the regions. This dynamical system is the subgradient field of the piecewise linear convex function $\Phi(x) = \max(-\mu_1^T x,\, -\mu_2^T x,\, 0) = \max\{x_1, x_2, 0\}$, and hence is an FPCS system. Part (b) depicts an unperturbed (dashed line) and a perturbed (solid line) trajectory. Our main result argues that the perturbed trajectory stays within a distance of the unperturbed trajectory bounded by a constant multiple of the size of the integral of the perturbation; cf. (3).

For a convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$, we denote its subdifferential by $\partial\Phi(x)$. We say that F is a *subgradient dynamical system* if there exists a convex function $\Phi(\cdot)$, such that for any $x \in \mathbb{R}^n$, $F(x) = -\partial\Phi(x)$. Furthermore, if Φ is of the form

$$\Phi(x) = \max_i \big(-\mu_i^T x + b_i \big),$$

for some $\mu_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, and with i ranging over a **finite** set, we say that F is a *Finitely Piecewise Constant Subgradient* (FPCS, for short) system; cf. Fig. 1.

Subgradient systems are known to have several useful properties: they are automatically non-expansive (cf. Part 5 of Theorem 4.4 in [26]), a fact that we will be using in the sequel. Existence and uniqueness results are also available [26].

Lemma 1 (Existence and Uniqueness of Solutions)**.** *For any subgradient dynamical system F and any $x_0 \in \mathbb{R}^n$, there exists a unique trajectory of F initialized at x_0.*

Proof. It follows from Lemma 2.30 of [26] that any subgradient dynamical system is a maximal monotone map[4]. The lemma then follows from Corollary 4.6 of [26]. □

III. Main Result

We now state the main result of the paper. Its proof is given in Section IV.

Theorem 1 (Input Sensitivity of FPCS Systems)**.** *Consider an FPCS system F. Then, there exists a constant C such that for any unperturbed trajectory $x(\cdot)$, and for any perturbed trajectory $\widetilde{x}(\cdot)$ with corresponding perturbation $U(\cdot)$ and the same initial conditions $\widetilde{x}(0) = x(0)$,*

$$\big\|\widetilde{x}(t) - x(t)\big\| \;\le\; C \sup_{\tau \le t} \big\|U(\tau)\big\|, \qquad \forall\, t \in \mathbb{R}_+. \tag{6}$$

Moreover, for any $\lambda \in \mathbb{R}^n$, the bound (6) *applies to the (necessarily FPCS) system $F(\cdot) + \lambda$ with the same constant C.*

Theorem 1 is limited to FPCS systems: if any of the assumptions in the definition of FPCS systems is removed, then a similar result is no longer possible. In [1] we discuss several examples of dynamical systems for which no constant C satisfies (6); cf. Section V.

We finally note that the vector λ in the dynamical system $F(\cdot) + \lambda$ can be viewed as a constant external field. Thus, the second part of the theorem asserts that the same bound holds uniformly for all constant external fields.

The proof of Theorem 1, presented in the next section, is fairly involved and so it is useful to provide some perspective on the challenges that are involved. For a constant-drift system, of the form $\dot{x}(t) = \mu + u(t)$, the result is immediate, because the state is fully determined by the integral $\int_0^t u(\tau)\, d\tau$. More generally, the unperturbed system goes through successive constant-drift regions, and one might expect that the result can be obtained by deriving and patching together bounds for each region encountered. There is however a difficulty, because the unperturbed trajectory often lies at the intersection of the boundaries of two or more constant drift regions. When that happens, the perturbed trajectory may chatter between different regions. As a consequence, the number of pieces and bounds that would have to be patched together can become arbitrarily large, and a bound of the desired form does not follow. For this reason, we need a much more refined analysis of the trajectories in the vicinity of the intersection of different regions, as will be seen in the next section.

IV. Proof

In this section we present the proof of Theorem 1, organized in a sequence of three subsections. In Subsection IV-A we present some notation, definitions, and lemmas, mostly concerning the geometric properties of FPCS systems and

[4] A set valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a *monotone map* if for any $x_1, x_2 \in \mathbb{R}^n$ and any $v_1 \in F(x_1)$ and $v_2 \in F(x_2)$, we have $\big(v_1 - v_2\big)^T \big(x_1 - x_2\big) \le 0$. It is called a *maximal monotone map* if it is monotone, and for any monotone map $\widetilde{F}$, that satisfies $F(x) \subseteq \widetilde{F}(x)$ for all x, we have $\widetilde{F} = F$.

unperturbed trajectories. In particular, we define critical points (Definition 2) as the extreme points of constant-drift regions.

In Subsection IV-B we consider a time interval during which the perturbed trajectory is far from the set of critical points. Such an interval can be divided into subintervals with an important property: the set of drifts encountered is low-dimensional, in a sense to be defined below. Within each such subinterval, we show in Lemma 5 that the local dynamics are equivalent to the dynamics of a lower-dimensional FPCS system, and employ a suitable induction on the system dimension to obtain a certain upper bound. Then, in Proposition 1, we piece together the bounds for the different subintervals to obtain an upper bound that applies as long as the perturbed trajectory remains far from the set of critical points.

In Subsection IV-C we consider the case where the perturbed trajectory comes close to a critical point: we show, in Proposition 2, that the unperturbed trajectory stays close to the perturbed trajectory, as long as the perturbed trajectory remains sufficiently close to that critical point. Finally, in Subsection IV-D we combine the two cases and bound the distance of the trajectories at all times.

From now on, we assume that $x(0) = \widetilde{x}(0)$ and that

$$\sup_t \big\|U(t)\big\| \leq \theta. \tag{7}$$

We will show that for any $t \geq 0$, we have $\big\|\widetilde{x}(t) - x(t)\big\| \leq C\theta$, for some constant C independent of U, θ, and $x(0)$. It is not hard to see that this implies the theorem in its original form.

The proof proceeds by induction on the system dimension n. In particular, we make the following **induction hypothesis**, which we assume to be in effect throughout the rest of this section.

$$\begin{array}{ll} \text{Induction} & \text{Theorem 1 holds for all} \\ \text{hypothesis} \;\;\textbf{:} & (n-1)\text{-dimensional FPCS systems.} \end{array} \tag{8}$$

We then rely on the induction hypothesis to prove the theorem for n-dimensional systems. For the basis of the induction we consider the case of zero-dimensional systems. In this case, the state space consists of a single point (the zero vector), we have $x(t) = \widetilde{x}(t) = 0$ at all times, and the result in Theorem 1 holds trivially.

A. Properties of Unperturbed Dynamics

In this subsection we present some notation and definitions, and prove some properties of unperturbed trajectories. We then define and study critical points. Throughout the proof, we assume that F is an FPCS system on $\mathbb{R}^n$, with $F = -\partial\Phi$, where $\Phi(x) = \max_{i=1,\ldots,m}\big(-\mu_i^T x + b_i\big)$. We assume that the vectors μ_i in the definition of Φ are distinct. This entails no loss of generality, because if $\mu_i = \mu_j$ and $b_i > b_j$, then $-\mu_j^T x + b_j$ is always dominated by $-\mu_i^T x + b_i$ and has no effect on $\Phi(\cdot)$.

Each vector μ_i is called a *drift* and we define $\mathcal{M}$ to be the set $\big\{\mu_i\big\}_{i=1}^m$ of all drifts. For each drift $\mu \in \mathcal{M}$, we use the notation b_μ to refer to the corresponding constant in the expression for Φ. With these conventions, we have

$$\Phi(x) = \max_{\mu\in\mathcal{M}} \big(-\mu^T x + b_\mu\big). \tag{9}$$

For every $x \in \mathbb{R}^n$, we define the set of active drifts at x as

$$\mathcal{M}(x) \triangleq \big\{\mu \in \mathcal{M} \,\big|\, \Phi(x) = -\mu^T x + b_\mu\big\}. \tag{10}$$

If at some x the corresponding set $\mathcal{M}(x)$ consists of a single element μ, we have $\dot{x} = \mu$. However, the dynamics become more interesting when $\mathcal{M}(x)$ contains multiple elements. For that case, it follows from the definition of the subdifferential that for any $x \in \mathbb{R}^n$, $F(x)$ is the convex hull of $\mathcal{M}(x)$.

For each $\mu \in \mathcal{M}$, we define its *effective region* R_μ by

$$R_\mu \triangleq \big\{x \in \mathbb{R}^n \,\big|\, \mu \in \mathcal{M}(x)\big\}. \tag{11}$$

Equivalently,

$$R_\mu = \big\{x \,\big|\, -\mu^T x + b_\mu \geq -\nu^T x + b_\nu,\ \forall\, \nu \in \mathcal{M}\big\},$$

which establishes that each region R_μ is a polyhedron and, in particular, closed and convex. We will be using $\mathcal{R}$ to denote the collection of all effective regions: $\mathcal{R} \triangleq \big\{R_\mu \,\big|\, \mu \in \mathcal{M}\big\}$.

From now on, and with some abuse of traditional notation, we will use $\dot{x}(t)$ to denote the **right-derivative** of $x(t)$, whenever it exists. The lemma that follows shows that for unperturbed trajectories this right derivative always exists and has some remarkable properties.

Lemma 2 (Properties of Unperturbed Trajectories)**.** *Let $x(\cdot)$ be an unperturbed trajectory of an FPCS system F. Then,*

(a) (Minimum Norm) For every $t \geq 0$, the right derivative of $x(t)$ exists and is given by

$$\dot{x}(t) = \operatorname*{argmin}_{v\in F(x(t))} \big\|v\big\|, \tag{12}$$

with the minimizer being unique.

(b) (Decreasing Drift Size) If $t > s$, then $\big\|\dot{x}(t)\big\| \leq \big\|\dot{x}(s)\big\|$, and the inequality is strict if $\dot{x}(t) \neq \dot{x}(s)$. Furthermore, an unperturbed trajectory traverses a connected sequence of at most $2^m - 2$ line segments, possibly followed by a half-line.

Proof. The first part of the lemma is an immediate consequence of Part 3 of Theorem 4.4 in [26]. For Part (b), we invoke Part 4 of Theorem 4.4 in [26] which states that $\big\|\dot{x}(t)\big\|$ is a non-increasing function of time. Since for any x, $F(x)$ is the convex hull of $\mathcal{M}(x)$, it follows from (12) that $\dot{x}(t)$ is uniquely determined by $\mathcal{M}\big(x(t)\big)$. There are at most $2^m - 1$ non-empty subsets $\mathcal{M}(x)$ of $\mathcal{M}$. Hence, $\dot{x}(t)$ can take at most $2^m - 1$ different values.

Fix a time $s \geq 0$ and let t be the infimum of the times $\tau > s$ for which $\dot{x}(\tau) \neq \dot{x}(s)$. The time function $\dot{x}(\cdot)$ is piecewise constant and right-continuous (Part 4 of Theorem 4.4 in [26]). This implies that $t > s$ and $\dot{x}(t) \neq \dot{x}(s)$. Furthermore, from the strict convexity of the Euclidean norm we obtain

$$\big\|\big(\dot{x}(s) + \dot{x}(t)\big)/2\big\| < \max\big(\big\|\dot{x}(s)\big\|,\, \big\|\dot{x}(t)\big\|\big). \tag{13}$$

Since every region R_μ is closed, there exists a sufficiently small neighbourhood $\mathcal{B}$ of $x(t)$ such that if $x(t) \notin R_\mu$, then $\mathcal{B}$ does not intersect R_μ. Equivalently, for any $y \in \mathcal{B}$, we have $\mathcal{M}(y) \subseteq \mathcal{M}\big(x(t)\big)$, and $F(y) \subseteq F\big(x(t)\big)$. In particular, consider a $\tau \in \big[s, t\big)$ such that $x(\tau) \in \mathcal{B}$. Then, $\dot{x}(s) = \dot{x}(\tau) \in F\big(x(\tau)\big) \subseteq F\big(x(t)\big)$. Since $F\big(x(t)\big)$ is convex, $\big(\dot{x}(s) + \dot{x}(t)\big)/2 \in F\big(x(t)\big)$. Therefore, (12) implies that $\|\dot{x}(t)\| \le \|(\dot{x}(s) + \dot{x}(t))/2\|$. Together with (13), this shows that $\|\dot{x}(t)\| < \|\dot{x}(s)\|$.

For the last statement in Part (b) of the lemma, note that there are at most $2^m - 1$ number of different possible sets $\mathcal{M}(x)$, and therefore as many choices for $F(x)$. Using (12), there are at most $2^m - 1$ possible values for $\dot{x}(t)$. As we have already shown that $\|\dot{x}(t)\|$ decreases strictly each time that it changes, an unperturbed trajectory consists of at most $2^m - 1$ pieces, with a constant derivative on each piece. This implies that the trajectory traverses a connected sequence of at most $2^m - 2$ line segments, possibly followed by a half-line. □

For any $x \in \mathbb{R}^n$, consider the unperturbed trajectory $z(\cdot)$ initialized with $z(0) = x$. We define the *actual drift* at x as $\xi(x) \triangleq \dot{z}(0)$, where we continue using the convention that $\dot{z}$ stands for the right derivative. According to Lemma 2(a), the actual drift always exists and is uniquely determined by x.

