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Lecture notes for Physics 8.324: Relativistic Quantum field theory II, Fall 2024

# Contents

# 1 Introduction and overview

## 1.1 Where we are now

In the previous semester we developed the basic structure of relativistic quantum field theory. We saw that constructing a relativistic theory of interacting quantum particles which respects causality requires the fundamental degrees of the theory to be **fields**. Restricting for a moment to theories with no fermions, quantum fields are operators $\Phi^a(x)$ obeying the following two conditions:

(1) **Poincare symmetry:** they transform in representations

$$U(\Lambda, a)^\dagger \Phi^a(x) U(\Lambda, a) = \sum_{a'} D_{aa'}(\Lambda) \Phi^{a'}(\Lambda^{-1}(x-a)) \tag{1.1}$$

of the Poincaré group, where $U(\Lambda, a)$ are the unitary operators implementing Poincaré symmetry on Hilbert space. The representation matrices $D_{aa'}$ obey

$$\sum_{a'} D_{aa'}(\Lambda_1) D_{a'a''}(\Lambda_2) = D_{aa''}(\Lambda_1 \Lambda_2). \tag{1.2}$$

(2) **Microcausality:** they commute at spacelike separation:

$$[\Phi^a(x), \Phi^b(y)] = 0 \qquad (x-y)^2 > 0. \tag{1.3}$$

Recall that the the Poincaré group is the set of spacetime transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{1.4}$$

with $\Lambda$ obeying

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta} \tag{1.5}$$

and also

$$\begin{aligned} \det \Lambda &= 1 \\ \Lambda^0{}_0 &\geq 1. \end{aligned} \tag{1.6}$$

The condition (1.5) ensures the transformation (1.4) preserves the Minkowski space inner product, or equivalently is an isometry of the Minkowski metric $\eta_{\mu\nu}$, while the conditions (1.6) ensure that it preserves the orientations of space and time (i.e. it does not include time reversal and/or spatial reflection). The spacetime vector $a$ is a spacetime translation while $\Lambda$ is a Lorentz transformation built out of some combination of boosts and rotations. Setting $a = 0$ the Poincaré group reduces to what is sometimes called the **proper orthochronous Lorentz group**, usually denoted $SO^+(d-1,1)$. Incorporating fermions into this formalism is not difficult; we need to modify it in two ways:

1. We allow for fields which transform in representations of the spin double cover of the Poincare group. This is a group that locally looks like the Poincare group, but globally allows for the fact that on fields of half-integer spin a rotation by $2\pi$ is equal to $-1$ instead of 1. We'll explain this in more detail in the next few sections.

2. Fermionic fields are required to anticommute instead of commute at spacelike separation.

Last semester we saw that this formalism leads to some rather general consequences for relativistic quantum mechanics:

- **Particle non-conservation:** In quantum field theories the interactions always allow for particles to be created and destroyed, essentially because the interaction terms always have the form $(a + a^\dagger)^n$ and this always includes terms with different numbers of creation and annihilation operators.

- **Existence of antiparticles:** Given any particle of mass $m$ and charge $q$, we saw that to construct a field operator obeying microcausality we need to also have a particle of mass $m$ and charge $-q$.
- **Spin-statistics theorem:** By using the rotational symmetry of the Euclidean path integral, we saw that particles/fields of integer spin must be bosons and particles/fields of half-integer spin must be fermions. This is essentially because we can exchange two fields by rotating each by $\pi$, and for an appropriate basis of fields this is the same as a rotation by $2\pi$.
- **CRT theorem:** Again using rotational symmetry in Euclidean signature, we saw that any relativistic quantum field theory has a special symmetry called $\mathcal{CRT}$ which reverses time, reflects one spatial direction, and exchanges particles and antiparticles.

An example of a relativistic quantum field theory that we studied in great detail last semester is the **free scalar field**, with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2. \tag{1.7}$$

We solved this theory by introducing the Heisenberg field operator

$$\Phi(x) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}} e^{ip\cdot x} + a^\dagger_{\vec{p}} e^{-ip\cdot x}\right], \tag{1.8}$$

introducing the canonical momentum

$$\Pi \equiv \dot{\Phi}, \tag{1.9}$$

and imposing the canonical commutation relations

$$\begin{aligned} &[\Phi(t,\vec{x}),\Phi(t,\vec{y})] = 0 \\ &[\Pi(t,\vec{x}),\Pi(t,\vec{y})] = 0 \\ &[\Phi(t,\vec{x}),\Pi(t,\vec{y})] = i\delta^{d-1}(\vec{x}-\vec{y}). \end{aligned} \tag{1.10}$$

These imply that the creation and annihilation operators obey

$$\begin{aligned} &[a_{\vec{p}}, a_{\vec{p}'}] = 0 \\ &[a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}] = 0 \\ &[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^{d-1}\delta^{d-1}(\vec{p}-\vec{p}'), \end{aligned} \tag{1.11}$$

which is essentially just the algebra of a continuous set of harmonic oscillators labeled by the spatial momentum $\vec{p}$. Moreover we showed that (after a renormalization of the cosmological constant) the Hamiltonian is given by

$$H = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \omega_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}}. \tag{1.12}$$

Thus the ground state of the theory, also called the vacuum, is the state $|\Omega\rangle$ which is annihilated by all $a_{\vec{p}}$, and by acting on the vacuum with the $a^\dagger_{\vec{p}}$s we can create particles, each of which has momentum $\vec{p}$ and energy $\omega_{\vec{p}}$. These particles are bosons, since e.g. we have

$$a^\dagger_{\vec{p}} a^\dagger_{\vec{p}'}|\Omega\rangle = a^\dagger_{\vec{p}'} a^\dagger_{\vec{p}}|\Omega\rangle. \tag{1.13}$$

We also considered interacting theories, in particular spending a lot of time on the $\lambda\phi^4$ theory with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.14}$$

This theory has many physical applications, for example it can be used to study the critical points of magnets and the Higgs boson in particle physics (this is the complex version). Unfortunately it cannot be solved exactly, but we saw we could learn quite a bit about it using perturbation theory. Perturbation theory is especially nice to construct starting from the path integral formalism. For example the path integral computes the time-ordered $n$-point correlation function of $\phi$ as

$$\langle T\phi(x_1)\dots\phi(x_n)\rangle = \frac{\int \mathcal{D}\phi\phi(x_1)\dots\phi(x_n)e^{iS_\epsilon}}{\int \mathcal{D}\phi e^{iS_\epsilon}}, \tag{1.15}$$

where $S_\epsilon$ is the action evaluated using the $i\epsilon$-prescription

$$S_\epsilon = \int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt \int d^{d-1}x\mathcal{L}. \tag{1.16}$$

We can evaluate these path integrals perturbatively by Taylor-expanding the integrand in $\lambda$, which reduces all integrals to moments of Gaussian integrals that can be done explicitly. The result is the **Feynman rules** for computing correlation functions.

In quantum field theory it is the fields that are fundamental, not the particles, and indeed in some quantum field theories there are no particles at all. On the other hand many quantum field theories do have particles, such as the standard model of particle physics, and in those theories we are often interested in the scattering theory of the particles. We formalized this scattering theory by introducing "in" and "out" eigenstates of the Hamiltonian, whose wave packets are non-interacting at early or late times respectively, and we defined the S-matrix as the inner product between the in and out states. We were careful to emphasize that the in and out states in general do not have any simple relationship to the fields appearing in the Lagrangian, for example in QCD the fields in the Lagrangian describe quarks and gluons but the scattering states are hadrons. We further explained how the S-matrix can be extracted from the correlation functions of fields using the **LSZ reduction formula**, which works by taking the Fourier transform of the correlator and then taking all external momenta to go on shell and looking at the residue of the resulting pole.

We closed the semester by introducing the **renormalization group**, initially as a way of interpreting the various short-distance divergences which arise in computing Feynman diagrams and then as a general philosophy of life. What we found is that the operation of going to low energies (or equivalently long distances) has a strong focusing behavior, formalized as Polchinski's theorem, which "forgets" high-energy/short-distance information about the theory. This focusing behavior is what leads to the extraordinarily predictive nature of quantum field theory: we can realize the generic low-energy behavior by including only "relevant" and "marginal" terms in the Lagrangian (these are terms whose energy dimension is less than or equal to $d$). So for example for $d = 4$ the most general Lagrangian we need to write down for a scalar field with a symmetry $\phi' = -\phi$ is precisely the Lagrangian (1.14).[1]

## 1.2 What's next?

So far our discussion of concrete quantum field theories has been limited to scalar theories, where the Poincaré representation $D$ appearing in equation (1.1) is just the trivial representation (meaning $D_{aa'}(\Lambda) = \delta_{aa'}$ for all $\Lambda$). Most of the interesting quantum field theories in nature however have nontrivial $D$, and in particular we have the following examples:

- **Spinor fields:** these fields transform in nontrivial Poincaré representations where a rotation by $2\pi$ acts as $-1$, so by the spin-statistics theorem they anticommute at spacelike separation. The particles they create/annihilate are fermions. Fermions are an essential part of nature, for example the Pauli exclusion principle is what accounts for the stability of solid matter, so we clearly had better learn how to include them in field theory.

[1] Strictly speaking we should also include a cosmological constant term, but this has no physical effect until we consider gravity.

- **Gauge fields:** these are one-form fields $A_\mu(x)$ which are also charged under a local gauge symmmetry. The simplest example is the vector potential of electromagnetism, which has the familiar gauge transformation
$$A'_\mu = A_\mu + \partial_\mu \Lambda \tag{1.17}$$
with $\Lambda$ an arbitrary scalar function of space and time. We will see that such a gauge symmetry is necessary whenever we have a massless particle of helicity one. There are also more sophisticated "non-abelian" gauge fields, where $A_\mu$ is a matrix of one-forms instead of just a single one-form. Gauge fields are the mediators of all of the non-gravitational forces in nature, and they also appear in the description of interesting condensed matter phenomena such as the fractional quantum hall effect.

- **Gravity:** in Einstein's theory of gravity the geometry of spacetime is described by a dynamical metric tensor $g_{\mu\nu}(x)$. This theory also has a local gauge symmetry, called **general coordinate invariance** or **diffeomorphism symmetry**, which says that the theory looks the same in arbitrary coordinate systems. Just as having a consistent theory of a massless particle of helicity one (the photon) requires the gauge symmetry (1.17), having a consistent theory of a massless particle of helicity two (the graviton) requires general coordinate invariance.

Gravitational theories are sufficiently different from non-gravitational theories that it is conventional to view them as falling outside of the standard framework of quantum field theory. There are several reasons for this:

1. **No local operators:** we will learn later that gauge symmetries typically must be viewed as redundancies rather than physical transformations, which is why in electromagnetism it is the gauge-invariant field strength tensor
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.18}$$
which measurable rather than the vector potential $A_\mu$.[2] In a gravitational theory general coordinate transformations are gauge transformations, so no local operator $O(x)$ can be gauge-invariant (since gauge transformations would move it around). In a gravitational theory gauge-invariant observables need to be defined in a *relational* manner, with their locations defined relative to some feature in the system or else the asymptotic boundary. This is rather inconvenient from the point of view of our field theory machinery, which is mostly built around correlation functions of local operators.

2. **No energy-momentum tensor:** in quantum field theory an essential object is the energy-momentum tensor, which we defined as the derivative of the action with respect to a background metric. The various Poincaré symmetry generators are expressed as integrals involving the energy-momentum tensor, and its correlation functions appear in many important quantum field theory applications (for example the energy flux in a particle detector or the flux of energy out of an evaporating black hole). In gravitational theories there is no energy-momentum tensor since the metric is a dynamical field rather than a background field.

3. **Less important role for Poincaré symmetry:** in ordinary quantum field theory the Poincaré group arises as the symmetry group of the background Minkowski metric. In gravity the metric is dynamical, and indeed depending on the matter configuration it need not look anything like Minkowski space. And moreover even when we do restrict to situations where the metric approaches the Minkowski metric at large distances, Poincaré symmetry mixes into a larger structure involving "BMS transformations" which even today is not completely understood.

4. **No renormalizable interactions:** we saw at the end of last semester that in general relativity the interactions between gravitons and also the interactions of gravitons and matter are non-renormalizeable, or in the Wilsonian parlance they are irrelevant. This means that they vanish at energies that are low

[2] An exception to this statement is the Aharonov-Bohm effect, which measures the gauge-invariant line integral of $A$ around a closed loop. This however is only really a separate degree of freedom if the loop is topologically non-contractible (or more carefully non-contractible within the region of spacetime where the gauge field is a good description of the physics).

compared to the Planck scale, which is why gravity is so weak, and it also means that gravitational interactions are not constrained by the focusing behavior of the renormalization group: as we work to higher and higher orders in Newton's constant, we need to include more and more terms in the Lagrangian to parametrize the possible interactions. This means that we need some "UV-complete" theory of quantum gravity if we are ever to understand in detail what happens in high-energy graviton scattering.[3]

5. **No theory yet:** building off of the previous point, so far we do not have any candidate theory of quantum gravity that is consistent with everything we know about the world. There is a promising framework called string theory which seems to contain many of the right ingredients, but so far we only know how to formulate string theory precisely in unrealistic situations with unbroken supersymmetry and a non-positive cosmological constant.

6. **Black holes and holography:** in the special corners where we do understand string theory, the precise formulation looks nothing like the quantum field theory we have pursued in this class. Rather than quantizing some matter fields and a metric on spacetime, we instead quantize some strongly-interacting non-gravitational lower-dimensional theory living in some auxiliary spacetime. The conventional picture of spacetime is *emergent*, rather than fundamental. This is a radical change from our quantum field theory understanding of non-gravitational physics, and it is usually referred to as holography. Holography seems to be necessary in order to have a consistent theory of quantum black holes.

For these reasons we will therefore mostly not consider gravity any further in this class. Spinor and gauge fields however will squarely be our business, and indeed the rest of the semester will more or less consist of developing the formalism for these and introducing some of their most interesting applications. Here are some of the applications we will consider:

- **Quantum electrodynamics (QED):** the fundamental theory of electrons, positrons, and nuclei interacting via quantized photons, QED underlies most of atomic physics and thus accounts for much of the world we see around us. In particular phenomena such as the 2s$\to$1s decay of hydrogen, the anomalous magnetic moment of the electron, and the Lamb shift cannot be understood without a quantum theory of light.

- **Yukawa theory of the nuclear force:** How are the protons and neutrons in the atomic nucleus held together? The answer is that there is an attractive force between nucleons which is mediated by scalar particles called pions, and we will study this force using quantum field theory. Next semester we will see how this force arises from quantum chromodynamics (QCD), which is the fundamental theory of the strong force.

- **2D Ising model:** one of the landmark developments of statistical physics was Onsager's solution of the classical Ising model in two spatial dimensions. A key accomplishment of Onsager's approach was the calculation of critical exponents in the vicinity of the transition point. Onsager's original solution is quite difficult to understand, but it was later realized that the essential reason why the model is solvable is that at the critical point it can be rewritten in terms of a free fermion field theory. This enables fairly straightforward calculations of the critical exponents.

- **Superconductivity:** one of the most remarkable phenonema in solid state physics is the possibility of conductivity with zero resistance. The available microscopic explanation of this, called BCS theory, is fairly sophisticated and is not strong enough to account for all observed examples of superconductivity. A general phenomenological theory is available however via the idea that a superconductor is merely a system in which the gauge symmetry of electromagnetism is spontaneously broken, or "Higgsed".

[3] It must be emphasized however that other non-renormalizeable theories, such as that which controls the physics of pions and nucleons, can be fit squarely within the framework quantum field theory. So unlike the other problems in this list, this one is not unique to gravity.

The main topic we will *not* cover this semester is "non-abelian gauge fields", which are a generalization of QED that is needed to describe the strong and weak nuclear forces; this will be the main topic of the third and final semester of QFT.

**Problems:**

1. Review anything above "what's next" which you aren't comfortable with, especially the equations.

# 2 Quantum Fermions

We now begin our systematic study of fermions in quantum field theory. By definition fermions are particles with the property that if you have more than one of them then the quantum state is antisymmetric under exchanging any two of them. This is to be distinguished from the bosons we have encountered so far, for example in our free scalar field theory the creation operators were commuting so

$$a^\dagger_{\vec{p}} a^\dagger_{\vec{p}'} |\Omega\rangle = a^\dagger_{\vec{p}'} a^\dagger_{\vec{p}} |\Omega\rangle. \tag{2.1}$$

In order to get fermions we need the creation and annihilation operators to be **anticommuting**, meaning that we want to get something like

$$\begin{aligned} \{a_{\vec{p}}, a_{\vec{p}',\sigma'}\} &= 0 \\ \{a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}',\sigma'}\} &= 0 \\ \{a_{\vec{p}}, a^\dagger_{\vec{p}',\sigma'}\} &= \delta_{\sigma,\sigma'} (2\pi)^{d-1} \delta^{d-1}(\vec{p} - \vec{p}'), \end{aligned} \tag{2.2}$$

where we have introduced the **anticommutator**[4]

$$\{A, B\} = AB + BA. \tag{2.3}$$

We've also included a spin index $\sigma$ out of regard for the spin-statistics theorem: fermions cannot have spin zero since a rotation by $\pi$ must act on them as $-1$. Acting on the vacuum with creation operators obeying the algebra (2.2) automatically gives us fermions, for example we now have

$$a^\dagger_{\vec{p}} a^\dagger_{\vec{p}',\sigma'} |\Omega\rangle = -a^\dagger_{\vec{p}',\sigma'} a^\dagger_{\vec{p}} |\Omega\rangle. \tag{2.4}$$

The Pauli exclusion principle is automatic as well, since we have

$$(a^\dagger_{\vec{p}})^2 = \frac{1}{2}\{a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}}\} = 0 \tag{2.5}$$

so it is impossible to create two fermions with the same momentum and spin.

Of course this is quantum field theory class and not quantum particle class, so rather than simply postulating the algebra (2.2) we need to derive it starting from some field theory. The commutativity of bosonic creation and annihilation operators came from the fact that bosonic fields obey the **canonical commutation relations**:

$$\begin{aligned} [\Phi^a(\vec{x}), \Phi^b(\vec{y})] &= 0 \\ [\Pi_a(\vec{x}), \Pi_b(\vec{y})] &= 0 \\ [\Phi^a(\vec{x}), \Pi_b(\vec{y})] &= i\delta^a_b \delta^{d-1}(\vec{x} - \vec{y}). \end{aligned} \tag{2.6}$$

[4]Not to be confused with the classical Poisson bracket, which we wrote using the same notation.

To get the fermionic algebra we therefore need fields which obey **canonical anticommutation relations**:

$$\begin{aligned}\{\Phi^a(\vec{x}),\Phi^b(\vec{y})\} &= 0\\ \{\Pi_a(\vec{x}),\Pi_b(\vec{y})\} &= 0\\ \{\Phi^a(\vec{x}),\Pi_b(\vec{y})\} &= i\delta^a_b\delta^{d-1}(\vec{x}-\vec{y}).\end{aligned} \tag{2.7}$$

The canonical commutation relations are an essential part of the procedure for constructing quantum theories starting from a classical Lagrangian, so these anticommutation relations may seem somewhat mysterious. We will therefore now study a simplified version of them, learning that a finite number of quantum fermions has a finite-dimensional Hilbert space and thus does not have a classical limit in the same way a finite number of bosonic variables (such as the harmonic oscillator) does. This will end up being the reason why you are familiar with the classical electromagnetic field but not with a "classical electron field". Once we have better intuition for quantum fermions, we will return to the problem of constructing fermionic quantum fields that are consistent with Lorentz invariance and causality.

## 2.1 One fermion

Let's first consider the quantum mechanics of a single fermionic variable $\Psi$ and its canonical conjugate $\Pi$. It is convenient to redefine the latter via $\Pi = i\Xi$ to remove the factor of $i$ from the canonical anticommutator:

$$\begin{aligned}\{\Psi,\Psi\} &= 0\\ \{\Xi,\Xi\} &= 0\\ \{\Psi,\Xi\} &= 1.\end{aligned} \tag{2.8}$$

This is the fermionic analogue of the canonical commutation relations for a single quantum particle moving in one spatial dimension. The first two lines merely say that

$$\Psi^2 = \Xi^2 = 0, \tag{2.9}$$

so in particular we can always find a state $|0\rangle$ which is annihilated by $\Psi$:

$$\Psi|0\rangle = 0. \tag{2.10}$$

This is because given any state $|\phi\rangle$ which isn't annihilated by $\Psi$ we simply act on it once with $\Psi$ to get a state $\Psi|\phi\rangle$ which is annihilated by $\Psi$ since $\Psi^2 = 0$. Acting on $|0\rangle$ with $\Xi$ we then get another state

$$|1\rangle = \Xi|0\rangle. \tag{2.11}$$

This state can't vanish since we have

$$\Psi|1\rangle = \Psi\Xi|0\rangle = \Big(\{\Psi,\Xi\} + \Xi\Psi\Big)|0\rangle = |0\rangle. \tag{2.12}$$

On the other hand we have

$$\Xi|1\rangle = \Xi^2|0\rangle = 0. \tag{2.13}$$

Altogether we have thus found a two-dimensional representation of the canonical anticommutation relations:

$$\Psi = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \qquad \Xi = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \qquad |0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}. \tag{2.14}$$

In modern parlance, the Hilbert space of a single fermion degree of freedom is just that of one qubit. I emphasize that this is much simpler than the Hilbert space for a single bosonic variable obeying canonical

commutation relations, which has an infinite-dimensional Hilbert space spanned by position eigenstates $|x\rangle$. We can write this representation in terms of Pauli matrices:

$$\begin{aligned}\Psi &= \frac{1}{2}\left(\sigma_x + i\sigma_y\right) \\ \Xi &= \frac{1}{2}\left(\sigma_x - i\sigma_y\right).\end{aligned} \tag{2.15}$$

Here of course the Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.16}$$

It is convenient to introduce a fermion number operator

$$F = \Xi\Psi, \tag{2.17}$$

which obeys

$$\begin{aligned}F|0\rangle &= \Xi\Psi|0\rangle = 0 \\ F|1\rangle &= \Xi\Psi\Xi|0\rangle = \{\Xi, \Psi\}\Xi|0\rangle = |1\rangle.\end{aligned} \tag{2.18}$$

In the Pauli representation the fermion parity operator $(-1)^F$, which acts on $|0\rangle$ as 1 and $|1\rangle$ as $-1$, is simply given by

$$(-1)^F = \sigma_z. \tag{2.19}$$

## 2.2 N fermions

All this generalizes nicely to a system of $N$ fermions $\Psi^a$ and their conjugates $\Xi_a$, which obey the canonical anticommutation relations

$$\begin{aligned}\{\Psi^a, \Psi^b\} &= 0 \\ \{\Xi_a, \Xi_b\} &= 0 \\ \{\Psi^a, \Xi_b\} &= \delta^a_b.\end{aligned} \tag{2.20}$$

