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Allocational flexibility in constrained supply chains

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ABSTRACT

We show that when a one-supplier/one-newsvendor supply chain is capacity-constrained, wholesale price contracts have some flexibility in allocating the channel-optimal profit. We analyze how this flexibility changes as we change the supply chain's capacity constraint and market demand. We also explore the allocation that is achieved in equilibrium in a newsvendor procurement game. Finally, we generalize our results to risk-sharing contracts and show that those contracts also gain additional flexibility in allocating the channel-optimal profit.

1. Introduction

There is a wealth of supply contracts available that coordinate a newsvendor's decision for unconstrained supplier–retailer channels: buy-back contracts, revenue-sharing contracts, etc. (Cachon, 2003). A contract *coordinates* the actions of a newsvendor *for* a supply channel if the contract causes the newsvendor to take actions when solving his *own* decision problem that are also optimal for the *channel*. Our paper starts from the fact that simpler contracts, namely linear wholesale price contracts, (which are thought to be unable to coordinate a newsvendor's decision for unconstrained channels) can, in fact, coordinate a newsvendor's procurement decision for resource-constrained channels. This is relevant for supply channels in which capacity of some resource is limited. For example, shelf space at retail stores, warehouse space, procurement budgets, time available for manufacturing, raw materials, etc. We show that in addition to having this coordination capability, in constrained supply channels, wholesale price contracts also have flexibility in allocating profit while maintaining coordination.

However, this extra gain in *allocational flexibility* is not limited to wholesale price contracts. We also show that when the channel is constrained, buyback contracts also gain some additional allocational flexibility. In particular, we show that buyback contracts gain a feature that they do not have in the unconstrained setting: the flexibility in allocating channel optimal profit, for any fixed level of risk.

Wholesale price contracts are commonplace since they are straightforward and easy to implement. While risk-sharing contracts such as revenue-sharing agreements can coordinate a retailer's decision in a newsvendor setting, Cachon and Lariviere (2005) note that these alternative contracts impose a heavier administrative burden. For example, these alternative contracts may require an investment in information technology or a higher level of trust between the trading partners due to the additional processes involved. In this paper, we show that the flexibility gained by wholesale price contracts in allocating the channel-optimal profit makes these simpler contracts more efficient and appropriate for a wider variety of supply channels than previously known.

Furthermore, after analyzing the allocational flexibility of wholesale price contracts, we analyze an equilibrium setting, where choosing the wholesale price is an initial stage of a game for the supplier. In the equilibrium setting we explore conditions for the game's equilibrium wholesale price to coordinate the newsvendor's procurement decision for the channel (i.e., necessary and sufficient conditions so that the game's equilibrium is included in the set of coordinating wholesale price contracts) and find the equilibrium profit allocation achieved.

The organization of this paper is as follows. In Section 2, we provide an overview of the supply contracts literature and in Section 3, we provide a stylized 1-supplier/1-retailer model. In Section 4, we show that wholesale price contracts have some flexibility in allocating the channel-optimal profit between the supplier and retailer (a flexibility that does not exist in the unconstrained setting). We also conduct some comparative statics and analyze how this flexibility changes as a function of capacity and market demand. Then in Section 5 we move on and consider risk-sharing contracts for the same supply chain model. We show that they still coordinate a capacity-constrained channel and,

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furthermore, there is even more flexibility in the choice of risk-sharing contracts (for coordinating the channel). In particular, for any given level of risk (represented by the buyback parameter of a buyback contract), there is now flexibility in allocating the channel profit (without sacrificing coordination), a flexibility that is not present in the unconstrained setting. Then, in Section 6 we analyze the equilibrium of a newsvendor procurement game in order to find and analyze the equilibrium profit allocation. Finally, we summarize our findings and provide managerial insights in Section 7.

2. Literature review

The supply contracts literature has been based on the observation, pointed out, for example, by Lariviere and Porteus (2001), that wholesale price contracts are simple but *do not* coordinate the retailer's order quantity decision for a supplier–retailer supply chain in a newsvendor setting and have no flexibility in allocating the channel-optimal profit. This observation has led to the study of an assortment of alternative contracts. For example, buy back contracts (Pasternack, 1985), quantity flexibility contracts (Tsay, 1999), and many others. Cachon (2003) provides an excellent survey of the many contracts and models that have been studied in the supply contracts literature. The mindset surrounding wholesale price contract's inability to channel-coordinate is true under appropriate assumptions—which the supply contracts literature has been implicitly assuming: that there are no capacity constraints (e.g., shelf space and budget).

Considering capacity constraints in a supply channel is not new to the supply contracts literature. However, most other papers in the literature consider choosing capacity as one stage of a game (before downstream demand is realized) that also involves a production decision after demand is finally realized (Cachon and Lariviere, 2001; Gerchak and Wang, 2004; Wang and Gerchak, 2003; Tomlin, 2003). Our paper, although complementary to this stream of literature, does not involve an endogenous capacity choice for any party but rather analyzes how an exogenous capacity constraint determines the set of wholesale prices that can coordinate the retailer's decision for the channel and the possible allocations of channel-optimal profit. Pasternack (2001) considers an exogenous budget constraint, but not for the purposes of studying coordination or allocational flexibility. Rather, he analyzes a retailer's optimal procurement decision when the retailer has two available strategies: buying on consignment and outright purchase.

Also our paper is not the first to reconsider wholesale price contracts and their benefits beyond simplicity. Cachon (2004) looks at how inventory risk is allocated according to wholesale price contracts and the resulting impact on supply chain efficiency. As far as we are aware, our paper is the first to consider the allocational flexibility of linear wholesale price contracts under a simple capacity-constrained production/procurement newsvendor model.

3. Model

A risk-neutral retailer r faces a newsvendor problem in ordering from a risk-neutral supplier for a single good: there is a single

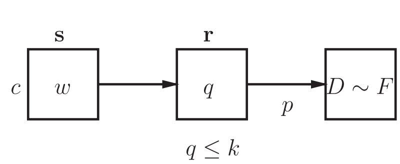

Fig. 1. "Single supplier & single capacity constrained retailer" model.

sales season, the retailer decides on an order quantity q and orders well in advance of the season, the entire order arrives before the start of the season, and finally demand is realized, resulting in sales for the retailer (without an opportunity for replenishment). Without loss of generality, we assume that units remaining at the end of the season have no salvage value and that there is no cost for stocking out.

The model's parameters are summarized in Fig. 1 with the arrows denoting the direction of product flow. In particular, the supplier has a fixed marginal cost of c per unit supplied and charges the retailer a wholesale price $w \geq c$ per unit ordered. The retailer's price p per unit to the market is fixed, and we assume that $p > w$. For that price, the demand D is random with probability density function (p.d.f.) f and cumulative distribution function (c.d.f.) F. We also define $\overline{F}(x) \stackrel{\text{def}}{=} 1 - F(x) = P(D > x)$. We say that a c.d.f. F has the *IGFR property* (increasing generalized failure rate), if $g(x) \stackrel{\text{def}}{=} x \cdot f(x)/\overline{F}(x)$ is weakly increasing on the set of all x for which $\overline{F}(x) > 0$ (Lariviere and Porteus, 2001). Most distributions used in practice (such as the Normal, the Uniform, the Gamma, and the Weibull distribution) have the IGFR property.

We assume that the retailer's capacity is constrained by some $k > 0$; for example, the retailer can only hold k units of inventory, or accept a shipment not larger than k. For a different interpretation, k could represent a constraint on the capacity of the channel or a budget constraint.

Assumption 1. The probability density function (p.d.f.) f for the demand D has support $[0, l]$, with $l > k$, on which it is positive and continuous.