We now proceed to define *critical points*, which will play a central role in the sequel.

Definition 2 (Critical Points)**.** A point $p \in \mathbb{R}^n$ is called a *critical point* if $\operatorname{span}(\{\mu - \mu' \mid \mu, \mu' \in \mathcal{M}(p)\}) = \mathbb{R}^n$. The set of critical points is denoted by $\mathcal{C}$.

An equivalent condition is that for a critical point p, the affine span of $\mathcal{M}(p)$, i.e., the smallest affine space that contains $\mathcal{M}(p)$, is equal to the entire set $\mathbb{R}^n$. For this to happen, $\mathcal{M}(p)$ must have at least $n+1$ elements, and therefore p must lie at the intersection of at least $n+1$ regions (although the converse is not always true). For the example in Fig. 1, $p = 0$ is the only candidate and is in fact a critical point because the affine span condition is satisfied. Furthermore, it will be shown in Lemma 3(a) that the critical points are the extreme points of the regions R_μ.

Definition 3 (Basin of a Critical Point)**.** Consider some $\rho \in \mathbb{R}_+ \cup \{\infty\}$ and a critical point p, with actual drift $\xi(p)$ equal to ξ. The closed ball $\mathcal{B}$ of radius ρ centered at p is called a basin of p (and ρ is called a basin radius for p) if for every $x \in \mathcal{B}$ and every $y \in F(x)$, we have $\xi^T y \ge \|\xi\|^2$.

Note that the inequality $\xi^T y \ge \|\xi\|^2$ implies that $\|\xi\| \le \|y\|$. As a result, $\xi(p)$ has the minimum norm among all possible drifts within the basin of a critical point p. Also note that basins of a critical point p are not necessarily unique: if the radius ρ is positive, another basin is obtained by reducing the radius. Figure 2 shows an example of a two-dimensional system with three critical points and some associated basins. Basins of critical points will appear frequently in the sequel.

Before moving to study the properties of critical points, we introduce one last definition.

Definition 4 (Conic Neighbourhood Constant)**.** We define the *Conic Neighbourhood Constant* (CNC), denoted by $\rho_{\min}$, as

$$\rho_{\min} \triangleq \frac{1}{2} \min \left\{ d(p, R) \;\middle|\; p \in \mathcal{C},\, R \in \mathcal{R},\, p \notin R \right\}, \qquad (14)$$

i.e., $\rho_{\min}$ is half of the minimum over all critical points, of the distance of a critical point from the regions that do not contain it. We use the convention that the minimum of an empty set is infinite.

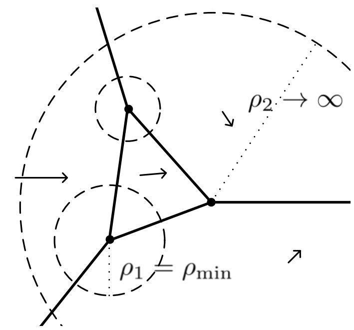


Fig. 2. A two-dimensional FPCS system with four regions and three critical points. The balls around the critical points show examples of associated basins. Here, the basin of the rightmost critical point can be taken equal to $\mathbb{R}^n$, thus containing the basins of all other critical points; cf. Lemma 3(c). The CNC $\rho_{\min}$ (cf. Definition 4) is also shown.

Note that $\rho_{\min}$ is always positive (and possibly infinite). We say that a dynamical system F is *conic* if $F = -\partial\Phi$, where Φ is of the form $\Phi(x) = \max_i \{-\mu_i^T (x-p)\}$, for some $p \in \mathbb{R}^n$. It is not hard to see that for such a conic system, either p is the only critical point or no critical points exist. It turns out that the "local" dynamics in the CNC-neighbourhood of a critical point of a general system are conic, hence the name CNC.

The lemma that follows lists a number of useful properties of critical points.

Lemma 3 (Properties of Critical Points)**.** *Consider an FPCS system F, with an associated set of critical points $\mathcal{C}$.*

(a) *A point in a region R_ν is a critical point if and only if it is an extreme point of R_ν. In particular, there are finitely many critical points.*

(b) *Consider a critical point $p \in \mathcal{C}$ and a basin radius ρ for p. Let $z(\cdot)$ be the unperturbed trajectory with initial point $z(0) = p$, and let $\xi = \dot{z}(0)$ be the actual drift at p. Then, before the time that $z(\cdot)$ exits the basin, $\dot{z}(t)$ is constant, and $z(t) = p + t\xi$, for all $t \in [0, \rho/\|\xi\|]$.*

(c) *If $\mathcal{C}$ is non-empty, then there exists a critical point $p \in \mathcal{C}$ such that the entire set $\mathbb{R}^n$ is a basin of p. In the special case where F is conic with a unique critical point p, the entire set $\mathbb{R}^n$ is a basin of p.*

(d) *The CNC, $\rho_{\min}$, defined in (14), is a basin radius for every critical point.*

(e) *Consider a basin radius ρ of a critical point $p \in \mathcal{C}$, an unperturbed trajectory $x(\cdot)$, and times $t_1 < t_2$. Suppose that $\|x(t_1) - p\| \le \rho/3$ and $\|x(t_2) - p\| > \rho$. Then, for any $t \ge t_2$, $\|x(t) - p\| > \rho/3$.*

(f) *Fix some $\lambda \in \mathbb{R}^n$ and consider $F'(\cdot) \triangleq F(\cdot) + \lambda$, which is also an FPCS system. Then, F and F' have the same set of regions $\mathcal{R}$, the same set of critical points $\mathcal{C}$, and the same CNC $\rho_{\min}$.*

In words, part (e) states that an unperturbed trajectory that starts near a critical point p and later goes sufficiently far from p, will never come back close to p. The proof of Lemma 3 is given in Appendix A.

B. Bounding the Deviation when the Trajectories are Far from the Set of Critical Points

In this subsection we bound the distance between perturbed and unperturbed trajectories, for the case where the perturbed trajectory stays far from the set of critical points. To do this, we will show that when far from the set of critical points, the local dynamics are similar to those of a lower-dimensional system, and then use the induction hypothesis (8).

We start with some definitions. For any $x \in \mathbb{R}^n$ and $r > 0$, let

$$\mathcal{U}_r(x) \triangleq \bigcup_{y:\, \|y-x\|_2 \le r} \mathcal{M}(y), \tag{15}$$

which is the set of possible drifts in the r-neighbourhood of x.

Definition 5 (Low-Dimensional Sets)**.** We call a subset $\mathcal{U} \subseteq \mathbb{R}^n$ *low-dimensional* if $\operatorname{span}\{x - y \,|\, x, y \in \mathcal{U}\} \neq \mathbb{R}^n$.

Equivalently, a set is low-dimensional if its affine span is not the entire space. If x is a critical point, then, by definition, the vectors in $\{\mu_i - \mu_j \mid \mu_i, \mu_j \in \mathcal{M}(x)\}$ span $\mathbb{R}^n$ and the set $\mathcal{U}_r(x)$ is *not* low-dimensional, for any $r > 0$. On the other hand, as asserted by the next lemma, which is proved in Appendix B, $\mathcal{U}_r(x)$ is low-dimensional when x is sufficiently far from critical points.

Lemma 4. *Consider an FPCS system with an associated set of critical points $\mathcal{C}$. There exists $\gamma \ge 1$ such that if $r > 0$ and $d(x, \mathcal{C}) > \gamma r$, then $\mathcal{U}_r(x)$ is low-dimensional.*

In the sequel, it will be convenient to compare the perturbed trajectory with an unperturbed trajectory that starts at the same state at some intermediate time. This motivates the following terminology.

Definition 6 (Coupled Trajectories)**.** Let $\theta, T \ge 0$ be some constants. Let $x(\cdot)$ be an unperturbed trajectory. Let $\widetilde{x}(\cdot)$ be a perturbed trajectory with a perturbation $U(\cdot)$ that satisfies (7). If in addition we have $\widetilde{x}(T) = x(T)$, we then say that $x(\cdot)$ and $\widetilde{x}(\cdot)$ are *θ-coupled at time T.*

The proof will now continue along the following lines. When far enough from the set of critical points, the set $\mathcal{U}_r(x)$ is low-dimensional (Lemma 4). This yields a description of the dynamics as the superposition of an essentially $(n-1)$-dimensional FPCS system and a constant drift.

Lemma 5. *Consider an FPCS system. There exists a constant $\sigma \ge 1$ such that the following statement holds for all $T, \theta > 0$. Let $x(\cdot)$ and $\widetilde{x}(\cdot)$ be a pair of θ-coupled trajectories at time 0. Suppose that $\mathcal{U} \subseteq \mathcal{M}$ is low-dimensional, and that $\mathcal{U}_{\sigma\theta}\big(x(t)\big) \subseteq \mathcal{U}$, for all $t \in [0, T]$. Then,*

$$\big\|\widetilde{x}(t) - x(t)\big\| \le \sigma\theta, \qquad \forall\, t \in [0, T]. \tag{16}$$

Moreover, for any $\lambda \in \mathbb{R}^n$, the same constant σ also applies to the FPCS system $F(\cdot) + \lambda$.

Proof. Let us fix some $\mu \in \mathcal{U}$. Let $\overline{\mathcal{U}}$ be the affine span of $\mathcal{U}$:

$$\overline{\mathcal{U}} \triangleq \mu + \operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\}. \tag{17}$$

Note that any choice of $\mu \in \mathcal{U}$ leads to the same set $\overline{\mathcal{U}}$. Let w be the projection of 0 onto $\overline{\mathcal{U}}$, i.e., the smallest norm element of $\overline{\mathcal{U}}$; cf. Fig. 3(a). Since $w \in \overline{\mathcal{U}}$, we have

$$w - \mu \in \operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\}. \tag{18}$$

Furthermore, by the orthogonality of projections, w is orthogonal to the difference of any two elements of $\overline{\mathcal{U}}$. In particular,

$$\begin{aligned} &w^T(\mu - \nu) = 0, \qquad \forall\, \nu \in \overline{\mathcal{U}}, \\ &w^T(\mu - w) = 0. \end{aligned} \tag{19}$$

Since $\mathcal{U}$ is low-dimensional, $\operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\}$ is a proper subset of $\mathbb{R}^n$. Let Y be a subspace of dimension $n-1$ that contains $\operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\}$ and is orthogonal to w.[5] Note that by the definition of $\overline{\mathcal{U}}$,

$$\begin{aligned} \overline{\mathcal{U}} - w &= \mu - w + \operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\} \\ &= \operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\} \\ &\subseteq Y, \end{aligned} \tag{20}$$

where the second equality is due to (18).

Any vector has an orthogonal decomposition as the sum of its projections on $Y^{\perp}$ (the orthogonal complement of Y) and Y; we use the subscripts w and Y to indicate the corresponding components, e.g.,

$$\begin{aligned} x &= x_w + x_Y, \\ \widetilde{x} &= \widetilde{x}_w + \widetilde{x}_Y, \\ U &= U_w + U_Y. \end{aligned} \tag{21}$$

We will now show that the w and Y components of a trajectory evolve without interacting, according to a one-dimensional system with drift w, and an $(n-1)$-dimensional system F_Y, respectively; see Fig. 3(c) for an illustration.

Claim 1. *Consider some $x \in \mathbb{R}^n$, and suppose that $\mathcal{M}(x) \subseteq \mathcal{U}$. Then, $F(x) = w + F_Y(x_Y)$, where $F_Y : Y \to 2^Y$ is an FPCS system on the $(n-1)$-dimensional subspace Y.*

Proof of Claim. Let $\overline{F}_Y(x) = F(x) - w$. Since $F(x)$ is contained in the convex hull of $\mathcal{U}$, it is also in the affine span of $\mathcal{U}$, i.e., $F(x) \subseteq \overline{\mathcal{U}}$. Then, (20) implies that

$$\overline{F}_Y(x) = F(x) - w \subseteq \overline{\mathcal{U}} - w \subseteq Y. \tag{22}$$

On the other hand, $\overline{F}_Y(x)$, being equal to $F(x) - w$, is the negative of the subdifferential of

$$\overline{\Phi}_Y(x) \triangleq \max_{\mu \in \mathcal{U}} \Big[\big(-(\mu - w)\big)^T x + b_\mu\Big]; \tag{23}$$

we used here the assumption $\mathcal{M}(x) \subseteq \mathcal{U}$. For any $x \in \mathbb{R}^n$,

$$\begin{aligned} \overline{\Phi}_Y(x) &= \overline{\Phi}_Y(x_w + x_Y) \\ &= \max_{\mu \in \mathcal{U}} \Big[-\big(\mu - w\big)^T\big(x_w + x_Y\big) + b_\mu\Big] \\ &= \max_{\mu \in \mathcal{U}} \Big[-\big(\mu - w\big)^T x_Y + b_\mu\Big] \\ &\triangleq \Phi_Y(x_Y), \end{aligned} \tag{24}$$

[5] If $w \neq 0$, then Y must be the orthogonal complement of the one-dimensional space spanned by w. If $w = 0$, then any $(n-1)$-dimensional subspace that contains $\operatorname{span}\big\{\nu - \mu \,\big|\, \nu \in \mathcal{U}\big\}$ will do, and the choice of Y need not be unique.