Namely we first construct a state $|0\rangle$ which is annihilated by all of the $\Psi^a$:

$$\Psi^a|0\rangle = 0. \tag{2.21}$$

We can always find such a state by picking any state which isn't annihilated by some subset of the $\Psi_a$ and then acting on it with them. The order in which we act doesn't matter since the $\Psi_a$ all anticommute and thus can be moved past each other up to a sign. We then construct a basis for the Hilbert space of the form

$$|s_1 \dots s_N\rangle = \Xi_1^{s_1} \dots \Xi_N^{s_N}|0\rangle, \tag{2.22}$$

with each $s_a$ equal to zero or one. Note that by definition we have

$$|0\rangle = |0 \dots 0\rangle. \tag{2.23}$$

The action of $\Psi^a$ and $\Xi_a$ on these basis states is not hard to work out from the algebra (2.20), for example if $s_a = 0$ then the state is annihilated by $\Psi^a$ while if $s_a = 1$ then it is annihilated by $\Xi_a$. The Hilbert space for $N$ fermionic variables is thus just the Hilbert space of $N$ qubits. The fermion number operator $F$ is now defined to be

$$F = \sum_{a=1}^{N} \Xi_a \Psi^a, \tag{2.24}$$

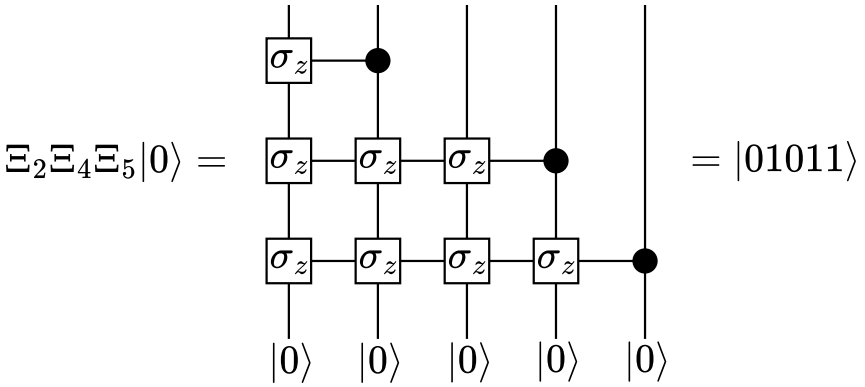


Figure 1: Creating a five-qubit basis state by acting on the $|0\rangle$ state with fermion operators. The black dots indicate the $\frac{\sigma_x - i\sigma_y}{2}$ factors, and the Jordan-Wigner string operators $\sigma_z$ always act on the zero state and thus only contribute factors of $+1$.

and acting on the basis states we have

$$F|s_1 \dots s_N\rangle = (s_1 + \dots s_N)|s_1 \dots s_N\rangle. \tag{2.25}$$

We can give a more explicit representation of the fermion algebra using Pauli matrices. Considering first the case $N = 2$, it not hard to check that the operators

$$\begin{aligned}
\Psi^1 &= \left(\frac{\sigma_x + i\sigma_y}{2}\right) \otimes I \\
\Xi_1 &= \left(\frac{\sigma_x - i\sigma_y}{2}\right) \otimes I \\
\Psi^2 &= \sigma_z \otimes \left(\frac{\sigma_x + i\sigma_y}{2}\right) \\
\Xi_2 &= \sigma_z \otimes \left(\frac{\sigma_x - i\sigma_y}{2}\right)
\end{aligned} \tag{2.26}$$

obey the algebra (2.20). For example we have

$$\{\Psi^1, \Psi^2\} = 0 \tag{2.27}$$

since $\sigma_x$ and $\sigma_y$ both anticommute with $\sigma_z$, and we have

$$\{\Psi^2, \Xi_2\} = 1 \tag{2.28}$$

since $\sigma_z^2 = 1$ and we already showed that $\{\frac{\sigma_x + i\sigma_y}{2}, \frac{\sigma_x - i\sigma_y}{2}\} = 1$ in the one-fermion case. It is also straightforward to confirm (2.22), for example

$$\Xi_1\Xi_2|00\rangle = |11\rangle \tag{2.29}$$

since $\sigma_z|0\rangle = |0\rangle$ and $\frac{\sigma_x - i\sigma_y}{2}|0\rangle = |1\rangle$. This construction generalizes to $N$ fermions in a nice way: we simply have

$$\begin{aligned}
\Psi^a &= \sigma_z \otimes \dots \sigma_z \otimes \frac{\sigma_x + i\sigma_y}{2} \otimes I \otimes \dots \otimes I \\
\Xi_a &= \sigma_z \otimes \dots \sigma_z \otimes \frac{\sigma_x - i\sigma_y}{2} \otimes I \otimes \dots \otimes I,
\end{aligned} \tag{2.30}$$

where there are $N$ tensor factors in both expressions (for the $N$ qubits) and the operator $\frac{\sigma_x \pm i\sigma_y}{2}$ appears in the $a$th factor. In the homework you will show in detail that these expressions obey (2.20). The chain of $\sigma_z$ operators appearing to the left of $\frac{\sigma_x \pm i\sigma_y}{2}$ is typically called the **Jordan-Wigner string**, and this representation of the $N$-fermion algebra on the Hilbert space of $N$ qubits is called the **Jordan-Wigner representation**. The expression (2.22) for the fermion basis has a nice graphical representation in terms of the Jordan-Wigner strings, see figure 1. Fermion parity also has a nice representation along these lines, it is the complete Jordan-Wigner string:

$$(-1)^F = \sigma_z \otimes \ldots \otimes \sigma_z. \tag{2.31}$$

The easiest way to see this is to exponentiate the action of $F$ on basis states:

$$(-1)^F |s_1 \ldots s_N\rangle = (-1)^{\sum_a s_a} |s_1 \ldots s_N\rangle. \tag{2.32}$$

It is worth emphasizing a rather subtle point about this construction. In quantum mechanics class we typically learn that the composition of two physical systems is mathematically described by a tensor product. So it is here: the Hilbert space of $N_1 + N_2$ fermions is the tensor product of an $N_1$-qubit Hilbert space and an $N_2$-qubit Hilbert space. On the other hand the fermionic *operators* are NOT tensor product operators: a fermion operator on the first system needs to anticommute with a fermion operator on the second system! Taking care of this anticommutation is precisely the reason for including the Jordan-Wigner string. On the other hand including the string seems a bit arbitrary: why did we put the $\sigma_z$ operators on the left instead of the right? And for that matter why did we have to introduce an ordering of the fermions at all? The algebra (2.20) makes no reference to such an ordering, and for bosonic degrees of freedom we didn't need to pick an ordering. The place where the ordering appeared is when we introduced the particular basis (2.22) for the system; other orderings would lead to bases which differ from this one by signs. Keeping track of the signs associated to this choice is one of the main headaches of dealing with fermions: most of the time it doesn't matter and cancels from observable quantities, but every now and then it leads to something deep.

## 2.3 Some comments on the inner product and hermiticity

I'll now make some comments about the inner product on the $N$-fermion Hilbert space. The natural inner product that you presumably were already assuming I am using is to define

$$\langle s'_1 \ldots s'_N | s_1 \ldots s_N \rangle = \delta_{s'_1 s_1} \ldots \delta_{s'_N s_N}. \tag{2.33}$$

With this inner product $\Psi^a$ and $\Xi_a$ are hermitian conjugates:

$$\Psi^{a\dagger} = \Xi_a \tag{2.34}$$

$$\Xi_a^\dagger = \Psi^a, \tag{2.35}$$

as you can confirm directly from (2.30). I'll refer to this inner product as the **physical inner product**, and in most applications of fermions this is the one we want. We will eventually however meet some more mysterious fermions called "Fadeev-Poppov ghosts", which will arise in the path integral formulation of non-Abelian gauge theories. If one attempts to give these ghosts a Hilbert space interpretation one finds that they instead have the **ghost inner product**

$$\langle s'_1 \ldots s'_N | s_1 \ldots s_N \rangle = \delta_{s'_1, s_1+1} \ldots \delta_{s'_N, s_N+1}, \tag{2.36}$$

where the addition in the Kronecker $\delta$ is mod 2 (i.e. $0+1=1$ and $1+1=0$). So in particular for $N=1$ the ghost inner product is

$$\langle 0|1\rangle = \langle 1|0\rangle = 1 \qquad\qquad \langle 0|0\rangle = \langle 1|1\rangle = 0. \tag{2.37}$$

Note that this isn't really a genuine inner product, as there are nonzero states of vanishing norm. These states of course do not appear in the physical Hilbert space of the non-Abelian gauge theory, but they are sometimes useful in intermediate steps of calculations. With the ghost inner product $\Psi^a$ and $\Xi_a$ are hermitian. For now however I recommend you don't worry about ghosts and stick to the physical inner product.

## 2.4 Real fermions

There is an alternative way of presenting the fermion algebra (2.20) where we make the redefinitions

$$\chi_a = \frac{1}{\sqrt{2}} \left( \Psi^a + \Xi_a \right) \tag{2.38}$$

$$\widetilde{\chi}_a = \frac{1}{i\sqrt{2}} \left( \Psi^a - \Xi_a \right), \tag{2.39}$$

which then obey the algebra

$$\begin{aligned} \{\chi_a, \chi_b\} = \{\widetilde{\chi}_a, \widetilde{\chi}_b\} = \delta_{ab} \\ \{\chi_a, \widetilde{\chi}_b\} = 0. \end{aligned} \tag{2.40}$$

In the physical inner product both $\chi_a$ and $\widetilde{\chi}_a$ are hermitian:

$$\chi_a^\dagger = \chi_a \qquad\qquad \widetilde{\chi}_a^\dagger = \chi_a, \tag{2.41}$$

so they are conventionally referred to as **real fermions**. The $\Psi^a$ are referred to as **complex fermions**, and as you would expect one complex fermion is equal to two real fermions. The Jordan-Wigner representation of real fermions is particularly simple:

$$\begin{aligned} \chi_a &= \sigma_z \otimes \ldots \sigma_z \otimes \frac{\sigma_x}{\sqrt{2}} \otimes I \otimes \ldots \otimes I \\ \widetilde{\chi}_a &= \sigma_z \otimes \ldots \sigma_z \otimes \frac{\sigma_y}{\sqrt{2}} \otimes I \otimes \ldots \otimes I. \end{aligned} \tag{2.42}$$

In the condensed matter and quantum information communities real fermions are sometimes called "Majorana fermions", but this term is really a relativistic notion that we will meet presently so we will reserve it for then.

It is interesting to note that this construction always produces an even number of real fermions. It is a topic of some controversy whether or not it makes sense to have a quantum system with an odd number of real fermions. My vote is no: the constructions which I've seen always amount to either taking a system with one more real fermion (to make the total number even) and then "forgetting" about one of the fermions, which doesn't give an irreducible representation of the anticommutation relations, or taking a system with one fewer real fermions (to again get an even number) and then interpreting fermion parity $(-1)^F$ as an "extra" real fermion.

**Problems:**

1. Confirm the action (2.25) of fermion number on the basis states of $N$ fermions.
2. Confirm the canonical anticommutation relations (2.20) using the explicit representation (2.30).
3. Confirm the Majorana algebra (2.40) starting from the canonical anticommutation relations (2.20), and also confirm them from the explicit representation (2.42).

# 3 Spinors

We would now like to construct Lorentz representations for fields that create and annihilate fermions. To set up this task more precisely, it is useful to first review some facts about angular momentum in quantum mechanics.

## 3.1 Angular momentum review

In three spatial dimensions a rotation is a $3 \times 3$ matrix $R$ with unit determinant whose transpose is also its inverse, or in other words it is an element of the Lie group $SO(3)$.[5] In quantum mechanics each spatial rotation is represented on Hilbert space by a unitary operator $U(R)$, with the unitary operators obeying the representation condition

$$U(R)U(R') = U(RR'). \tag{3.1}$$

The rotation group can be represented in many different ways, and trying to work out all the representations directly by constructing the unitary operators $U(R)$ is somewhat challenging. A better idea is to first think about how to represent the rotation group in an infinitesimal neighborhood of the identity, by way of the angular momentum operators $\mathcal{J}_x$, $\mathcal{J}_y$, and $\mathcal{J}_z$. To introduce these we observe that we can write a general element of $SO(3)$ as

$$R = e^{iS}, \tag{3.2}$$

where $S$ is purely imaginary (to make sure that $R$ is real) and the orthogonality condition $R^{-1} = R^T$ implies that $S$ is antisymmetric:

$$S^T = -S. \tag{3.3}$$

Setting the determinant of $R$ to one also tells us that

$$1 = \det R = e^{i\mathrm{Tr}S}, \tag{3.4}$$

so we also want $S$ to be traceless:

$$\mathrm{Tr}\,(S) = 0. \tag{3.5}$$

In fact this follows already from the antisymmetry of $S$, so we need only require that $S$ is imaginary and antisymmetric. The set of $3 \times 3$ imaginary antisymmetric matrices is a three-dimensional vector space, spanned by the generators

$$\mathcal{J}_x = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathcal{J}_y = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \mathcal{J}_z = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.6}$$

These are of course the angular momenta about the $x$, $y$, and $z$ axes, the signs are chosen so that a counterclockwise rotation by $\theta$ about the unit vector $\hat{n}$ is

$$R(\theta, \hat{n}) = e^{-i\theta\hat{n}\cdot\vec{\mathcal{J}}}. \tag{3.7}$$

It is simple to check that these generators obey the algebra

$$[\mathcal{J}_i, \mathcal{J}_j] = i\sum_k \epsilon_{ijk}\mathcal{J}_k, \tag{3.8}$$

where $\epsilon_{ijk}$ is the completely antisymmetric tensor with $\epsilon_{xyz} = 1$. Equation (3.8) is called the **Lie algebra** of $SO(3)$. We can think of the Lie algebra as encoding the multiplication rules of $SO(3)$ "near the identity", since if we multiply two elements of $SO(3)$ which are near the identity we have

$$\begin{aligned} e^{i\epsilon S_1}e^{i\epsilon S_2} &= \left(1 + i\epsilon S_1 - \frac{1}{2}\epsilon^2 S_1^2 + \ldots\right)\left(1 + i\epsilon S_2 - \frac{1}{2}\epsilon^2 S_2^2 + \ldots\right) \\ &= 1 + i\epsilon(S_1 + S_2) - \frac{1}{2}\epsilon^2(S_1 + S_2)^2 - \frac{1}{2}\epsilon^2[S_1, S_2] + \ldots \\ &= e^{i\epsilon(S_1+S_2) - \frac{1}{2}\epsilon^2[S_1,S_2] + O(\epsilon^3)}. \end{aligned} \tag{3.9}$$

[5] A **Lie Group** is a group which is also a smooth manifold, which roughly speaking means it locally looks like a piece of $\mathbb{R}^n$ for some $n$, and for which the group multiplication and inversion operations are infinitely differentiable. In physics we are most often interested in matrix Lie groups, meaning Lie groups that can be faithfully represented using finite-dimensional matrices. For matrix Lie groups the Lie algebra is the vector space of infinitesimal generator matrices, which we will see in a moment needs to be closed under taking commutators.

This formula can be extended to all orders in $\epsilon$, which gives something called the Baker-Campbell-Hausdorff formula, and the higher order terms in the exponent on the right hand side all have the form of nested commutators of $S_1$ and $S_2$. Therefore if we know how to compute the commutators of the generators, then we know how to compute the products of arbitrary group elements near the identity.[6] Thus if we want to construct a unitary representation of $SO(3)$ near the identity, it is enough to find three hermitian matrices $J_i$ of any dimension obeying the Lie algebra (3.8), since exponentiating them then gives a representation of $SO(3)$ with that dimension (at least near the identity). Such a set of hermitian matrices is called a representation of the Lie algebra of $SO(3)$.

In your quantum mechanics class you presumably used ladder operators to construct all the finite-dimensional irreducible representations of the Lie algebra (3.8), seeing that they are labeled by an integer or half-integer $j = 0, \frac{1}{2}, 1, 3/2, \ldots$ called the **spin** and that they have dimension $2j + 1$. The representations with integer spin exponentiate to genuine representations of $SO(3)$ (not just near the identity), and they can all be realized by taking tensor products of the defining spin-one vector representation and then restricting to invariant subspaces. The half-integer spin representations are more interesting, here we will focus on the spin-1/2 representation which is furnished by the Pauli matrices:

$$J_x = \frac{\sigma_x}{2} \qquad J_y = \frac{\sigma_y}{2} \qquad J_z = \frac{\sigma_z}{2}. \tag{3.10}$$

Exponentiating this representation we have the famous formula

$$e^{-i\theta\hat{n}\cdot\vec{J}} = \cos\left(\frac{\theta}{2}\right) I - i(\hat{n}\cdot\vec{\sigma})\sin\left(\frac{\theta}{2}\right), \tag{3.11}$$

where $\hat{n}$ is a unit vector. In particular for a rotation by $2\pi$ in any direction we have

$$e^{-i2\pi\hat{n}\cdot\vec{J}} = -I. \tag{3.12}$$

This is perhaps surprising, because a rotation by $2\pi$ is equal to the identity in $SO(3)$. In other words a spin-1/2 particle doesn't really transform in a representation of $SO(3)$, even though we saw in the previous paragraph that it will do so near the identity. The problem is global in nature: the product of two rotations by $\pi$ is equal to the identity in $SO(3)$, but it is equal to $-1$ in the spin-1/2 representation. The reason this can happen is that the group $SO(3)$ is not **simply-connected** - it has closed loops which cannot be contracted to a point.

To see how non-contractible loops can cause trouble, let's write the group multiplication rule as

$$e^{-i\vec{\theta}_1\cdot\vec{\mathcal{J}}}e^{-i\vec{\theta}_2\cdot\vec{\mathcal{J}}} = e^{-i\vec{\phi}(\vec{\theta}_1,\vec{\theta}_2)\cdot\vec{\mathcal{J}}}. \tag{3.13}$$

We would like to show that

$$e^{-i\vec{\theta}_1\cdot\vec{J}}e^{-i\vec{\theta}_2\cdot\vec{J}} = e^{-i\vec{\phi}(\vec{\theta}_1,\vec{\theta}_2)\cdot\vec{J}}, \tag{3.14}$$

which would mean that we have a genuine representation of $SO(3)$. We can build up the left-hand side of (3.14) using only multiplication near the identity by starting at the identity and "scaling up" $\vec{\theta_1}$ using a path $\vec{\theta}(s) = 2s\vec{\theta_1}$ with $s \in [0, 1/2]$, and then "scaling up" $\vec{\theta_2}$ via a path $\vec{\theta} = \vec{\theta_1} + (2s-1)\vec{\theta_2}$ with $s \in [1/2, 1]$. Similarly we can "scale up" the right-hand side of (3.14) using a path $\vec{\theta} = s\vec{\phi}(\vec{\theta_1}, \vec{\theta_2})$ with $s \in [0, 1]$. These two paths start and end in the same place in $SO(3)$, and if they are continuously deformable into each other then we can get from one to the other by adjusting the path using multiplications near the identity. If so then the two sides of (3.14) must be equal. This would always be the case of $SO(3)$ were simply-connected, but alas it isn't and the spin-1/2 representation has taken advantage of it. Formalizing this argument (see appendix 2.B of Weinberg's book) gives the first of two very useful mathematical facts:[7]

[6] This statement is true for any Lie group, it is not special to $SO(3)$. This argument also shows that the Lie algebra has to be closed under taking commutators, as we saw is the case for $SO(3)$.

[7] For a proof of the second see (for example) "Introduction to smooth manifolds" by Lee.

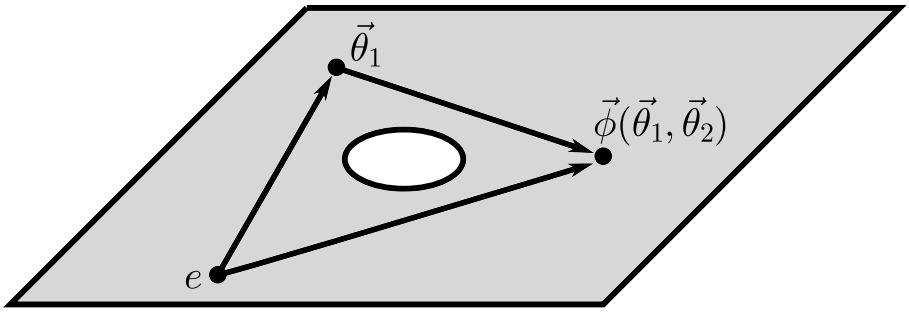


Figure 2: Failure of a group representation due to a non-contractible loop: using multiplications near the identity we can get from the identity to $e^{i\vec{\theta}_1 \cdot \vec{J}}$ and then $e^{i\vec{\theta}_1 \cdot \vec{J}} e^{i\vec{\theta}_2 \cdot \vec{J}}$, but this isn't the same path in the group as the one we'd use to get from the identity to $e^{i\vec{\phi}(\vec{\theta}_1, \vec{\theta}_2)}$ and if they can't be deformed continuously into each other then they might not give the same answer.

- For any connected simply-connected Lie group $G$, every representation of the Lie algebra exponentiates to a representation of $G$.
- Every connected Lie group $G$ is the quotient $\widetilde{G}/\Gamma$ of a unique connected simply-connected Lie group $\widetilde{G}$ which is called its **universal covering group**, with $\Gamma$ a discrete central subgroup of $\widetilde{G}$. Moreover $G$ and $\widetilde{G}$ have the same Lie algebra, since their group structure near the identity is identical.

What is going on with the rotation group is that we really have[8]

$$SO(3) \cong SU(2)/\mathbb{Z}_2, \tag{3.15}$$

where $\mathbb{Z}_2$ is the discrete central subgroup of $SU(2)$ consisting of $I$ and $-I$. The Lie algebra of $SU(2)$ consists of traceless hermitian $2 \times 2$ matrices, which of course are spanned by the Pauli matrices, and thus the Lie algebra of $SU(2)$ is indeed the same as the Lie algebra of $SO(3)$. Non-contractible loops in $SO(3)$ are precisely those which are obtained from paths in $SU(2)$ which start at an element $\widetilde{g}$ and end at $-\widetilde{g}$, see figure 3 for an illustration. This situation is usually described by saying that $SU(2)$ is a **double cover** of $SO(3)$. The half-integer spin representations of the Lie algebra of $SO(3)$ really exponentiate to representations of $SU(2)$ rather than $SO(3)$.[9]

Before leaving the rotation group there is one other pedagogical point which is worth making. This is that if we act on the Pauli matrices by conjugation with the spin-$\frac{1}{2}$ representation matrices we have the vector transformation[10]

$$e^{i\theta\hat{n}\cdot\vec{J}} \sigma_i e^{-i\theta\hat{n}\cdot\vec{J}} = \sum_j \left( e^{-i\theta\hat{n}\cdot\vec{\mathcal{J}}} \right)_{ij} \sigma_j, \tag{3.16}$$

where I emphasize that on the left-hand side we have the $2 \times 2$ spin-1/2 generators $\vec{J}$ while on the right-hand side we have the $3 \times 3$ spin-1 generators $\vec{\mathcal{J}}$. In other words the states in the Hilbert space transform in

[8] For people who like topology, the topology of $SU(2)$ is actually just that of the three-sphere $\mathbb{S}^3$, and the $\mathbb{Z}_2$ map we quotient by identifies opposite points on this sphere. The topology of $SO(3)$ is therefore that of the real projective space $\mathbb{RP}^3$. One way to see the topology of $SU(2)$ is to notice that a general complex $2 \times 2$ matrix can be written as $n_4 I + i\vec{n} \cdot \vec{\sigma}$, and then requiring this matrix to have unit determinant implies that $n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1$ and requiring it to be unitary requires that $n_1, \ldots, n_4$ are real.

[9] There is a more old-fashioned description of this situation, where one says that the half-integer spins furnish **projective representations** of $SO(3)$, meaning representations where the multiplication rule (3.1) holds only up to a phase.

[10] This illustrates a general construction in Lie theory: by conjugating the elements of the Lie algebra by the group transformation we can construct a representation of any Lie group acting on its own Lie algebra. This is called the **adjoint representation**, so what we learn here is that the adjoint representation of $SU(2)$ is the $j = 1$ vector representation.