As a consequence, $\overline{F}(0) = 1$ and $\overline{F}$ is continuously differentiable, strictly decreasing, and invertible on $(0, l)$. There is no additional restriction on the value of l. This is not a restrictive assumption and is made for technical reasons as shown in our proofs.

3.1. Retailer's problem

Faced with uncertain sales $S(q) \stackrel{\text{def}}{=} \min\{q, D\}$ (when ordering q units) and a wholesale price w (from the supplier), the retailer decides on a quantity to order from the supplier in order to maximize expected profit $\pi_r(q) \stackrel{\text{def}}{=} E[pS(q)] - wq$ while satisfying the capacity constraint k. Namely, it solves the following concave optimization problem with linear constraints in the decision variable, q:

RETAILER(k,w)

$$\begin{aligned} &\text{maximize} \quad pE[S(q)] - wq \\ &\quad \text{subject to} \quad k - q \geq 0 \\ &\qquad q \geq 0. \end{aligned} \tag{1}$$

Because of our assumptions on the c.d.f. F, it can be shown that *RETAILER(k,w)* has a unique solution which we denote by $q^r(w)$.

3.2. Channel's problem

Denote the channel's expected profit by $\pi_s(q) \stackrel{\text{def}}{=} E[pS(q) - cq]$. Under capacity constraint k, the optimal order quantity q^s for the system/channel is the solution to concave optimization problem (2), *CHANNEL(k)*. Note that *CHANNEL(k)* has identical linear constraints but a slightly altered objective function when compared to *RETAILER (k,w)*:

CHANNEL(k)

$$\begin{aligned} &\text{maximize} \quad pE[S(q)] - cq \\ &\quad \text{subject to} \quad k - q \geq 0 \\ &\qquad q \geq 0 \end{aligned} \tag{2}$$

Again because of our assumptions on the c.d.f. F it can be shown that *CHANNEL(k)* also has a unique solution which we denote by q^s.

We denote the unique solution, $\arg\max_{0 \le q < \infty} \pi_S(q)$, for the unconstrained channel problem by q^*. It is well known that $q^* = \overline{F}^{-1}(c/p)$ (e.g., Cachon and Terwiesch, 2006). Because of convexity, it is also easily seen that $q^s = \min\{q^*, k\}$.

3.3. Definition: coordinating the retailer's action

A wholesale price contract *w coordinates* the retailer's ordering decision *for* the supply channel when it causes the retailer to order the channel-optimal amount, i.e., $q^r(w) = q^s$. If there is no capacity constraint (or equivalently if k is very large), 'double marginalization' results in the retailer not ordering enough (i.e., $q^r(w) < q^s$) under any wholesale price contract, $w > c$. In the next section, we will show that when the capacity constraint k is small relative to demand, there exist a set of wholesale price contracts $w > c$ that can coordinate the retailer's order quantity, i.e., $q^r(w) = q^s$ and we analyze the achievable allocations of channel-optimal profit.

4. Achievable allocations of channel-optimal profit with wholesale price contracts

The follow lemma describes the set of coordinating wholesale prices under a capacity constraint.

Lemma 1. *In a 1-supplier/1-retailer configuration where the retailer faces a newsvendor problem and has a capacity constraint k, any wholesale price*

$$w \in \mathcal{W}(k) \overset{\text{def}}{=} [c, p\overline{F}(\min\{q^*, k\})]$$

will coordinate the retailer's ordering decision for the supply channel, i.e., $q^r(w) = q^s$. *Furthermore, if* $q^r(w) = q^s$ *and* $c \le w \le p$, *then* $w \in \mathcal{W}(k)$.

Proof. See Appendix A.

Notice that if the capacity constraint k is larger than or equal to the unconstrained channel's optimal order quantity, q^*, then $p\overline{F}(\min\{q^*, k\}) = p\overline{F}(q^*) = c$, reducing to the 'classic' result in the supply contracts literature. However, this is true only when *the capacity constraint is not binding for the channel* (i.e., $q^* \le k$). When *the capacity constraint k is binding for the channel* (i.e., $q^* > k$), then any wholesale price $w \in [c, p\overline{F}(k)]$ will coordinate the retailer's action *and* only wholesale prices in the range $[c, p\overline{F}(k)]$ can coordinate the retailer's action.

Many factors such as 'power in the channel', 'outside alternatives', 'inventory risk exposure', and 'competitive environment' ultimately influence the actual wholesale price (selected from the set $[c, p]$) charged by the supplier. In the unconstrained setting, regardless of these factors, coordination is not possible with a linear wholesale price contract (because the supplier presumably would not agree to price at cost). However, when the capacity constraint is binding for the channel, coordination becomes *possible* (because the set of coordinating wholesale price contracts becomes $[c, p\overline{F}(k)]$ (rather than $\{c\}$) and ultimately depends on these other factors. Theorem 6 in Section 6 considers a equilibrium setting where the retailer takes on all the inventory risk (akin to the 'Stackelberg game' in Lariviere and Porteus, 2001 and 'push mode' in Cachon, 2004), and provides additional conditions that must be met so that the 'equilibrium' wholesale price contract is a member of the set of coordinating wholesale price contracts, $[c, p\overline{F}(k)]$.

By agreeing to focus on the set $\mathcal{W}(k)$ in negotiating over a wholesale price for coordination purposes, the supplier and retailer are implicitly agreeing to a '*minimum* share of expected revenue' requirement for the retailer and thus a '*maximum* share of expected revenue' restriction for the supplier. This notion is formalized in Lemma 2.

Lemma 2. *If the capacity constraint k is binding for the channel (i.e.,* $q^* > k$*), then any coordinating linear wholesale price contract* $w \in \mathcal{W}(k)$ *guarantees that the retailer receive at least a fraction* $(\int_0^k \overline{F}(x)\,dx - k \cdot \overline{F}(k)) / \int_0^k \overline{F}(x)\,dx$ *of the channel's expected revenue, and that the supplier receive at most a fraction* $k \cdot \overline{F}(k) / \int_0^k \overline{F}(x)\,dx$ *of the channel's expected revenue. Furthermore, if F has the IGFR property, then the supplier's maximum revenue share is weakly decreasing as k increases.*

Proof. See Appendix B.

An important distinction regarding the supplier and retailer 'share of expected revenue' guarantees formalized in Lemma 2 is that the supplier's share results in a guaranteed income (i.e., no uncertainty) whereas the retailer's share results in an uncertain income. For example, from Lemma 2 there exists some wholesale price $w \in \mathcal{W}(k)$, where the supplier receives a fraction $k \cdot \overline{F}(k) / \int_0^k \overline{F}(x)\,dx$ of the expected channel revenue, $pE[S(k)]$. But the supplier's income is certain, wk, whereas the retailer's income is an uncertain amount, $pS(k) - wk$.

As a numerical example, if $k \cdot \overline{F}(k) / \int_0^k \overline{F}(x)\,dx = 1/2$, the supplier can receive *up to* fifty percent of the expected channel revenue and still keep the channel coordinated, whereas we require that the retailer receive *at least* fifty percent of the revenue in order for the wholesale price to coordinate the actions of the retailer.

The benefits of risk sharing contracts in the unconstrained setting include the ability to channel-coordinate the retailer's decision *as well as* flexibility (due to the extra contract parameters) that allows for any allocation of the optimal channel profit between the supplier and retailer. Cachon (2003) provides excellent examples of the 'channel-profit allocation flexibility' inherent in these more complex contracts.

Theorem 1 demonstrates that in a resource constrained setting, wholesale price contracts also have flexibility in allocating the channel-optimal profit. Namely, these simpler contracts allow for a range of divisions of the optimal channel profit among the firms. The divisions allowed (without losing coordination) depend on the channel's capacity, k. Similar to our observations regarding Lemma 2 for the implicit revenue requirements, the supplier's share results in a guaranteed income (i.e., no uncertainty) whereas the retailer's share results in an uncertain income.