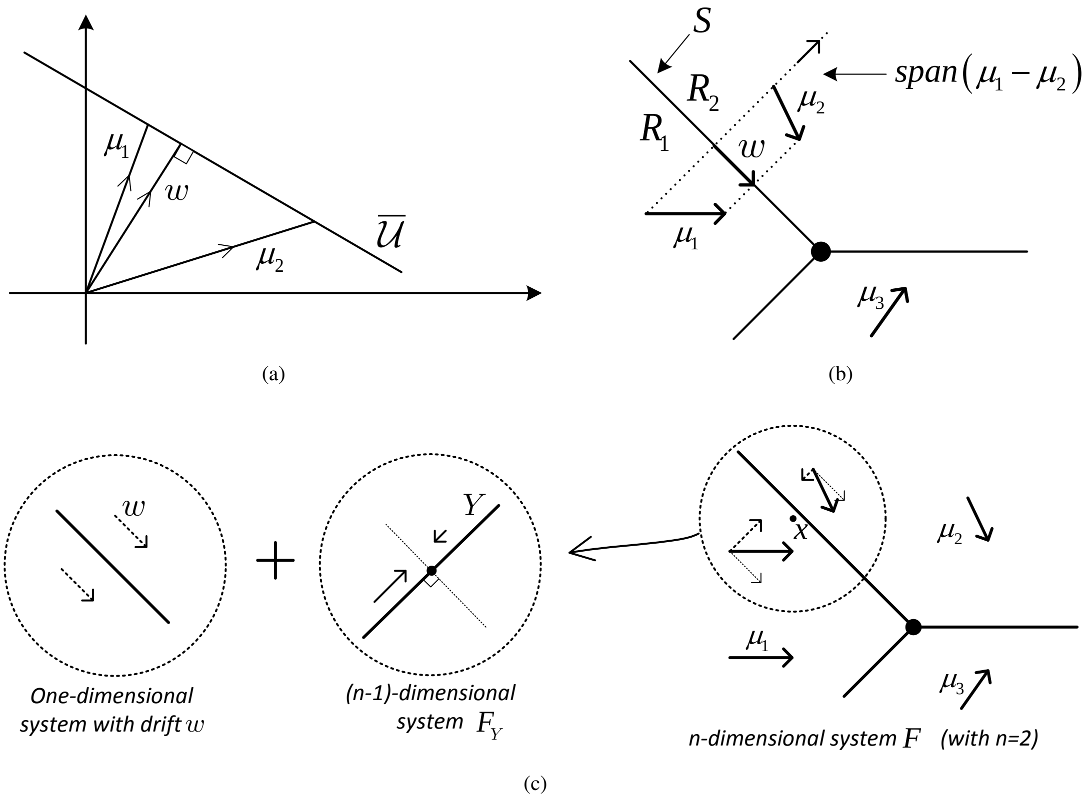


Fig. 3. (a) Illustration of the affine space $\overline{\mathcal{U}}$ and its minimum norm element w, when $\mathcal{U}$ consists of two elements μ_1 and μ_2. (b) Consider the intersection $S = \{x \mid -\mu_1^T x + b_{\mu_1} = -\mu_2^T x + b_{\mu_2}\}$ of the two regions R_{μ_1} and R_{μ_2}, and note that S is orthogonal to the span of $\mu_1 - \mu_2$. In this case, w is a direction of motion along S. (c) The dynamics can be decomposed as the superposition of a movement along S, in the direction of w, together with lower-dimensional hybrid dynamics in directions orthogonal to S.

where the third equality is due to (19). Let $F_Y : Y \to 2^Y$ be equal to $-\Phi_Y(\cdot)$. Then, F_Y is an FPCS system on the $(n-1)$-dimensional subspace Y. It follows from (24) that $\overline{\Phi}_Y(x)$ only depends on x_Y. Therefore, its negative subdifferential $\overline{F}_Y(x)$ also only depends on x_Y, and $\overline{F}_Y(x) = F_Y(x_Y)$. Hence, the definition $\overline{F}_Y(x) = F(x) - w$ implies that $F(x) = w + \overline{F}_Y(x) = w + F_Y(x_Y)$, which establishes the claim. ☐

We now appeal to the induction hypothesis (8), and let $C_{\mathcal{U}}$ be equal to the constant C_Y of Theorem 1 for the $(n-1)$-dimensional FPCS system F_Y. Let σ be the maximum of all such constants $C_{\widetilde{\mathcal{U}}}$ plus 4, over all low-dimensional subsets $\widetilde{\mathcal{U}} \subseteq \mathcal{M}$:

$$\sigma \triangleq \max\left\{C_{\widetilde{\mathcal{U}}} \;\middle|\; \widetilde{\mathcal{U}} \subseteq \mathcal{M} \text{ and } \widetilde{\mathcal{U}} \text{ is low-dimensional}\right\} + 4. \tag{25}$$

Suppose now that we add a constant drift λ to F. We observe that for any given low-dimensional $\mathcal{U}$, the resulting set-valued mapping F_Y only changes through the addition of a constant drift λ_Y; its structure remains otherwise the same. Hence, according to the induction hypothesis (8), $C_{\mathcal{U}}$ is not affected when we add a constant drift $\lambda \in \mathbb{R}^n$ to the dynamics. As a consequence, the value of σ associated with a system $F(\cdot)$ remains the same when we consider the system $F(\cdot) + \lambda$.

We now return to the main part of the proof of the lemma. We argue by contradiction, and assume that (16) fails to hold. Then, from the right-continuity of $x(t)$ and $\widetilde{x}(t)$, there exists a time $\widetilde{T} \leq T$ such that

$$\begin{aligned} &\big\|\widetilde{x}(\widetilde{T}) - x(\widetilde{T})\big\| \geq \sigma\theta, \\ &\big\|\widetilde{x}(t) - x(t)\big\| < \sigma\theta, \quad \forall\, t < \widetilde{T}. \end{aligned} \tag{26}$$

It follows from (26) and the assumption $\mathcal{U}_{\sigma\theta}\big(x(t)\big) \subseteq \mathcal{U}$ that, for any $t < \widetilde{T}$, $\mathcal{M}\big(x(t)\big) = \mathcal{U}_0\big(x(t)\big) \subseteq \mathcal{U}_{\sigma\theta}\big(x(t)\big) \subseteq \mathcal{U}$. Furthermore, $\mathcal{M}\big(\widetilde{x}(t)\big) \subseteq \mathcal{U}_{\sigma\theta}\big(x(t)\big) \subseteq \mathcal{U}$.

Consider some $t < \widetilde{T}$ and let $\widetilde{\zeta}(t) \in F\big(\widetilde{x}(t)\big)$ be a perturbed drift associated with the perturbed trajectory $\widetilde{x}(\cdot)$ (cf. Definition 1). It follows from Claim 1 that $\widetilde{\zeta}(t) - w \in F\big(\widetilde{x}(t)\big) - w = F_Y\big(\widetilde{x}(t)\big) \subseteq Y$. Thus, the orthogonal decomposition of $\widetilde{\zeta}(t)$ yields $\widetilde{\zeta}_Y(t) = \widetilde{\zeta}(t) - w$. This allows us to develop an orthogonal decomposition of $\widetilde{x}(t)$, as follows:

$$\begin{aligned} \widetilde{x}(t) &= \widetilde{x}(0) + \int_0^t \widetilde{\zeta}(\tau)\, d\tau + U(t) \\ &= \widetilde{x}(0) + \int_0^t \big(w + \widetilde{\zeta}_Y(\tau)\big)\, d\tau + U(t) \\ &= \Big[\widetilde{x}_w(0) + wt + U_w(t)\Big] \\ &\quad + \Big[\widetilde{x}_Y(0) + \int_0^t \widetilde{\zeta}_Y(\tau)\, d\tau + U_Y(t)\Big] \\ &= \widetilde{x}_w(t) + \widetilde{x}_Y(t), \end{aligned} \tag{27}$$

where the last equality follows because the two terms inside brackets belong to $Y^{\perp}$ and Y, respectively, and therefore provide the orthogonal decomposition of $\widetilde{x}(t)$.

Similarly, using also the assumption $x(0)=\widetilde{x}(0)$,

$$\begin{aligned} x(t) &= \Big[\widetilde{x}_w(0)+wt\Big] + \Big[\widetilde{x}_Y(0) + \int_0^t \zeta_Y(\tau)\,d\tau\Big] \\ &= x_w(t) + x_Y(t), \end{aligned} \tag{28}$$

with $\zeta_Y(t) \in F_Y\big(x(t)\big)$.

Note that x_Y is an unperturbed trajectory of the FPCS system F_Y, on the $(n-1)$-dimensional subspace Y. Moreover, since $\widetilde{\zeta}_Y(t) \in F_Y\big(\widetilde{x}(t)\big)$, $\widetilde{x}_Y$ is a perturbed trajectory of the same system, associated with the perturbation $U_Y(t)$. Since, $\big\|U_Y(\tau)\big\| \le \big\|U(\tau)\big\| \le \theta$, for all $\tau \ge 0$, it follows from the induction hypothesis that $\big\|\widetilde{x}_Y(t)-x_Y(t)\big\| \le C_{\mathcal{U}}\theta$ for $t<\widetilde{T}$. Then, for $t<\widetilde{T}$,

$$\begin{aligned} \big\|\widetilde{x}(t)-x(t)\big\| &\le \big\|\widetilde{x}_w(t)-x_w(t)\big\| + \big\|\widetilde{x}_Y(t)-x_Y(t)\big\| \\ &\le \big\|U_w(t)\big\| + C_{\mathcal{U}}\theta \\ &\le \theta + \big(\sigma-4\big)\theta \\ &= \big(\sigma-3\big)\theta. \end{aligned} \tag{29}$$

The proof at this point would have been complete, except that in order to bound $\big\|\widetilde{x}(\widetilde{T})-x(\widetilde{T})\big\|$, we need to account for the possibility that $U(t)$ has a jump at time $\widetilde{T}$. We have

$$\begin{aligned} \big\|\widetilde{x}(\widetilde{T})-x(\widetilde{T})\big\| \le\ & \limsup_{t\uparrow\widetilde{T}} \Big(\big\|\widetilde{x}(\widetilde{T})-\widetilde{x}(t)\big\| \\ &+ \big\|\widetilde{x}(t)-x(t)\big\| + \big\|x(t)-x(\widetilde{T})\big\|\Big) \\ \le\ & \limsup_{t\uparrow\widetilde{T}} \big\|\widetilde{x}(\widetilde{T})-\widetilde{x}(t)\big\| \\ &+ \limsup_{t\uparrow\widetilde{T}} \big\|\widetilde{x}(t)-x(t)\big\| \\ &+ \limsup_{t\uparrow\widetilde{T}} \big\|x(t)-x(\widetilde{T})\big\| \\ \le\ & \limsup_{t\uparrow\widetilde{T}} \left(\int_t^{\widetilde{T}} \widetilde{\zeta}(\tau)\,d\tau + U(\widetilde{T}) - U(t)\right) \\ &+ \big(\sigma-3\big)\theta + 0 \\ \le\ & 2\theta + \big(\sigma-3\big)\theta \\ =\ & \sigma\theta - \theta, \end{aligned} \tag{30}$$

where the third inequality is due to (29) and the continuity of $x(t)$, and the last inequality is due to (7) and the integrability of $\widetilde{\zeta}(\tau)$. Equation (30) contradicts (26), and the lemma follows. □

Suppose now that the perturbed trajectory stays far from the set of critical points throughout the time interval $[0,T]$. In light of Lemma 2(b), we can divide $[0,T]$ into a finite number of subintervals during which the unperturbed system has a constant drift, use Lemma 5 to obtain bounds on the distance of the perturbed and unperturbed trajectories during each subinterval, and then combine them to obtain a bound over the entire interval $[0,T]$.

Proposition 1. *Fix an FPCS system F. Consider the constant γ in Lemma 4, the constant σ in Lemma 5, and let $\eta = m2^{m+1}\sigma$, where m is the number of elements of the set $\mathcal{M}$ of drifts. Let $x(\cdot)$ and $\widetilde{x}(\cdot)$ be a pair of θ-coupled trajectories at time 0. If $d\big(\widetilde{x}(t),\mathcal{C}\big) \ge \gamma\eta\theta$ for all $t\in[0,T]$, then $\big\|\widetilde{x}(t)-x(t)\big\| \le \eta\theta$ for all $t\in[0,T]$.*

Proof. As already mentioned, we will divide the interval $[0,T]$ into at most $m2^m$ subintervals. We will then use Lemma 5 to show that the distance between the two trajectories can only increase by an additive factor of $\sigma\theta$ in each subinterval.

We define a sequence of times τ_k by letting $\tau_1=0$ and

$$\tau_{k+1} \triangleq \inf\Big\{ t\in(\tau_k,T] \;\Big|\; \mathcal{U}_{k\sigma\theta}\big(x(t)\big) \not\subseteq \mathcal{U}_{k\sigma\theta}\big(x(\tau_k)\big)\Big\}, \tag{31}$$

for $k\ge 1$, with the convention that $\tau_{k+1}=T$ if the set on the right-hand side of (31) is empty. In words, τ_{k+1} is the time that the $(k\sigma\theta)$-neighbourhood of the unperturbed trajectory touches a new region, which does not intersect with the $(k\sigma\theta)$-neighbourhood of $x(\tau_k)$. Let $K_{\max}$ be the maximum k such that $\tau_k<T$, so that $\tau_{K_{\max}+1}=T$. For $k\le K_{\max}$, we refer to the interval $[\tau_k,\tau_{k+1}]$ as phase k; see Fig. 4 for an illustration.