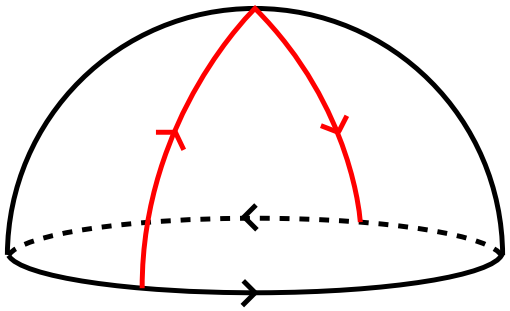

Figure 3: Visualizing a non-contractible loop in $SO(3)$. $SU(2)$ has the topology of the three-sphere $\mathbb{S}^3$, and $SO(3)$ is obtained by identifying antipodal points (see footnote 8). We can therefore think of $SO(3)$ as the "northern hemisphere" of $\mathbb{S}^3$ (here represented as $\mathbb{S}^2$), with opposite points identified on the equator (which is an $\mathbb{S}^2$ but is here represented as an $\mathbb{S}^1$). We can make a noncontractible loop starting at the north pole by going down to the equator, coming out on the opposite side of the equator, and going back up to the north pole. Try as you might, you can't contract this loop to the north pole. It is fun to check however that if you have a loop which traverses this path twice, you *can* contract it to the north pole! If you know about the fundamental group, this shows that $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$.

the spin-1/2 represenation of $SU(2)$, while the Pauli operators transform under conjugation in the spin-one representation. This is a finite-dimensional model for our field Lorentz transformation formula

$$U(\Lambda)^\dagger \Phi^a(x) U(\Lambda) \sum_{a'} D^a{}_{a'}(\Lambda) \Phi^{a'}(\Lambda^{-1}x). \tag{3.17}$$

You should always be careful to distinguish the transformation of the states from the transformation of the operators!

## 3.2 Lie algebra of the Lorentz group

We'll now discuss the Lie algebra of the (proper orthochronous) Lorentz group $SO^+(d-1,1)$. Recall that this is the set of $d \times d$ matrices $\Lambda^\mu{}_\nu$ obeying

$$\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \eta_{\mu\nu} = \eta_{\alpha\beta} \tag{3.18}$$

and also

$$\begin{aligned} \det \Lambda &= 1 \\ \Lambda^0{}_0 &\geq 1. \end{aligned} \tag{3.19}$$

Near the identity we can expand

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon \omega^\mu{}_\nu, \tag{3.20}$$

and at $O(\epsilon)$ the condition (3.18) implies that

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0. \tag{3.21}$$

To identify the Lorentz generator matrices $\mathcal{J}^{\mu\nu}$ we therefore want to have that

$$e^{\frac{i}{2}\epsilon\omega_{\mu\nu}\mathcal{J}^{\mu\nu}} = \delta^\mu_\nu + \epsilon\omega^\mu{}_\nu + O(\epsilon^2). \tag{3.22}$$

Expanding the exponential, we see that we want

$$\frac{i}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^\mu{}_\nu = \omega^\mu{}_\nu, \tag{3.23}$$

which (imposing antisymmetry in $\alpha$ and $\beta$) tells us that the Lorentz generator matrices are

$$(\mathcal{J}^{\alpha\beta})^{\mu}{}_{\nu} = -i\left(\eta^{\alpha\mu}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}_{\nu}\right). \tag{3.24}$$

Here that the $\alpha$ and $\beta$ indices tell us which generator we are talking about, while the $\mu$ and $\nu$ indices are the matrix indices. Via a slightly tedious computation (which you will do on the homework) we can then compute the Lie algebra of the Lorentz group:

$$[\mathcal{J}^{\alpha\beta}, \mathcal{J}^{\rho\sigma}] = i\Big(\eta^{\alpha\rho}\mathcal{J}^{\beta\sigma} + \eta^{\beta\sigma}\mathcal{J}^{\alpha\rho} - \eta^{\alpha\sigma}\mathcal{J}^{\beta\rho} - \eta^{\beta\rho}\mathcal{J}^{\alpha\sigma}\Big). \tag{3.25}$$

Here I've suppressed the matrix indices on $\mathcal{J}^{\alpha\beta}$, and the commutator is a matrix commutator. Thus if we wish to construct a representation of $SO^+(d-1,1)$ near the identity, then following our experience with $SO(3)$ we should look for a set of matrices $J^{\alpha\beta}$ which obey the Lorentz algebra (3.25) and then exponentiate them. This will always give a representation near the identity, and for $d \geq 4$ at worst it will give us a representation of the double cover $Spin(d-1,1)$ of $SO^+(d-1,1)$. We'll give an explicit construction of $Spin(d-1,1)$ in the next subsection.

## 3.3 The spinor representation 1: general idea

The construction of spinor representations begins with a remarkably clever observation going back to Dirac. Say that we can somehow find $d$ $n \times n$ matrices $\gamma^\mu$ obeying the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{3.26}$$

Then the matrices

$$J^{\mu\nu} = \frac{-i}{4}[\gamma^\mu, \gamma^\nu] \tag{3.27}$$

obey the Lorentz algebra (3.25)! We can check this using the commutator and anticommutator identities

$$\begin{aligned}[A, BC] &= B[A, C] + [A, B]C \\ [AB, C] &= A\{B, C\} - \{A, C\}B.\end{aligned} \tag{3.28}$$

We first note that

$$\begin{aligned}[J^{\mu\nu}, \gamma^\alpha] &= \frac{-i}{4}\left([\gamma^\mu\gamma^\nu, \gamma^\alpha] - [\gamma^\nu\gamma^\mu, \gamma^\alpha]\right) \\ &= \frac{-i}{4}\left(\gamma^\mu\{\gamma^\nu, \gamma^\alpha\} - \{\gamma^\mu, \gamma^\alpha\}\gamma^\nu - \gamma^\nu\{\gamma^\mu, \gamma^\alpha\} + \{\gamma^\nu, \gamma^\alpha\}\gamma^\mu\right) \\ &= -i\left(\eta^{\nu\alpha}\gamma^\mu - \eta^{\mu\alpha}\gamma^\nu\right),\end{aligned} \tag{3.29}$$

from which we have

$$\begin{aligned}[J^{\mu\nu}, [\gamma^\alpha, \gamma^\beta]] &= [J^{\mu\nu}, \gamma^\alpha]\gamma^\beta + \gamma^\alpha[J^{\mu\nu}, \gamma^\beta] - [J^{\mu\nu}, \gamma^\beta]\gamma^\alpha - \gamma^\beta[J^{\mu\nu}, \gamma^\alpha] \\ &= i\Big((\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu)\gamma^\beta + \gamma^\alpha(\eta^{\mu\beta}\gamma^\nu - \eta^{\nu\beta}\gamma^\mu) - (\eta^{\mu\beta}\gamma^\nu - \eta^{\nu\beta}\gamma^\mu)\gamma^\alpha - \gamma^\beta(\eta^{\mu\alpha}\gamma^\nu - \eta^{\nu\alpha}\gamma^\mu)\Big) \\ &= i\Big(\eta^{\mu\alpha}[\gamma^\nu, \gamma^\beta] + \eta^{\nu\beta}[\gamma^\mu, \gamma^\alpha] - \eta^{\mu\beta}[\gamma^\nu, \gamma^\alpha] - \eta^{\nu\alpha}[\gamma^\mu, \gamma^\beta]\Big)\end{aligned} \tag{3.30}$$

and thus

$$[J^{\mu\nu}, J^{\alpha\beta}] = i\Big(\eta^{\mu\alpha}J^{\nu\beta} + \eta^{\nu\beta}J^{\mu\alpha} - \eta^{\mu\beta}J^{\nu\alpha} - \eta^{\nu\alpha}J^{\mu\beta}\Big). \tag{3.31}$$

By exponentiating these generators we therefore can construct a matrix Lie group $Spin(d-1,1) \subset GL(n,\mathbb{C})$ with the same Lie algebra as the Lorentz group. We will see soon that $Spin(d-1,1)$ is a double cover of $SO^+(d-1,1)$ in just the same way as $SU(2)$ is a double cover of $SO(3)$, and it is simply-connected for $d \geq 4$.

It will be useful later to note that using (3.24) we can rewrite our expression (3.29) as

$$[J^{\mu\nu}, \gamma^\alpha] = -\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta \gamma^\beta, \tag{3.32}$$

which shows that $\gamma^\mu$ transforms as a vector under conjugation by the spinor representation matrices:

$$e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\gamma^\alpha e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}} = \left(e^{\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}\right)^\alpha{}_\beta \gamma^\beta. \tag{3.33}$$

## 3.4 The spinor representation 2: explicit matrices

We now give some explicit constructions of the $\gamma$-matrices. The key observation is that in fact when $d$ is even we have already essentially done so: if we define

$$\begin{aligned}
\gamma^0 &= i\sqrt{2}\chi_0 \\
\gamma^1 &= \sqrt{2}\widetilde{\chi}_0 \\
\gamma^{2a} &= \sqrt{2}\chi_a \\
\gamma^{2a+1} &= \sqrt{2}\widetilde{\chi}_a,
\end{aligned} \tag{3.34}$$

with $a = 1, 2, \ldots, \frac{d-2}{2}$, then substituting into (3.26) we find precisely our real fermion algebra

$$\begin{aligned}
\{\chi_a, \chi_b\} = \{\widetilde{\chi}_a, \widetilde{\chi}_b\} &= \delta_{ab} \\
\{\chi_a, \widetilde{\chi}_b\} &= 0
\end{aligned} \tag{3.35}$$

now with $a, b = 0, 1, \ldots, \frac{d-2}{2}$. We therefore can give an explicit representation of the $\gamma$-matrices:

$$\begin{aligned}
\gamma^0 &= i\sigma_y \otimes I \otimes \ldots \otimes I \\
\gamma^1 &= \sigma_x \otimes I \otimes \ldots \otimes I \\
\gamma^{2a} &= \sigma_z \otimes \ldots \otimes \sigma_z \otimes \sigma_x \otimes I \otimes \ldots \otimes I \\
\gamma^{2a+1} &= \sigma_z \otimes \ldots \otimes \sigma_z \otimes \sigma_y \otimes I \otimes \ldots \otimes I,
\end{aligned} \tag{3.36}$$

where in the third and fourth lines we've gone back to $a = 1, 2, \ldots, \frac{d-2}{2}$ and the $\sigma_x/\sigma_y$ appears in the $(a+1)$st tensor factor. For even $d$ we have thus found a $2^{\frac{d}{2}}$-dimensional representation of $Spin(d-1,1)$. We can check that it is indeed not a representation of the Lorentz group $SO^+(d-1,1)$, for example from this explicit representation we have

$$J^{23} = \frac{-i}{4}[\gamma^2, \gamma^3] = I \otimes \frac{\sigma_z}{2} \otimes I \otimes \ldots \otimes I, \tag{3.37}$$

so then

$$e^{-i2\pi J^{23}} = -1 \tag{3.38}$$

just as for the spin $1/2$ representation of $SO(3)$.

We argued in the previous section that this representation gives an irreducible representation of the canonical anticommutation relations, and so it gives an irreducible representation of the Dirac algebra (3.26) as well. On the other hand it is actually a *reducible* representation of $Spin(d-1,1)$. The reason is our old friend fermion parity, which we will here give new name:

$$\gamma = i^{-\frac{d-2}{2}}\gamma^0\gamma^1\ldots\gamma^{d-1} = \sigma_z \otimes \ldots \otimes \sigma_z. \tag{3.39}$$

This matrix commutes with the Lorentz generators $J^{\mu\nu}$ since they are quadratic in $\gamma$ and therefore have even fermion parity, and so our $2^{d/2}$-dimensional representation of $Spin(d-1,1)$ breaks up into a $2^{(d-2)/2}$-dimensional block where $\gamma = 1$ and a $2^{(d-2)/2}$-dimensional block where $\gamma = -1$. These two representations (when $d$ is even) are called the **Weyl representations** of $Spin(d-1,1)$.

What about when $d$ is odd? There is then a simple trick: we simply take the $\gamma$-matrices we just constructed for dimension $d-1$, and then append $\gamma^{d-1} = \gamma$. Since $\gamma$ anticommutes with all of the other $\gamma^\mu$ matrices and squares to one, it is indeed a valid candidate to complete the Dirac algebra for odd $d$! Since $\gamma$ is now one of the Dirac matrices, it no longer commutes with the Lorentz generators which act on $x^{d-1}$. The spinor representation is therefore irreducible for odd $d$, and there are no Weyl representations. In general we can write the dimensionality of the spinor representation as

$$n = 2^{\lfloor \frac{d}{2} \rfloor}, \tag{3.40}$$

where $\lfloor x \rfloor$ is the "floor" function that gives the largest integer which is less than or equal to $x$.

It is worth emphasizing that the representation we have constructed isn't unique: given any invertible matrix $W$ we can always define

$$\gamma^{\mu\prime} = W\gamma^\mu W^{-1}, \tag{3.41}$$

which will obey (3.26) provided that $\gamma^\mu$ does. Typically the $\gamma$-matrices are chosen so that $\gamma^0$ is antihermitian and $\gamma^i$ is hermitian however, in which case we should restrict to unitary $W$.

To give some concrete expressions, for $d = 2$ the representation we've constructed is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.42}$$

with the $\gamma = 1$ block being called "left-moving" and the $\gamma = -1$ block being called "right-moving" (we'll see why in the next section). For $d = 4$ our representation is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{pmatrix} \qquad \gamma^3 = \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix} \qquad \gamma = \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix}. \tag{3.43}$$

For $d = 4$ this is not the most popular representation however, a more popular one, related to this one by $\gamma^\mu_{new} = W\gamma^\mu_{old}W^\dagger$ with

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & -1 & i & 0 \end{pmatrix}, \tag{3.44}$$

is given by

$$\gamma^0 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \gamma^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{3.45}$$

Here the block with $\gamma = 1$ is called "left-handed" and the block with $\gamma = -1$ is called "right-handed", again for reasons we will see next time. I'll refer to (3.45) as the **standard representation** of the $\gamma$-matrices in $d = 4$. You can check that in the standard representation the Lorentz generators are given by

$$J^{0i} = \frac{i}{2} \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \qquad J^{ij} = \sum_k \frac{1}{2}\epsilon^{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{3.46}$$

which are indeed block diagonal in the $\gamma = \pm 1$ blocks.

## 3.5 Hermiticity and complex conjugation properties of the Dirac matrices

In constructing Lagrangians for fields transforming in the spinor representation, it is important to know how the $\gamma$-matrices behave under hermitian and complex conjugation.

We've already seen that we can take $\gamma^0$ to be antihermitian and $\gamma^i$ to be hermitian: using the Dirac algebra we can describe this as

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{3.47}$$

Indeed we have

$$(\gamma^0)^3 = -\gamma^0$$
$$\gamma^0\gamma^i\gamma^0 = -\gamma^i(\gamma^0)^2 = \gamma^i. \tag{3.48}$$

The complex conjugation properties are more annoying. Returning to even dimensions for a moment, in our representation (3.36) you can see that $\gamma^0, \gamma^1, \gamma^2, \gamma^4, \ldots$ are real, $\gamma^3, \gamma^5, \ldots$ are imaginary, and $\gamma$ is real. Therefore the matrices[11]

$$B_1 \equiv \gamma^3\gamma^5\ldots\gamma^{d-1} \qquad\qquad B_2 \equiv \gamma B_1 \tag{3.49}$$

obey

$$\begin{aligned} B_1\gamma^\mu B_1^\dagger &= (-1)^{\frac{d-2}{2}}\gamma^{\mu *} \\ B_2\gamma^\mu B_2^\dagger &= (-1)^{d/2}\gamma^{\mu *} \\ B_1\gamma B_1^\dagger &= (-1)^{\frac{d-2}{2}}\gamma^{*} \\ B_2\gamma B_2^\dagger &= (-1)^{\frac{d-2}{2}}\gamma^{*}. \end{aligned} \tag{3.50}$$

Either of these act as

$$B_{1,2}J_{\mu\nu}B_{1,2}^\dagger = -J^*_{\mu\nu}, \tag{3.51}$$

so we have

$$\left(e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\right)^* = \left(e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu *}}\right) = B_{1,2}e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}B_{1,2}^\dagger. \tag{3.52}$$

Thus the Dirac representation of $Spin(d-1,1)$ is unitarily equivalent to its complex conjugate. The two Weyl representations will also be self-conjugate if $\frac{d-2}{2}$ is even, while they will be exchanged if $\frac{d-2}{2}$ is odd (these statements follow from the last two lines of (3.50)).

To understand conjugation in odd dimensions we again define $\gamma^{d-1} = \gamma$, but now the only choice is to define

$$B \equiv \gamma^3\gamma^5\ldots\gamma^{d-2} \tag{3.53}$$

and we then have

$$B\gamma^\mu B^{-1} = (-1)^{\frac{d-3}{2}}\gamma^{\mu *}. \tag{3.54}$$

Finally I'll mention that in dimensions $d = 0, 1, 2, 3, 4 \bmod 8$, we can consistently impose a **Majorana constraint**

$$\Psi^* = B\Psi, \tag{3.55}$$

on an a field $\Psi$ transforming in the spinor representation of $Spin(d-1,1)$. Here $B = B_1$ for $d = 0, 2 \bmod 8$, $B = B_2$ for $d = 2, 4 \bmod 8$ (either choice is allowed for $d = 2 \bmod 8$, and in fact they are unitarily equivalent), and $B = \gamma^3\ldots\Gamma^{d-2}$ for $d = 1, 3 \bmod 8$. These choices follow from finding a $B$ that obeys $B^*B = 1$, which is needed for the consistency of the constraint since we must have

$$\psi = B^*\psi^* = B^*B\psi. \tag{3.56}$$

A spinor obeying such a constraint is called a **Majorana spinor**. In fact whenever $B^*B = 1$, $B$ is a symmetric matrix (remember it is unitary) and we can do a unitary redefinition of the $\gamma$ matrices such that $B' = 1$ or $B' = \Gamma$; this is called the Majorana representation.

[11] These definitions depend on our choice of representation of the $\gamma$ matrices, but none of the following equations do. Under a representation change $\gamma^{\mu'} = W\gamma^\mu W^\dagger$ we have $B'_{1,2} = W^*B_{1,2}W^\dagger$.

**Problems:**

1. Confirm the expression (3.16) relating the spin-1/2 and spin-one representations of the rotation group. Feel free to use mathematica to help with the matrix exponentials.

2. Confirm the Lorentz algebra (3.25) of the Lorentz group starting from the generators (3.24).

3. Show that for $d = 4$ if we define $\mathcal{J}^x = \mathcal{J}^{23}$, $\mathcal{J}^y = \mathcal{J}^{31}$, and $\mathcal{J}^z = \mathcal{J}^{12}$, then the Lorentz algebra (3.25) implies that $\mathcal{J}^i$ obeys the $SO(3)$ Lie algebra (3.8).

4. Confirm the equivalence of the two expressions for $\gamma$ in equation (3.39).

5. Confirm the equivalence of the two representations (3.43) and (3.45) that we've given for the four-dimensional $\gamma$-matrices.

6. Check the conjugation equations (3.50), and find explicit representations of $B_1$ and $B_2$ for the $\gamma$-matrix representations (3.42), (3.43), and (3.45). Hint: for the last representation you will need to use the transformation in footnote (11).

# 4 Spinor field theory

We've now learned about the defining spinor representation of $Spin(d-1,1)$, which in the way we defined things is just the set of $2^{\lfloor \frac{d}{2} \rfloor}$-dimensional matrices which can be obtained by multiplying things of the form[12]

$$M^a{}_b = \left( e^{\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}} \right)^a{}_b \tag{4.1}$$

with $\omega_{\mu\nu}$ real and antisymmetric and

$$J^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]. \tag{4.2}$$

We'd like to now introduce quantum fields that transform in the spinor representation. Such a field is called a **spinor field**, and if there is only one of them it is conventionally denoted $\Psi^a(x)$. We can write the Lorentz transformation of $\Psi$ as

$$U(\Lambda)^\dagger \Psi^a(x) U(\Lambda) = \sum_b D^a{}_b(\Lambda) \Psi^b(\Lambda^{-1}x), \tag{4.3}$$

where the symbol $\Lambda$ now means an element of $Spin(d-1,1)$ rather than an element of $SO^+(d-1,1)$. $U(\Lambda)$ is the unitary tranformation which represents $\Lambda$ on Hilbert space, $D^a{}_b(\Lambda)$ is the matrix representing $\Lambda$ in the defining spinor representation of $Spin(d-1,1)$, and $\Lambda^{-1}x$ somewhat heuristically indicates acting on $x$ with the inverse of the Lorentz transformation which corresponds to the $\mathbb{Z}_2$ equivalence class of $\Lambda$ in $Spin(d-1,1)$. To write this last part more honestly, we should first recall in more detail the double cover relationship between $Spin(d-1,1)$ and the proper orthochronous Lorentz group $SO^+(d-1,1)$:

$$SO^+(d-1,1) \cong Spin(d-1,1)/\mathbb{Z}_2. \tag{4.4}$$

The $\mathbb{Z}_2$ we quotient by here is just the overall sign of the matrix in the defining representation of $Spin(d-1,1)$. We can therefore introduce a **quotient map**

$$q : Spin(d-1,1) \to SO^+(d-1,1), \tag{4.5}$$

[12] It would be easier if we could just define $Spin(d-1,1)$ to be the set of matrices of this form, but unfortunately for general noncompact Lie groups the exponential map from the Lie algebra to the Lie group is not surjective so it is necessary to include products of exponentials to get a group. This is not just pedantic nitpicking, for the physically relevant cases of $Spin(2,1)$ and $Spin(3,1)$ the exponential map is indeed not surjective.

which sends each element $\Lambda \in Spin(d-1,1)$ to the element of $SO^+(d-1,1)$ corresponding to the equivalence class $\{\Lambda, -\Lambda\}$. We can then write the field transformation more accurately as

$$U(\Lambda)^\dagger \Psi^a(x) U(\Lambda) = \sum_b D^a{}_b(\Lambda)\Psi^b(q(\Lambda^{-1})x), \tag{4.6}$$

where now we honestly have $\Lambda \in Spin(d-1,1)$ everywhere. Typically however we will follow common practice and lazily omit the quotient map in the field transformation. Suppressing also the matrix indices, we have the more elegant expression

$$U(\Lambda)^\dagger \Psi(x) U(\Lambda) = D(\Lambda)\Psi(\Lambda^{-1}x). \tag{4.7}$$

## 4.1 The Dirac Lagrangian

Our goal is now to write down a Lagrangian for the spinor field $\Psi$ which is invariant under (4.7), and then to use it to construct a relativistic quantum field theory with fermions. As we discussed a few sections ago fermion fields do not have classical limits in the usual sense, since the Hilbert space for a finite number of fermions is finite-dimensional, so rather than writing down a classical Lagrangian we will aim to write down a Lagrangian which is already a quantum operator, from which we can construct a quantum Hamiltonian. In the next section we will see that actually it *is* possible to have a path integral formulation of fermionic theories where we integrate over some kind of "classical fermions", but these are rather tricky objects so to avoid confusion we will stick with operators for now.