Theorem 1. *If the capacity constraint is binding for the channel (i.e.,* $q^* > k$*), there exists a wholesale price contract* $w \in \mathcal{W}(k)$ *that can allocate a fraction* t_s *of the channel-optimal profit to the supplier and a fraction* $1 - t_s$ *to the retailer, if and only if* $t_s \in [0, t_s^{\max}(k; \overline{F})]$, *where*

$$t_s^{\max}(k; \overline{F}) \overset{\text{def}}{=} \frac{k \cdot (\overline{F}(k) - c/p)}{\int_0^k (\overline{F}(x) - c/p)\,dx}.$$

Furthermore, if F has the IGFR property, then $t_s^{\max}(k; \overline{F})$ *is weakly decreasing as k increases in the range* $[0, q^*)$.

Proof. See Appendix C.

Let us interpret Theorem 1 at two extremes values for the capacity k. As k approaches q^*, $t_s^{\max}(k; \overline{F})$ approaches zero. Thus the supplier cannot get any fraction of the channel-optimal profit with any wholesale price contract from $\mathcal{W}(k)$ (this was to be expected because $\mathcal{W}(k) = \{c\}$ when $k \ge q^*$). At the other extreme, as k tends to zero, $t_s^{\max}(k; \overline{F})$ tends to one. Thus any allocation of the channel-optimal profit becomes possible with some wholesale price contract from $\mathcal{W}(k)$ (this is natural, because as k tends to zero, the interval $\mathcal{W}(k)$ becomes $[c, p]$). See Fig. 2.

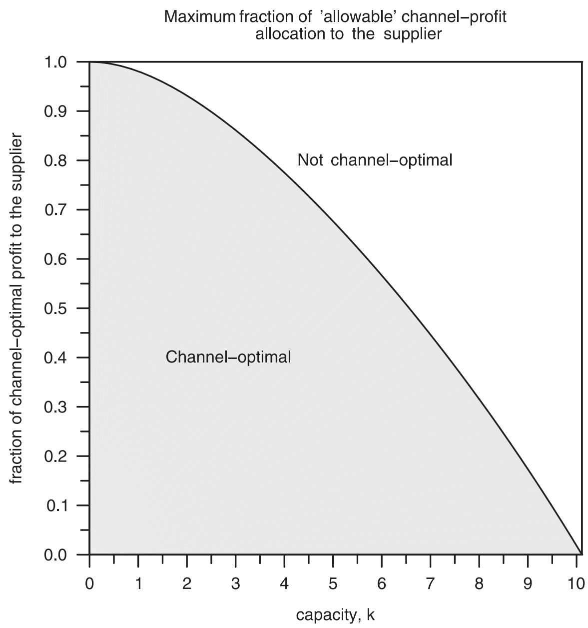

Fig. 2. Flexibility in allocating channel-optimal profit as a function of the capacity constraint. *Note*: Demand is distributed according to a Gamma distribution with mean 10 and coefficient of variation $2^{-1/2} \approx 0.707$. The retail price is $p=10$, and the cost is $c=4$. (these are similar to parameters used in Cachon, 2004). Thus, $q^* \approx 10.112$. The shaded region denotes the fractions of profit to the supplier consistent with a channel-optimal outcome (i.e., the set $[0, t_s^{\max}(k;\overline{F})]$). Or in other words, the shaded region represents the fractional allocations of channel-optimal profit to the supplier that are achievable with some wholesale price contract $w \in \mathcal{W}(k)$.

4.1. Comparative statics

It can be shown that the set of coordinating wholesale price contracts $\mathcal{W}(k)$ increases with the probability $\overline{F}(k)$ of excess demand, when k is held fixed. Lemma 3 formalizes a related idea: the larger the expected excess demand, the greater the maximum possible share of revenue at the supplier without sacrificing channel-coordination.

Lemma 3. *Consider two different demands D_1 and D_2, with each D_i associated with a c.d.f. F_i, that have the same mean and such that $\overline{F}_1(k) \geq \overline{F}_2(k)$. Suppose that (a) the capacity constraint k is binding for the channel under both distributions (i.e., $\min\{q_1^*, q_2^*\} > k$), and (b) $E[(D_1-k)^+] \geq E[(D_2-k)^+]$ (i.e., the expected excess demand under D_1 is higher than that under D_2). Then,*

$$\frac{k \cdot \overline{F}_1(k)}{\int_0^k \overline{F}_1(x)\,dx} \geq \frac{k \cdot \overline{F}_2(k)}{\int_0^k \overline{F}_2(x)\,dx}.$$

Proof. See Appendix D.

Theorem 2 uses Lemma 3 and makes precise the idea that when we serve a larger market the 'flexibility' in allocating the channel-optimal profit 'increases'.

Theorem 2. *Under the same assumptions as in Lemma 3, we have*

$$t_s^{\max}(k;\overline{F}_1) \geq t_s^{\max}(k;\overline{F}_2).$$

Proof. See Appendix E.

Theorem 2 suggests that a supplier (and retailer) can find flexibility in profit allocation by joining a supply channel that serves a larger market.

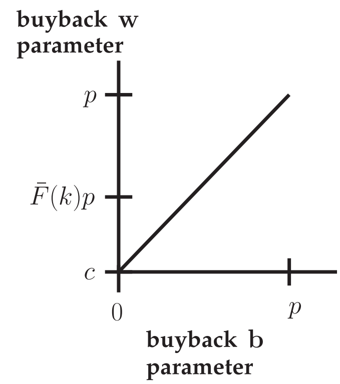

Fig. 3. Some buyback contracts (w,b) that channel-coordinate a constrained newsvendor. *Note*: The buyback contracts (w,b) that channel-coordinate an unconstrained newsvendor's ordering decision (the ones graphed in this figure) still coordinate a capacity-constrained newsvendor. $\overline{F}(k)p$ is labelled on the y-axis purely for comparison with Fig. 4.

5. Achievable allocations of channel-optimal profit with risk-sharing contracts

In Lemma 4, we show that buyback contracts, which are known to coordinate an *unconstrained* newsvendor's procurement decision, continue to coordinate a constrained newsvendor's procurement decision.

Lemma 4. *Consider a 1-supplier/1-retailer configuration in the presence of a capacity constraint $k \geq 0$. Buyback and revenue sharing contracts coordinate the retailer's ordering decision for the channel, and allow for any profit allocation. In particular, the buyback and revenue sharing contracts that coordinate an unconstrained retailer (in the corresponding unconstrained channel) continue to coordinate the constrained retailer's order decision and allow for any profit allocation.*

Proof. See Appendix F.

Fig. 3 illustrates the set of buyback contracts (w,b) that channel-coordinate a capacity-constrained newsvendor (as well as unconstrained retailer) as described in Lemma 4. The buyback contracts in Fig. 3 are the *only* buyback contracts that can coordinate an unconstrained newsvendor. However, the buyback contracts in Fig. 3 are *not* the only buyback contracts that can coordinate a constrained newsvendor. There are more.

In Lemma 5 we provide necessary and sufficient conditions for a buyback contract (w,b) to coordinate a capacity-constrained newsvendor. Furthermore, we show that the set of buyback contracts that coordinate an constrained newsvendor's procurement decision is a superset of the set of buyback contracts that coordinate an unconstrained newsvendor's procurement decision.

Lemma 5. *Consider a 1-supplier/1-retailer configuration in the presence of a capacity constraint $k \geq 0$, and assume that $\overline{F}(k) > c/p$. A buyback contract $(w,b) \in \{(u,v) | c \leq u \leq p,\ v \leq u\}$ coordinates a newsvendor's procurement decision for the channel if and only if*

$$(w,b) \in \mathcal{B}(k) \stackrel{\text{def}}{=} \{(u,v) | u = (1-\lambda)v + \lambda p,\ \lambda \in [c/p, \overline{F}(k)]\}.$$

Proof. See Appendix G.