First, we show that the number of phases, $K_{\max}$, is less than $m2^m$. According to Lemma 2(b), the time interval $[0,T]$ can be partitioned into at most 2^m-1 subintervals $[z_j,z_{j+1})$, $1\le j\le 2^m-1$, during each of which the unperturbed trajectory $x(t)$ is a line segment; that is, there exists a sequence of vectors ξ_j, $1\le j\le 2^m-1$ such that

$$x(t) = x(z_j) + (t-z_j)\xi_j, \qquad \forall\, t\in[z_j,z_{j+1}]. \tag{32}$$

We argue that at most m phase changes are possible during a subinterval $[z_j,z_{j+1}]$, i.e., at most m of the times τ_ks lie in the interval $[z_j,z_{j+1}]$. Suppose that there are l phase changes (for some $l\ge 0$), at times $\tau_{k_j+1},\dots,\tau_{k_j+l}\in(z_j,z_{j+1}]$. For each $k\in\big\{k_j+1,\dots,k_j+l\big\}$, let $\mu_k\in\mathcal{M}$ be a drift that caused the phase change at time τ_k, i.e.,

$$\mu_k \in \mathcal{U}_{(k-1)\sigma\theta}\big(x(\tau_k)\big) \setminus \mathcal{U}_{(k-1)\sigma\theta}\big(x(\tau_{k-1})\big). \tag{33}$$

Equivalently,

$$d\big(x(\tau_{k-1}),\,R_k\big) \,>\, d\big(x(\tau_k),\,R_k\big) \,=\, (k-1)\sigma\theta, \tag{34}$$

where $R_k \triangleq R_{\mu_k}$ is the effective region of μ_k. We will now show that these regions R_k, for $k\in\big\{k_j+1,\dots,k_j+l\big\}$, are distinct for different k. In order to draw a contradiction, suppose that there are $k_1,k_2\in\big\{k_j+1,\dots,k_j+l\big\}$ with $k_1<k_2$ such that $\mu_{k_1}=\mu_{k_2}$, or equivalently $R\triangleq R_{k_1}=R_{k_2}$. Let $f:[z_j,z_{j+1}]\to\mathbb{R}_+$ be the distance between $x(t)$ and the region R:

$$f(t) \triangleq d\big(x(t)\,,\,R\big), \quad \forall\, t\in[z_j,z_{j+1}]. \tag{35}$$

The region R is a convex set. Therefore $f(\cdot)$ is the composition of a convex function (the distance from R) and an affine function $x(t):[z_j,z_{j+1}]\to\mathbb{R}^n$ (see (32)). Hence, f is also convex. Moreover, it follows from (34) and the assumption $k_1<k_2$ that

$$f(\tau_{k_2-1}) \,>\, f(\tau_{k_2}) \,=\, (k_2-1)\sigma\theta \,>\, (k_1-1)\sigma\theta \,=\, f(\tau_{k_1}). \tag{36}$$

However, since f is convex and $\tau_{k_1}\le\tau_{k_2-1}\le\tau_{k_2}$, we must have $f(\tau_{k_2-1}) \le \max\big(f(\tau_{k_1})\,,\,f(\tau_{k_2})\big)$, which contradicts (36). Hence, each distinct $k_i\in\big\{k_j+1,\dots,k_j+l\big\}$ is associated with a disticnt region R_{k_i}. On the other hand, since

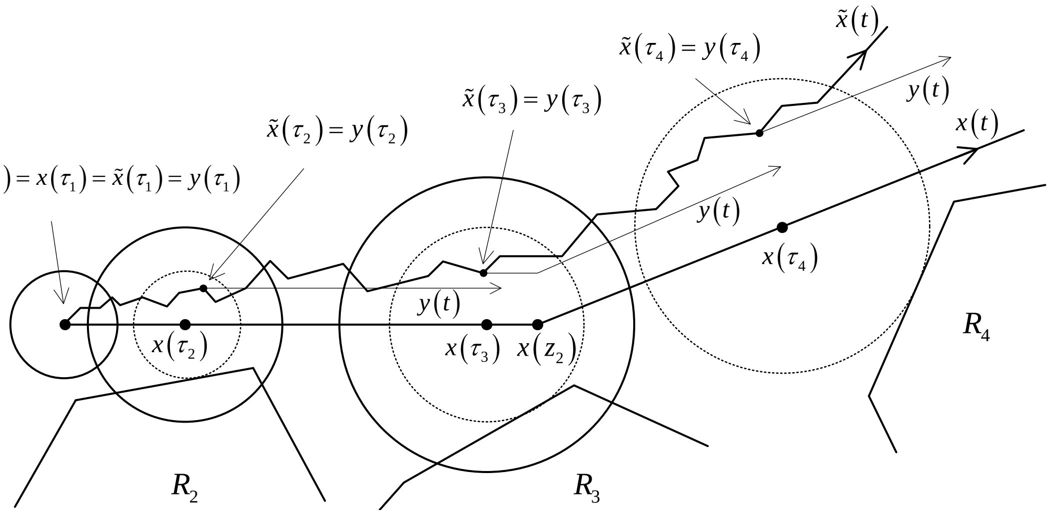


Fig. 4. An illustration of the different phases and variables used in the proof of Proposition 1 (not all regions are shown). A new phase begins at time τ_{k+1} when the $(k\sigma\theta)$-neighbourhood of the unperturbed trajectory, $x(\cdot)$, touches a new region, R_{k+1}. At the beginning of a phase, an auxiliary unperturbed trajectory $y(\cdot)$ is coupled with the perturbed trajectory $\widetilde{x}(\cdot)$. The dotted circle is a translation of a solid circle, centered at $x(t)$. As soon as the dotted circle touches a new region boundary, a new solid circle, with larger radius is created.

the number of different regions is at most m, there are at most m phase changes during each of the at most $2^m - 1$ line segments in the trajectory of $x(\cdot)$, and the total number of phases, $K_{\max}$, is smaller than $m2^m$.

In our next step, we use induction on the phases to show that if $k \le K_{\max}$, then

$$\big\|\widetilde{x}(t) - x(t)\big\| \le k\sigma\theta, \quad \forall\, t \in [\tau_k, \tau_{k+1}]. \tag{37}$$

For any $k \ge 1$, we consider the induction hypothesis

$$\big\|\widetilde{x}(\tau_k) - x(\tau_k)\big\| \le (k-1)\sigma\theta. \tag{38}$$

Note that (38) is automatically true for $k = 1$, because $\tau_1 = 0$ and $\widetilde{x}(0) - x(0)$ has been assumed to be zero. This provides the basis of the induction. Using the triangle inequality and the inequalities $\gamma \ge 1$ and $\eta = m2^{m+1}\sigma \ge 2K_{\max}\sigma \ge 2k\sigma$, we obtain

$$\begin{aligned} d\big(x(\tau_k), \mathcal{C}\big) &\ge d\big(\widetilde{x}(\tau_k), \mathcal{C}\big) - \big\|x(\tau_k) - \widetilde{x}(\tau_k)\big\| \\ &\ge \gamma\eta\theta - (k-1)\sigma\theta \\ &\ge 2\gamma k\sigma\theta - (k-1)\sigma\theta \\ &\ge \gamma k\sigma\theta. \end{aligned} \tag{39}$$

Let $\mathcal{U} = \mathcal{U}_{k\sigma\theta}\big(x(\tau_k)\big)$. It follows from (39) and Lemma 4, with $r = k\sigma\theta$, that $\mathcal{U}$ is low-dimensional. Furthermore, the definition of τ_{k+1} in (31) implies that

$$\mathcal{U}_{k\sigma\theta}\big(x(t)\big) \subseteq \mathcal{U}, \qquad \forall\, t \in [\tau_k, \tau_{k+1}). \tag{40}$$

Let $y(\cdot)$ be an unperturbed trajectory with initial condition $y(\tau_k) = \widetilde{x}(\tau_k)$. Since the unperturbed dynamics are non-expansive, for any $t \ge \tau_k$, we have

$$\begin{aligned} \big\|x(t) - y(t)\big\| &\le \big\|x(\tau_k) - y(\tau_k)\big\| \\ &= \big\|x(\tau_k) - \widetilde{x}(\tau_k)\big\| \\ &\le (k-1)\sigma\theta. \end{aligned} \tag{41}$$

It is not hard to see that (41) and (40) imply that

$$\mathcal{U}_{\sigma\theta}\big(y(t)\big) \subseteq \mathcal{U}, \qquad \forall\, t \in [\tau_k, \tau_{k+1}), \tag{42}$$

Hence, the conditions of Lemma 5 hold, with the initial time being τ_k instead of zero. Therefore, $\big\|\widetilde{x}(t) - y(t)\big\| \le \sigma\theta$, for all $t \in [\tau_k, \tau_{k+1}]$. As a result, for $t \in [\tau_k, \tau_{k+1}]$,

$$\begin{aligned} \big\|\widetilde{x}(t) - x(t)\big\| &\le \big\|\widetilde{x}(t) - y(t)\big\| + \big\|y(t) - x(t)\big\| \\ &\le \sigma\theta + (k-1)\sigma\theta \\ &= k\sigma\theta, \end{aligned} \tag{43}$$

where the second inequality is due to (41). This establishes (37) and, in particular, that (38) holds with k replaced by $k+1$ (the induction step). Finally, the proposition follows from (37) and the fact that $k \le K_{\max} < m2^m$. $\square$

C. Proof of the Bound when Close to a Critical Point

In Proposition 1, we presented a bound on the distance between the trajectories when there are no nearby critical points. The next proposition deals with the other extreme, where the trajectories are in a basin of a critical point.

Proposition 2. *Consider two constants $\theta, T > 0$, a critical point $p \in \mathcal{C}$ and a basin $\mathcal{B}_\rho$ of radius ρ for p. Let $x(\cdot)$ and $\widetilde{x}(\cdot)$ be a pair of θ-coupled trajectories at time 0, with $\widetilde{x}(t) \in \mathcal{B}_\rho$, for all $t \in [0, T]$. Suppose that $0 < r < \rho$, with ρ possibly infinite, and that for the ball $\mathcal{B}_r$ of radius r centered at p,*

$$d\big(\mathcal{B}_\rho \backslash \mathcal{B}_r\,,\, \mathcal{C}\big) \;\ge\; (\gamma + 1)\eta\theta, \tag{44}$$

where $\eta = m2^{m+1}\sigma$ and γ are the constants defined in Proposition 1 and Lemma 4, respectively. Then, $\big\|\widetilde{x}(t) - x(t)\big\| \le 4r$, for all $t \in [0, T]$.

Proof. Since the balls $\mathcal{B}_\rho$ and $\mathcal{B}_r$ are centered at p, we have $d\big(\mathcal{B}_\rho \backslash \mathcal{B}_r\,,\, \mathcal{C}\big) \le d\big(\mathcal{B}_\rho \backslash \mathcal{B}_r\,,\, p\big) = r$ and, from (44), $r > \eta\theta$. Let

$$r_1 \triangleq r - \eta\theta, \quad r_2 \triangleq r_1 + 3\theta, \tag{45}$$

and consider two balls $\mathcal{B}_{r_1}$ and $\mathcal{B}_{r_2}$ centered at p, with radii r_1 and r_2, respectively. Since p is a critical point, it is in the intersection of at least two regions. Therefore, the number m of elements of the set $\mathcal{M}$ of drifts is at least two, and

$$\eta = m2^{m+1}\sigma \geq m2^{m+1} \geq 16. \tag{46}$$

As a result, $r_2 \leq r$ and $\mathcal{B}_{r_1} \subset \mathcal{B}_{r_2} \subset \mathcal{B}_r \subset \mathcal{B}_\rho$; see Fig. 5.

Consider the region $\mathcal{B}_\rho \backslash \mathcal{B}_{r_1}$ between $\mathcal{B}_{r_1}$ and $\mathcal{B}_\rho$. It follows from (44) that

$$\begin{aligned} d\big(\mathcal{B}_\rho \backslash \mathcal{B}_{r_1}, \mathcal{C}\big) &\geq d\big(\mathcal{B}_\rho \backslash \mathcal{B}_r, \mathcal{C}\big) - \big(r - r_1\big) \\ &\geq \big(\gamma + 1\big)\eta\theta - \eta\theta \\ &= \gamma\eta\theta. \end{aligned} \tag{47}$$

The high level idea is that for any $t \in [0, T]$, the perturbed solution is either in $\mathcal{B}_{r_2}$ or in $\mathcal{B}_\rho \backslash \mathcal{B}_{r_1}$ (or possibly in both). When $\widetilde{x}(t) \in \mathcal{B}_{r_2}$ we will show that the unperturbed trajectory is also close to the critical point p. But the more interesting case is when $\widetilde{x}(t) \in \mathcal{B}_\rho \backslash \mathcal{B}_{r_1}$. The idea here is to look at the perturbed solution, and at certain times that it hits the boundary of $\mathcal{B}_{r_2}$, consider an auxiliary unperturbed trajectory that is coupled with $\widetilde{x}(t)$ at that time. Using Proposition 1, we can then show that these coupled trajectories stay close to each other, as long as the perturbed trajectory stays in $\mathcal{B}_\rho \backslash \mathcal{B}_{r_1}$. As a result, and using also the fact that the dynamics are non-expansive, the auxiliary trajectory $\widetilde{x}(t)$ will remain close to $x(t)$, and the distance $\big\|\widetilde{x}(t) - x(t)\big\|$ stays bounded. The various parameters and trajectories are illustrated in Fig. 5.