The basic challenge in writing down an invariant Lagrangian is that we need to cancel the spinor transformation of $\Psi$ by contracting it with something. An obvious guess would be to contract it with $\Psi^\dagger$, where here I introduce a convention that the symbol "†" takes the Hilbert space adjoint of $\Psi$ and *also* the transpose on the spin index. When I write $\Psi^*$ I will mean the Hilbert space adjoint *without* a transpose on the spin index. In other words from the point of view of the spinor index $a$ we should think of $\Psi$ and $\Psi^*$ both as column vectors, while we should think of $\Psi^\dagger$ as a row vector. We therefore might guess that the quantity

$$\Psi^\dagger \Psi \tag{4.8}$$

is Lorentz (or really Spin) invariant. This would indeed be the case if the spinor representation matrices $D(\Lambda)$ were unitary, but alas they aren't in general. The culprit is $J^{0i}$, which is antihermitian since we have

$$(J^{0i})^\dagger = -\frac{i}{4}[\gamma^{0\dagger}, \gamma^{i\dagger}] = \frac{i}{4}[\gamma^0, \gamma^i] = -J^{0i}. \tag{4.9}$$

We therefore need to write the transformation of $\Psi^\dagger$ as

$$\Psi^{\dagger\prime} = \Psi^\dagger e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu\dagger}}, \tag{4.10}$$

which won't in general cancel the transformation

$$\Psi' = e^{\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\Psi \tag{4.11}$$

of $\Psi$. The way out of this quandary is for us to remember our expression

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{4.12}$$

for the hermitian conjugates of the $\gamma$-matrices. This implies that we have

$$\gamma^{\mu\dagger}\gamma^0 = -\gamma^0\gamma^\mu, \tag{4.13}$$

and thus

$$J^{\mu\nu\dagger}\gamma^0 = -\frac{i}{4}[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}]\gamma^0 = -\frac{i}{4}\gamma^0[\gamma^\mu, \gamma^\nu] = \gamma^0 J^{\mu\nu}. \tag{4.14}$$

Therefore we can convert a $J^{\mu\nu\dagger}$ to a $J^{\mu\nu}$ by moving a $\gamma^0$ across it. In particular this means we have

$$\Psi^{\dagger\prime}\gamma^0 = \Psi^\dagger e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu\dagger}}\gamma^0 = \Psi^\dagger\gamma^0 e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}, \tag{4.15}$$

so the quantity

$$\overline{\Psi} \equiv \Psi^\dagger\gamma^0 \tag{4.16}$$

has the simple transformation

$$\overline{\Psi}' = \overline{\Psi}e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}. \tag{4.17}$$

In terms of the group elements and including the position-dependence, we have

$$\begin{aligned}\Psi'(x) &= D(\Lambda)\Psi(\Lambda^{-1}x)\\ \overline{\Psi}'(x) &= \overline{\Psi}(\Lambda^{-1}x)D(\Lambda^{-1}),\end{aligned} \tag{4.18}$$

so these transformations are primed to cancel each other. Spinor Lagrangians are thus typically constructed out of $\Psi$ and $\overline{\Psi}$ rather than $\Psi$ and $\Psi^\dagger$. In particular we have discovered our first Lorentz-invariant quantity:

$$\overline{\Psi}\Psi. \tag{4.19}$$

Before putting this quantity in a Lagrangian however, we should check if it is hermitian:

$$(\overline{\Psi}\Psi)^\dagger = \Psi^\dagger\gamma^{0\dagger}\Psi = -\overline{\Psi}\Psi. \tag{4.20}$$

This of course is easily fixed, we can just take our candidate Lagrangian term to be $i\overline{\Psi}\Psi$. As you might guess this will end up being a mass term for the fermions which are created and annihilated by $\Psi$.

To get a sensible quantum field theory we of course also want to have terms involving derivatives of the fields (otherwise the fields at different points don't talk to each other). Based on our experience with scalar fields you might guess that a good term to write down is

$$\partial_\mu\overline{\Psi}\partial^\mu\Psi, \tag{4.21}$$

but there is a more relevant (in the Wilsonian sense) term that we can write down that has only one derivative instead of two:

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi. \tag{4.22}$$

To see that this term is Lorentz invariant, we first recall that in the last section we showed (using our new notation) that[13]

$$D(\Lambda^{-1})\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu. \tag{4.23}$$

We thus have

$$\begin{aligned}\overline{\Psi}'\gamma^\mu\partial_\mu\Psi' &= \overline{\Psi}D(\Lambda^{-1})\gamma^\mu D(\Lambda)\Lambda_\mu{}^\nu\partial_\nu\Psi\\ &= \overline{\Psi}\Lambda^\mu{}_\sigma\gamma^\sigma\Lambda_\mu{}^\nu\partial_\nu\Psi\\ &= \overline{\Psi}\gamma^\nu\partial_\nu\Psi,\end{aligned} \tag{4.24}$$

so (4.22) is indeed a Lorentz scalar. We should also check its hermiticity:

$$\begin{aligned}\left(\overline{\Psi}\gamma^\mu\partial_\mu\Psi\right)^\dagger &= \partial_\mu\Psi^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}\Psi\\ &= -\partial_\mu\Psi^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0\Psi \qquad (4.25)\\ &= \partial_\mu\overline{\Psi}\gamma^\mu\Psi. \qquad (4.26)\end{aligned}$$

[13]Here we have again abused notation by dropping the quotient map on the right-hand side, where we really should write $q(\Lambda)^\mu{}_\nu$. From now on we will continue this abuse without further comment.

This certainly doesn't look hermitian, or even antihermitian, but it is antihermitian up to a total derivative:

$$\overline{\Psi}\gamma^\mu\partial_\mu\Psi + \left(\overline{\Psi}\gamma^\mu\partial_\mu\Psi\right)^\dagger = \partial_\mu\left(\overline{\Psi}\gamma^\mu\Psi\right). \tag{4.27}$$

Thus $i\overline{\Psi}\gamma^\mu\partial_\mu\Psi$ will integrate to a real action (and more importantly lead to a Hamiltonian density which integrates to a hermitian Hamiltonian) provided that our boundary conditions at infinity ensure this boundary term does not contribute (for example by setting $\Psi$ or $\overline{\Psi}$ to zero at the spatial boundary).

We therefore can write down our first viable candidate Lagrangian density for a spinor field:

$$\mathcal{L} = -i\left(\overline{\Psi}\gamma^\mu\partial_\mu\Psi + m\overline{\Psi}\Psi\right). \tag{4.28}$$

The overall minus sign is included with the benefit of hindsight: it ensures we will get a Hamiltonian which is bounded from below. To work out the units of the parameter $m$, we can use the derivative term (which must have energy dimension $d$) to see that the energy dimension of a spinor field is

$$[\Psi] = \frac{d-1}{2}. \tag{4.29}$$

Since $m\overline{\Psi}\Psi$ should have dimension $d$, this means that $m$ has units of mass/energy:

$$[m] = 1. \tag{4.30}$$

The term (4.21) that we considered above has energy dimension $d+1$, and is thus irrelevant as promised. The Lagrangian (4.28) is called the **Dirac Lagrangian**, and its equation of motion (for example obtained by varying with respect to $\overline{\Psi}$) is the **Dirac equation**

$$\gamma^\mu\partial_\mu\Psi + m\Psi = 0. \tag{4.31}$$

In a moment we will study solutions of the Dirac equation and then use them to construct a quantum spinor field theory, but it is worthwhile to first mention a convenient notation due to Feynman. In spinor computations we quite often want to contract some vector or one-form with the $\gamma$=matrix vector. Feynman tells us to indicate this by drawing a slash through the object:

$$\not{v} \equiv v_\mu\gamma^\mu = v^\mu\gamma_\mu. \tag{4.32}$$

With this notation we can write the Dirac Lagrangian and the Dirac Equation as

$$\begin{aligned} \mathcal{L} &= -i\left(\overline{\Psi}\not{\partial}\Psi + m\overline{\Psi}\Psi\right) \\ \not{\partial}\Psi + m\Psi &= 0. \end{aligned} \tag{4.33}$$

There is a historical comment which is worth making at this point. In some quantum mechanics courses the Dirac equation (4.31) is introduced as a kind of relativistic generalization of the Schrodinger equation, and is solved e.g. in the presence of a background Coulomb potential to understand relativistic corrections to the spectrum of the hydrogen atom. This idea is deeply wrong however, for several reasons:

- The Dirac equation accounts for some but not all of the relativistic corrections to the hydrogen atom: it treats the electron quantum mechanically but not the photon. At higher orders in the fine structure constant it therefore misses important contributions such as the Lamb shift.
- In this archaic interpretation the Dirac equation was seen as successfully predicting that electrons must have spin 1/2, for example because attempting to have a spin-zero relativistic Schrodinger equation leads to the Klein-Gordon equation which does not make any sense when viewed as governing a wave function. In quantum field theory however there is absolutely no problem with having relativistic scalars, and there is no reason a priori for the electron to have spin 1/2 (except of course that we know this to be the case experimentally).

- Taken literally, the Dirac equation predicts the existence of "negative energy electrons", for which there is of course no experimental evidence. In quantum field theory these instead become (positive energy) positrons.

For these reasons (and also others) it is best to discard any "wave function" interpretation of the Dirac equation: its true physical interpretation is as the equation of motion for a quantum field.

I'll also mention that from here it is easy enough to write down interacting spinor theories as well. For example in the Yukawa model of nuclear interactions one has a real scalar field $\phi$ interacting with a Dirac spinor $\Psi$, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2 - i\left(\overline{\Psi}\not{\partial}\Psi + m\overline{\Psi}\Psi\right) - ig\phi\overline{\Psi}\Psi. \tag{4.34}$$

Here $\Psi$ creates/annihilates a nucleon (i.e. a proton or neutron) and $\phi$ creates/annihilates a pion.[14] In a few sections we will learn how to do perturbative scattering calculations in this theory, seeing that the pion creates an attractive force between the nucleons.

## 4.2 Solutions of the Dirac equation

We'll now spend some time studying the classical solutions $\psi(x)$ of the Dirac equation. The first thing to note is that

$$\not{\partial}^2\psi = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu\psi = \partial^2\psi, \tag{4.35}$$

and thus by applying the Dirac equation twice we have

$$\partial^2\psi = \not{\partial}^2\psi = -m\not{\partial}\psi = m^2\psi. \tag{4.36}$$

Thus each component of $\psi$ obeys the Klein-Gordon equation! We therefore can look for positive and negative frequency solutions of the form

$$\psi_+ = u(\vec{p})e^{i\vec{p}\cdot\vec{x}-i\omega_{\vec{p}}t} \tag{4.37}$$

$$\psi_- = v(\vec{p})e^{-i\vec{p}\cdot\vec{x}+i\omega_{\vec{p}}t}, \tag{4.38}$$

where $u(\vec{p})$ and $v(\vec{p})$ are $2^{\lfloor d/2\rfloor}$-dimensional vectors obeying

$$\begin{aligned}\left(i\not{p}+m\right)u &= 0\\ \left(i\not{p}-m\right)v &= 0.\end{aligned} \tag{4.39}$$

We can also define $\overline{u} = u^\dagger\gamma^0$ and $\overline{v} = v^\dagger\gamma^0$, and a little thought shows that these obey

$$\begin{aligned}\overline{u}\left(i\not{p}+m\right) &= 0\\ \overline{v}\left(i\not{p}-m\right) &= 0.\end{aligned} \tag{4.40}$$

Solving (4.39) explicitly is a bit tricky. We can make life easier by noting that if $u(p)$ and $v(p)$ are solutions of (4.39) for some particular $p$, then we have

$$\begin{aligned}\left(i\not{\Lambda p}+m\right)D(\Lambda)u(\vec{p}) &= \left(i\gamma^\nu\Lambda_\nu{}^\mu p_\mu + m\right)D(\Lambda)u(\vec{p})\\ &= D(\Lambda)\left(i\gamma^\mu p_\mu + m\right)D(\Lambda^{-1})D(\Lambda)u(\vec{p})\\ &= D(\Lambda)\left(i\not{p}+m\right)u(\vec{p})\\ &= 0\end{aligned} \tag{4.41}$$

[14] This model doesn't quite get the pion physics right however, a better model takes the interaction term to be $g\phi\overline{\Psi}\gamma\Psi$. This reflects the fact that pions are pseudoscalars, meaning that they are odd under parity instead of even. Either kind of interaction is still called a Yukawa interaction, and we will mostly study the version with no $\gamma$ since it is a bit easier to calculate in.

and

$$\begin{aligned}(i\not{\Lambda p} - m)\, D(\Lambda)v(\vec{p}) &= (i\gamma^\nu \Lambda_\nu{}^\mu p_\mu - m)\, D(\Lambda)v(\vec{p}) \\ &= D(\Lambda)\,(i\gamma^\mu p_\mu - m)\, D(\Lambda^{-1})D(\Lambda)v(\vec{p}) \\ &= D(\Lambda)\left(i\not{p} - m\right) v(\vec{p}) \\ &= 0. \end{aligned} \tag{4.42}$$

In both cases in the second line we used the $\gamma$-matrix transformation (4.23) with the substitution $\Lambda \to \Lambda^{-1}$. Therefore we can solve (4.39) for some particular "reference" momentum $k$, write each other timelike (or null in the case of $m = 0$) momentum $p$ as some standard Lorentz (or really spin) transformation $L_p$ of $k$

$$p = L_p k, \tag{4.43}$$

and then construct solutions of (4.39) for general $p$ as

$$\begin{aligned} u(\vec{p}) &= D(L_p)u(\vec{k}) \\ v(\vec{p}) &= D(L_p)v(\vec{k}). \end{aligned} \tag{4.44}$$

**Massive spinor solutions**

Let's first consider the massive case. Then, as in our discussion of the little group back in section 9, we can take the reference momentum to be in the rest frame:

$$k^\mu = (m, \vec{0}). \tag{4.45}$$

The equations (4.39) then reduce to

$$\begin{aligned} i\gamma^0 u(\vec{k}) &= u(\vec{k}) \\ i\gamma^0 v(\vec{k}) &= -v(\vec{k}), \end{aligned} \tag{4.46}$$

so we should take $u(\vec{k})$ to live in the $+1$ eigenspace of the hermitian matrix $i\gamma^0$ and $v(\vec{k})$ to live in its $-1$ eigenspace. We can get a physical interpretation of these subspaces by considering the action of the little group $Spin(d-1) \subset Spin(d-1,1)$ of spin transformations which fix $k$. Since $k$ is in its rest frame the little group is generated by the rotation matrices

$$J^{ij} = \frac{-i}{4}[\gamma^i, \gamma^j], \tag{4.47}$$

which commute with $\gamma^0$ since they are quadratic in $\gamma^i$ and $\gamma^j$. If we pick bases $u^a(\vec{k}, \sigma)$, $v^a(\vec{k}, \sigma)$ for the $i\gamma^0 = \pm 1$ subspaces, then the little group must rotate us within these bases:

$$\begin{aligned} \sum_{a'} D^a{}_{a'}(\Lambda) u^{a'}(\vec{k}, \sigma) &= \sum_\sigma u^a(\vec{k}, \sigma') \hat{D}^{\{u\}}_{\sigma',\sigma}(\Lambda) \qquad\qquad (\Lambda k = k) \\ \sum_{a'} D^a{}_{a'}(\Lambda) v^{a'}(\vec{k}, \sigma) &= \sum_\sigma v^a(\vec{k}, \sigma') \hat{D}^{\{v\}*}_{\sigma',\sigma}(\Lambda) \qquad\qquad (\Lambda k = k). \end{aligned} \tag{4.48}$$

The quantities $\hat{D}^{\{u,v\}}_{\sigma'\sigma}(\Lambda)$ must themselves form representations of the little group,

$$\hat{D}^{\{u,v\}}(\Lambda_1)\hat{D}^{\{u,v\}}(\Lambda_2) = \hat{D}^{\{u,v\}}(\Lambda_1\Lambda_2) \qquad\qquad (\Lambda k = k). \tag{4.49}$$

I included the complex conjugate in the definition of the $v$ transformation because in section 9.2 last semester we saw that for a free quantum field

$$\Phi^a(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[ u^a(\vec{p}, \sigma) a_{\vec{p},\sigma} e^{ip\cdot x} + v^a(\vec{p}, \sigma) b^\dagger_{\vec{p},\sigma} e^{-ip\cdot x} \right], \tag{4.50}$$

the equations (4.48) precisely give the "intertwining" relations between the little group representation acting on the fields and the little group representation acting on the particles annihilated by $a_{\vec{p},\sigma}$ or created by $b^\dagger_{\vec{p},\sigma}$. I'll remind you that a choice of representation of the little group is what we really mean by the spin/helicity of a particle in relativistic quantum mechanics, so a choice of basis for $u$ and $v$ is thus equivalent to a choice of spin basis for these particles. In particular a nice choice is to diagonalize a commuting set of the rotation generators, for example $J^{12}, J^{34}, \ldots$.

We can make this all more explicit by restricting to $d = 4$ and using the standard representation of the $\gamma$-matrices. We then have

$$i\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{4.51}$$

and

$$J^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{4.52}$$

in particular with

$$J^{12} = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}. \tag{4.53}$$

We thus can pick basis vectors

$$u(\vec{k},+) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad u(\vec{k},-) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad v(\vec{k},+) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \qquad v(\vec{k},-) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{4.54}$$

which you can confirm obey

$$\begin{aligned} i\gamma^0 u(\vec{k},\pm) &= u(\vec{k},\pm) \\ i\gamma^0 v(\vec{k},\pm) &= -v(\vec{k},\pm) \\ J^{12}u(\vec{k},\pm) &= \pm\frac{1}{2}u(\vec{k},\pm) \\ J^{12}v(\vec{k},\pm) &= \mp\frac{1}{2}v(\vec{k},\pm). \end{aligned} \tag{4.55}$$

The last two lines here come from (4.48): they show that $u(\vec{k},\pm)$ will multiply an annihilation operator for a particle of $J_z = \pm 1/2$ and $v(\vec{k},\pm)$ will multiply a creation operator for a particle of $J_z = \pm 1/2$. Thus a spinor field in $d = 4$ creates/annihilates particles of spin 1/2.

It will be useful later for us to note that our spinor basis vectors obey

$$\begin{aligned} \sum_\sigma u(\vec{p},\sigma)\overline{u}(\vec{p},\sigma) &= -(\not{p} + im) \\ \sum_\sigma v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma) &= -(\not{p} - im). \end{aligned} \tag{4.56}$$

The easiest way to confirm these for $d = 4$ is by direct computation using the standard representation: in the rest frame we have

$$\begin{aligned} \sum_\sigma u(\vec{k},\sigma)\overline{u}(\vec{k},\sigma) &= -im\begin{pmatrix} I & I \\ I & I \end{pmatrix} = m\gamma^0 - im = -\not{k} - im \\ \sum_\sigma v(\vec{k},\sigma)\overline{v}(\vec{k},\sigma) &= -im\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} = m\gamma^0 + im = -\not{k} + im, \end{aligned} \tag{4.57}$$

and then we can boost to a general $\vec{p}$ by using the reference Lorentz transformation:

$$\sum_\sigma u(\vec{p},\sigma)\overline{u}(\vec{p},\sigma) = D(L_p)(-\not{k} - im)D(L_p^{-1}) = -\not{p} - im$$
$$\sum_\sigma v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma) = D(L_p)(-\not{k} + im)D(L_p^{-1}) = -\not{p} + im. \tag{4.58}$$

Here we have again used (4.23).

In fact we can also determine these spin sums (up to normalization) purely by using group theory and the Dirac equation, which generalizes this argument to any dimension and choice of $\gamma$-matrix representation. Let's first consider the spin sum at the reference momentum. Interpreting $u$ and $v$ each as matrix whose row index is $a$ and whose column index is $\sigma$, we can write (4.48) as

$$Du(\vec{k}) = u(\vec{k})\hat{D}^{\{u\}}$$
$$Dv(\vec{k}) = v(\vec{k})\hat{D}^{\{v\}*} \tag{4.59}$$

and the spin sums as $u\overline{u}$ and $v\overline{v}$. Taking the adjoints of these and multiplying by $\gamma^0$ we have

$$\overline{u}(\vec{k})D^{-1} = u(\vec{k})^\dagger D^\dagger\gamma^0 = \hat{D}^{\{u\}\dagger}\overline{u}(\vec{k})$$
$$\overline{v}(\vec{k})D^{-1} = v(\vec{k})^\dagger D^\dagger\gamma^0 = \hat{D}^{\{v\}T}\overline{v}(\vec{k}). \tag{4.60}$$

The spin sums therefore must be invariant under conjugation by $D(\Lambda)$ with $\Lambda$ in the little group of $k$:

$$D(\Lambda)u(\vec{k})\overline{u}(\vec{k})D(\Lambda^{-1}) = u(\vec{k})\hat{D}\hat{D}^\dagger\overline{u}(\vec{k}) = u(\vec{k})\overline{u}(\vec{k})$$
$$D(\Lambda)v(\vec{k})\overline{v}(\vec{k})D(\Lambda^{-1}) = v(\vec{k})\hat{D}^*\hat{D}^T\overline{v}(\vec{k}) = v(\vec{k})\overline{v}(\vec{k}), \tag{4.61}$$

where we have used the fact that $\hat{D}$ is a unitary representation of the little group. I now claim that this implies that when we must have

$$u(\vec{k})\overline{u}(\vec{k}) = A + B\gamma^0 + C\gamma + D\gamma^0\gamma$$
$$v(\vec{k})\overline{v}(\vec{k}) = A' + B'\gamma^0 + C'\gamma + D'\gamma^0\gamma \tag{4.62}$$

for some constants $A, B, C, D, A', B', C', D'$. This is because the set of products of $\gamma$-matrices is actually a basis (with complex coefficients) for all $2^{\lfloor d/2\rfloor}$-dimensional matrices. One way to see this is to recall that the set of all products of Pauli operators is such a basis, and it is not hard to see that by using our tensor-product-of-Paulis representation of the $\gamma$-matrices we can make any such product. The only such products which will be invariant under arbitrary spatial rotations are those which involve only $\gamma^0$ and those which involve the product of all of the spatial $\gamma^i s$. Recalling that $\gamma = i^{-\frac{d-2}{2}}\gamma^0\ldots\gamma^{d-1}$ and also that $\gamma^2 = 1$ and $(\gamma^0)^2 = -1$, the only options are the identity, $\gamma^0$, $\gamma$, and $\gamma^0\gamma$. We can then use the Dirac equation in the forms (4.39) and (4.40) to see that $C = C' = D = D' = 0$ and to determine the ratio of $A$ to $B$ and $A'$ to $B'$, confirming (4.57) (and thus (4.56)) up to an overall normalization.

It will also be useful for us next time to know that there is a second set of sum rules involving $u$ and $v$:

$$\sum_a \overline{u}^a(\vec{p},\sigma)u^a(\vec{p},\sigma') = -2im\delta_{\sigma\sigma'}$$
$$\sum_a \overline{v}^a(\vec{p},\sigma)v^a(\vec{p},\sigma') = 2im\delta_{\sigma\sigma'}$$
$$\sum_a \overline{u}^a(\vec{p},\sigma)v^a(\vec{p},\sigma') = 0$$
$$\sum_a \overline{v}^a(\vec{p},\sigma)u^a(\vec{p},\sigma') = 0. \tag{4.63}$$

In the matrix language we just introduced, we can write these as

$$\begin{aligned}\overline{u}u &= -2im\\ \overline{v}v &= 2im\\ \overline{u}v &= 0\\ \overline{v}u &= 0.\end{aligned} \tag{4.64}$$

These rules can of course be confirmed directly from our explicit expressions for $u$ and $v$, but it is more fun to get them from group theory and the Dirac equation. The third and fourth are easy: in the rest frame $u$ and $v$ are eigenstates of the hermitian operator $i\gamma^0$ with different eigenvalues, so they must be orthogonal:

$$u^\dagger v = v^\dagger u = 0. \tag{4.65}$$

Moreover in the rest frame $\overline{u}$ is proportional to $u^\dagger$ and $\overline{v}$ is proportional to $v^\dagger$ by (4.46). The result for general $\vec{p}$ then follows from (4.44). To get the first two lines we need to do a bit more. Indeed note that by (4.59) and (4.60) we have

$$\begin{aligned}\hat{D}^{\{u\}\dagger}\overline{u}u\hat{D}^{\{u\}} &= \overline{u}u\\ (\hat{D}^{\{v\}*})^\dagger\overline{v}v\hat{D}^{\{v\}*} &= \overline{v}v.\end{aligned} \tag{4.66}$$

In other words $\overline{u}u$ and $\overline{v}v$ must be invariant under conjugation by an irreducible representation of the little group; by Schur's lemma they therefore must both be proportional to the identity:

$$\begin{aligned}\overline{u}u &= A\\ \overline{v}v &= B.\end{aligned} \tag{4.67}$$

We can determine the coefficients of proportionality using our previous spin sums:

$$\begin{aligned}Au = u\overline{u}u &= -(\not{p} + im)u = -2imu\\ Bv = v\overline{v}v &= -(\not{p} - im)v = 2imv,\end{aligned} \tag{4.68}$$

where in the last equality for each line we used the Dirac equation (4.39).