Notice that if capacity becomes large enough (so that $k \geq q^*$), then the set of coordinating buyback contracts implied by Lemma 5 and Fig. 4 simplifies to the 'classical' set of coordinating buyback contracts implied by Lemma 4 and Fig. 3.

For the constrained newsvendor, notice from Fig. 4, that for any given buyback parameter b, there is a set of wholesale price

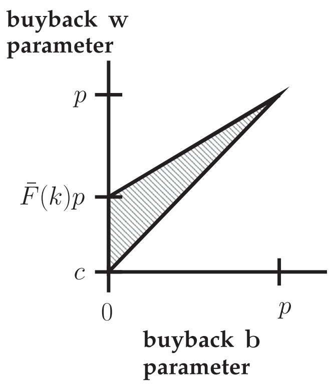

Fig. 4. Necessary and sufficient conditions for a buyback contract (w,b) to channel-coordinate a constrained newsvendor. *Note*: The shaded area represents $\mathcal{B}(k)$, all the buyback contracts (w,b) that channel-coordinate a capacity-constrained newsvendor when $k \le q^c$. Compare with Fig. 3.

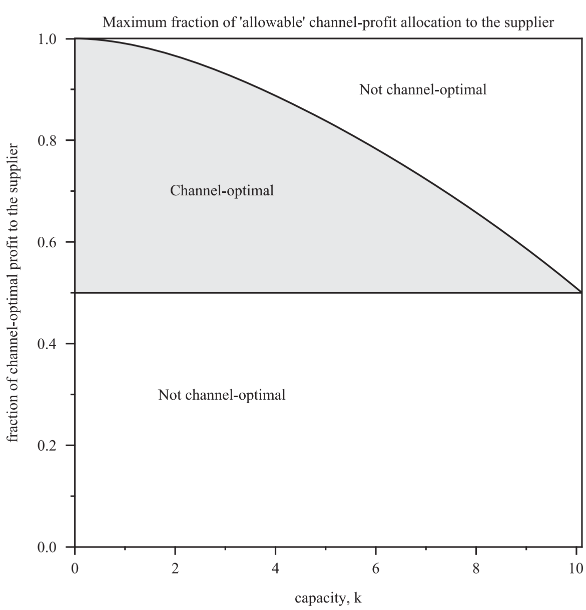

Fig. 5. Flexibility in allocating channel-optimal profit as a function of the capacity constraint. *Note*: We use the same parameters as in Fig. 2, resulting in $q^* \approx 10.112$. Furthermore, the buyback parameter is $b = p/2$, representing the level of inventory risk the supplier accepts. The shaded region denotes the fractions of profit to the supplier consistent with a channel-optimal outcome (i.e., the set $[b/p, t_s^{\max}(k;\bar{F},b)]$). Or in other words, the shaded region represents the fractional allocations of channel-optimal profit to the supplier that are achievable with some buyback contract that has a buyback parameter $p/2$ and comes from the set of coordinating contracts defined in Theorem 5.

parameters such that the resulting buyback contract coordinates the retailer's ordering decision for the channel. However for the unconstrained newsvendor, from Fig. 3, we see that for any fixed buyback parameter b, there is only one wholesale price parameter that coordinates the channel. In other words, in the unconstrained setting, for any given level of inventory risk that the supplier takes on (represented by the buyback parameter b), there is no flexibility in allocating the channel profit while maintaining coordination. However, in the constrained setting, for any level of inventory risk that the supplier accepts, there is still flexibility in allocating the channel profit. For revenue-sharing contracts, a similar flexibility exists in the constrained setting that is absent in the unconstrained, when the revenue share parameter is held fixed.

Theorem 3 formalizes the idea that in a resource constrained setting, buyback contracts have flexibility in allocating the channel-optimal profit when the inventory risk (of loss) is held fixed for the supplier (i.e., the buyback parameter is held fixed). These contracts allow for a range of divisions of the optimal channel profit among the firms. The divisions allowed (without losing coordination) depend on the channel's capacity, k. Unlike our observations for wholesale price contracts in Lemma 2 and Theorem 1 for the implicit revenue requirements, the supplier's share results in an uncertain income similar to the retailer, whose share also results in an uncertain income.

Theorem 3. *Consider a buyback parameter $b \le p$. If the capacity constraint is binding for the channel (i.e., $q^* > k$), there exists a buyback contract $(w,b) \in \mathcal{B}(k)$ that can allocate a fraction t_s of the channel-optimal profit to the supplier and a fraction $1-t_s$ to the retailer, if and only if $t_s \in [t_s^{\min}(k;\bar{F},b), t_s^{\max}(k;\bar{F},b)]$, where $t_s^{\min}(k;\bar{F},b) \overset{\text{def}}{=} b/p$ and*

$$t_s^{\max}(k;\bar{F},b) \overset{\text{def}}{=} (1-b/p) \cdot \frac{(\bar{F}(k)-c/p) \cdot k}{\int_0^k (\bar{F}(x)-c/p)\,dx} + b/p.$$

Furthermore, if F has the IGFR property, then $t_s^{\max}(k;\bar{F},b)$ is weakly decreasing as k increases in the range $[0,q^*)$.

Proof. See Appendix H.

Let us interpret Theorem 3 at two extremes values for the capacity k. As k approaches q^*, $t_s^{\max}(k;\bar{F},b)$ approaches $b/p = t_s^{\min}(k;\bar{F},b)$. Thus the supplier can only obtain one particular fraction of the channel-optimal profit with any wholesale price contract from the set of coordinating buyback contracts that has a fixed level of inventory risk b (this was to be expected because $\bar{F}(k)p = c$ when $k = q^*$ so that Figs. 3 and 4 are identical and for any b there is only w). At the other extreme, as k tends to zero, $t_s^{\max}(k;\bar{F},b)$ tends to one. Thus, for a buyback parameter b, *any* allocation of the channel-optimal profit that allocates *at least* b/p of the channel-optimal profit to the supplier becomes possible with some buyback contract from the set of coordinating contracts (this is natural, because as k tends to zero, the set of coordinating contracts becomes the entire region above the rectangle's diagonal in Fig. 3). See Fig. 5 for an example illustrating feasible allocations of the channel-optimal profit at intermediate capacity values.

Corollary 1 points out that as the supplier takes larger inventory risk (by increasing the buyback parameter), the fraction of optimal channel profit that the supplier can obtain while keeping the channel coordinated increases. This corollary follows directly from Theorem 3.

Corollary 1. *Both $t_s^{\min}(k;\bar{F}_1,b)$ and $t_s^{\max}(k;\bar{F}_1,b)$ are strictly increasing and continuous in b when $b \in [0,p)$.*

Theorem 4 parallels Theorem 2 and formalizes the idea that when we serve a larger market the 'flexibility' in allocating the channel-optimal profit 'increases'.

Theorem 4. *Consider a buyback parameter $b \le p$. Under the same assumptions as in Lemma 3, we have*

$$t_s^{\max}(k;\bar{F}_1,b) \ge t_s^{\max}(k;\bar{F}_2,b).$$

Proof. See Appendix I.

Theorem 4 suggests that a supplier (and retailer) can find flexibility in profit allocation by joining a supply channel that serves a larger market.