Let $T_0^{out} = 0$, and for any $i \geq 1$ let

$$T_i^{in} \triangleq \inf\Big\{t \in (T_{i-1}^{out}, T] \;\Big|\; \widetilde{x}(t) \in \mathcal{B}_{r_1}\Big\}, \tag{48}$$

$$T_i^{out} \triangleq \inf\Big\{t \in (T_i^{in}, T] \;\Big|\; \widetilde{x}(t) \notin \mathcal{B}_{r_2}\Big\}. \tag{49}$$

If either set is empty, we let the left hand side be equal to T. We consider a number of rounds. Round i starts at time T_i^{out} and ends at time T_{i+1}^{in}. Note that the union of these rounds does not necessarily cover $[0, T]$. Also, note that since there is a gap of size $3\theta > 2\theta$ between the boundaries of $\mathcal{B}_{r_1}$ and $\mathcal{B}_{r_2}$, it takes some lower-bounded positive time for the perturbed trajectory to travel from one boundary to the other, and hence the length of each round is lower bounded by a positive constant. So, the number of rounds during $[0, T]$ is finite.

To each round i we associate an unperturbed trajectory, denoted by $x^i(t)$, $t \in [T_i^{out}, T_{i+1}^{in}]$, with initial point $x^i(T_i^{out}) = \widetilde{x}(T_i^{out})$, i.e., $x^i(\cdot)$ is coupled with the perturbed trajectory at time T_i^{out}. For any $t \in [T_i^{out}, T_{i+1}^{in}]$, since $\widetilde{x}(t) \in \mathcal{B}_\rho \backslash \mathcal{B}_{r_1}$, (47) asserts that $d\big(\widetilde{x}(t), \mathcal{C}\big) \geq \gamma\eta\theta$. Therefore, it follows from Proposition 1 that for any $t \in [T_i^{out}, T_{i+1}^{in}]$,

$$\big\|\widetilde{x}(t) - x^i(t)\big\| \leq \eta\theta. \tag{50}$$

Note that $x^0(t) = x(t)$, for all $t \geq 0$. Thus, if $T_1^{in} = T$, the inequality (50) together with the fact $\eta\theta < r < 4r$ imply that $\big\|\widetilde{x}(t) - x(t)\big\| < 4r$, for all $t \in [0, T]$, as desired. So, in the following we assume that $T_1^{in} < T$. Note that the right-continuity of $\widetilde{x}(\cdot)$ implies that $\big\|\widetilde{x}(T_1^{in}) - p\big\| \leq r_1$. Let $z(\cdot)$ be an unperturbed trajectory that starts at the critical point p at time T_1^{in}, i.e., $z\big(T_1^{in}\big) = p$. It follows from (50) and the non-expansive property of the dynamics that for any $t \geq T_1^{in}$,

$$\begin{aligned} \big\|x(t) - z(t)\big\| &\leq \big\|x(T_1^{in}) - z(T_1^{in})\big\| \\ &= \big\|x(T_1^{in}) - p\big\| \\ &\leq \big\|p - \widetilde{x}(T_1^{in})\big\| + \big\|\widetilde{x}(T_1^{in}) - x(T_1^{in})\big\| \\ &\leq r_1 + \eta\theta \\ &= r, \end{aligned} \tag{51}$$

where in the last inequality we used (50) with $i = 0$ and $t = T_1^{in}$. We now proceed to derive a bound on $\big\|\widetilde{x}(t) - x(t)\big\|$, for $t \geq T_1^{in}$, by developing a bound on $\big\|\widetilde{x}(t) - z(t)\big\|$. Let $\xi = \xi(p)$ be the actual drift at p. We consider two cases.

Case 1. (p is an equilibrium point, i.e., $\xi = 0$)**.** In this case, $z(t) = p$, for all $t \geq T_1^{in}$. Comparing with the unperturbed trajectory $x^i(\cdot)$ and using the non-expansive property and the definition of T_i^{out}, it follows that for any round $i \geq 1$ and any $t \in [T_i^{out}, T_{i+1}^{in}]$,

$$\begin{aligned} \big\|x^i(t) - z(t)\big\| &\leq \big\|x^i(T_i^{out}) - z(T_i^{out})\big\| \\ &= \big\|\widetilde{x}(T_i^{out}) - p\big\| \\ &= r_2 \;\leq\; 3r_2. \end{aligned} \tag{52}$$

Combining this with (50) and (51), we get the following bound for all $i \geq 1$ and for all $t \in [T_i^{out}, T_{i+1}^{in}]$,

$$\begin{aligned} \big\|\widetilde{x}(t) - x(t)\big\| &\leq \big\|\widetilde{x}(t) - x^i(t)\big\| + \big\|x^i(t) - z(t)\big\| \\ &\quad + \big\|z(t) - x(t)\big\| \\ &\leq \eta\theta + 3r_2 + r \\ &= \eta\theta + 3\big(r - \eta\theta + 3\theta\big) + r \\ &= 4r + 2\big(4.5 - \eta\big)\theta \\ &< 4r, \end{aligned} \tag{53}$$

where the second inequality is due to (50), (52), and (51), and the last inequality is due to (46).

Furthermore, for any $t \in [T_i^{in}, T_i^{out})$, our definitions imply that $\big\|\widetilde{x}(t) - z(t)\big\| = \big\|\widetilde{x}(t) - p\big\| < r_2$. Hence, for such t,

$$\big\|\widetilde{x}(t) - x(t)\big\| \leq \big\|\widetilde{x}(t) - z(t)\big\| + \big\|z(t) - x(t)\big\| \leq r_2 + r < 4r,$$

where the second inequality is due to (51). Thus, the proposition holds in this case.

Case 2. (p is not an equilibrium point, i.e., $\xi \neq 0$)**.** The dynamics in this case are illustrated in Fig. 5. Here, we need to find an alternative derivation of (52), and also derive a new bound for $\big\|\widetilde{x}(t) - z(t)\big\|$ when $t \in [T_i^{in}, T_i^{out})$.

Let $\zeta(\cdot)$ be a perturbed drift associated with the perturbed trajectory $\widetilde{x}(\cdot)$. Since $\mathcal{B}_\rho$ is a basin and $\zeta(t) \in F\big(\widetilde{x}(t)\big)$, it follows from the definition of basins that for any $t \in [0, T]$,

$$\xi^T \zeta(t) \geq \big\|\xi\big\|^2. \tag{54}$$

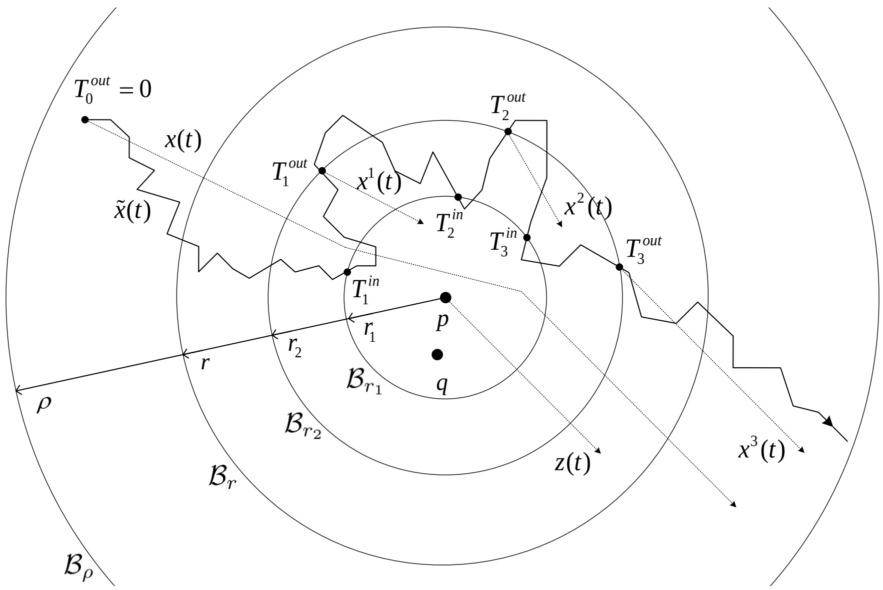


Fig. 5. An illustration of the balls, the different rounds, and the variables used in the proof of Proposition 2. Here, p and q are two critical points, and T_i^{in} and T_i^{out} are defined in (48) and (49), respectively. There are four balls $\mathcal{B}_{r_1}$, $\mathcal{B}_{r_2}$, $\mathcal{B}_r$, and $\mathcal{B}_\rho$ all centered at p, with radii r_1, r_2, r, and ρ, respectively. Also, $x(\cdot)$ is an unperturbed trajectory that is coupled with the perturbed trajectory $\widetilde{x}(\cdot)$ at time 0, $z(\cdot)$ is an unperturbed trajectory that starts at p at time T_1^{in}, and each $x^i(\cdot)$ is an unperturbed trajectory that is coupled with $\widetilde{x}(\cdot)$ at time T_i^{out}.

Hence, for any $t \in [T_1^{in}, T]$,

$$\begin{aligned}\big\|\widetilde{x}(t)-p\big\| &\geq \frac{1}{\|\xi\|}\xi^T\big(\widetilde{x}(t)-p\big)\\ &= \frac{1}{\|\xi\|}\Bigg(\xi^T\Big(\widetilde{x}\big(T_1^{in}\big)-p\Big) + \int_{T_1^{in}}^{t}\xi^T\zeta(\tau)\,d\tau\\ &\qquad + \xi^T\big(U(t)-U(T_1^{in})\big)\Bigg)\\ &\geq -\big\|\widetilde{x}\big(T_1^{in}\big)-p\big\| + \frac{1}{\|\xi\|}\int_{T_1^{in}}^{t}\|\xi\|^2\,d\tau \,-\, 2\theta\\ &\geq -r_1 + (t-T_1^{in})\|\xi\| - 2\theta.\end{aligned} \tag{55}$$

The first inequality above is the Cauchy-Schwarz inequality; the equality follows from the definition of perturbed trajectories (cf. Definition 1); the next inequality uses the Cauchy-Schwarz inequality for the first term, (54) for the second, and the bounds on $U(\cdot)$ for the third; the last inequality uses the defining property $\big\|\widetilde{x}\big(T_1^{in}\big)-p\big\| = r_1$ of T_1^{in}.

We define an escape time $T^{esc} = T_1^{in} + \big(r_1 + r_2 + 3\theta\big)/\|\xi\|$. The following claim suggests that if $\widetilde{x}(t)$ ever escapes $\mathcal{B}_{r_2}$, it happens before time T^{esc}.

Claim 2. *If $T_i^{in} < T$ for some $i \geq 1$, then $T_i^{out} \leq T^{esc}$.*

Proof of Claim. If $T^{esc} \geq T$, then $T_i^{out} \leq T \leq T^{esc}$. Suppose now that $T^{esc} < T$. It follows from (55) and the definition of T^{esc} that $\big\|\widetilde{x}\big(T^{esc}\big) - p\big\| \geq r_2 + \theta > r_2$. Hence $\widetilde{x}(t)$ is outside of $\mathcal{B}_{r_2}$ at time T^{esc}, and by definition, $T_i^{out} \leq T^{esc}$. □

Since $z(T_1^{in}) = p$, we have $\big\|\dot{z}(0)\big\| = \xi$. According to Lemma 2(b), $\big\|\dot{z}(t)\big\|$ is a non-increasing function of time, and therefore $\big\|\dot{z}(t)\big\| \leq \big\|\xi\big\|$, for $t \geq T_1^{in}$. Hence, if $t \in \big[T_1^{in}, T^{esc}\big]$, then

$$\begin{aligned}\big\|z(t)-p\big\| &\leq \big(t-T_1^{in}\big)\big\|\xi\big\|\\ &\leq \big(T^{esc}-T_1^{in}\big)\big\|\xi\big\|\\ &= r_2+r_1+3\theta\\ &= 2r_2.\end{aligned} \tag{56}$$

We now proceed by considering two cases: $t \in \big[T_i^{out}, T_{i+1}^{in}\big)$ and $t \in \big[T_i^{in}, T_i^{out}\big)$. We first suppose that $T_i^{out} < T$, and consider a $t \in \big[T_i^{out}, T_{i+1}^{in}\big)$. Since $T_i^{in} \leq T_i^{out} < T$, Claim 2 implies that $T_i^{out} \leq T^{esc}$. It then follows from (56) that $\big\|z\big(T_i^{out}\big) - p\big\| \leq 2r_2$. Then,

$$\begin{aligned}\big\|x^i(t)-z(t)\big\| &\leq \big\|x^i\big(T_i^{out}\big)-z\big(T_i^{out}\big)\big\|\\ &\leq \big\|x^i\big(T_i^{out}\big)-p\big\| + \big\|p-z\big(T_i^{out}\big)\big\|\\ &\leq r_2 + 2r_2\\ &= 3r_2.\end{aligned} \tag{57}$$

where the first inequality is due to the non-expansive property, and the last inequality is due to the definition of T_i^{out}. Thus, the bound (52) and the subsequent derivation of (53) remain valid for this case as well, so that for every round i,

$$\big\|\widetilde{x}(t)-x(t)\big\| < 4r, \qquad \forall\ t \in \big[T_i^{out}, T_{i+1}^{in}\big). \tag{58}$$

We now discuss the case where t does not belong to a round, i.e., $t \in \big[T_i^{in}, T_i^{out}\big)$, for some i such that $T_i^{in} < T$. It follows from Claim 2 that $t < T_i^{out} \leq T^{esc}$. Therefore, (56) implies that if $t \in \big[T_i^{in}, T_i^{out}\big)$, then $\big\|p - z(t)\big\| \leq 2r_2$. Therefore,

$$\begin{aligned}\big\|\widetilde{x}(t)-x(t)\big\| &\leq \big\|\widetilde{x}(t)-p\big\| + \big\|p-z(t)\big\| + \big\|z(t)-x(t)\big\|\\ &\leq r_2 + 2r_2 + r\\ &< 4r,\end{aligned}$$

where the second inequality is due to $t \in \big[T_i^{in}, T_i^{out}\big)$ and (51). Together with (58), this completes the argument for Case 2, and the proof of the proposition. $\square$

D. Completing the Proof of the Theorem

We now use the machinery developed in this section and combine the results for the various cases to complete the proof of Theorem 1.