It is also useful to consider the quantities of the form $\overline{u}\gamma^\mu u$, $\overline{v}\gamma^\mu v$, etc. These can be computed using the *Gordon identities*:

$$\begin{aligned}2im\overline{u}(\vec{p}',\sigma')\gamma^\mu u(\vec{p},\sigma) &= \overline{u}(\vec{p}',\sigma')\Big(p'^\mu + p^\mu + 2i(p'_\alpha - p_\alpha)J^{\alpha\mu}\Big)u(\vec{p},\sigma)\\ 2im\overline{v}(\vec{p}',\sigma')\gamma^\mu v(\vec{p},\sigma) &= -\overline{v}(\vec{p}',\sigma')\Big(p'^\mu + p^\mu + 2i(p'_\alpha - p_\alpha)J^{\alpha\mu}\Big)v(\vec{p},\sigma)\\ 2im\overline{u}(\vec{p}',\sigma')\gamma^\mu v(\vec{p},\sigma) &= \overline{u}(\vec{p}',\sigma')\Big(p'^\mu - p^\mu + 2i(p'_\alpha + p_\alpha)J^{\alpha\mu}\Big)v(\vec{p},\sigma).\end{aligned} \tag{4.69}$$

In particular setting $p^\mu = p'^\mu$ in the first two of these and $\vec{p} = -\vec{p}'$ and taking the $\mu = 0$ component in the last one these become

$$\begin{aligned}\overline{u}(\vec{p},\sigma')\gamma^\mu u(\vec{p},\sigma) &= -2p^\mu\delta_{\sigma\sigma'}\\ \overline{v}(\vec{p},\sigma')\gamma^\mu v(\vec{p},\sigma) &= -2p^\mu\delta_{\sigma\sigma'}\\ u^\dagger(\vec{p},\sigma')v(-\vec{p},\sigma) &= 0,\end{aligned} \tag{4.70}$$

which we will use next time. To derive the first Gordon identity, note that by the Dirac equation (4.39) we have

$$\begin{aligned}im\overline{u}(\vec{p}',\sigma')\gamma^\mu u(\vec{p},\sigma) &= \overline{u}(\vec{p}',\sigma')\gamma^\mu\not{p}u(\vec{p},\sigma)\\ &= \frac{1}{2}p_\alpha\overline{u}(\vec{p}',\sigma')\Big(\{\gamma^\mu,\gamma^\alpha\} + [\gamma^\mu,\gamma^\alpha]\Big)u(\vec{p},\sigma)\\ &= \overline{u}(\vec{p}',\sigma')\Big(p^\mu - 2ip_\alpha J^{\alpha\mu}\Big)u(\vec{p},\sigma)\end{aligned} \tag{4.71}$$

and

$$\begin{aligned} im\overline{u}(\vec{p}',\sigma')\gamma^\mu u(\vec{p},\sigma) &= \overline{u}(\vec{p}',\sigma')\not{p}'\gamma^\mu u(\vec{p},\sigma) \\ &= \frac{1}{2}p'_\alpha \overline{u}(\vec{p}',\sigma')\Big(\{\gamma^\alpha,\gamma^\mu\}+[\gamma^\alpha,\gamma^\mu]\Big)u(\vec{p},\sigma) \\ &= \overline{u}(\vec{p}',\sigma')\Big(p'^\mu + 2ip'_\alpha J^{\alpha\mu}\Big)u(\vec{p},\sigma). \end{aligned} \tag{4.72}$$

Adding these together gives the first line of (4.69). The other two lines are derived similarly.

**Massless spinor solutions**

We'll now repeat this analysis for the massless case, being somewhat sketchier. Taking the reference momentum to be

$$k^\mu = (k,0,0,\ldots,k) \tag{4.73}$$

with $k>0$, the Dirac equation (4.39) tells us that we want

$$\left(\gamma^{d-1}-\gamma^0\right)u(\vec{k}) = \left(\gamma^{d-1}-\gamma^0\right)v(\vec{k}) = 0. \tag{4.74}$$

The matrix $\gamma^{d-1}-\gamma^0$ is not hermitian, and in fact it does not have a complete set of eigenvectors. It does however always have a set of $2^{\lfloor d/2\rfloor -1}$ linearly-independent eigenvectors with eigenvalue zero. One way to see this is to instead consider $\gamma^1-\gamma^0$, which is related to $\gamma^{d-1}-\gamma^0$ by a rotation, and which in our tensor product representation is just

$$(\sigma_x - i\sigma_y)\otimes I\otimes\ldots\otimes I. \tag{4.75}$$

The matrix in the first factor is just $\begin{pmatrix}0&0\\2&0\end{pmatrix}$, which annihilates the vector $\begin{pmatrix}0\\1\end{pmatrix}$, so the tensor product of $\begin{pmatrix}0\\1\end{pmatrix}$ with any basis vector on the other factors is an eigenvector of eigenvalue zero. These $2^{\lfloor d/2\rfloor-1}$ eigenvectors transform in the spinor representation of the little group $SO(d-2)$.

We can make this more concrete by considering specific dimensions. For $d=2$ (4.74) just reads

$$\left(\gamma^1-\gamma^0\right)u(\vec{k}) = \left(\gamma^1-\gamma^0\right)v(\vec{k}) = 0. \tag{4.76}$$

Multiplying on both sides by $\gamma^0$, we get

$$\begin{aligned} \gamma u &= -u \\ \gamma v &= -v, \end{aligned} \tag{4.77}$$

which confirms our notation from last time that a right-moving solution has $\gamma=-1$. If we instead take the reference momentum to be $k^\mu=(k,-k)$ then we see that a left-moving solution has $\gamma=1$.[15] For $d=4$ a basis of solutions for (4.74) is

$$u(\vec{k},-) = v(\vec{k},+) = \sqrt{2k}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad u(\vec{k},+) = v(\vec{k},-) = \sqrt{2k}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{4.78}$$

The signs are again chosen so that

$$\begin{aligned} J^{12}u(\vec{k},\pm) &= \pm\frac{1}{2}u(\vec{k},\pm) \\ J^{12}v(\vec{k},\pm) &= \mp\frac{1}{2}v(\vec{k},\pm), \end{aligned} \tag{4.79}$$

[15]The $d=2$ case is a bit special because the little group is $SO(0)=\mathbb{Z}_2$, so the nontrivial element can't be reached by exponentiating an infinitesimal rotation. We thus need to solve the equation seperately for right and left moving solutions.

which ensures that $u(\vec{k},\pm)$ will multiply an annihilation operator for a particle of helicity $\pm 1/2$, while $v(\vec{k},\pm)$ will multiply a creation operator for a particle of helicity $\pm 1/2$ (remember that by definition the helicity of a massless particle in $d=4$ is the angular momentum about its direction of motion). By convention we define the charge of a field to be the charge of the particle it annihilates, so this is why the upper two components of a Dirac spinor in $d=4$ are called a "left-handed" Weyl spinor while the lower two components are called a "right-handed" Weyl spinor. The helicity sums again give

$$\sum_\sigma u(\vec{p},\sigma)\overline{u}(\vec{p},\sigma) = \sum_\sigma v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma) = -\not{p}, \tag{4.80}$$

as you will check directly from (4.78) in the homework, and one can also again establish this generally (up to normalization) from group theory.

## 4.3 Summary of spin sum rules

It may be useful to collect in one place a list of the sum identities we've derived.

$$\begin{aligned}
\sum_\sigma u(\vec{p},\sigma)\overline{u}(\vec{p},\sigma) &= -(\not{p}+im)\\
\sum_\sigma v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma) &= -(\not{p}-im)\\
\overline{u}(\vec{p},\sigma)u(\vec{p},\sigma') &= -2im\delta_{\sigma\sigma'}\\
\overline{v}(\vec{p},\sigma)v(\vec{p},\sigma') &= 2im\delta_{\sigma\sigma'}\\
\overline{u}(\vec{p},\sigma)v(\vec{p},\sigma') &= 0\\
\overline{v}(\vec{p},\sigma)u(\vec{p},\sigma') &= 0.\\
\overline{u}(\vec{p},\sigma')\gamma^\mu u(\vec{p},\sigma) &= -2p^\mu\delta_{\sigma\sigma'}\\
\overline{v}(\vec{p},\sigma')\gamma^\mu v(\vec{p},\sigma) &= -2p^\mu\delta_{\sigma\sigma'}\\
u^\dagger(\vec{p},\sigma')v(-\vec{p},\sigma) &= 0\\
v^\dagger(\vec{p},\sigma')u(-\vec{p},\sigma) &= 0.
\end{aligned} \tag{4.81}$$

All of these hold in general dimensions and arbitrary representations of the $\gamma$-matrices, both for massive and massless spinors.

**Problems:**

1. Show that in even dimensions the interaction $\Phi\overline{\Psi}\gamma\Psi$ between a real scalar and a Dirac spinor is Lorentz-invariant and hermitian. Show the same for the modified kinetic term $i\overline{\Psi}\gamma\partial\!\!\!/\Psi$, with the hermiticity now holding up to a total derivative.

2. Confirm the null spin sums (4.80) starting from our explicit representations for $u$ and $v$ in $d=4$, and if you are feeling brave also try making a group theory argument determining them up to normalization.

3. (Extra Credit) In $3+1$ dimensions we can classify all the irreducible representations of $Spin(3,1)$ using a a clever trick. Defining $J_x = J^{23}$, $J_y = J^{31}$, and $J_z = J^{12}$, and also $K_i = J^{0i}$, show that the quantities $A_i = \frac{1}{2}(J_i + iK_i)$ and $B_i = \frac{1}{2}(J_i - iK_i)$ obey the Lie algebra of two copies of $SU(2)$:

$$\begin{aligned}
[A_i, A_j] &= i\epsilon_{ijk}A_k\\
[B_i, B_j] &= i\epsilon_{ijk}B_k\\
[A_i, B_j] &= 0.
\end{aligned} \tag{4.82}$$

   Therefore any irreducible representation of the Lorentz algebra can be converted to an irreducible representation of $SU(2)\times SU(2)$ and vice versa. Fortunately for us we already know how to think

about irreducible representations of $SU(2)$, they are labeled by the spin $j$ and have dimension $2j+1$. To get an irreducible representation of $SU(2) \times SU(2)$ we just need an object with a pair of $SU(2)$ indices, one transforming in the spin $j_A$ representation and one transforming in the spin $j_B$ representation. In other words we can represent $\vec{A}$ and $\vec{B}$ as

$$\begin{aligned} \vec{A}_{a'b',ab} &= \delta_{bb'} \vec{J}^{j_A}_{a'a} \\ \vec{B}_{a'b',ab} &= \delta_{aa'} \vec{J}^{j_B}_{b'b}, \end{aligned} \tag{4.83}$$

where $\vec{J}^j$ are the three generators of the spin $j$ representation of $SU(2)$. What is the dimensionality of the spin $(j_A, j_B)$ representation of the Lorentz group? Can you make a guess for which values of $j_A$ and $j_B$ give the scalar, Weyl spinor, and vector representations of $Spin(3,1)$?

# 5 Quantization of the Dirac field

We've now set up the formalism for a quantum field $\Psi$ transforming in the spinor representation of $Spin(d-1,1)$, with the Dirac Lagrangian

$$\mathcal{L} = -i\overline{\Psi}\left(\partial\!\!\!/ + m\right)\Psi. \tag{5.1}$$

In this section we will use this Lagrangian to construct a free quantum field theory and compute its correlation functions.

## 5.1 Canonical anticommutation relations

According to the Dirac Lagrangian, the canonical conjugate to $\Psi$ is[16]

$$\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = -i\Psi^\dagger (\gamma^0)^2 = i\Psi^\dagger. \tag{5.2}$$

We can therefore guess that $\Psi$ and $\Psi^\dagger$ at equal time should obey the canonical anticommutation relations

$$\begin{aligned} \{\Psi^a(t,\vec{x}), \Psi^b(t,\vec{y})\} &= 0 \\ \{\Psi^{a\dagger}(t,\vec{x}), \Psi^{b\dagger}(t,\vec{y})\} &= 0 \\ \{\Psi^a(t,\vec{x}), \Psi^{b\dagger}(t,\vec{y})\} &= \delta^{ab}\delta^{d-1}(\vec{x}-\vec{y}), \end{aligned} \tag{5.3}$$

and also that the Hamiltonian density should be given by

$$\mathcal{H} = \Pi^a \dot{\Psi}^a - \mathcal{L} = i\overline{\Psi}\left(\gamma^i \partial_i + m\right)\Psi. \tag{5.4}$$

It isn't clear that these guesses are correct, since after all there isn't really a sensible classical theory of the Dirac field, but we will now see that the quantum theory we construct from these anticommutation relations and the Hamiltonian is quite sensible indeed. When we discuss the fermionic path integral in the next section we will see a more systematic way to go between classical Lagrangians and quantum Hamiltonians in fermionic systems.

As a first sanity check we can confirm that with this Hamiltonian $\Psi$ indeed obeys the Dirac equation:

$$\begin{aligned} \dot{\Psi}^a(t,\vec{x}) &= i[H, \Psi^a(t,\vec{x})] \\ &= i\int d^{d-1}x' [\mathcal{H}(t,\vec{x'}), \Psi^a(t,\vec{x})] \\ &= \int d^{d-1}x' \{\Psi^{\dagger b}(t,\vec{x}'), \Psi^a(t,\vec{x})\} \left(\gamma^0(\gamma^i\partial_i + m)\right)^{bc} \Psi^c(t,\vec{x}) \\ &= \left(\gamma^0(\gamma^i\partial_i + m)\right)^{ac} \Psi^c(t,\vec{x}), \end{aligned} \tag{5.5}$$

[16] This is less obvious than it appears, since $\Psi$ and $\overline{\Psi}$ are anticommuting so we might worry about the overall sign of this derivative. The correct rule, as we will see in a moment when we check the Dirac equation, is that we should think of the derivative as acting from the right and re-order any anticommuting variables so that the one we are differentiating with respect to is on the right. We will see such manipulations in more detail when we discuss the fermionic path integral.

and thus

$$\gamma^0\partial_0\Psi + (\gamma^i\partial_i + m)\Psi = 0. \tag{5.6}$$

In going from the second line to the third line of (5.5) we use the anticommutator identity

$$[AB, C] = A\{B, C\} - \{A, C\}B. \tag{5.7}$$

Since $\Psi$ obeys the Dirac equation, we can expand it in the set of solutions we constructed in the previous section:

$$\Psi^a(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[u^a(\vec{p},\sigma)a_{\vec{p},\sigma}e^{i(\vec{p}\cdot\vec{x}-\omega_{\vec{p}}t)} + v^a(\vec{p},\sigma)b^\dagger_{\vec{p},\sigma}e^{-i(\vec{p}\cdot\vec{x}-\omega_{\vec{p}}t)}\right], \tag{5.8}$$

where $a_{\vec{p},\sigma}$ and $b_{\vec{p},\sigma}$ are operator coefficients for our solutions and as usual $\omega_{\vec{p}} = \sqrt{|p|^2 + m^2}$. To isolate the operator coefficients, let's first take the Fourier transforms of $\Psi$ and $\Psi^\dagger$:

$$\int d^{d-1}x\Psi(0,\vec{x})e^{-i\vec{p}\cdot\vec{x}} = \sum_{\sigma'} \frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big(u(\vec{p},\sigma')a_{\vec{p},\sigma'} + v(-\vec{p},\sigma')b^\dagger_{-\vec{p},\sigma'}\Big) \tag{5.9}$$

$$\int d^{d-1}x\Psi^\dagger(0,\vec{x})e^{i\vec{p}\cdot\vec{x}} = \sum_{\sigma'} \frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big(u^\dagger(\vec{p},\sigma')a^\dagger_{\vec{p},\sigma'} + v^\dagger(-\vec{p},\sigma')b_{-\vec{p},\sigma'}\Big). \tag{5.10}$$

Using our spin sum rules

$$\begin{aligned} u^\dagger(\vec{p},\sigma)u(\vec{p},\sigma') &= 2\omega_{\vec{p}}\delta_{\sigma,\sigma'} \\ v^\dagger(\vec{p},\sigma)v(\vec{p},\sigma') &= 2\omega_{\vec{p}}\delta_{\sigma,\sigma'} \\ u^\dagger(\vec{p},\sigma)v(-\vec{p},\sigma') &= 0 \\ v^\dagger(-\vec{p},\sigma)u(\vec{p},\sigma') &= 0, \end{aligned} \tag{5.11}$$

we can extract the operator coefficients as

$$\begin{aligned} a_{\vec{p},\sigma} &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\int d^{d-1}x\, u^\dagger(\vec{p},\sigma)\Psi(0,\vec{x})e^{-i\vec{p}\cdot\vec{x}} \\ a^\dagger_{\vec{p},\sigma} &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\int d^{d-1}x\, \Psi^\dagger(0,\vec{x})u(\vec{p},\sigma)e^{-i\vec{p}\cdot\vec{x}} \\ b_{\vec{p},\sigma} &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\int d^{d-1}x\, \Psi^\dagger(0,\vec{x})v(\vec{p},\sigma)e^{-i\vec{p}\cdot\vec{x}} \\ b^\dagger_{\vec{p},\sigma} &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\int d^{d-1}x\, v^\dagger(\vec{p},\sigma)\Psi(0,\vec{x})e^{i\vec{p}\cdot\vec{x}}. \end{aligned} \tag{5.12}$$

Our goal is now to compute the anticommutators of the $a$'s and $b$'s using the canonical anticommutation relations (5.3). Many of these vanish automatically by virtue of only involving $\Psi$'s or only involving $\Psi^\dagger$'s. There are four nontrivial anticommutators that we need to compute:

$$\begin{aligned} \{a_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}\} &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\frac{1}{\sqrt{2\omega_{\vec{p}'}}}\int d^{d-1}x d^{d-1}x'\, u^\dagger(\vec{p},\sigma)\{\Psi(0,\vec{x}), \Psi^\dagger(0,\vec{x}')\}u(\vec{p}',\sigma')e^{i\vec{p}'\cdot\vec{x}'-i\vec{p}\cdot\vec{x}} \\ &= \frac{1}{2\omega_{\vec{p}}}(2\pi)^{d-1}\delta^{d-1}(\vec{p}'-\vec{p})u^\dagger(\vec{p},\sigma)u(\vec{p},\sigma') \\ &= (2\pi)^{d-1}\delta^{d-1}(\vec{p}'-\vec{p})\delta_{\sigma,\sigma'} \end{aligned} \tag{5.13}$$

and

$$\begin{aligned} \{a_{\vec{p},\sigma}, b_{\vec{p}',\sigma'}\} &= \frac{1}{\sqrt{2\omega_{\vec{p}}}}\frac{1}{\sqrt{2\omega_{\vec{p}'}}}\int d^{d-1}x\, e^{-i(\vec{p}+\vec{p}')\cdot\vec{x}}u^\dagger(\vec{p},\sigma)v(\vec{p}',\sigma') \\ &= \frac{1}{2\omega_{\vec{p}}}(2\pi)^{d-1}\delta^{d-1}(\vec{p}+\vec{p}')u^\dagger(\vec{p},\sigma)v(-\vec{p},\sigma') \\ &= 0, \end{aligned} \tag{5.14}$$

and similarly

$$\{b_{\vec{p},\sigma}, b^\dagger_{\vec{p}',\sigma'}\} = (2\pi)^{d-1}\delta^{d-1}(\vec{p}' - \vec{p})\delta_{\sigma,\sigma'}$$
$$\{a^\dagger_{\vec{p},\sigma}, b^\dagger_{\vec{p}',\sigma'}\} = 0. \tag{5.15}$$

Thus we see that the $a_{\vec{p},\sigma}$ and $b_{\vec{p},\sigma}$ obey the anticommutation relations for two independent sets of fermionic particles, with the convention being to say that $a^\dagger_{\vec{p},\sigma}$ creates particles and $b^\dagger_{\vec{p},\sigma}$ creates antiparticles. The one-particle states created by acting with these on the vacuum carry a little group index $\sigma$ just as we anticipated in the last section, and in particular for $d = 4$ they are spin-1/2 particles. In quantum electrodynamics $a^\dagger_{\vec{p},\sigma}$ creates an electron and $b^\dagger_{\vec{p},\sigma}$ creates a positron.

## 5.2 Calculation of the Hamiltonian

To validate this particle interpretation of the canonical anticommutation relations, we need to compute the Hamiltonian. Indeed we have

$$\begin{aligned}
H =& i\int d^{d-1}x\, \overline{\Psi}(\gamma^i\partial_i + m)\Psi \\
=& i\int d^{d-1}x\, \Psi^\dagger(0,\vec{x})\dot{\Psi}(0,\vec{x}) \\
=& i\int d^{d-1}x \sum_{\sigma,\sigma'}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\int \frac{d^{d-1}p'}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\frac{1}{\sqrt{2\omega_{\vec{p}'}}}\Big(u^\dagger(\vec{p}',\sigma')a^\dagger_{\vec{p}',\sigma'}e^{-i\vec{p}'\cdot\vec{x}'} + v^\dagger(\vec{p}',\sigma')b_{\vec{p}',\sigma'}e^{i\vec{p}'\cdot\vec{x}'}\Big) \\
& \times i\omega_{\vec{p}} \times \Big(-u(\vec{p},\sigma)a_{\vec{p},\sigma}e^{i\vec{p}\cdot\vec{x}} + v(\vec{p},\sigma)b^\dagger_{\vec{p},\sigma}e^{-i\vec{p}\cdot\vec{x}}\Big) \\
=& \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\omega_{\vec{p}}\Big(a^\dagger_{\vec{p},\sigma}a_{\vec{p},\sigma} - b_{\vec{p},\sigma}b^\dagger_{\vec{p},\sigma}\Big) \\
=& \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\omega_{\vec{p}}\Big(a^\dagger_{\vec{p},\sigma}a_{\vec{p},\sigma} + b^\dagger_{\vec{p},\sigma}b_{\vec{p},\sigma}\Big) - \text{“}\infty\text{”},
\end{aligned} \tag{5.16}$$

where in the last line we have used the anticommutation relation for $b$ and $b^\dagger$ to exchange their order at the cost of an infinite negative contribution to the vacuum energy. This divergence is familiar from our quantization of a free scalar field, we can absorb it into a renormalization of the cosmological constant. Once we have done so, we see that the Hamiltonian is bounded from below and we can find the ground state, usually called the vacuum state in field theory, by imposing the condition

$$a_{\vec{p},\sigma}|\Omega\rangle = b_{\vec{p},\sigma}|\Omega\rangle = 0. \tag{5.17}$$

Particles and antiparticles are then created by acting on the vacuum with creation operators, each carrying energy $\omega_{\vec{p}}$.

It is interesting to note that the infinite contribution to the cosmological constant is negative for fermions and positive for bosons. In theories where there is a symmetry between bosons and fermions, one might hope that these contributions could cancel. This is indeed the case, and theories with such a symmetry are called **supersymmetric theories**. Supersymmetric field theories are quite interesting for several reasons:

(1) The automatic cancellation of divergences we saw here arises in many places in supersymmetric field theories, leading to theories which are much simpler to analyze than their non-supersymmetric counterparts. This makes supersymmetric field theories a wonderful laboratory for exploring quantum field theory phenomena, especially in the context of strongly-interacting dynamics where traditional perturbative approaches are useless.

(2) This cancellation also has the possibility of addressing the Higgs hierarchy problem in the standard model that we mentioned last semester - if bosonic and fermionic contributions to the Higgs mass cancel in loops, then it is not so puzzling that the Higgs boson is light compared to fundamental energy scales such as the Planck mass. The experimental constraints on SUSY at the weak scale are now quite strong however, so this idea seems less appealing than it did a few decades ago.