6. Equilibrium setting

The equilibrium setting we analyze is a two-stage (Stackelberg) game. In the *first* stage, the supplier (the '*leader*') sets a wholesale price w. In the *second* stage, the retailer (the '*follower*') chooses an optimal response q, given the wholesale price w. The supplier produces and delivers q units before the sales season starts and offers no replenishments. Both the supplier and retailer aim to maximize their own profit. The supplier's payoff function is $\pi_s(w;q)=(w-c)q$ and the retailer's payoff function is $\pi_r(q;w)=E[pS(q)-wq]$. Lariviere and Porteus (2001) analyze this Stackelberg game, for an unconstrained channel with one supplier and one retailer. They find that when F has the IGFR property, the game results in a unique outcome (q^e, w^e) defined implicitly in terms of the equations:

$$p\overline{F}(q^e)(1-g(q^e))-c=0, \tag{3}$$

$$p\overline{F}(q^e)-w^e=0, \tag{4}$$

where g is the generalized failure rate function $g(y)\overset{\text{def}}{=}yf(y)/\overline{F}(y)$. Furthermore, they show that the outcome is not channel optimal. In this section, we explore the profit allocation of the outcome when the channel has a capacity constraint (i.e., $q\le k$).

Lemma 6 provides necessary and sufficient conditions on the channel's capacity constraint k for the Stackelberg game to result in a channel-optimal equilibrium.

Lemma 6. *Assume F has the IGFR property. Consider the above described game, when the channel capacity is k units. This game has a unique equilibrium, given by $q^{eq}(k)=\min\{k,q^e\}$ and $w^{eq}(k)=\max\{p\overline{F}(k),w^e\}$, where q^e and w^e are defined by Eqs. (3) and (4), respectively. This equilibrium is channel optimal if and only if*

$$k\le q^e. \tag{5}$$

Under this condition, we have $q^{eq}=k$ and $w^{eq}=p\overline{F}(k)$.

Proof. See Appendix J.

The function $p\overline{F}(y)(1-g(y))-c$ represents the supplier's marginal profit on the yth unit, when $y<k$. When F has the IGFR property, the supplier's marginal profit is decreasing in y, while the marginal profit is nonnegative. This fact and Eq. (3) imply that inequality (5) is equivalent to the inequality $p\overline{F}(k)(1-g(k))-c\ge 0$, which can be interpreted as a statement that the supplier's marginal profit (when relaxing the capacity constraint) on the kth unit is greater than zero. Therefore, inequality (5) suggests that when the capacity constraint is binding for the supplier's problem (the ‘leader’ in the Stackelberg game), then the outcome of the game is channel optimal and vice-versa.

Theorem 5 follows from Theorem 1 and Lemma 6 and describes the equilibrium allocation of channel-optimal profit in the above described game.

Theorem 5. *Assume F has the IGFR property. Consider the above described game, when the channel capacity is k units and satisfies the inequality $k\le q^e$. The equilibrium wholesale price contract allocates the fraction $t_s^{\max}(k;\overline{F})$ of the channel-optimal profit to the supplier and the fraction $1-t_s^{\max}(k;\overline{F})$ to the retailer. Furthermore, $t_s^{\max}(k;\overline{F})$ is weakly decreasing as k increases in the range $[0,q^e)$.*

Therefore, in conjunction with Theorem 2, Theorem 5 suggests that in equilibrium the larger the expected excess demand, the greater the fraction of channel-optimal profit allocated to the supplier.

7. Discussion

We have shown that when a one-supplier/one-newsvendor supply chain is capacity-constrained at the supplier or the newsvendor, wholesale price contracts have some flexibility in allocating the channel-optimal profit (a flexibility that does not exist in the unconstrained setting). This implies that two firms have some degree of flexibility in negotiating wholesale price contracts in order to achieve the channel-optimal profit while simultaneously satisfying incentive compatibility constraints when, for example, there is limited shelf space. Furthermore, we analyzed how this flexibility changes as we change the supply chain's capacity and find that as the supply chain's capacity decreases the flexibility in allocating the channel-optimal profit increases. Furthermore, we find that as the market size increases, this allocational flexibility also increases. We also analyzed the profit allocation that is achieved in equilibrium in a newsvendor procurement game and find that suppliers attain a larger fraction of the channel's profit as the expected excess demand increases. Finally, we generalized our results to risk-sharing contracts and show that those contracts also gain additional flexibility in allocating the channel-optimal profit when the risk parameter, e.g., the buyback price, is held constant.

Appendix

In order to not disrupt the flow of presentation, the proofs for our results are contained here.

Appendix A. Proof: 1-supplier/1-retailer, Set of wholesale prices $\mathcal{W}(k)$

Proof of Lemma 1. We start by proving that if $w\in\mathcal{W}(k)$, then $q^r(w)=q^s$. Suppose first that $q^*\le k$. We then have $p\overline{F}(\min\{q^*,k\})=p\overline{F}(q^*)=c$. Therefore, $\mathcal{W}(k)=\{c\}$. Thus, for any $w\in\mathcal{W}(k)$, the problems *RETAILER(k,w)* and *CHANNEL(k)* are the same and $q^r(w)=q^s$.

Suppose now that $q^*>k$. We then have $q^s=k$ and, furthermore, $p\overline{F}(\min\{q^*,k\})=p\overline{F}(k)>p\overline{F}(q^*)=c$. (The strict inequality is obtained because $\overline{F}$ is strictly decreasing.) Therefore, $\mathcal{W}(k)=[c,p\overline{F}(k)]$. Solving $(\partial/\partial x)(E[pS(x)]-p\overline{F}(k)x)=0$ for $x\in[0,l]$ and noting $\partial S(x)/\partial x=\overline{F}(x)$, we obtain $q^r(p\overline{F}(k))=k$. Since $q^r(w)$ is nondecreasing as we decrease w, we see that for all $w\in\mathcal{W}(k)$, $q^r(w)=k=q^s$.

Suppose now that $q^r(w)=q^s$ and $c\le w\le p$. We have shown that

$$\mathcal{W}(k)=\begin{cases}\{c\}, & \text{if } q^*\le k;\\ [c,p\overline{F}(k)] & \text{if } q^*>k.\end{cases}$$

When $q^*\le k$, the first order conditions imply that $p\overline{F}(q^r(w))-w=0=p\overline{F}(q^s)-c$ for any $w\ge c$, which implies w must equal c. When $q^*>k$, we know that $q^s=k$. Assume $w>p\overline{F}(k)$ when $q^r(w)=q^s$. Due to invertibility around k, $q^r(w)<k$. This is a contradiction because $q^s=q^r(w)<k$. □

Appendix B. Proof: Revenue requirement implicit in $\mathcal{W}(k)$

Proof of Lemma 2. If the capacity constraint k is binding for the channel (i.e., $q^*>k$), then $\mathcal{W}(k)=[c,p\overline{F}(k)]$. For any wholesale price, the supplier's fraction of expected revenue is $r_s(w)\overset{\text{def}}{=}wq(w)/E[pS(q(w))]$ where $q(w)$ is the retailer's order quantity for a wholesale price w. Thus for any coordinating linear wholesale price contract $w\in\mathcal{W}(k)$,

$$r_s(w)=\frac{wk}{E[pS(k)]}=\frac{wk}{p\int_0^k\overline{F}(x)\,dx}.$$

The maximum possible value for $r_s(w)$, when $w\in\mathcal{W}(k)$, is

$$r_s^{\max}(k;\overline{F})=\frac{(p\overline{F}(k))k}{pE[S(k)]}=\frac{k\cdot\overline{F}(k)}{\int_0^k\overline{F}(x)\,dx}.$$

Accordingly, the expected revenue that the retailer receives with any linear wholesale price contract $w \in \mathcal{W}(k)$ is at least a fraction

$$1-\frac{k \cdot \overline{F}(k)}{\int_0^k \overline{F}(x)\,dx}=\frac{\int_0^k \overline{F}(x)\,dx-k \cdot \overline{F}(k)}{\int_0^k \overline{F}(x)\,dx}$$

of the total.