If there are no critical points, then the perturbed trajectory never gets close to a critical point, and the theorem follows from Proposition 1. In the following, we assume that the set of critical points is non-empty. Let M be the number of critical points, let $\rho_{\min}$ be the CNC, and let $D^{\mathcal{C}}$ be the diameter of the set of critical points:

$$D^{\mathcal{C}} \triangleq \max_{p,q\in\mathcal{C}} \big\|p-q\big\|. \tag{59}$$

According to Lemma 3(a), there are finitely many critical points, so that $D^{\mathcal{C}}$ is well-defined and finite. We define a threshold parameter θ^* as follows:

$$\theta^* \triangleq \frac{\rho_{\min}}{40\big(M+2\big)\big(\gamma+1\big)\eta}, \tag{60}$$

where γ and η are the constants defined in Lemma 4 and Proposition 1, respectively. In what follows, we use Proposition 2 to prove that the following constant satisfies Theorem 1,

$$C \;=\; \begin{cases} 4D^{\mathcal{C}}/\theta^* \,+\, 5\big(M+2\big)\big(\gamma+1\big)\eta, & \text{if } \theta^* \neq 0,\\ 4\big(\gamma+1\big)\eta+1, & \text{if } \theta^*=0. \end{cases} \tag{61}$$

As an example, for the system illustrated in Fig. 1, it can be checked that the above constants are as follows: $D^{\mathcal{C}} = \theta^* = 0$, $M=1$, $\gamma=1$, $\sigma=5$, $\eta=240$, and $C=1921$.

We consider two cases, depending on whether the perturbation bound θ is larger or smaller than the threshold θ^*.

Case 1 ($\theta \geq \theta^*$)**.** According to Lemma 3(c), there exists a critical point p^*, for which the entire set $\mathbb{R}^n$ is a basin. We let $r = D^{\mathcal{C}} + \big(\gamma+1\big)\eta\theta$ and $\rho=\infty$. This choice of p^*, r, and ρ observes the conditions of Proposition 2. Note that if $\theta^*=0$, then $\rho_{\min}=0$, in which case there is at most one critical point. Then, $\theta^*=0$ implies $D^{\mathcal{C}}=0$. Therefore, $4r < C\theta$, for all values of θ^*. It then follows from Proposition 2 that

$$\big\|\widetilde{x}(t)-x(t)\big\| \leq 4r < C\theta, \qquad \forall t\geq 0, \tag{62}$$

which establishes the desired result.

Case 2 ($\theta < \theta^*$)**.** Once again, we rely on Proposition 2, but in a local manner. We consider a "small" basin of size $\rho_{\min}/2$ for each critical point, and define a number of phases $\big[T_i^{in}, T_i^{out}\big]$ so that throughout any particular phase, the perturbed trajectory lies in one of these basins. We then use Proposition 2 to bound the distance between the two trajectories in each phase, and use Proposition 1 to bound their distance while outside the basins. In the end, we use Lemma 3(e) to show that each basin is visited at most once, in a certain sense, and finally put everything together to prove the desired bound on the distance of the two trajectories. Figure 6 shows an illustration of the different trajectories and variables that we use in the argument that follows.

Let

$$r = \big(\gamma+1\big)\eta\theta, \qquad \rho = \rho_{\min}/2. \tag{63}$$

It follows from (60) and the assumption $\theta<\theta^*$ that $r<\rho$. Moreover, based on Lemma 3(d), ρ is a basin radius for each one of the critical points. For any critical point $p\in\mathcal{C}$, let $\mathcal{B}_r(p)$ and $\mathcal{B}_\rho(p)$ be the balls of radii r and ρ, respectively, centered at p. We define two sequences of times T_i^{in} and T_i^{out} as follows. Let $T_0^{out}=0$, and for any $i\geq 1$, let

$$T_i^{in} \triangleq \inf\Big\{t > T_{i-1}^{out} \;\Big|\; \exists\, p\in\mathcal{C}:\ \widetilde{x}(t)\in\mathcal{B}_r(p)\Big\}. \tag{64}$$

We denote by p^i the critical point p in the right-hand side of (64), so that $\widetilde{x}(T_i^{in})\in\mathcal{B}_r(p^i)$. Note that the different balls $\mathcal{B}_r(p)$ do not intersect and therefore p^i is uniquely defined; we refer to it as the *effective critical point* at phase i. We then define

$$T_i^{out} \triangleq \inf\Big\{t > T_i^{in} \;\Big|\; \widetilde{x}(t)\notin\mathcal{B}_\rho(p^i)\Big\}. \tag{65}$$

In the above, we let T_i^{in} or T_i^{out} be infinite in case the set on the right-hand side of (64) or (65) is empty.

Fix an $i\geq 1$. We first derive a bound on $\big\|\widetilde{x}(t)-x(t)\big\|$ for $t\in\big[T_{i-1}^{out}, T_i^{in}\big]$. Let $y^i(\cdot)$ be an unperturbed trajectory with $y^i\big(T_{i-1}^{out}\big)=\widetilde{x}\big(T_{i-1}^{out}\big)$. By definition, for any $t\in\big[T_{i-1}^{out}, T_i^{in}\big]$, $d\big(\widetilde{x}(t),\mathcal{C}\big) \;\geq\; r \;=\; \big(\gamma+1\big)\eta\theta$. Therefore, it follows from Proposition 1 that

$$\big\|\widetilde{x}(t)-y^i(t)\big\| \leq \eta\theta < r, \quad \forall\, t\in\big[T_{i-1}^{out}, T_i^{in}\big]. \tag{66}$$

Hence, from the non-expansive property of the unperturbed dynamics, for any $t\in\big[T_{i-1}^{out}, T_i^{in}\big]$, we obtain

$$\begin{aligned} \big\|\widetilde{x}(t)-x(t)\big\| \;&\leq\; \big\|x(t)-y^i(t)\big\| + \big\|\widetilde{x}(t)-y^i(t)\big\| \\ &\leq\; \big\|x\big(T_{i-1}^{out}\big)-y^i\big(T_{i-1}^{out}\big)\big\| + r \\ &=\; \big\|x\big(T_{i-1}^{out}\big)-\widetilde{x}\big(T_{i-1}^{out}\big)\big\| + r. \end{aligned} \tag{67}$$

On the other hand, if $t\in\big[T_i^{in}, T_i^{out}\big]$, for some $i\geq 1$, we have for any critical point $p\in\mathcal{C}$ other than p^i,

$$\begin{aligned} d\big(p,\,\mathcal{B}_\rho(p^i)\big) \;&\geq\; \big\|p-p^i\big\| - \rho \\ &\geq\; \rho_{\min} - \rho \\ &=\; \frac{\rho_{\min}}{2} \\ &\geq\; \big(\gamma+1\big)\eta\theta^* \\ &>\; r, \end{aligned} \tag{68}$$

where the second and third inequalities follow from the definitions of $\rho_{\min}$ and θ^* in (14) and (60), respectively. Using also the fact that $d\big(p^i,\ \mathcal{B}_\rho(p^i)\backslash\mathcal{B}_r(p^i)\big)\geq r$, we obtain

$$d\big(\mathcal{B}_\rho(p^i)\backslash\mathcal{B}_r(p^i)\,,\,\mathcal{C}\big) \;\geq\; r \;=\; \big(\gamma+1\big)\eta\theta. \tag{69}$$

Moreover, for any $t\in\big[T_i^{in}, T_i^{out}\big]$, $\widetilde{x}(t)\in\mathcal{B}_\rho(p^i)$. Hence, the conditions of Proposition 2 are observed. Let $x^i(t)$ be an unperturbed trajectory with $x^i\big(T_i^{in}\big)=\widetilde{x}\big(T_i^{in}\big)$. Proposition 2 implies that, for any $t\in\big[T_i^{in}, T_i^{out}\big]$, $\big\|\widetilde{x}(t)-x^i(t)\big\|\leq 4r$. Hence, from the non-expansiveness of the dynamics, we have for any $t\in\big[T_i^{in}, T_i^{out}\big]$,

$$\begin{aligned} \big\|\widetilde{x}(t)-x(t)\big\| \;&\leq\; \big\|x(t)-x^i(t)\big\| + \big\|\widetilde{x}(t)-x^i(t)\big\| \\ &\leq\; \big\|x\big(T_i^{in}\big)-x^i\big(T_i^{in}\big)\big\| + 4r \\ &=\; \big\|x\big(T_i^{in}\big)-\widetilde{x}\big(T_i^{in}\big)\big\| + 4r. \end{aligned} \tag{70}$$

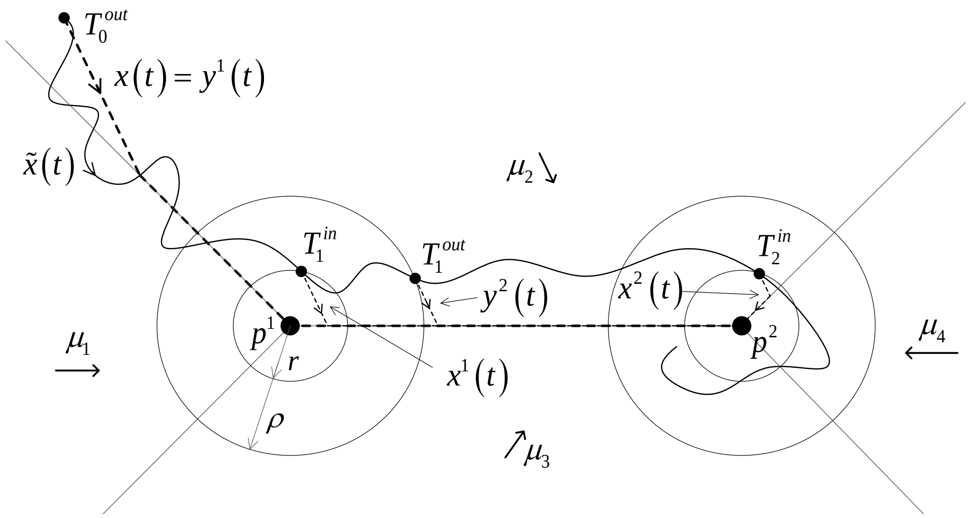


Fig. 6. Illustration of the different trajectories and variables in Case 2 of the proof of Theorem 1. The figure shows a two-dimensional FPCS system with four regions and two critical points, p^1 and p^2. There are two balls of radii r and ρ, defined in (63), centered at each critical point. The solid curved line $\widetilde{x}(\cdot)$ is a perturbed trajectory, coupled with an unperturbed trajectory $x(\cdot)$ at time 0. Times T_i^{in} and T_i^{out}, defined in (64) and (65), are the first times that the perturbed trajectory hits the ball of radius r or leaves the larger ball of radius ρ around p^i, respectively. In this example, $T_2^{out} = \infty$, because the perturbed trajectory never leaves the ρ-neighbourhood of p^2. For each i, $x^i(\cdot)$ and $y^i(\cdot)$ are unperturbed trajectories coupled with $\widetilde{x}(\cdot)$ at times T_i^{in} and T_{i-1}^{out}, respectively. These unperturbed trajectories are shown by dashed lines.