(3) Supersymmetry arises very naturally in string theory, which so far is our best-understood candidate for a theory of quantum gravity. Unfortunately this supersymmetry might be broken at very high energy scales however, so it need not be accessible with near-term experiments.

One could teach a whole class on supersymmetric field theories and supergravity, and in fact Jesse Thaler is teaching one here at MIT right now! In this class however we will content ourselves with exploring the simplest supersymmetric field theory in $d = 4$ spacetime dimensions, the Wess-Zumino model, which you will do on the homework this week.

## 5.3 Symmetries of the Dirac Lagrangian

What are the symmetries of the Dirac theory (5.1)? Poincare symmetry certainly, by construction, but what else do we have? Perhaps the most obvious symmetry is the $U(1)$ **charge rotation symmetry**

$$\Psi'(x) = e^{i\theta q}\Psi(x), \tag{5.18}$$

which we can write infinitesimally as

$$\delta_S\Psi(x) = iq\Psi(x). \tag{5.19}$$

This leaves the Lagrangian invariant, since $\Psi$ and $\overline{\Psi}$ transform oppositely. The charge $q$ must be an integer so that a rotation by $2\pi$ is equal to doing nothing. The Noether current for this symmetry is

$$J^\mu = -\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}\delta_S\Psi = -q\overline{\Psi}\gamma^\mu\Psi, \tag{5.20}$$

where in computing the derivative we (somewhat heuristically since we haven't discussed Noether's theorem for fermionic fields) again take the derivative to act on the Lagrangian density from the right. We can check this sign by computing the action of the symmetry charge on $\Psi$:

$$\begin{aligned}
[Q, \Psi^a(0,\vec{x})] &= q\int d^{d-1}x[\Psi^{b\dagger}(0,\vec{x}')\Psi^b(0,\vec{x}'), \Psi^a(0,\vec{x})] \\
&= -q\int d^{d-1}x\{\Psi^{b\dagger}(0,\vec{x}'), \Psi^a(0,\vec{x})\}\Psi^b(0,\vec{x}') \\
&= -q\Psi^a(0,\vec{x}) \\
&= i\delta_S\Psi^a(0,\vec{x}).
\end{aligned} \tag{5.21}$$

We can also check the conservation of the current:

$$\partial_\mu J^\mu = -q\left(\partial_\mu\overline{\Psi}\gamma^\mu\Psi + \overline{\Psi}\not{\partial}\Psi\right) = -q\overline{\Psi}\Psi(m-m) = 0, \tag{5.22}$$

where in the second equality we used the Dirac equation both for $\Psi$ and its adjoint

$$\partial_\mu\overline{\Psi}\gamma^\mu = m\overline{\Psi}. \tag{5.23}$$

It is interesting to compute the charge operator in terms of the raising and lowering operators, this gives

$$\begin{aligned}
Q =& q\int d^{d-1}x \Psi^\dagger \Psi \\
=& q\int d^{d-1}x \sum_{\sigma,\sigma'} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \int \frac{d^{d-1}p'}{(2\pi)^{d-1}} \frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{p}}}} \left( u^\dagger(\vec{p}',\sigma') a^\dagger_{\vec{p}',\sigma'} e^{-i\vec{p}'\cdot\vec{x}} + v^\dagger(\vec{p}',\sigma') b_{\vec{p}',\sigma'} e^{i\vec{p}'\cdot\vec{x}} \right) \\
& \times \left( u(\vec{p},\sigma) a_{\vec{p},\sigma} e^{i\vec{p}\cdot\vec{x}} + v(\vec{p},\sigma) b^\dagger_{\vec{p},\sigma} e^{i\vec{p}\cdot\vec{x}} \right) \\
=& q\sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left( a^\dagger_{\vec{p},\sigma} a_{\vec{p},\sigma} + b_{\vec{p},\sigma} b^\dagger_{\vec{p},\sigma} \right) \\
=& q\sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left( a^\dagger_{\vec{p},\sigma} a_{\vec{p},\sigma} - b^\dagger_{\vec{p},\sigma} b_{\vec{p},\sigma} \right) + \text{“}\infty\text{”}.
\end{aligned} \tag{5.24}$$

Thus after a renormalization we see that $a^\dagger_{\vec{p},\sigma}$ creates particles of charge $q$ and $b^\dagger_{\vec{p},\sigma}$ creates particles of charge $-q$, so they are indeed antiparticles. In quantum electrodynamics $q$ is the electric charge measured in units of the elementary charge

$$e = 1.602176634 \times 10^{-19}\,\text{C}, \tag{5.25}$$

so $q = -1$ for the electron field and $q = 1$ for the proton field. In more civilized units we write the elementary charge in terms of the **fine structure constant**

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}. \tag{5.26}$$

We will see later in the semester that the perturbation theory of QED is a perturbation series in $\alpha$, and since $\alpha \ll 1$ this perturbation series is quite good for many observables.

Another interesting symmetry we can consider is **chiral symmetry**, which in even spacetime dimensions acts on a Dirac spinor as

$$\Psi'(x) = e^{i\theta q \gamma} \Psi(x). \tag{5.27}$$

Here recall that

$$\gamma = i^{-\frac{d-2}{2}} \gamma^0 \dots \gamma^{d-1}. \tag{5.28}$$

The derivative term in the Dirac Lagrangian is invariant under this transformation:

$$-i\overline{\Psi}' \partial\!\!\!/ \Psi' = -i\Psi^\dagger e^{-i\theta q\gamma} \gamma^0 \gamma^\mu e^{i\theta q\gamma} \partial_\mu \Psi = -i\Psi^\dagger e^{-i\theta q\gamma} \gamma^0 e^{-i\theta q\gamma} \gamma^\mu \partial_\mu \Psi = -i\Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi, \tag{5.29}$$

where we've used that $\gamma$ anticommutes with $\gamma^\mu$, but the mass term isn't:

$$-i\overline{\Psi}'\Psi' = -i\overline{\Psi} e^{-i\theta q\gamma} \gamma^0 e^{i\theta q\gamma} \Psi = -i\overline{\Psi} e^{2i\theta q\gamma} \Psi. \tag{5.30}$$

Chiral symmetry is therefore only a symmetry of the Dirac Lagrangian if $m = 0$, in which case you can think of it as a transformation which rotates the upper Weyl component of $\Psi$ as $e^{i\theta q}$ and the lower Weyl component of $\Psi$ as $e^{-i\theta q}$. In the homework you will work out the Noether current and charge for chiral symmetry.

Finally we'll discuss $\mathcal{CRT}$ symmetry, which we showed last semester is a symmetry of any relativistic quantum field theory. Our general expression for the action of $\mathcal{CRT}$ symmetry on the set of dynamical fields is

$$\Theta^\dagger_{\mathcal{CRT}} \Phi^a(x) \Theta_{\mathcal{CRT}} = i^{f_a} \left( D_E(\mathcal{RT})^a{}_b \Phi^b(\mathcal{RT}x) \right)^\dagger, \tag{5.31}$$

where $D_E$ is the representation of $Spin(d)$ which the fields transform under in the Euclidean path integral, $\mathcal{R}$ is the coordinate transformation which reflects $x^1 \to -x^1$, and $\mathcal{T}$ is the coordinate transformation which reverses time. We haven't yet discussed Euclidean fermions (see the next section), but their transformation

under Euclidean rotations is easy enough to guess: the spinor representation of $SO(d)$ is simply that generated by the Euclidean $\gamma$-matrices

$$\begin{aligned}\gamma_E^0 &= i\gamma^0 \\ \gamma_E^i &= \gamma^i\end{aligned} \tag{5.32}$$

via the rotation generators

$$J_E^{\mu\nu} = -\frac{i}{4}[\gamma_E^\mu, \gamma_E^\nu]. \tag{5.33}$$

In particular a Euclidean rotation in the 01 plane is generated by

$$J_E^{01} = \frac{1}{4}[\gamma^0, \gamma^1], \tag{5.34}$$

so we have

$$D_E(\mathcal{RT}) = e^{-i\frac{\pi}{4}[\gamma^0,\gamma^1]} = -i\gamma^0\gamma^1, \tag{5.35}$$

as can be confirmed somewhat tediously by using the $\gamma$-matrix algebra or more easily by working with our product-Pauli representation. Therefore acting on a Dirac spinor we have the $\mathcal{CRT}$ transformation

$$\Theta_{\mathcal{CRT}}^\dagger \Psi(x) \Theta_{\mathcal{CRT}} = -(\gamma^0\gamma^1\Psi(\mathcal{RT}x))^*, \tag{5.36}$$

where here I've used the convention of writing $*$ for the Hilbert space adjoint that doesn't transpose Dirac indices. On the homework you will confirm that the Dirac Lagrangian is invariant under this transformation. It is also sometimes possible to define separate $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{T}$ symmetry transformations on Dirac spinors; you will explore this in the next problem set.

## 5.4 Correlation functions

Last semester we learned that the most natural observables in any quantum field theory are its **correlation functions** - vacuum expectation values of products of Heisenberg local operators. In a free theory with quadratic Lagrangian such as the Dirac theory we have been considering in this section, all higher-point correlation functions can be decomposed into two-point functions. To understand what the theory does it is therefore enough to understand its two-point functions.

Perhaps the most obvious two-point function we could consider is the expectation value of the product of two spinor fields:

$$\langle\Omega|\Psi^a(x_2)\Psi^b(x_1)|\Omega\rangle. \tag{5.37}$$

This however vanishes by the selection rule for charge rotation symmetry (i.e. that the charges of the operators in the correlation function must add up to zero for the correlator to be nonvanishing). A correlation function which is non-vanishing is

$$S^{ab}(x_2 - x_1) = \langle\Omega|\Psi^a(x_2)\overline{\Psi}^b(x_1)|\Omega\rangle, \tag{5.38}$$

where we have anticipated that by translation invariance $S^{ab}$ will only depend on the difference of $x_2$ and $x_1$. We can compute $S^{ab}$ as follows:

$$\begin{aligned}S^{ab}(x_2 - x_1) &= \sum_{\sigma,\sigma'}\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\int \frac{d^{d-1}p'}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}\sqrt{2\omega_{\vec{p}'}}}e^{ip\cdot x_2 - ip'\cdot x_1}\langle\Omega|a_{\vec{p},\sigma}a^\dagger_{\vec{p}',\sigma'}|\Omega\rangle u^a(\vec{p},\sigma)\overline{u}^b(\vec{p}',\sigma') \\ &= \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{2\omega_{\vec{p}}}e^{ip\cdot(x_2-x_1)}\sum_\sigma u^a(\vec{p},\sigma)\overline{u}^b(\vec{p},\sigma) \\ &= -\int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{2\omega_{\vec{p}}}(\not{p} + im)^{ab}e^{ip\cdot(x_2-x_1)} \\ &= i(\not{\partial}_2 - m)^{ab}G(x_2 - x_1),\end{aligned} \tag{5.39}$$

where

$$G(x_2 - x_1) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{2\omega_{\vec{p}}} e^{ip\cdot(x_2-x_1)} \tag{5.40}$$

is the free scalar two-point function we computed last semester (see section four for an evaluation of this integral in terms of a Bessel function). This means that the Dirac two-point function $S^{ab}(x_2 - x_1)$ has the same qualitative features as the scalar two-point function: exponential decay at spacelike separation when $m > 0$, power-law decay at spacelike separation when $m = 0$, and a power-law divergence as $x_2$ and $x_1$ come together. As a sanity check we can confirm that when $x_2 \neq x_1$ then $S^{ab}(x)$ obeys the Dirac equation:

$$(\partial\!\!\!/_2 + m)^{ab} S^{bc}(x_2 - x_1) = i(\partial\!\!\!/_2 + m)^{ab}(\partial\!\!\!/_2 - m)^{bc} G(x_2 - x_1) = i\delta^{ac}(\partial_2^2 - m^2)G(x_2 - x_1) = 0. \tag{5.41}$$

We can also compute the two-point function with the operators in the other order, leading to

$$\begin{aligned}\langle\Omega|\overline{\Psi}^b(x_1)\Psi^a(x_2)|\Omega\rangle &= -\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{2\omega_{\vec{p}}} (p\!\!\!/ - im)^{ab} e^{-ip\cdot(x_2-x_1)} \\ &= -i(\partial\!\!\!/_2 - m)^{ab} G(x_1 - x_2).\end{aligned} \tag{5.42}$$

In the scalar case, we saw that in addition to the two-point function it was also natural to discuss the time-ordered Feynman propagator

$$G_F(x_2 - x_1) = \langle\Omega|T\Phi(x_2)\Phi(x_1)|\Omega\rangle := \Theta(t_2 - t_1)\langle\Omega|\Phi(x_2)\Phi(x_1)|\Omega\rangle + \Theta(t_1 - t_2)\langle\Omega|\Phi(x_1)\Phi(x_2)|\Omega\rangle. \tag{5.43}$$

For example we saw that this is what the path integral formulation naturally computes, and also that this is what shows up naturally when evaluating Feynman diagrams. For fermionic operators however this definition of the time-ordered product is not so natural: fermions anticommute at spacelike separation, so if the time-ordering does not respect this then the quantity we'd define would be singular when $x_2$ and $x_1$ lie on the same time slice. To fix this, we define the time-ordered spinor propagator to be

$$S_F^{ab}(x_2 - x_1) = \Theta(t_2 - t_1)\langle\Omega|\Psi^a(x_2)\overline{\Psi}^b(x_1)|\Omega\rangle - \Theta(t_1 - t_2)\langle\Omega|\overline{\Psi}^b(x_1)\Psi^a(x_2)|\Omega\rangle. \tag{5.44}$$

Using (5.39) and (5.42) we then have

$$\begin{aligned}S_F^{ab}(x_2 - x_1) &= i(\partial\!\!\!/_2 - m)^{ab}\Big(\Theta(t_2 - t_1)G(x_2 - x_1) + \Theta(t_1 - t_2)G(x_1 - x_2)\Big) \\ &= i(\partial\!\!\!/_2 - m)^{ab} G_F(x_2 - x_1).\end{aligned} \tag{5.45}$$

Finally recalling our covariant expression

$$G_F(x_2 - x_1) = \int \frac{d^d p}{(2\pi)^d} \frac{-i}{p^2 + m^2 - i\epsilon} e^{ip\cdot(x_2-x_1)} \tag{5.46}$$

for the scalar Feynman propagator, we have a covariant expression for the spinor propagator:

$$S_F^{ab}(x_2 - x_1) = \int \frac{d^d p}{(2\pi)^d} \frac{i(p\!\!\!/ + im)^{ab}}{p^2 + m^2 - i\epsilon} e^{ip\cdot(x_2-x_1)}. \tag{5.47}$$

We will use this expression often in perturbative calculations involving fermions.

## 5.5 Majorana fermions

In this section we'll briefly discuss the quantization of Majorana fermions. Recall that a Majorana fermion is one which obeys the constraint

$$\Psi^* = B\Psi \tag{5.48}$$

with $B$ being whichever of the two complex conjugation matrices $B_1$ or $B_2$ obeys $B^T = B$ (and thus $B^*B = 1$ since both $B_1$ and $B_2$ are unitary). For example for $d = 2$ and $d = 3$ we have

$$B = I, \tag{5.49}$$

so the Majorana constraint just says the components of $\Psi$ are real, while for $d = 4$ you will see in the homework that we want

$$B = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \tag{5.50}$$

Note that for $d = 2$ the Majorana constraint separately constrains the left-moving and right-moving components of $\Psi$, while for $d = 4$ it relates the left-handed and right-handed components.

Taking the transpose of the constraint (5.48) we see that a Majorana fermion obeys

$$\overline{\Psi} = \Psi^T \mathcal{C}, \tag{5.51}$$

with

$$\mathcal{C} \equiv B\gamma^0. \tag{5.52}$$

The Lagrangian for a massive Majorana fermion is

$$\mathcal{L} = -\frac{i}{2}\overline{\Psi}(\partial\!\!\!/ + m)\Psi = -\frac{i}{2}\Psi^T\mathcal{C}(\partial\!\!\!/ + m)\Psi. \tag{5.53}$$

The factor of $1/2$ is included because the usual Dirac action counts each independent component of a Majorana spinor twice. Working out the canonical anticommutation relations and the Hamiltonian from this action is somewhat subtle due to the presence of the Majorana constraint (5.48), to handle it properly we need a version of the Hamiltonian formalism that deals with constraints. My favorite approach to this is called "covariant phase space", which you can read about in my first paper with Jie-qiang Wu, and there is another approach called "Dirac brackets" which you can read about in Weinberg's book. Either way the result is that the canonical anticommutation relations for this theory are the same as for the Dirac theory:

$$\{\Psi^a(t,\vec{x}), \Psi^{*b}(t,\vec{y})\} = \{\Psi^a(t,\vec{x}), B^{bc}\Psi^c(t,\vec{y})\} = \delta^{ab}\delta^{d-1}(\vec{x}-\vec{y}), \tag{5.54}$$

and thus

$$\{\Psi^a(t,\vec{x}), \Psi^b(t,\vec{y})\} = B^{ab*}\delta^{d-1}(\vec{x}-\vec{y}), \tag{5.55}$$

and that the Hamiltonian density is given by half of the Dirac result:

$$\mathcal{H} = \frac{i}{2}\overline{\Psi}\left(\gamma^i\partial_i + m\right)\Psi = \frac{i}{2}\Psi^T\mathcal{C}\left(\gamma^i\partial_i + m\right)\Psi. \tag{5.56}$$

To get a sense of what the Majorana constraint means for the particle content of the theory, we can write both sides in terms of our field decomposition:

$$\begin{aligned}\Psi^* &= \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[u^*(\vec{p},\sigma)a^\dagger_{\vec{p},\sigma}e^{-ip\cdot x} + v^*(\vec{p},\sigma)b_{\vec{p},\sigma}e^{ip\cdot x}\right] \\ &= \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[Bu(\vec{p},\sigma)a_{\vec{p},\sigma}e^{ip\cdot x} + Bv(\vec{p},\sigma)b^\dagger_{\vec{p},\sigma}e^{-ip\cdot x}\right],\end{aligned} \tag{5.57}$$

so comparing coefficients of $e^{ip\cdot x}$ and $e^{-ip\cdot x}$ we see that the Majorana constraint imposes a relation between $a_{\vec{p},\sigma}$ and $b_{\vec{p},\sigma}$. Indeed if we choose the phase of $B$ appropriately, we can arrange for the relationship to simply be that

$$a_{\vec{p},\sigma} = b_{\vec{p},\sigma}. \tag{5.58}$$

In other words we can arrange so that

$$\begin{aligned}Bu(\vec{p},\sigma) &= v^*(\vec{p},\sigma) \\ Bv(\vec{p},\sigma) &= u^*(\vec{p},\sigma).\end{aligned} \tag{5.59}$$

You will confirm this in the homework for $d = 4$. A Majorana spinor thus creates a fermion which is its own antiparticle.

**Problems:**

1. What is the Noether current $J^\mu$ for the chiral symmetry transformation (5.27) of a massless Dirac fermion? Write an expression for the Noether charge $Q = \int d^{d-1}x J^0$ in $d = 4$ in terms of the creation and annihilation operators for massless fermions. Hint: You will need to work out spin sums such as $u^\dagger \gamma u = -\overline{u}\gamma^0\gamma u$, which you can do in the reference frame by noting that $u(\vec{k}, \sigma)$ and $v(\vec{k}, \sigma)$ are eigenvectors of $\gamma$.

2. Check that the Dirac Lagrangian is invariant (up to a total derivative) under the $\mathcal{CRT}$ transformation (5.36). Hint: you will need to remember that $\Theta_{\mathcal{CRT}}$ is antiunitary, but if you do things right you shouldn't have to know the complex conjugates of the $\gamma$-matrices (i.e. you shouldn't have to use $B_1$ or $B_2$). Make sure you also take into account the anticommuting nature of $\Psi$ and $\Psi^\dagger$.

3. You should have found on the last homework that for $d = 4$ we have

$$B_2 = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \tag{5.60}$$

   Show that this obeys

$$\begin{aligned} B_2 u(\vec{p}, \sigma) &= -iv^*(\vec{p}, \sigma) \\ B_2 v(\vec{p}, \sigma) &= -iu^*(\vec{p}, \sigma) \end{aligned} \tag{5.61}$$

   for the $u$ and $v$ we defined in section four. Therefore to get a simple representation of the Majorana field in terms of these basis vectors we should take our Majorana constraint to be $\Psi^* = B\Psi$ with $B = iB_2$ so that (5.58) holds. With this choice, compute the Hamiltonian of a Majorana spinor in terms of the creation and annihilation operators.

4. When working four-dimensional spinors many people like to use a notation which more explicitly recognizes that the spinor representation is reducible. In this notation, called **two-component spinor notation**, the fundamental objects are left-handed Weyl spinors $\psi_\alpha$, with $\alpha = 1, 2$. Right-handed spinor indices are then written with a dot on them, so for example the complex conjugate of $\psi_\alpha$ is denoted $\overline{\psi}_{\dot{\alpha}}$. Note that here the bar just means complex conjugate. One then decomposes a Dirac spinor in terms of two left-handed Weyl spinors as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \epsilon^{\dot{\alpha}\dot{\beta}} \overline{\chi}_{\dot{\beta}} \end{pmatrix}, \tag{5.62}$$

   with

$$\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2. \tag{5.63}$$

   This looks a bit nicer if we define

$$\overline{\chi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \overline{\chi}_{\dot{\beta}}, \tag{5.64}$$

   and in general two-component indices can be raised and lowered using $\epsilon$ and its inverse $-\epsilon$ (there is also an $\epsilon^{\alpha\beta}$ with the same components at $\epsilon^{\dot{\alpha}\dot{\beta}}$). The $\gamma$-matrices are then decomposed as

$$\gamma^\mu = -i \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \overline{\sigma}^{\mu,\dot{\alpha}\beta} & 0 \end{pmatrix}, \tag{5.65}$$

   with $\sigma^\mu = (I, \vec{\sigma})$ and $\overline{\sigma}^\mu = (I, -\vec{\sigma})$, except when $\gamma^0$ appears in the definition of $\overline{\Psi} = \Psi^\dagger \gamma^0$, in which case it is written as

$$\gamma^0 = -i \begin{pmatrix} 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix}. \tag{5.66}$$

What is the Dirac Lagrangian written in terms of $\psi_\alpha$ and $\chi_\alpha$? Your expression should only have dotted indices contracted with dotted indices and undotted indices contracted with undotted indices. If we impose the Majorana constraint $\Psi^* = B\Psi$, what does this say about the relationship between $\psi_\alpha$ and $\chi_\alpha$? This notation is particularly convenient when discussing supersymmetry, if you are brave you can try solving the next problem using two-component notation.