Next we show that if F has the IGFR property, then $r_s^{\max}(k;\overline{F})$ is weakly decreasing as k increases. We first note that

$$\frac{\partial r_s^{\max}(k;\overline{F})}{\partial k}=\frac{\overline{F}(k)}{\int_0^k \overline{F}(x)\,dx}\cdot(1-g(k)-r_s^{\max}(k;\overline{F})), \tag{6}$$

where $g(x) \stackrel{\text{def}}{=} xf(x)/\overline{F}(x)$ is the generalized failure rate function. From L'Hôpital's rule, we also have $\lim_{k\to 0} r_s^{\max}(k;\overline{F})=1$. Furthermore, the function $r_s^{\max}(k;\overline{F})$ is bounded above by 1 and goes to zero as $k\to\infty$. If this function is not weakly decreasing, there must exist some value t such that the derivative of $r_s^{\max}(k;\overline{F})$ at t is zero, and positive for values slightly larger than t. We then have

$$r_s^{\max}(t;\overline{F})=1-g(t) \tag{7}$$

since the derivative of $r_s^{\max}(k;\overline{F})$ at t is zero. For k slightly larger than t, the function $r_s^{\max}(k;\overline{F})$ increases, and $g(k)$ is nondecreasing, by the IGFR assumption. But then, Eq. (6) implies that the derivative of $r_s^{\max}(k;\overline{F})$ is negative, which is a contradiction. □

Appendix C. Proof: $\mathcal{W}(k)$'s flexibility in allocating the channel-optimal profit

Proof of Theorem 1. We first recall that given our assumption $k<q^*$, the set of coordinating wholesale price contracts is $\mathcal{W}(k)=[c,p\overline{F}(k)]$.

First we prove that $t_s \in [0,t_s^{\max}(k;\overline{F})]$, if and only if there exists a wholesale price contract $w \in \mathcal{W}(k)$ such that w allocates a fraction t_s of the channel-optimal profit to the supplier (and thus a fraction $1-t_s$ to the retailer).

For any wholesale price w, the supplier's fraction of the channel's expected profit is

$$t_s(w) \stackrel{\text{def}}{=} \frac{(w-c)q(w)}{E[pS(q(w))-cq(w)]}$$

where $q(w)$ is the retailer's order quantity for a wholesale price w. For any coordinating linear wholesale price contract $w \in \mathcal{W}(k)$, the retailer orders k units; thus we can simplify $t_s(w)$:

$$t_s(w)=\frac{(w-c)k}{E[pS(k)]-ck}=\frac{k(w/p-c/p)}{\int_0^k(\overline{F}(x)-c/p)\,dx}. \tag{8}$$

Observe that $t_s(c)=0$, $t_s(p\overline{F}(k))=t_s^{\max}(k;\overline{F})$, and $t_s(w)$ is strictly increasing and continuous in w for $w \in [c,p\overline{F}(k)]$. Thus, $t_s(w)$ is a one-to-one and onto map from the domain $[c,p\overline{F}(k)]$ to the range $[0,t_s^{\max}(k;\overline{F})]$.

Next we show that if F has the IGFR property, then

$$t_s^{\max}(k;\overline{F}) \stackrel{\text{def}}{=} \frac{k\cdot(\overline{F}(k)-c/p)}{\int_0^k(\overline{F}(x)-c/p)\,dx}$$

is weakly decreasing as k increases. Define $\overline{H}(x)=\overline{F}(x)-c/p/1-c/p$. Since $\overline{F}(q^*)=c/p$, $\overline{H}(x)$ restricted to the domain $[0,q^*)$ is equal to $1-H(x)$, where H is a c.d.f. with support $[0,q^*)$.

The generalized failure rate function $g_H(x)$ for H, defined in Eq. (9) below, can be rewritten in terms of the generalized failure rate function $g_F(x)$ for F, as follows:

$$g_H(x) \stackrel{\text{def}}{=} -\frac{x\frac{\partial \overline{H}(x)}{\partial x}}{\overline{H}(x)} \tag{9}$$

$$=\frac{xf(x)}{\overline{F}(x)-c/p}$$

$$=\frac{\overline{F}(x)}{\overline{F}(x)-c/p}\cdot\frac{xf(x)}{\overline{F}(x)}$$

$$=\frac{\overline{F}(x)}{\overline{F}(x)-c/p}\cdot g_F(x). \tag{10}$$

Furthermore,

$$\frac{\partial}{\partial x}\left(\frac{\overline{F}(x)}{\overline{F}(x)-c/p}\right)=\frac{f(x)\cdot c/p}{(\overline{F}(x)-c/p)^2}\geq 0, \tag{11}$$

which implies that $\overline{F}(x)/\overline{F}(x)-c/p$ is weakly increasing (over the domain $[0,\, q^*)$).

Since $\overline{F}(x)/\overline{F}(x)-c/p$ is positive and weakly increasing and F has the IGFR property, we can deduce that H also has the IGFR property when restricted to the domain $[0,q^*)$ (because of Eq. (10)).

Then, Theorem 2 (applied to $\overline{H}$) implies that $k\cdot\overline{H}(k)/\int_0^k \overline{H}(x)\,dx$ is weakly decreasing as k increases (while k is restricted to the domain $[0,q^*)$). But $t_s^{\max}(k;\overline{F})=k\cdot\overline{H}(k)/\int_0^k \overline{H}(x)\,dx$, which proves that $t_s^{\max}(k;\overline{F})$ is weakly decreasing as k increases (and $k<q^*$). □

Appendix D. Proof: Revenue requirement as we 'vary' F

Proof of Lemma 3. Note that $\int_0^k \overline{F}_i(x)\,dx=(\int_0^\infty \overline{F}_i(x)\,dx)-(\int_k^\infty \overline{F}_i(x)\,dx)=E[D_i]-E[(D_i-k)\cdot 1_{\{D_i>k\}}]$.

Thus,

$$\int_0^k \overline{F}_1(x)\,dx=E[D_1]-E[(D_1-k)\cdot 1_{\{D_1>k\}}]$$

$$=E[D_2]-E[(D_1-k)\cdot 1_{\{D_1>k\}}]$$

$$\leq E[D_2]-E[(D_2-k)\cdot 1_{\{D_2>k\}}]$$

$$=\int_0^k \overline{F}_2(x)\,dx. \tag{12}$$

The inequalities (12) and $\overline{F}_1(k)\geq\overline{F}_2(k)$ imply that $\overline{F}_1(k)/\int_0^k \overline{F}_1(x)\,dx\geq\overline{F}_2(k)/\int_0^k \overline{F}_2(x)\,dx$. □

Appendix E. Proof: Flexibility in allocating the channel-optimal profit as we 'vary' F

Proof of Theorem 2. Given the definition of $t_s^{\max}(k;\overline{F})$ (cf. Theorem 1), we need to prove that

$$\frac{\overline{F}_1(k)-c/p}{\int_0^k(\overline{F}_1(x)-c/p)\,dx}\geq\frac{\overline{F}_2(k)-c/p}{\int_0^k(\overline{F}_2(x)-c/p)\,dx}. \tag{13}$$

We know that $\overline{F}_1(k)\geq\overline{F}_2(k)$ and that the capacity constraint is binding for the channel's problem under both distributions. Thus,

$$\overline{F}_1(k)-c/p\geq\overline{F}_2(k)-c/p>0. \tag{14}$$

From inequality (12) in the proof of Lemma 3, we also know that $\int_0^k \overline{F}_1(x)\,dx\leq\int_0^k \overline{F}_2(x)\,dx$. Thus, we can deduce that

$$0<\int_0^k(\overline{F}_1(x)-c/p)\,dx\leq\int_0^k(\overline{F}_2(x)-c/p)\,dx. \tag{15}$$

Inequalities (14) and (15) imply that inequality (13) holds. □

Appendix F. Proof: Buyback and revenue-sharing contracts continue to coordinate

Proof of Lemma 4. Our proof follows the proof technique given in Cachon (2003) for the 1-supplier, 1-retailer channel in the absence of a capacity constraint.