Combining (70) and (67) and a straightforward inductive argument, it follows that for any $i \geq 0$, and for any $t \in [0, T_i^{out}]$ (where T_i^{out} can be infinite), we have

$$\left\|\widetilde{x}(t) - x(t)\right\| \leq 5ir. \tag{71}$$

Let i^* be the maximum i such that $T_i^{in} < \infty$, with the convention that $i^* = 0$ if $T_1^{in} = \infty$. Equivalently, when i^* is non-zero, it is the maximum i such that the set on the right hand side of (64) is non-empty. We show that i^* is finite and in fact upper bounded by the number of critical points, M. In order to draw a contradiction, suppose that $i^* \geq M+1$. In this case, there is a repeated critical point $p \in \mathcal{C}$ that is effective in at least two phases; that is, there exist i, j, and p, such that $1 \leq i < j \leq M+1$, $T_j^{in} < \infty$, and $p^i = p^j = p$. Then,

$$\begin{aligned}\left\|x\big(T_i^{in}\big) - p\right\| &\leq \left\|x\big(T_i^{in}\big) - \widetilde{x}\big(T_i^{in}\big)\right\| + \left\|\widetilde{x}\big(T_i^{in}\big) - p^i\right\| \\ &\leq 5ir + r \\ &< 5(M+2)r,\end{aligned} \tag{72}$$

where the second inequality is due to (71) and the definition of T_i^{in}. The same bound also holds for $\left\|x\big(T_j^{in}\big) - p\right\|$. On the other hand,

$$\begin{aligned}\left\|x\big(T_i^{out}\big) - p\right\| &\geq \left\|p^i - \widetilde{x}\big(T_i^{out}\big)\right\| - \left\|x\big(T_i^{out}\big) - \widetilde{x}\big(T_i^{out}\big)\right\| \\ &\geq \rho - 5ir \\ &\geq \frac{\rho_{\min}}{2} - 5(M+1)r \\ &= 20(M+2)(\gamma+1)\eta\theta^* - 5(M+1)r \\ &\geq 20(M+2)(\gamma+1)\eta\theta - 5(M+1)r \\ &= 20(M+2)r - 5(M+1)r \\ &> 15(M+2)r,\end{aligned} \tag{73}$$

where the second and third inequalities are due to (71) and the definition of ρ, respectively. Let $\alpha = 15(M+2)r$. Using the Definitions of θ^* and r, and the assumption $\theta < \theta^*$, we see that $\alpha < \rho_{\min}$. From Lemma 3(d), α is a basin radius for every critical point. It follows from (72), (73), and again (72) that $\left\|x\big(T_i^{in}\big) - p\right\| < \alpha/3$, $\left\|x\big(T_i^{out}\big) - p\right\| > \alpha$, and $\left\|x\big(T_j^{in}\big) - p\right\| < \alpha/3$, respectively. Since $T_i^{in} \leq T_i^{out} \leq T_j^{in}$, this contradicts Lemma 3(e). Therefore, the initial hypothesis $i^* \geq M+1$ cannot be true, and we conclude that $i^* \leq M$. Hence, $T_{M+1}^{out} = \infty$. It follows from (71) and the definition of C in (61) that for any $t \geq 0$,

$$\left\|\widetilde{x}(t) - x(t)\right\| \leq 5(M+1)r \leq C\theta, \tag{74}$$

which shows that the Theorem also holds for Case 2.

We now prove the last statement in Theorem 1, namely, that the bound (6) applies to the system $\dot{x} \in F(x) + \lambda$, with the same constant C. Let $F'(\cdot) = F(\cdot) + \lambda$. According to Lemma 3(f), the systems F and F' have identical effective regions and critical points. It follows that the constants $\rho_{\min}$, $D^{\mathcal{C}}$, M (number of critical points), and m (number of regions) are also the same. As a consequence, the constants γ and σ, defined in Lemmas 4 and 5, respectively, are also identical for the two systems, and the same is true for the constant $\eta = m2^{m+1}\sigma$, defined in Proposition 1, the constant θ^* defined in (60), and finally for the constant C in (61). In other words, the same constant C also works for the dynamical system F'. This completes the proof of Theorem 1. □

V. Discussion

In this section we review our main results and their implications, and also discuss the extent to which they can or cannot be generalized to broader classes of systems.

We have established a *bounded input sensitivity* property of FPCS (finitely piecewise constant subgradient) systems, in a strong sense. In particular, we have shown that the increase in the distance between perturbed and unperturbed trajectories is upper-bounded by a constant multiple of the *magnitude of the integral* of the instantaneous perturbations; cf. (6). As discussed in the introduction, this is much stronger than the elementary upper bounds which involve the integral of the magnitude of the instantaneous perturbations. As an example, for the system illustrated in Fig. 1, and with i.i.d. Bernoulli perturbations, the naive bound in (3) grows at the rate of $t/2$, whereas the bound in (6) only grows as $(C/2)\sqrt{t}\log t$, for some $C < 2000$, with high probability. Thus, over short time scales, the naive bound is stronger, but in the regime of large t (which is the relevant one for heavy-traffic asymptotic analysis) our bound is tighter. Furthermore, the best constant C for that example is likely to be much smaller.[6] In any case, our work carries out the important first step, that of showing that C is in fact finite. We finally note that our definitions are broad enough to include as possible perturbations the sample paths of jump or Brownian motion processes.

A. Implications

FPCS systems arise in many contexts. As discussed in Section I, a prominent example is the celebrated Max-Weight policy for scheduling in queueing networks. Having made this connection, we can (cf. [14]) apply a variant of our result to the Max-Weight policy, establish bounds on the distance between the actual discrete-time stochastic system and its fluid approximation, and also obtain state space collapse results that are stronger than available ones [13], [27].

More broadly, flows or algorithms that evolve along the subgradient of a potential function are a fairly natural model, likely to arise in many other contexts. Recall also that, as mentioned in Section I, the FPCS class has been shown [2] to contain all non-expansive finite-partition hybrid systems that obey some minimal well-formedness and uniqueness properties.

B. Generalizations

Broad generalizations that assume only a subset of the properties of FPCS systems are not possible. In [1] we provide (counter)examples that show that a sensitivity bound of the form (6) does **not** hold for various classes of systems. Our counterexamples include:

1) A non-expansive system; hence the non-expansiveness property is not sufficient by itself.
2) A system that moves along the gradient of a twice continuously differentiable strictly convex function; hence the subgradient property is not sufficient by itself.
3) A system that moves along the subgradient of a piecewise linear convex function with infinitely many number of pieces; hence the finiteness of the number of pieces is essential.

Even though our main result cannot be extended by weakening its assumptions, it may still be possible to derive similar sensitivity bounds for other classes of systems. For example, [1] provides necessary and sufficient conditions for linear systems $\dot{x} = Ax$, in terms of the spectrum of A. It will be interesting to explore whether there are some other natural classes of systems that do not have the non-expansiveness property but for which the conclusions in Theorem 1 are valid.

[6] Using a variant of our proof, tailored to that example, it can be shown that C can be set to 6.5.

C. Some open problems

Besides attempts to obtain bounded sensitivity results for other types of systems, there are some interesting open problems for FPCS systems specifically.

1) The bound in Theorem 1 involves a constant C which grows exponentially with the number of regions. It is not known whether this is unavoidable or whether a smaller (polynomial) constant is possible.
2) Theorem 1 studies the distance between a perturbed and an unperturbed trajectory, but this does not necessarily provide a strong bound on the distance between two perturbed trajectories. Consider an FPCS system and two different perturbations $U_1(\cdot)$ and $U_2(\cdot)$ that are close at all times. Whether the perturbed trajectories are guaranteed to stay close is an open problem.

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Appendix A
Proof of Lemma 3

Proof of Lemma 3(a). We fix some ν and $p \in R_\nu = \{x \mid -\nu^T x + b_\nu \geq -\mu^T x + b_\mu,\ \forall\ \mu\}$. Suppose that p is a critical point. By the definition of $\mathcal{M}(p)$, we have $-\nu^T x + b_\nu = -\mu^T x + b_\mu$ for every $\mu \in \mathcal{M}(p)$, i.e., these constraints are all active at p. Furthermore, by the definition of critical points, the vectors $\{\mu - \mu' \mid \mu, \mu' \in \mathcal{M}(p)\}$ span $\mathbb{R}^n$. It is not hard to see that this implies that the vectors $\{\mu - \nu \mid \mu \in \mathcal{M}(p)\}$ also span $\mathbb{R}^n$, so that n of them are linearly independent. Using linear programming terminology, out of the constraints that define R_ν, there are n linearly independent active constraints at p, and p is a "basic feasible solution" in R_ν. This is equivalent to p being an extreme point of R_ν; cf. Theorem 2.3 in [28].

For the converse implication, suppose that p is an extreme point of R_ν. Using again Theorem 2.3 in [28], n of the vectors $\mu - \nu$, associated with active constraints at p (i.e., with $\mu \in \mathcal{M}(p)$) are linearly independent. It follows that the vectors $\mu - \mu'$, for $\mu, \mu' \in \mathcal{M}(p)$ span $\mathbb{R}^n$, and (by the definition), p is a critical point. $\square$

Proof of Lemma 3(b). In order to draw a contradiction, consider a time $t > 0$ where $z(t)$ is in the basin and $\dot{z}(t) \neq \xi$. It follows from Lemma 2(b) that $\|\dot{z}(t)\| < \|\xi\|$. Hence, $\xi^T \dot{z}(t) \leq \|\xi\| \cdot \|\dot{z}(t)\| < \|\xi\|^2$, which contradicts the definition of a basin. $\square$

Proof of Lemma 3(c). We assume that the set $\mathcal{C}$ of critical points is non-empty. We will first show that there exists a critical point p such that $\|\xi(p)\| \leq \|\xi(x)\|$, for all $x \in \mathbb{R}^n$. We will then show that $\mathbb{R}^n$ is a basin for this particular p.

Since there exists a critical point, Part (a) implies that some R_ν has an extreme point. Using linear programming theory (cf. Theorem 2.6 in [28]) it follows that all of the non-empty regions R_μ also have extreme points[7].

[7] This is because the regions are defined in terms of constraints $a^T x \leq b$ or $a^T x \geq b$, where each a is of the form $a = \mu - \mu'$, for some μ, μ'; different regions correspond to different choices in the direction of the inequalities, but the vectors a are the same or every region.

Consider some x in some region R_ν. Let x' be an extreme point of that region, chosen so that all constraints that were active at x are also active at x'. (This can be done by moving inside R_ν while respecting active constraints, until additional constraints are made active, exactly as in the proof of Theorem 2.6 in [28].) The resulting extreme point x' satisfies $\mathcal{M}(x') \supseteq \mathcal{M}(x)$. Since $F(x)$ is the convex hull of $\mathcal{M}(x)$, it follows that $F(x') \supseteq F(x)$. From Lemma 2(a), $\xi(x)$ is the minimum norm element of $F(x)$, which implies that $\|\xi(x')\| \leq \|\xi(x)\|$. We conclude that when we minimize the function $\|\xi(x)\|$ over all $x \in \mathbb{R}^n$, it suffices to restrict to the (finite) set of extreme points of the different regions, or equivalently the set of critical points (cf. Part (a)). This concludes the proof that there exists a critical point, p, such that $\|\xi(p)\| \leq \|\xi(x)\|$, for all $x \in \mathbb{R}^n$. Let $\xi^* = \xi(p)$.

We now proceed to show that $\mathbb{R}^n$ is a basin of p. Let $z(t)$ be an unperturbed trajectory with initial point $z(0) = p$. Similar to the proof of Part (b), if for some $t > 0$, $\dot{z}(t) \neq \xi^*$, then it follows from Lemma 2(b) that $\|\xi\big(z(t)\big)\| = \|\dot{z}(t)\| < \|\dot{z}(0)\| = \|\xi^*\|$, which contradicts the definition of ξ^*. Hence, for any $t \geq 0$, $z(t) = p + t\xi^*$, which implies that $-\xi^* \in \partial\Phi\big(p + t\xi^*\big)$, where Φ is the convex function for which F is the subdifferential.

For any $x \in \mathbb{R}^n$, let $\widetilde{\Phi}(x) = \Phi(x) + t{\xi^*}^T(x - p)$, so that $\partial\widetilde{\Phi}(x) = \partial\Phi(x) + \xi^*$. Since $-\xi^* \in \partial\Phi(p + \xi^*)$, we have $0 \in \partial\widetilde{\Phi}\big(p + t\xi^*\big)$, which implies that $p + t\xi^*$ is a minimizer of $\widetilde{\Phi}$. Consider an $x \in \mathbb{R}^n$ and a $y \in F(x)$. Then, $y - \xi^* \in -\partial\widetilde{\Phi}(x)$. It follows from the supporting hyperplane theorem that, for any $t \geq 0$, $(\xi^* - y)^T(p + t\xi^* - x) \leq \widetilde{\Phi}(p + t\xi^*) - \widetilde{\Phi}(x) \leq 0$, where the last inequality is because $p + t\xi^*$ is a minimizer of $\widetilde{\Phi}$. Then, by letting t go to infinity, we obtain

$$\big(\xi^* - y\big)^T\xi^* = \lim_{t\to\infty} \frac{1}{t}\big(\xi^* - y\big)^T\big(p + t\xi^* - x\big) \leq 0. \quad (75)$$

Hence, $\|\xi^*\|^2 \leq {\xi^*}^T y$, which shows that $\mathbb{R}^n$ is a basin of p. In the special case where F is conic and has a critical point, then this is the only critical point and therefore has R^n for a basin. $\square$

Proof of Lemma 3(d). Consider a critical point $p \in \mathcal{C}$, and let

$$\widetilde{\Phi}(x) = \max_{\mu\in\mathcal{M}(p)} \big\{ -\mu^T\big(x - p\big)\big\}, \qquad \forall\, x \in \mathbb{R}^n. \quad (76)$$

Hence, the dynamical system $\dot{x} \in \widetilde{F}(x) \triangleq -\partial\widetilde{\Phi}(x)$ is conic. Since the vectors $\big\{\mu - \mu' \mid \mu \in \mathcal{M}(p)\big\}$ span $\mathbb{R}^n$, it follows that p is also a critical point of the system $\dot{x} \in \widetilde{F}(x)$. Lemma 3(c) then implies that the entire set $\mathbb{R}^n$ is a basin for p, for the system $\dot{x} \in \widetilde{F}(x)$.