5. The simplest supersymmetric theory in four spacetime dimensions is the Wess-Zumino model, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\Phi_1\partial^\mu\Phi_1 - \frac{1}{2}\partial_\mu\Phi_2\partial^\mu\Phi_2 - \frac{m^2}{2}\Phi_1^2 - \frac{m^2}{2}\Phi_2^2 - \frac{i}{2}\overline{\Psi}(\partial\!\!\!/ + m)\Psi. \tag{5.67}$$

Here $\Phi_1$ and $\Phi_2$ are a pair of real scalar fields, and $\Psi$ is a Majorana fermion. Show that the Wess-Zumino Lagrangian is invariant under the infinitesimal supersymmetry transformation

$$\begin{aligned}\delta_S\Phi_1 &= i\bar{\epsilon}\Psi \\ \delta_S\Phi_2 &= \bar{\epsilon}\gamma\Psi \\ \delta_S\Psi &= \partial_\mu(\Phi_1 + i\Phi_2\gamma)\gamma^\mu\epsilon - m(\Phi_1 - i\Phi_2\gamma)\epsilon.\end{aligned} \tag{5.68}$$

Here $\epsilon$ is an "infinitesimal Grassman Majorana spinor", which means you should view it as anticommuting with $\Psi$ and obeying the same Majorana constraint as $\Psi$ does. As is usual for a global symmetry you should take $\epsilon$ to be position-independent. You will find it useful to make use of the fact that

$$\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -\gamma^{\mu T}. \tag{5.69}$$

# 6 Path integrals for fermions

Last semester we saw that the path integral approach to quantum field theory is a powerful tool for deriving general results such as the spin-statistics and $\mathcal{CRT}$ theorems, and it is also for deriving the Feynman rules for perturbative computations of correlation functions and scattering amplitudes. The path integral we derived last semester however started with the canonical commutation relations for a set of bosonic variables $Q^a$ and their canonical conjugates $P_a$. In this section we will learn how to construct a path integral for fermionic variables $\Psi^a$ and $\Xi_a$ obeying canonical anticommutation relations. This is a tricky subject, as you will see, but it is also fun and pays off down the road.[17]

## 6.1 Review of the bosonic path integral

Let's first quickly recall how we derived the bosonic path integral. We have a set of $M$ bosonic coordinates $Q^a$ and their conjugate momenta $P_b$ obeying the canonical commutation relations

$$\begin{aligned}[Q^a, Q^b] &= 0 \\ [P_a, P_b] &= 0 \\ [Q^a, P_b] &= i\delta^a_b,\end{aligned} \tag{6.1}$$

and to start with we are interested in computing transition amplitudes of the form

$$G(q_f, q_i; t_f, t_i) = \langle q_f, t_f | q_i, t_i\rangle, \tag{6.2}$$

[17] Most quantum field theory textbooks just make up the rules for the fermion path integral and then check that they give the same Feynman rules one derives from the operator approach. The more logical approach we use here, which derives the fermionic path integral directly from the operator formalism of quantum mechanics, was as far as I know first developed in detail by Polchinski; you can read about it in appendix A of his string theory book or in section 9.5 of Weinberg. The basic idea however seems to go back to a 1956 paper by Candlin, with the Grassmann integral being developed more systematically by Berezin in 1966. My version is a bit more systematic than Polchinski's, for example in carefully distinguishing between $\langle\widetilde{\psi}|$ and $\langle\psi|$ and being explicit about the supermodule structure of $\widetilde{H}$.

where $|q,t\rangle$ is a simultaneous eigenstate of the $Q^a(t)$:

$$Q^a(t)|q,t\rangle = q^a|q,t\rangle. \tag{6.3}$$

Explicitly we have $|q,t\rangle = e^{iHt}|q,0\rangle$ (there isn't a sign mistake here, check it). We then constructed the path integral formalism by repeatedly inserting complete sets of states:

$$\langle q_f,t_f|q_i,t_i\rangle = \prod_{m=1}^{N-1}\left(\int dq_m\right)\langle q_f,t_f|q_{N-1},t_f-\epsilon\rangle\langle q_{N-1},t_f-\epsilon|q_{N-2},t_f-2\epsilon\rangle\ldots\langle q_2,t_2|q_1,t_1\rangle\langle q_1,t_i+\epsilon|q_i,t_i\rangle, \tag{6.4}$$

where we have split the time interval $t_f-t_i$ into $N$ pieces of size $\epsilon$. We then used the canonical commutation relations to show that when $\epsilon$ is small we have

$$\langle q',t+\epsilon|q,t\rangle \approx \int \frac{d^Mp}{(2\pi)^M}e^{i\epsilon\left(\sum_a p_a\frac{q'^a-q^a}{\epsilon}-H(q',p)\right)}, \tag{6.5}$$

and thus the path integral expression

$$\begin{aligned}\langle q_f,t_f|q_i,t_i\rangle &= \lim_{\epsilon\to 0}\prod_{m=1}^{N-1}\left(\int d^Mq_m\right)\prod_{n=0}^{N-1}\left(\int\frac{d^Mp_n}{(2\pi)^M}\right)\exp\left[i\epsilon\sum_{\ell=0}^{N-1}\left(\sum_a p_{\ell,a}\frac{q^a_{\ell+1}-q^a_\ell}{\epsilon}-H\left(q_{\ell+1},p_\ell\right)\right)\right]\\ &:= \int\mathcal{D}q|_{q_i}^{q_f}\int\mathcal{D}p\,\exp\left[i\int_{t_i}^{t_f}dt\left(\sum_a p_a(t)\dot{q}^a(t)-H(q(t),p(t))\right)\right].\end{aligned} \tag{6.6}$$

In deriving this we took the operator ordering in $H$ to put all $P$ to the right of all $Q$. We also learned that we can compute expectation values of time-ordered products as

$$\begin{aligned}\langle q_f,t_f|TO_1(Q(\bar{t}_1),P(\bar{t}_1))\ldots O_p(Q(\bar{t}_p),P(\bar{t}_p))|q_i,t_i\rangle = \int\mathcal{D}q|_{q_i}^{q_f}\int\mathcal{D}p\,O_1(q(\bar{t}_1),p(\bar{t}_1))\ldots O_p(q(\bar{t}_p),p(\bar{t}_p))\\ \times\exp\left[i\int_{t_i}^{t_f}dt\left(\sum_a p_a(t)\dot{q}^a(t)-H(q(t),p(t))\right)\right],\end{aligned} \tag{6.7}$$

where the $O_n$ are ordered so that all $P$ are to the *left* of all $Q$, and that we could replace the external states in this correlation function by the ground state provided that we take $t_i\to-\infty(1-i\epsilon)$ and $t_f\to\infty(1-i\epsilon)$ and then divide by the same with no operator insertions. We will now see how to write analogous formulas for fermions.

## 6.2 Grassman variables and fermionic eigenstates

Now let's consider a set of $M$ complex fermions, obeying the canonical anticommutation algebra

$$\begin{aligned}\{\Psi^a,\Psi^b\} &= 0\\ \{\Xi_a,\Xi_b\} &= 0\\ \{\Psi^a,\Xi_b\} &= \delta^a_b.\end{aligned} \tag{6.8}$$

I'll remind you that these operators act on a Hilbert space of $M$ qubits, which we can span using the basis states

$$|s_1\ldots s_M\rangle = \Xi_1^{s_1}\ldots\Xi_M^{s_M}|0\rangle, \tag{6.9}$$

where each $s_i$ can be zero or one and $|0\rangle = |0\ldots0\rangle$ is a state annihilated by all of the $\Psi^a$.

To imitate the path integral construction of the previous subsection, the first thing we need is a set of simultaneous eigenstates of the $\Psi^a$. Here we immediately have a problem however: if

$$\Psi^a|\psi\rangle = \psi^a|\psi\rangle, \tag{6.10}$$

then we must have $(\psi^a)^2 = 0$ since $(\Psi^a)^2 = 0$. If $\psi^a$ is a complex number then this of course just implies that $\psi^a = 0$. We do have one state, $|0\rangle$, which is annihilated by all of the $\Psi^a$, but this is very far from having a complete basis of eigenstates to insert in deriving a path integral. To solve this problem we need to invent a new kind of eigenvalue which can square to zero without being zero.

To build intuition, let's first consider the case where $M = 1$. The Hilbert space is just that of one qubit. The idea is to introduce a formal object $\psi$, called a **Grassmann variable**, which obeys

$$\{\psi, \psi\} = \{\psi, \Psi\} = \{\psi, \Xi\} = 0 \tag{6.11}$$

and also

$$\psi|0\rangle = |0\rangle\psi. \tag{6.12}$$

Note that this implies that

$$\psi|1\rangle = \psi\Xi|0\rangle = -\Xi\psi|0\rangle = -\Xi|0\rangle\psi = -|1\rangle\psi. \tag{6.13}$$

At this point people often ask "but what is the Grassman variable?" I don't know a particularly elegant answer to this question: it is simply a symbol $\psi$ which obeys the rules I just stated. Given such a variable we can then define a set of objects of the form

$$r = a + b\psi, \tag{6.14}$$

where $a$ and $b$ are complex numbers. The set $R$ of all such objects forms what mathematicians call a **ring**: we can add and multiply elements in the obvious way

$$\begin{aligned}(a + b\psi) + (c + d\psi) &= (a + b) + (c + d)\psi \\ (a + b\psi)(c + d\psi) &= ac + (ad + bc)\psi,\end{aligned} \tag{6.15}$$

but there is in general no multiplicative inverse (what would $\psi^{-1}$ be?). We then introduce an extension $\widetilde{\mathcal{H}}$ of the qubit Hilbert space $\mathcal{H}$ we started with, which allows superpositions of $|0\rangle$ and $|1\rangle$ with coefficients in $R$. Mathematicans would call $\widetilde{\mathcal{H}}$ a **module** over the ring $R$, and in fact it is what is sometimes called a **supermodule** since we can write it as the direct sum of a submodule with Grassmann weight zero and a submodule with Grassmann weight one. In this accounting commuting objects such as $|0\rangle$ and any complex number have weight zero, while anticommuting objects such as $\psi$ and $|1\rangle$ have weight one. Weight zero objects also commute with weight one objects. The reason we introduce the Grassmann supermodule $\widetilde{H}$ over the physical Hilbert space $\mathcal{H}$ is that it contains nontrivial eigenstates for $\Psi$. Indeed if we define

$$|\psi\rangle = |0\rangle - \psi|1\rangle, \tag{6.16}$$

we can easily check that

$$\Psi|\psi\rangle = \psi|0\rangle = \psi\left(|0\rangle - \psi|1\rangle\right) = \psi|\psi\rangle, \tag{6.17}$$

where we used that $\Psi|0\rangle = 0$, $\Psi|1\rangle = |0\rangle$, and the above rules for manipulating $\psi$. Note that $|\psi\rangle$ has weight zero, so we can commute it freely with anticommuting objects such as $\psi$.

Extending this construction to general $M$ is not difficult, we introduce $M$ Grassmann variables $\psi^a$ obeying

$$\{\psi^a, \psi^b\} = \{\psi^a, \Psi^b\} = \{\psi^a, \Xi_b\} = 0 \tag{6.18}$$

and

$$\psi^a|0\rangle = |0\rangle\psi^a. \tag{6.19}$$

We can construct a simultaneous eigenstate of the $\Psi^a$ in the Grassmann supermodule as

$$|\psi\rangle = (1 - \psi^1\Xi_1)\ldots(1 - \psi^M\Xi_M)|0\rangle, \tag{6.20}$$

which again has weight zero. The easiest way to show that this is an eigenstate is to first observe that for some fixed $b$ (not summed over) we have

$$\Psi^a(1 - \psi^b\Xi_b) = \Psi^a + \psi^b(\{\Psi^a, \Xi_b\} + \Xi_b\Psi^a) = (1 - \psi^b\Xi_b)\left(\Psi^a + \delta^a_b\psi^b\right). \tag{6.21}$$

Thus when we act on $|\psi\rangle$ with $\Psi^a$ it moves freely through all the factors not involving $\psi^a$, and when it meets the factor $(1-\psi^a\Xi_a)$ (again with no sum on $a$) then it generates two terms: one where $\Psi^a$ moves through freely and the other where it is replaced by a factor of $\psi^a$. The former term vanishes since when $\Psi^a$ moves through the rest of the factors it meets $|0\rangle$ and annihilates it. We thus have

$$\Psi^a|\psi\rangle = \psi^a|\psi\rangle. \tag{6.22}$$

It will also be useful for us to work out the action on $|\psi\rangle$ by a product of some of the $\Psi^a$, this is given by

$$\Psi^{a_1}\dots\Psi^{a_n}|\psi\rangle = \psi^{a_1}\dots\psi^{a_n}|\psi\rangle. \tag{6.23}$$

There is no sign on the right hand side since, although we have to move each $\Psi^{a_i}$ through the eigenvalues of the $\Psi$'s to its right, we then have to move the eigenvalues back.

The Hilbert space $\mathcal{H}$ has an inner product, and it is natural to also define an inner product on the Grassman supermodule. Given $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$ and $r_1, r_2 \in R$, the inner product on $\widetilde{\mathcal{H}}$ is defined to obey

$$(|\phi_1\rangle r_1, |\phi_2\rangle r_2) = r_1^*\langle\phi_1|\phi_2\rangle r_2. \tag{6.24}$$

The inner product on general elements of $\widetilde{\mathcal{H}}$ is then extended from this by linearity:

$$(|\psi_1\rangle + |\psi_1'\rangle, |\psi_2\rangle + |\psi_2'\rangle) = (|\psi_1\rangle, |\psi_2\rangle) + (|\psi_1\rangle, |\psi_2'\rangle) + (|\psi_1'\rangle, |\psi_2\rangle) + (|\psi_1'\rangle, |\psi_2'\rangle). \tag{6.25}$$

Note that this construction requires us to know how to take the complex conjugates of Grassmann variables. We thus need to designate whether each Grassman variable is real or complex: if $\psi$ is real then (of course) we have $\psi^* = \psi$, while if $\psi$ is complex then $\psi^*$ should be treated as an independent Grassman variable. Moreover the action of the complex conjugate on products of Grassman variables is defined to obey

$$(\psi^1\psi^2)^* = \psi^{2*}\psi^{1*}. \tag{6.26}$$

To construct the path integral we also need "bra" versions of the eigenstates of $\Psi^a$. A natural guess would be that these should be the duals of $|\psi\rangle$ with respect to the inner product we just defined on the Grassman supermodule $\widetilde{H}$, but this doesn't actually work. The reason is simple: in fermionic systems $\Psi$ doesn't necessarily anticommute with its adjoint, e.g. for a Dirac spinor $\Psi$ and $\Psi^\dagger$ are canonical conjugates, so its left eigenstates don't need to be the duals of its right eigenstates. What we need for the path integral construction, as we will see in a moment, are really the left eigenstates rather than the duals. A better approach is to first introduce a bra $\langle\widetilde{0}| \in \mathcal{H}$, which is defined by saying that it obeys

$$\langle\widetilde{0}|\Xi_a = 0 \tag{6.27}$$

for all $a$ and also

$$\langle\widetilde{0}|0\rangle = 1. \tag{6.28}$$

These equations imply that $\langle\widetilde{0}|$ has weight zero,

$$\psi^a\langle\widetilde{0}| = \langle\widetilde{0}|\psi^a, \tag{6.29}$$

since we have

$$\psi^a\langle\widetilde{0}|0\rangle = \langle\widetilde{0}|0\rangle\psi^a = \langle\widetilde{0}|\psi^a|0\rangle. \tag{6.30}$$

We can get a complete basis for the bras as

$$\langle\widetilde{s_1\dots s_M}| = \langle\widetilde{0}|\Psi_M^{s_M}\dots\Psi_1^{s_1}, \tag{6.31}$$

so in particular note that $\langle\widetilde{0}| = \langle\widetilde{0\dots0}|$. The relationship between $\langle\widetilde{0}|$ and the dual of $|0\rangle$ depends on our choice of inner product. In the standard inner product, where $\Xi_a = \Psi^{a\dagger}$, we simply have $\langle\widetilde{0}| = \langle 0|$. In the ghost inner product, where $\Psi^a$ and $\Xi_a$ are hermitian, we instead have $\langle\widetilde{0}| = \langle 1\dots1|$. We will organize our

presentation so that the only properties we need of $\langle\widetilde{0}|$ are (6.27) and (6.28), so the formulas we write will be valid for either choice of inner product. We therefore won't actually need to use the inner product we constructed on $\widetilde{\mathcal{H}}$, although it is still good to know about it.

We'll now construct a left eigenstate of $\Psi$ in the single-fermion case with $M = 1$. Defining[18]

$$\langle\widetilde{\psi}| = -\langle\widetilde{0}|\Psi(1 - \Xi\psi) = -\langle\widetilde{1}| + \langle\widetilde{0}|\psi, \tag{6.32}$$

we can check that

$$\langle\widetilde{\psi}|\Psi = -\langle\widetilde{1}|\psi = \langle\widetilde{\psi}|\psi, \tag{6.33}$$

So $\langle\widetilde{\psi}|$ is indeed a left eigenstate of $\Psi$. We can generalize this to $M$ fermions by defining

$$\langle\widetilde{\psi}| = (-1)^M\langle\widetilde{0}|\Psi^M \dots \Psi^1(1 - \Xi_1\psi^1)\dots(1 - \Xi_M\psi^M), \tag{6.34}$$

which obeys

$$\langle\widetilde{\psi}|\Psi^a = \langle\widetilde{\psi}|\psi^a \tag{6.35}$$

as you will check on the homework. As for the case of a right eigenstate, we also have

$$\langle\widetilde{\psi}|\Psi^{a_1}\dots\Psi^{a_n} = \langle\widetilde{\psi}|\psi^{a_1}\dots\psi^{a_n}. \tag{6.36}$$

The weight of $\langle\widetilde{\psi}|$ is $M \bmod 2$, so we have to be careful moving it past anticommuting objects. It will also be useful for us to know that

$$\langle\widetilde{\psi}'|\psi\rangle = (\psi'^M - \psi^M)\dots(\psi'^1 - \psi^1), \tag{6.37}$$

which you will also check on the homework. We will see in the next subsection that the right-hand side of this equation is the definition of the Grassmann $\delta$-function $\delta^M(\vec{\psi}' - \vec{\psi})$.

## 6.3 Grassmann calculus

In our construction of the bosonic path integral, the actual integrals are over eigenvalues of the bosonic operators $Q^a$ and $P_a$. In a fermionic theory we thus we need to learn how to integrate over our Grassmann eigenvalues $\psi^a$. It is also useful to think about how to differentiate with respect to a Grassmann variable, for example since we will introduce Grassmann sources for fermionic operators in the path integral and then want to differentiate with respect to these sources to bring down fermionic operators in correlation functions.

Let's first consider the calculus of a function $f$ of a single Grassman variable $\theta$. Since $\theta^2 = 0$ this function will automatically be linear:

$$f(\theta) = a + \theta b. \tag{6.38}$$

Here $a$ and $b$ can be complex numbers, but we'll also allow them to depend on other Grassmann variables. It is quite natural to define the derivative to be

$$\frac{df}{d\theta} = b. \tag{6.39}$$

What about the integral? This is less clear, but the right definition is that it is the same as the derivative:

$$\int d\theta f(\theta) = b. \tag{6.40}$$

One way to motivate this that we would like the integral to be linear in the sense that

$$\int d\theta\,(f(\theta) + g(\theta)) = \int d\theta f(\theta) + \int d\theta g(\theta), \tag{6.41}$$

[18]Note that in the standard inner product $\langle\widetilde{\psi}|$ indeed is not the dual of the $|\psi\rangle$ we constructed in (6.16). It *is* however the dual in the ghost inner product, since in the ghost inner product $\Psi$ is hermitian. Note also that with the ghost inner product we should take $\psi$ to be a real Grassman variable, while for the standard inner product it is complex.

and we'd also like the integral of a total derivative to vanish:

$$\int d\theta \frac{df}{d\theta} = 0. \tag{6.42}$$

These two conditions tell us that

$$\int d\theta f(\theta) = \lambda b \tag{6.43}$$

with $\lambda$ some general $f$-independent constant. We might as well take $\lambda = 1$, since otherwise there is no natural way to choose it.

There is an interesting feature of the behavior of the Grassmann integral under linear redefinitions of the integration variable. Indeed say we have a change of variables

$$\theta = \theta' c + d, \tag{6.44}$$

with $c, d \in \mathbb{C}$. We'd like the integral to be invariant under this change of variables, i.e. to have

$$\int d\theta' f(\theta') = \int d\theta f(\theta). \tag{6.45}$$

We can write the right-hand side as

$$\int d\theta f(\theta' c + d) = \int d\theta \left(a + db + \theta' cb\right), \tag{6.46}$$

so to get this to be equal to $b$ we need to define the measure transformation to be

$$d\theta = \frac{1}{c} d\theta'. \tag{6.47}$$

This is the opposite of the usual transformation of the measure under a linear change of variables.

Let's now generalize to $N$ Grassman variables $\theta_1, \ldots, \theta_N$. A function of these variables now has the more complicated Taylor expansion

$$f(\theta_1, \ldots, \theta_N) = a_0 + \theta_1 a_{1,1} + \ldots + \theta_N a_{1,N} + \ldots + \theta_N \ldots \theta_1 a_N. \tag{6.48}$$

The partial derivatives $\frac{\partial}{\partial \theta_n}$ are defined to act on $f$ from the left, anticommuting with each $\theta_m$ with $m \neq n$ until they meet a $\theta_n$, which they then replace by one. So for example

$$\frac{\partial}{\partial \theta_2} 2\theta_1 \theta_2 \theta_3 = -2\theta_1 \theta_3. \tag{6.49}$$

On terms with no $\theta_n$, $\frac{\partial}{\partial \theta_n}$ gives zero. The integral is defined as

$$\int d\theta_1 \ldots d\theta_N f(\theta_1, \ldots, \theta_N) = \partial_{\theta_1} \ldots \partial_{\theta_N} f = a_N. \tag{6.50}$$

Alternatively we can define the integral by iterating the single-variable integral:

$$\int d\theta_1 \ldots d\theta_N f(\theta_1, \ldots, \theta_N) = \int d\theta_1 \left( \int d\theta_2 \left( \ldots \int d\theta_N f \right) \ldots \right). \tag{6.51}$$

We can work out the change of variables rule under a linear transformation

$$\theta_n = \sum_m L_{nm} \theta'_m + b_n \tag{6.52}$$

in the same way as we did for $N = 1$: we want to ensure that

$$\int d\theta_1 \dots d\theta_N f(\theta_1, \dots, \theta_N) = \int d\theta_1 \dots d\theta_N f((L\theta' + b)_1, \dots, (L\theta' + b)_N) = \int d\theta'_1 \dots d\theta'_N f(\theta'_1, \dots, \theta'_N). \tag{6.53}$$

The highest-order term in the integrand of the middle quantity is

$$\sum_{m_1,\dots,m_N} L_{N,m_N} \dots L_{1,m_1} \theta'_{m_N} \dots \theta'_{m_1} a_N = \sum_{\pi \in S_N} (-1)^{p(\pi)} L_{N,\pi(N)} \dots L_{1,\pi(1)} \theta'_N \dots \theta'_1 a_N = \det L \, \theta'_N \dots \theta'_1 a_N, \tag{6.54}$$

where $S_N$ is the permutation group on $N$ elements and $(-1)^{p(\pi)}$ is the sign of $\pi$. Thus for (6.53) to hold we need to have

$$d\theta_1 \dots d\theta_N = \frac{1}{\det L} d\theta'_1 \dots d\theta'_N, \tag{6.55}$$

where again the factor of $\det L$ is inverted compared to what it would be for the transformation of an ordinary integration measure.

It is useful to introduce a $\delta$-function for Grassmann integration. This is given by

$$\delta^N(\vec{\theta}' - \vec{\theta}) = (\theta'_N - \theta_N) \dots (\theta'_1 - \theta_1), \tag{6.56}$$

which you will confirm in the homework obeys

$$\int d\theta'_1 \dots d\theta'_N \delta^N(\vec{\theta}' - \vec{\theta}) f(\theta'_1, \dots, \theta'_N) = f(\theta_1, \dots, \theta_N). \tag{6.57}$$

for any function $f$. This $\delta$-function also has a useful integral representation,

$$\delta^N(\vec{\theta}' - \vec{\theta}) = (-1)^{N(N+1)/2} \int d\xi_1 \dots d\xi_N e^{\sum_a \xi_a(\theta_a - \theta'_a)}. \tag{6.58}$$

To derive this note we can rewrite the right-hand side as

$$(-1)^{N(N+1)/2} \int d\xi_1 \dots d\xi_N \big(1 - \xi_N(\theta'_N - \theta_N)\big) \dots \big(1 - \xi_1(\theta'_1 - \theta_1)\big), \tag{6.59}$$

so to get a nonvanishing contribution to the integral we should pick the second term in each factor. The sign works out from moving each $\xi_n$ past the $\theta$'s to its left.