Our proof has two parts. The first part shows that buyback contracts coordinate a capacity-constrained newsvendor, allocating any fraction of the channel optimal profit among the parties. The second part shows that buyback contracts are equivalent to revenue sharing contracts in a constrained setting.

Under a buyback contract (w,b) the newsvendor pays w per unit to the supplier for each unit ordered and is compensated b per unit for any unit unsold at the end of the sales season. We show that if

$$w = b + c(p-b)/p, \quad b \in [0,p], \tag{16}$$

then the buyback contract (w,b) coordinates the capacity-constrained newsvendor's ordering decision, giving the newsvendor $(p-b)/p$ fraction of the channel-optimal profit and the supplier b/p fraction of the channel-optimal profit.

We show that under the above buyback contract, (w,b), the channel-optimal order quantity, q^c, equals the retailer-optimal order quantity, q^r, as well as the supplier-optimal order quantity (i.e., the retailer's order quantity that is optimal from the supplier's point of view), q^s. Indeed,

$$\begin{aligned} q^c &\overset{\text{def}}{=} \underset{0 \le q \le k}{\arg\max}\; pS(q) - cq \\ &= \underset{0 \le q \le k}{\arg\max}((p-b)/p)(pS(q)-cq) \\ &= \underset{0 \le q \le k}{\arg\max}(p-b)S(q) - (w-b)q \quad \text{(Using buyback contract (16))} \\ &= \underset{0 \le q \le k}{\arg\max}\; pS(q) - wq + b(q - S(q)) \\ &\overset{\text{def}}{=} q^r \end{aligned} \tag{17}$$

and

$$\begin{aligned} q^c &\overset{\text{def}}{=} \underset{0 \le q \le k}{\arg\max}\; pS(q) - cq \\ &= \underset{0 \le q \le k}{\arg\max}(b/p)(pS(q)-cq) \\ &= \underset{0 \le q \le k}{\arg\max}\; bS(q) - (c-w+b)q \quad \text{(Using buyback contract (16))} \\ &= \underset{0 \le q \le k}{\arg\max}\; wq - cq - b(q - S(q)) \\ &\overset{\text{def}}{=} q^s \end{aligned} \tag{18}$$

Eqs. (17) and (18) prove that the newsvendor and supplier receive $((p-b)/p)$ and (b/p) fractions, respectively, of the channel-optimal profit.

Next, we remind the reader that buyback contracts and revenue sharing contracts are equivalent (regardless of the channel's capacity constraint). Under a revenue sharing contract the newsvendor purchases each unit from a supplier at a price of w_r per unit, keeps a fraction f of the revenue, and shares a fraction $(1-f)$ of the revenue with the supplier. A given buyback contract, (w,b), is a revenue sharing contract where the newsvendor purchases at $w-b$ per unit from the supplier and in return shares a fraction b/p of the revenue with the supplier. Similarly, a given revenue sharing contract, (w_r, f), is a buyback contract where the newsvendor purchases at $w_r + (1-f)p$ per unit and is compensated $(1-f)p$ per unit by the supplier for any unsold items at the end of the sales season. Since there is a one-to-one mapping from buyback contracts to revenue sharing contracts and because buyback contracts coordinate a constrained newsvendor's ordering decision, we conclude that revenue sharing contracts also coordinate a constrained newsvendor's ordering decision. □

Appendix G. Proof: Necessary and sufficient conditions for risk-sharing contracts to coordinate

Proof of Lemma 5. Let

$$B \overset{\text{def}}{=} \{(u,v) | u = (1-\lambda)v + \lambda p, \lambda \in [c/p, F(k)]\}$$

and

$$A \overset{\text{def}}{=} \{(u,v) | c \le u \le p, v \le u\}.$$

The proof has two parts. First we show every buyback contract $(w,b) \in B \subseteq A$ channel-coordinates the newsvendor's decision. Then, we show that there are no other buyback contracts in the set A that can channel-coordinate the newsvendor's decision. Before we proceed note that the optimal order quantity for the constrained channel is k (because $\overline{F}(k) > c/p$). Thus, the capacity constraint is tight.

First we show that every buyback contract $(w,b) \in B$ channel-coordinates. If $(w,b) \in B$, then $w - b = \lambda(p-b)$ for some $\lambda \in [c/p, \overline{F}(k)]$. The newsvendor orders $\min\{k, \overline{F}^{-1}(w-b/p-b)\}$. But $w-b/p-b \in [c/p, \overline{F}(k)]$, therefore $\overline{F}^{-1}(w-b/p-b) \ge k$ and $\min\{k, \overline{F}^{-1}(w-b/p-b)\} = k$. The newsvendor thus orders the channel-optimal order quantity for this capacity-constrained channel.

Next we show that there is no buyback contract (w,b) outside of B but in set A that channel-coordinates the newsvendor's action. Assume the contrary. Namely, assume a buyback contract $(w,b) \in A \backslash B$ channel-coordinates the newsvendor's action. Under buyback contract (w,b), the constrained newsvendor orders $\min\{k, \overline{F}^{-1}(w-b/p-b)\}$. But since (w,b) channel-coordinates the newsvendor's decision, we have $\min\{k, \overline{F}^{-1}(w-b/p-b)\} = k$, since the newsvendor's constraint is tight. Therefore, $\overline{F}^{-1}(w-b/p-b) \ge k$, implying $w-b/p-b \le \overline{F}(k)$. Furthermore, $\min_{(w,b) \in A}(w-b)/(p-b) = c/p$, implying $(w-b)/(p-b) \ge c/p$. Thus, $(w,b) \in B$, because $w-b = \lambda(p-b)$ for some $\lambda \in [c/p, \overline{F}(k)]$. But this is a contradiction. □

Appendix H. Proof: Buyback flexibility in allocating the channel-optimal profit

Proof of Theorem 3. We first recall that given our assumption $k < q^*$, the set of coordinating buyback contracts is $\mathcal{B}(k) \overset{\text{def}}{=} \{(u,v) | u = (1-\lambda)v + \lambda p, \lambda \in [c/p, \overline{F}(k)]\}$.

First we prove that $t_s \in [t_s^{\min}(k; \overline{F}, b), t_s^{\max}(k; \overline{F}, b)]$, if and only if there exists a buyback contract $(w,b) \in \mathcal{B}(k)$ such that (w,b) allocates a fraction t_s of the channel-optimal profit to the supplier (and thus a fraction $1-t_s$ to the retailer).