Let $\mathcal{B}$ be the ball of radius $\rho_{\min}$ centred at p, where $\rho_{\min}$ is the CNC. By the definition of the CNC, if $\mathcal{B} \bigcap R_\mu$ is non-empty for some $\mu \in \mathcal{M}$, then $p \in R_\mu$. Hence, for any $x \in \mathcal{B}$, we must have $\mathcal{M}(x) \subseteq \mathcal{M}(p)$. Therefore, for any $x \in \mathcal{B}$,

$$\begin{aligned}
\Phi(x) &= \max_{\mu\in\mathcal{M}} \big(-\mu^T x + b_\mu\big) \\
&= \max_{\mu\in\mathcal{M}(p)} \big(-\mu^T x + b_\mu\big) \\
&= \max_{\mu\in\mathcal{M}(p)} \Big(-\mu^T\big(x - p\big) - \mu^T p + b_\mu\Big) \quad (77) \\
&= \Phi(p) + \max_{\mu\in\mathcal{M}(p)} \Big(-\mu^T\big(x - p\big)\Big) \\
&= \Phi(p) + \widetilde{\Phi}(x).
\end{aligned}$$

where the second equality is because the set $\mathcal{M}(x)$ of maximizers of $-\mu^T x + b_\mu$ is a subset of $\mathcal{M}(p)$. Hence, for any $x \in \mathcal{B}$, $F(x) = \widetilde{F}(x)$. As a result, for $x \in \mathcal{B}$, $\xi(x)$ for the system $\dot{x} \in F(x)$ is equal to $\xi(x)$ for the system $\dot{x} \in \widetilde{F}(x)$. Since $\mathbb{R}^n$ is a basin of p for the system $\dot{x} \in \widetilde{F}(x)$, it follows that for any $x \in \mathcal{B}$ and any $y \in F(x) = \widetilde{F}(x)$, we have $y^T\xi(p) \geq \|\xi(p)\|$. Hence, $\mathcal{B}$ is a basin for the system $\dot{x} \in F(x)$. $\square$

Proof of Lemma 3(e). The result will be derived by comparing the trajectory $x(t)$ of interest to another unperturbed trajectory, $z(t)$, initialized with $z(t_1) = p$. According to the non-expansive property of the dynamics, we have $\|x(t) - z(t)\| \leq \|x(t_1) - z(t_1)\| \leq \rho/3$, for every $t \geq t_1$. Hence,

$$\|z(t_2) - p\| \geq \|x(t_2) - p\| - \|x(t_2) - z(t_2)\| > \rho - \frac{\rho}{3} = 2\rho/3.$$

In order to draw a contradiction, suppose that there is a time $t_3 > t_2$ such that $\|x(t_3) - p\| \leq \rho/3$. In this case,

$$\|z(t_3) - p\| \leq \|z(t_3) - x(t_3)\| + \|x(t_3) - p\| \leq \frac{\rho}{3} + \frac{\rho}{3} = \frac{2}{3}\rho.$$

Hence, $z(t_3)$ is in the basin of p, which implies that $\xi(p)^T\xi\big(z(t_3)\big) \geq \|\xi(p)\|$ and

$$\|\xi\big(z(t_3)\big)\| \geq \|\xi(p)\|. \quad (78)$$

The trajectory $z(t)$ starts inside the $2\rho/3$-neighbourhood of p at time t_1, leaves this neighbourhood before time t_2, and returns back to it by time t_3. Since the $2\rho/3$-neighbourhood is convex, $z(t)$ must have changed its direction in the meanwhile, and there exists a time $t' \in (t_1, t_3)$ such that $\dot{z}(t') \neq \dot{z}(t_1) = \xi(p)$. Then, using Lemma 2(b),

$$\|\xi\big(z(t_3)\big)\| \leq \|\dot{z}(t')\| < \|\dot{z}(t_1)\| = \|\xi(p)\|. \quad (79)$$

This contradicts (78) and concludes the proof. $\square$

Proof of Lemma 3(f). For every drift $\mu \in \mathcal{M}$ of F, $\mu + \lambda$ is a drift of F'. The associated effective region $R'_{\mu+\lambda}$ of F' is given by $R'_{\mu+\lambda} = \big\{x \in \mathbb{R}^n \bigm| -(\mu+\lambda)^T x + b_\mu \geq -(\nu+\lambda)^T x + b_\nu,\ \forall \nu \in \mathcal{M}\big\} = \big\{x \in \mathbb{R}^n \bigm| -\mu^T x + b_\mu \geq -\nu^T x + b_\nu,\ \forall \nu \in \mathcal{M}\big\} = R_\mu$. Hence, the regions associated with F and F' are the same. Consider a point $p \in \mathbb{R}^n$ and let $\mathcal{M}'(p) = \mathcal{M}(p) + \lambda$ be the set of active drifts of p in system F'. The affine span of $\mathcal{M}'(p)$ is $\mathbb{R}^n$ if and only if the affine span of $\mathcal{M}(p)$ is $\mathbb{R}^n$. Hence, p is a critical point for the system F' if and only if it is a critical point for the system F. Finally, by the definition of the CNC, since F and F' have the same set of regions and the same set of critical points, they also have the same CNC. $\square$

Appendix B
Proof of Lemma 4

We provide here the proof of Lemma 4. We will make use of an auxiliary result, proved in [29], which states that if a point is close to each of several half-spaces, then that point is also close to the intersection of those half-spaces.

Lemma 6 ([29], Lemma 5.1)**.** *Given a finite collection of half-spaces $W_i \subset \mathbb{R}^n$, with non-empty intersection, there exists a finite constant $c > 0$ such that*

$$d\Big(x\,,\bigcap_i W_i\Big) \le c \cdot \max_i d\big(x\,,\,W_i\big), \qquad \forall\ x \in \mathbb{R}^n. \tag{80}$$

Proof of Lemma 4. For any $x \in \mathbb{R}^n$, let $r(x) = \sup\big\{r : \mathcal{U}_r \text{ is low-dimensional}\big\}$. By definition, if x is not a critical point, then $r(x) > 0$, and if $r \ge r(x)$, then $\mathcal{U}_r$ is not low-dimensional. We will show that

$$\overline{\gamma} \triangleq \inf_{x \notin \mathcal{C}} \frac{r(x)}{d\big(x, \mathcal{C}\big)} \,>\, 0. \tag{81}$$

In order to draw a contradiction, suppose that there exists a sequence of points $y_k \in \mathbb{R}^n \backslash \mathcal{C}$ such that

$$\frac{r(y_k)}{d\big(y_k, \mathcal{C}\big)} \xrightarrow[k\to\infty]{} 0. \tag{82}$$

Since $\mathcal{U}_{r(y_k)}(y_k)$ is not low-dimensional, there exist $n+1$ drifts $\mu_1, \dots, \mu_{n+1} \in \mathcal{U}_{r(y_k)}(y_k)$ such that

$$\operatorname{span}\big\{\mu_i - \mu_j \,\big|\, i,j \le n+1\big\} = \mathbb{R}^n. \tag{83}$$

Because the set $\mathcal{M}$ of all drifts is finite, there exists an infinite subsequence $\big\{x_k\big\}$ of $\big\{y_k\big\}$ for which (83) holds for the same set of drifts. We fix this set of drifts $\big\{\mu_i\big\}_{i=1}^{n+1}$. Then, for any k, $\big\{\mu_i\big\}_{i=1}^{n+1} \subseteq \mathcal{U}_{r(x_k)}(x_k)$. It follows from the definition of $r(x)$ that for any k,

$$r(x_k) = \max_{i \le n+1} d\big(x_k\,,\,R_i\big), \tag{84}$$

where $R_i = R_{\mu_i}$ is the effective region of μ_i. We define $n(n+1)$ half-spaces $W_{i,j}$ as follows. For any $i, j \le n+1$ with $i \ne j$, let

$$W_{i,j} \triangleq \Big\{x \in \mathbb{R}^n \;\Big|\; -(\mu_i - \mu_j)^T x \,+\, b_i - b_j \,\ge\, 0\Big\}, \tag{85}$$

where b_i is a shorthand for b_{μ_i}. Then, for any $i \le n+1$,

$$R_i \subseteq \bigcap_{j \ne i} W_{i,j}. \tag{86}$$

Hence, for any $i \le n+1$ and any $x \in \mathbb{R}^n$, $d\big(x, R_i\big) \ge \max_{j \le n+1} d\big(x, W_{i,j}\big)$. Then, it follows from (84) that for any $k \ge 1$,

$$r(x_k) \,=\, \max_{i \le n+1} d\big(x_k\,,\,R_i\big) \,\ge\, \max_{\substack{i,j \le n+1 \\ i \ne j}} d\big(x, W_{i,j}\big) \tag{87}$$

It follows from (83) that the following system of n linear equations is non-degenerate:

$$-(\mu_i - \mu_{n+1})^T x \,+\, b_i - b_{n+1} \,=\, 0, \quad i = 1, \dots, n. \tag{88}$$

Hence, it has a unique solution, which we denote by p. Note that $W_{i,j}$ and $W_{j,i}$ are different, and their intersection is $\big\{x \;\big|\; -(\mu_i - \mu_j)^T x \,+\, b_i - b_j \,=\, 0\big\}$. Therefore,

$$\{p\} = \bigcap_{\substack{i,j \le n+1 \\ i \ne j}} W_{i,j}. \tag{89}$$

It follows from Lemma 6, with $\delta = 1/c$, that there exists a constant $\delta > 0$ such that for any $x \in \mathbb{R}^n$,

$$\max_{\substack{i,j \le n+1 \\ i \ne j}} d\big(x, W_{i,j}\big) \,\ge\, \delta\, d\Big(x, \bigcap_{\substack{i,j \le n+1 \\ i \ne j}} W_{i,j}\Big) \,=\, \delta\, d\big(x, p\big). \tag{90}$$

Combining (87) and (90), we have for any k,

$$r(x_k) \,\ge\, \max_{\substack{i,j \le n+1 \\ i \ne j}} d\big(x_k, W_{i,j}\big) \,\ge\, \delta\, d\big(x_k, p\big). \tag{91}$$

Back to the hypothesis (82), there are two possible cases: (a) $\big\{x_k\big\}$ has a subsequence $\big\{z_k\big\}$ with $d\big(z_k, \mathcal{C}\big) \to \infty$, or (b) x_k has a subsequence z_k with $r(z_k) \to 0$.

In the first case, where $d\big(z_k, \mathcal{C}\big) \to \infty$, it follows from (91) that

$$\begin{aligned} \lim_{k\to\infty} \frac{r(z_k)}{d\big(z_k, \mathcal{C}\big)} &\ge \delta \lim_{k\to\infty} \frac{d\big(z_k, p\big)}{d\big(z_k, \mathcal{C}\big)} \\ &\ge \delta \lim_{k\to\infty} \frac{d\big(z_k, \mathcal{C}\big) - d\big(p, \mathcal{C}\big)}{d\big(z_k, \mathcal{C}\big)} \\ &= \delta \,>\, 0, \end{aligned} \tag{92}$$

which contradicts (82).

In the second case, where $r(z_k) \to 0$, it follows from (91) and (84) that for any $i \le n+1$,

$$d\big(p, R_i\big) \,\le\, d\big(p, z_k\big) + d\big(z_k, R_i\big) \,\le\, \frac{r(z_k)}{\delta} + r(z_k) \xrightarrow[k\to\infty]{} 0. \tag{93}$$

Then, since each R_i is a closed set, we must have $p \in R_i$. Hence, $p \in \bigcap_{i \le n+1} R_i$ which together with (83) implies that p is a critical point. Using this fact, and then (91), we obtain

$$\lim_{k\to\infty} \frac{r(z_k)}{d\big(z_k, \mathcal{C}\big)} \ge \lim_{k\to\infty} \frac{r(z_k)}{d\big(z_k, p\big)} \ge \delta > 0, \tag{94}$$

which again contradicts (82). Hence, (82) is contradicted in both cases, and (81) follows.

Let $\gamma = \max\{1,\, 1/\overline{\gamma}\}$. It follows from (81) that $\gamma r(x) \ge d\big(x, \mathcal{C}\big)$, for all $x \notin \mathcal{C}$. Hence, if $\gamma r < d\big(x, \mathcal{C}\big)$, then $r < r(x)$, and by the definition of $r(x)$, $\mathcal{U}_r(x)$ is low-dimensional. □