For any buyback contract (w,b), the supplier's fraction of the channel's expected profit is

$$t_s(w;b) \overset{\text{def}}{=} \frac{(w-c)q(w,b) - b(q - S(q(w,b)))}{E[pS(q(w,b)) - cq(w,b)]}$$

where $q(w,b)$ is the retailer's order quantity for a buyback contract (w,b). For any coordinating buyback contract $(w,b) \in \mathcal{B}(k)$, the retailer orders k units; thus we can simplify $t_s(w;b)$:

$$t_s(w;b) = \frac{(w-c)k - b(k - S(k))}{E[pS(k)] - ck} = \frac{1}{p}\frac{(w-c)k - b(\int_0^k (1-\overline{F}(x))\,dx)}{\int_0^k (\overline{F}(x) - c/p)\,dx}. \tag{19}$$

Therefore, for any $\lambda \in [c/p, \overline{F}(k)]$, we have

$$t_s((1-\lambda)b + \lambda p; b) = \frac{1}{p} \cdot \frac{((1-\lambda)b + \lambda p - c)\cdot k + b \cdot \int_0^k (\overline{F}(x) - 1)\,dx}{\int_0^k (\overline{F}(x) - c/p)\,dx}$$

$$=\frac{1}{p}\cdot\frac{(-\lambda b+\lambda p-c)\cdot k+b\cdot\int_0^k\overline{F}(x)\,dx}{\int_0^k(\overline{F}(x)-c/p)\,dx}$$
$$=\frac{(1-b/p)\lambda k-(c/p)k+(b/p)\cdot\int_0^k\overline{F}(x)\,dx}{\int_0^k(\overline{F}(x)-c/p)\,dx}$$
$$=\frac{(1-b/p)\cdot(\lambda-c/p)\cdot k+(b/p)\cdot\int_0^k(\overline{F}(x)-c/p)\,dx}{\int_0^k(\overline{F}(x)-c/p)\,dx}$$
$$=\frac{(1-b/p)\cdot(\lambda-c/p)\cdot k}{\int_0^k(\overline{F}(x)-c/p)\,dx}+b/p. \tag{20}$$

From Eq. (20), observe that $t_s((1-c/p)b+(c/p)p;b)=b/p$ and $t_s((1-\overline{F}(k))b+\overline{F}(k)p;b)=t_s^{\max}(k;\overline{F},b)$. Furthermore, from Eq. (19), we have that $t_s(w;b)$ is strictly increasing and continuous in w when w is in the set

$$[(1-c/p)b+(c/p)p,(1-\overline{F}(k))b+\overline{F}(k)p].$$

Thus, $t_s(w;b)$ is a one-to-one and onto map from the domain $\{(1-\lambda)b+\lambda p|\lambda\in[c/p,\overline{F}(k)]\}$ to the range $[t_s^{\min}(k;\overline{F},b),t_s^{\max}(k;\overline{F},b)]$.

From Theorem 1, we have that if F has the IGFR property, then

$$t_s^{\max}(k;\overline{F},b)\overset{\text{def}}{=}(1-b/p)\cdot\frac{(\overline{F}(k)-c/p)\cdot k}{\int_0^k(\overline{F}(x)-c/p)\,dx}+b/p$$

is weakly decreasing as k increases in the range $[0,q^*]$. □

Appendix I. Proof: Buyback flexibility in allocating the channel-optimal profit as we 'vary' *F*

Proof of Theorem 4. From Theorem 2, we have that

$$\frac{(\overline{F}_1(k)-c/p)\cdot k}{\int_0^k(\overline{F}_1(x)-c/p)\,dx}\geq\frac{(\overline{F}_2(k)-c/p)\cdot k}{\int_0^k(\overline{F}_2(x)-c/p)\,dx}. \tag{21}$$

Therefore, we have that

$$(1-b/p)\cdot\frac{(\overline{F}_1(k)-c/p)\cdot k}{\int_0^k(\overline{F}_1(x)-c/p)\,dx}+b/p\geq(1-b/p)\cdot\frac{(\overline{F}_2(k)-c/p)\cdot k}{\int_0^k(\overline{F}_2(x)-c/p)\,dx}+b/p.\quad\square \tag{22}$$

Appendix J. Proof: When is the equilibrium of the Stackelberg game channel optimal?

Proof of Lemma 6. The retailer's profit function $\pi_r(q;w)$ under a wholesale price contract w is defined as $\pi_r(q;w)\overset{\text{def}}{=}E[pS(q)-wq]$. Since $\pi_r(q;w)$ is concave, in q, we can use the first order conditions and conclude that for a wholesale price $w\in[c,p]$, the constrained retailer's order quantity $q^r(w)$ is given by

$$q^r(w)=\min\{k,\overline{F}^{-1}(w/p)\}. \tag{23}$$

The supplier's profit function $\pi_s(w;q)$ under a wholesale price contract w is defined as $\pi_s(w;q)\overset{\text{def}}{=}(w-c)q$. Since $q^r(w)$ is the retailer's best response in the second stage to a wholesale price w by the supplier in the first stage, Eq. (23) allows us to express the supplier's objective function as follows:

$$\pi_s(w)=\begin{cases}(w-c)k, & \text{if } c\leq w\leq\max\{c,p\overline{F}(k)\};\\ (p\overline{F}(q^r(w))-c)q^r(w) & \text{if } \max\{c,p\overline{F}(k)\}<w\leq p.\end{cases} \tag{24}$$

For $w>\max\{c,p\overline{F}(k)\}$, note that

$$\partial\pi_s(w)/\partial w=(p\overline{F}(q^r(w))(1-g(q^r(w)))-c)\cdot\frac{\partial q^r(w)}{\partial w}.$$

Since the function $p\overline{F}(y)(1-g(y))-c$ is strictly decreasing in y when it is nonnegative and equals zero at q^e (see Eq. (3)), we can deduce that $(p\overline{F}(q^r(w))(1-g(q^r(w)))-c)>0$ for $w>w^e$ (because $q^r(w)<q^e$). Furthermore, $\partial q^r(w)/\partial w<0$ for $w>p\overline{F}(k)$. Therefore, we can conclude that $\partial\pi_s(w)/\partial w<0$ for $w>\max\{w^e,p\overline{F}(k)\}$.

Either the inequality $p\overline{F}(k)<w^e$ holds or the inequality $w^e\leq p\overline{F}(k)$ holds. First assume that the inequality $p\overline{F}(k)<w^e$ holds. Eq. (24) implies that $\pi_s(w)$ is increasing linearly between c and $\max\{c,p\overline{F}(k)\}$. Furthermore, since

$$(p\overline{F}(q^r(w))(1-g(q^r(w)))-c)<0$$

for $w<w^e$ (because $q^r(w)>q^e$), we can deduce that

$$\frac{\partial\pi_s(w)}{\partial w}=(p\overline{F}(q^r(w))(1-g(q^r(w)))-c)\cdot\frac{\partial q^r(w)}{\partial w}>0$$

for $w\in(\max\{c,p\overline{F}(k)\},w^e)$. And we know

$$\frac{\partial\pi_s(w)}{\partial w}<0$$

for $w>\max\{w^e,p\overline{F}(k)\}=w^e$. Therefore, $w^{eq}(k)=w^e$ and Eqs. (23) and (4) imply $q^{eq}(k)=q^e$. The inequality $p\overline{F}(k)<w^e$ is equivalent to the inequality $q^e<k$ (see Eq. (4)). Therefore, when $q^e<k$ holds, the inequality $w^{eq}(k)=w^e>\max\{c,p\overline{F}(k)\}=p\overline{F}(\min\{q^*,k\})$ holds and we can deduce that $w^{eq}(k)\notin\mathcal{W}(k)$ (using Lemma 1).

Next assume $w^e\leq p\overline{F}(k)$ holds. Since $\partial\pi_s(w)/\partial w<0$ for $w>\max\{w^e,p\overline{F}(k)\}=\max\{c,p\overline{F}(k)\}$, Eq. (24) implies $w^{eq}(k)=p\overline{F}(k)$ and Eq. (23) implies $q^{eq}(k)=k$. The inequality $w^e\leq p\overline{F}(k)$ is equivalent to the inequality $k\leq q^e$ (see Eq. (4)). Therefore, when $k\leq q^e$ holds, the equality $w^{eq}(k)=p\overline{F}(k)=\max\{c,p\overline{F}(k)\}=p\overline{F}(\min\{q^*,k\})$ holds and we can deduce that $w^{eq}(k)\in\mathcal{W}(k)$ (again using Lemma 1). □

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