It will also be useful for us to have a formula for the action of a product of the $\Xi_a$ on $|\psi\rangle$. Starting from (6.20) we have

$$\Xi_{a_1} \dots \Xi_{a_n} |\psi\rangle = \left(-\frac{\partial}{\partial \psi^{a_1}}\right) \dots \left(-\frac{\partial}{\partial \psi^{a_n}}\right) |\psi\rangle, \tag{6.60}$$

since we can re-order the factors $(1 - \psi^n \Xi_n)$ in $|\psi\rangle$ so that they appear in the same order as the $\Xi_a$ and then use

$$\Xi_a(1 - \psi^a \Xi_a) = \Xi_a = -\frac{\partial}{\partial \psi^a}(1 - \psi^a \Xi_a). \tag{6.61}$$

Using our integral representation for the $\delta$-function and switching back to our $M$-fermion notation we then have

$$\begin{aligned}
\langle \widetilde{\psi'} | \Psi^{a_1} \dots \Psi^{a_n} \Xi_{b_1} \dots \Xi_{b_m} | \psi \rangle &= \langle \widetilde{\psi'} | \psi'^{a_1} \dots \psi'^{a_n} \left(-\frac{\partial}{\partial \psi^{b_1}}\right) \dots \left(-\frac{\partial}{\partial \psi^{b_m}}\right) |\psi\rangle \\
&= \big((-1)^M \psi'^{a_1}\big) \dots \big((-1)^M \psi'^{a_n}\big) \left((-1)^{M+1} \frac{\partial}{\partial \psi^{b_1}}\right) \dots \left((-1)^{M+1} \frac{\partial}{\partial \psi^{b_m}}\right) \langle \widetilde{\psi'} | \psi \rangle \\
&= (-1)^{M(M+1)/2} \int d\xi_1 \dots d\xi_M \, \psi'^{a_1} \dots \psi'^{a_n} \xi_{b_1} \dots \xi_{b_m} e^{\sum_a \xi_a(\psi^a - \psi'^a)}.
\end{aligned} \tag{6.62}$$

## 6.4 The fermion path integral

We now have all the tools we need to imitate our derivation of the bosonic path integral in fermionic theories. In particular our Grassmann technology has given us the completeness relation

$$\int d^M\psi|\psi\rangle\langle\widetilde{\psi}| = \int |\psi\rangle d^M\psi\langle\widetilde{\psi}| = 1, \tag{6.63}$$

where for future convenience in the second expression we use that $|\psi\rangle$ has weight zero to move the integration measure between the ket and the bra. We can check this completeness relation by acting on an eigenstate:

$$\left(\int |\psi'\rangle d^M\psi'\langle\widetilde{\psi'}|\right)|\psi\rangle = \int d\psi'\delta^M(\vec{\psi}' - \vec{\psi})|\psi'\rangle = |\psi\rangle. \tag{6.64}$$

Let's now use this to get a path integral representation for the transition amplitude:

$$\langle\widetilde{\psi_f, t_f}|\psi_i, t_i\rangle = \langle\widetilde{\psi_f, t_f}|\int |\psi_{N-1}, t_{N-1}\rangle d^M\psi_{N-1}\langle\widetilde{\psi_{N-1}, t_{N-1}}|\ldots\int |\psi_1, t_1\rangle d^M\psi_1\langle\widetilde{\psi_1, t_1}|\psi_i, t_i\rangle, \tag{6.65}$$

where we have again discretized the time interval $t_f - t_i$ into $N$ intervals of length $\epsilon$. We can approximate the inner products at small $\epsilon$ using equation (6.62):

$$\begin{aligned}
\langle\widetilde{\psi', t+\epsilon}|\psi, t\rangle &\approx \langle\widetilde{\psi', t}|\left(1 - i\epsilon H(\Psi, \Xi)\right)|\psi, t\rangle \\
&= (-1)^{M(M+1)/2}\int d^M\xi(1 - i\epsilon H(\psi', \xi))e^{\sum_a \xi_a(\psi^a - \psi'^a)} \\
&\approx (-1)^{M(M+1)/2}\int d^M\xi \exp\left[i\epsilon\left(i\sum_a \xi_a\frac{\psi'^a - \psi^a}{\epsilon} - H(\psi', \xi)\right)\right],
\end{aligned} \tag{6.66}$$

where we have ordered the operators in the Hamiltonian so that all $\Xi_a$ appear to the right of all $\Psi^a$. It only remains to work out the overall sign. Each inner product contributes a Grassmann measure $d^M\psi_m d^M\xi_{m-1}$, and since this has an even number of Grassman integrals we can move them all to the left without incurring any more signs. Thus we have

$$\begin{aligned}
\langle\widetilde{\psi_f, t_f}|\psi_i, t_i\rangle = &\lim_{\epsilon\to 0}(-1)^{NM(M+1)/2}\int d^M\xi_{N-1}d^M\psi_{N-1}\ldots d^M\xi_1 d^M\psi_1 d^M\xi_0 \\
&\times\exp\left[i\epsilon\sum_{\ell=0}^{N-1}\left(i\sum_a \xi_{a,\ell}\frac{\psi^a_{\ell+1} - \psi^a_\ell}{\epsilon} - H(\psi_{\ell+1}, \xi_\ell)\right)\right].
\end{aligned} \tag{6.67}$$

If you like we can re-order the measure to be

$$\begin{aligned}
\langle\widetilde{\psi_f, t_f}|\psi_i, t_i\rangle = &\lim_{\epsilon\to 0}(-1)^{NM(M+1)/2}\int d^M\xi_0 d^M\xi_1 d^M\psi_1\ldots d^M\xi_{N-1}d^M\psi_{N-1} \\
&\times\exp\left[i\epsilon\sum_{\ell=0}^{N-1}\left(i\sum_a \xi_{a,\ell}\frac{\psi^a_{\ell+1} - \psi^a_\ell}{\epsilon} - H(\psi_{\ell+1}, \xi_\ell)\right)\right].
\end{aligned} \tag{6.68}$$

In both of these expressions we take $\psi_0 = \psi_i$ and $\psi_N = \psi_f$. Typically we won't try to keep track of the overall sign, which anyways will cancel out when we use ratios of path integrals to compute correlation functions, so somewhat more heuristically we can write this as

$$\langle\widetilde{\psi_f, t_f}|\psi_i, t_i\rangle = \int \mathcal{D}\psi|_{\psi_i}^{\psi_f}\mathcal{D}\xi\exp\left[i\int_{t_i}^{t_f} dt\Big(i\sum_a \xi_a(t)\dot{\psi}^a(t) - H(\psi(t), \xi(t))\Big)\right]. \tag{6.69}$$

This is our path integral expression for a fermionic transition amplitude! Note in particular that $i\xi_a$ is the canonical momentum, so this confirms the operator ordering in the Legendre transformation we did to construct the Dirac Hamiltonian.

We can also introduce a path integral representation for time-ordered correlation functions. This requires us to compute

$$\begin{aligned}\langle\widetilde{\psi', t+\epsilon}|O\big(\Psi(t),\Xi(t)\big)|\psi\rangle &\approx \langle\widetilde{\psi', t}|(1-i\epsilon H(\Psi,\Xi))O\big(\Psi(t),\Xi(t)\big)|\psi,t\rangle \\ &= (-1)^{M(M+1)/2}\int d^M\xi\,(1-i\epsilon H(\psi',\xi))O(\psi,\xi)e^{\sum_a \xi_a(\psi^a-\psi'^a)} \\ &\approx (-1)^{M(M+1)/2}\int d^M\xi\,O(\psi,\xi)\exp\left[i\epsilon\left(i\sum_a \xi_a\frac{\psi'^a-\psi^a}{\epsilon}-H(\psi',\xi)\right)\right], \quad (6.70)\end{aligned}$$

where we have ordered the operators in $O$ so that all $\Xi_a$ appear to the *left* of all $\Psi^a$. We then have

$$\begin{aligned}\langle\widetilde{\psi_f, t_f}|TO_1(\Psi(\bar t_1),\Xi(\bar t_1))\dots O_p(\Psi(\bar t_p),\Xi(\bar t_p))|\psi_i,t_i\rangle = \lim_{\epsilon\to 0}(-1)^{NM(M+1)/2}&\int d^M\xi_0 d^M\xi_1 d^M\psi_1\dots d^M\xi_{N-1}d^M\psi_{N-1} \\ &\times O_1(\psi(\bar t_1),\xi(\bar t_1))\dots O_p(\psi(\bar t_p),\xi(\bar t_p)) \\ &\times\exp\left[i\epsilon\sum_{\ell=0}^{N-1}\left(i\sum_a\xi_{a,\ell}\frac{\psi^a_{\ell+1}-\psi^a_\ell}{\epsilon}-H(\psi_{\ell+1},\xi_\ell)\right)\right], \quad (6.71)\end{aligned}$$

or more heuristically

$$\begin{aligned}\langle\widetilde{\psi_f, t_f}|TO_1(\Psi(\bar t_1),\Xi(\bar t_1))\dots O_p(\Psi(\bar t_p),\Xi(\bar t_p))|\psi_i,t_i\rangle = &\int\mathcal{D}\psi|_{\psi_i}^{\psi_f}\mathcal{D}\xi\,O_1(\psi(\bar t_1),\xi(\bar t_1))\dots O_p(\psi(\bar t_p),\xi(\bar t_p)) \\ &\times\exp\left[i\int_{t_i}^{t_f}dt\Big(i\sum_a\xi_a(t)\dot\psi^a(t)-H(\psi(t),\xi(t))\Big)\right]. \quad (6.72)\end{aligned}$$

Note that if the operators $O$ are fermionic, the antisymmetry in the time-ordered product will automatically work out due to the anticommuting nature of the Grassmann variables. As in the bosonic case we can compute vacuum expectation values using an $i\epsilon$ prescription:

$$\langle\Omega|TO_1(\Psi(\bar t_1),\Xi(\bar t_1))\dots O_p(\Psi(\bar t_p),\Xi(\bar t_p))|\Omega\rangle = \frac{\int\mathcal{D}\psi|_0^0\mathcal{D}\xi\,O_1(\psi(\bar t_1),\xi(\bar t_1))\dots O_p(\psi(\bar t_p),\xi(\bar t_p))e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)}dtL(\psi(t),\xi(t))}}{\int\mathcal{D}\psi|_0^0\mathcal{D}\xi\,e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)}dtL(\psi(t),\xi(t))}}, \quad (6.73)$$

where you can see that the overall sign in the measure indeed drops out. Note that the quantity on the left-hand side is completely physical, not requiring any discussion of the Grassmann supermodule $\widetilde{H}$, but we would have had trouble coming up with the quantity on the right-hand side without it.

For our Dirac theory it is useful to write out the Lagrangian more explicitly. Looking at (6.72) and recalling that $\Xi = \Psi^\dagger$, we have

$$\mathcal{L} = i\psi^\dagger\dot\psi - i\psi^\dagger\gamma^0(\gamma^i\partial_i+m)\psi = -i\overline{\psi}(\partial\!\!\!/+m)\psi. \quad (6.74)$$

Thus we see at last that to have a "classical" Lagrangian for a Dirac field, the field variables must be Grassmann-valued. Note also that when we differentiate this Lagrangian with respect to $\dot\psi$ to extract the canonical momenta $\pi = i\xi$ we should take the derivative from the right (see again (6.72)), just as we did when we canonically quantized the Dirac Lagrangian.

## 6.5 Gaussian integrals

Following our experience with bosonic path integrals, let's now learn how to evaluate fermionic Gaussian integrals. The first kind of path integral we want to evaluate, analogous to the Dirac partition function, is

$$Z[A] = \int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, e^{-\xi^T A \psi} \tag{6.75}$$

where $A$ is an $M \times M$ matrix. Introducing a change of variables

$$\psi'^a = \sum_b A^a{}_b \psi^b, \tag{6.76}$$

and using the measure transformation (6.55), we have

$$\begin{aligned} Z[A] &= \det(A) \int d\xi_1 d\psi'^1 \dots d\xi_M d\psi'^M \, e^{-\xi^T \psi'} \\ &= \frac{1}{M!} \det(A) \int d\xi_1 d\psi'^1 \dots d\xi_M d\psi'^M \, (-\xi^T \psi')^M \\ &= \det(A) \int d\psi'^1 d\xi_1 \dots d\psi'^M d\xi_M \, \xi_M \psi^M \dots \xi_1 \psi^1 \\ &= \det(A). \end{aligned} \tag{6.77}$$

We can also introduce linear terms:

$$Z[A, b, \hat{b}] = \int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, e^{-\xi^T A \psi + b^T \psi + \hat{b}^T \xi}, \tag{6.78}$$

where $b$ and $\hat{b}$ are vectors whose components are Grassmann variables. Introducing the change of variables

$$\begin{aligned} \psi &= \psi' - A^{-1} \hat{b} \\ \xi &= \xi' + (A^{-1})^T b, \end{aligned} \tag{6.79}$$

this simplifies to

$$\begin{aligned} Z[A, b, \hat{b}] &= \int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, e^{-\xi'^T A \psi' - b^T A^{-1} \hat{b}} \\ &= \det(A) e^{-b^T A^{-1} \hat{b}}. \end{aligned} \tag{6.80}$$

By using this expression we can compute correlation functions in the Gaussian ensemble:

$$\frac{\int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, \psi^{a_m} \dots \psi^{a_1} \xi_{b_1} \dots \xi_{b_m} e^{-\xi^T A \psi}}{\int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, e^{-\xi^T A \psi}} = \frac{\partial}{\partial b_{a_m}} \dots \frac{\partial}{\partial b_{a_1}} \frac{\partial}{\partial \hat{b}^{b_1}} \dots \frac{\partial}{\partial \hat{b}^{b_m}} e^{-b^T A^{-1} \hat{b}} \bigg|_{b, \hat{b} = 0}. \tag{6.81}$$

In particular if we apply this to the two-point function we get

$$\frac{\int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, \psi^a \xi_b e^{-\xi^T A \psi}}{\int d\xi_1 d\psi^1 \dots d\xi_M d\psi^M \, e^{-\xi^T A \psi}} = (A^{-1})^a{}_b. \tag{6.82}$$

More generally we have

$$\frac{\partial}{\partial b_{a_m}} \dots \frac{\partial}{\partial b_{a_1}} \frac{\partial}{\partial \hat{b}^{b_1}} \dots \frac{\partial}{\partial \hat{b}^{b_m}} e^{-b^T A^{-1} \hat{b}} = e^{-b^T A^{-1} \hat{b}} \left( -(A^{-1} \hat{b})^{a_m} + \frac{\partial}{\partial b_{a_m}} \right) \dots \left( -(A^{-1} \hat{b})^{a_1} + \frac{\partial}{\partial b_{a_1}} \right) \tag{6.83}$$

$$\times \left( (b^T A^{-1})_{b_1} + \frac{\partial}{\partial \hat{b}^{b_1}} \right) \left( (b^T A^{-1})_{b_m} + \frac{\partial}{\partial \hat{b}^{b_m}} \right) \times 1, \tag{6.84}$$

so to get a contribution which is nonvanishing when $b = \hat{b} = 0$ we should only have $b$-derivative terms for the first $m$ factors and only have $b^T A^{-1}$ terms for the second $m$ factors. We have a choice however of which derivative acts on which $b^T A^{-1}$, so (as in the bosonic case) we will have a sum over pairings of the $a_i$ with the $b_i$. Due to the anticommuting nature of the $b$'s and $\hat{b}$'s however, the pairings will now be weighted by sign. One nice way to write the answer is as

$$\frac{\partial}{\partial b_{a_m}} \cdots \frac{\partial}{\partial b_{a_1}} \frac{\partial}{\partial \hat{b}^{b_1}} \cdots \frac{\partial}{\partial \hat{b}^{b_m}} e^{-b^T A^{-1} \hat{b}} \big|_{b,\hat{b}=0} = \sum_{\pi \in S_m} (-1)^{p(\pi)} (A^{-1})^{a_1}{}_{b_{\pi(1)}} \cdots (A^{-1})^{a_1}{}_{b_{\pi(m)}}. \tag{6.85}$$

As an illustration of these results we can use the path integral to easily compute the Feynman propagator of a Dirac fermion. Recalling that $\Xi = \Psi^\dagger$, and taking into account the $i\epsilon$ prescription, from the Dirac Lagrangian (6.74) we have

$$A = -\gamma^0 \left( \gamma^0 (1 + i\epsilon) \partial_\tau + \gamma^i \partial_i + m \right). \tag{6.86}$$

We can invert this to find the Feynman propagator by solving the equation

$$\gamma^0 \left( \gamma^0 (1 + i\epsilon) \partial_\tau + \gamma^i \partial_i + m \right) S_F(x) \gamma^0 = \delta^d(x), \tag{6.87}$$

where I've written $A^{-1} = -S_F \gamma^0$ since we defined $S_F(x) = \langle \Omega | T \Psi(x) \overline{\Psi}(0) | \Omega \rangle$. Writing

$$S_F(x) = \int \frac{d^d p}{(2\pi)^d} \hat{S}_F(p) e^{ip \cdot x}, \tag{6.88}$$

the equation we need to solve is

$$\left( i\gamma^0 (1 + i\epsilon) p_0 + i\gamma^i p_i + m \right) \hat{S}_F(p) = -1. \tag{6.89}$$

This is solved by

$$\hat{S}_F(p) = \frac{i(\not{p} + im)}{p^2 + m^2 - i\epsilon}, \tag{6.90}$$

which is precisely the covariant momentum-space propagator we found in the previous section but now with far less work!

We can also evaluate Majorana-type Gaussian integrals. Defining[19]

$$Z[A] = \int d\psi^1 \dots d\psi^{2N} e^{\frac{1}{2} \psi^T A \psi} \tag{6.91}$$

with $A$ antisymmetric, we have

$$Z[A] = \mathrm{Pf}(A) \tag{6.92}$$

where $\mathrm{Pf}(A) = \sqrt{\det A}$ is the **Pfaffian** of $A$. There is a sign ambiguity in the square root: it is fixed by noting that any antisymmetric matrix can be written as $A = U^T \Sigma U$, with

$$\Sigma = \begin{pmatrix} 0 & a_1 & \dots & 0 \\ -a_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \tag{6.93}$$

and $U$ unitary, in terms of which the Pfaffian is

$$\mathrm{Pf}(A) = \det(U) a_1 \dots a_N. \tag{6.94}$$

For an odd number of fermions we have $Z[A] = 0$, which is consistent with the fact that we only have a Hilbert space interpretation with an even number of real fermions. Introducing a linear term, we get

$$Z[A, b] = \int d\psi^1 \dots d\psi^{2N} e^{\frac{1}{2} \psi^T A \psi + b^T \psi} = \mathrm{Pf}(A) e^{\frac{1}{2} b^T A^{-1} b}. \tag{6.95}$$

[19] We restrict to an even number of real fermions, in keeping with our principle that an odd number doesn't make sense quantum mechanically. If we nonetheless evaluate this integral with an odd number of fermions we'll just get zero since the determinant of an odd-dimensional antisymmetric matrix always vanishes ($\det A = \det A^T = \det(-A) = -\det A$).

## 6.6 Euclidean path integral for fermions

As in the bosonic case, it is also interesting to consider a Euclidean version of the fermion path integral. We define Euclidean Heisenberg operators

$$\begin{aligned}\Psi^a(\tau) &= e^{H\tau}\Psi^a(0)e^{-H\tau}\\ \Xi_a(\tau) &= e^{H\tau}\Xi_a(0)e^{-H\tau}\end{aligned} \tag{6.96}$$

and then proceed in the same way as before. The only difference is that everywhere we had $i\epsilon H$ we should now write $\epsilon H$, so our final expression for a transition amplitude with operator insertions is now

$$\begin{aligned}\widetilde{\langle\psi_f,\tau_f|}TO_1(\Psi(\overline{\tau}_1),\Xi(\overline{\tau}_1))\dots O_p(\Psi(\overline{\tau}_p),\Xi(\overline{\tau}_p))|\psi_i,\tau_i\rangle = \int \mathcal{D}\psi\Big|_{\psi_i}^{\psi_f}\mathcal{D}\xi\, O_1(\psi(\overline{\tau}_1),\xi(\overline{\tau}_1))\dots O_p(\psi(\overline{\tau}_p),\xi(\overline{\tau}_p))\\ \times\exp\left[-\int_{\tau_i}^{\tau_f}dt\Big(\sum_a \xi_a(\tau)\dot{\psi}^a(\tau)+H(\psi(\tau),\xi(\tau))\Big)\right].\end{aligned} \tag{6.97}$$

The quantity appearing in the exponent is the Euclidean fermion action

$$S_E=\int_{\tau_i}^{\tau_f}dt\big(\sum_a \xi_a(\tau)\dot{\psi}^a(\tau)+H(\psi(\tau),\xi(\tau))\Big). \tag{6.98}$$

There is however one novelty in the fermionic Euclidean path integral related to the thermal trace. Recall that in the bosonic case we could compute the thermal trace as

$$\int d^Mq\langle q|e^{-\beta H}|q\rangle=\int d^Mq\int\mathcal{D}q|_q^q\mathcal{D}p\,e^{-S_E}, \tag{6.99}$$

with

$$S_E=\int_0^\beta d\tau\left(i\sum_a p_a(\tau)\dot{q}^a(\tau)-H(q(\tau),p(\tau))\right) \tag{6.100}$$

being the bosonic Euclidean action. In other words the thermal partition function is just the Euclidean partition function with periodic boundary conditions in Euclidean time, $\tau\sim\tau+\beta$. What about in the fermionic case? Here we are in for a surprise. Let's try to compute the trace of an arbitrary operator $O$ using fermionic eigenstates:

$$\mathrm{Tr}(O)=_?\int d\psi\langle\widetilde{\psi}|O|\psi\rangle. \tag{6.101}$$

This equation however has an immediate problem: if we compute the "trace" of the identity then we just get

$$\int d\psi\langle\widetilde{\psi}|\psi\rangle=\delta^M(0)=0. \tag{6.102}$$

How can the trace of the identity be zero? Of course it can't, to see what is really going on let's compute the right-hand side of (6.101) in the one-fermion case with the standard inner product (so that $\langle\widetilde{0}|=\langle 0|$ and $\langle\widetilde{1}|=\langle 1|$):

$$\begin{aligned}\int d\psi\langle\widetilde{\psi}|O|\psi\rangle &= \int d\psi\big(-\langle 1|+\langle 0|\psi\big)O\big(|0\rangle-\psi|1\rangle\big)\\ &= \int d\psi\,\psi\left(\langle 0|O|0\rangle-\langle 1|O|1\rangle\right)\\ &= \mathrm{Tr}\left((-1)^FO\right).\end{aligned} \tag{6.103}$$

On the other hand we have

$$\int d\psi \langle \widetilde{\psi}|O|-\psi\rangle = \int d\psi\, \psi \left(\langle 0|O|0\rangle - \langle 1|O|1\rangle\right) = \mathrm{Tr}(O). \tag{6.104}$$

Thus we see that to compute a genuine trace we need to use *antiperiodic* boundary conditions for fermions, while if we use periodic boundary conditions then we are computing the trace with a factor of $(-1)^F$ inserted. This statement is true in general, and so in particular to compute a thermal partition function for fermions we need to use antiperiodic boundary conditions around the thermal circle.

**Problems:**

1. Confirm equation (6.35) starting from the definition (6.34).
2. Confirm equation (6.37). Hint: only terms which are proportional to $\langle \widetilde{0}|\Psi^M \dots \Psi^1 \Xi_1 \dots \Xi_M|\widetilde{0}\rangle = 1$ contribute, you just need to be careful about the signs.
3. Confirm equation (6.57). Hint: you will make your life a lot easier if you first note that $f(\theta'_1, \dots, \theta'_N) = f(\theta_1 + (\theta'_1 - \theta_1), \dots, \theta_N + (\theta'_N - \theta_N))$. Now Taylor expand $f$ in the quantities $(\theta'_n - \theta_n)$. Do any terms beyond the lowest order contribute to the integral?
4. Show for $M$ fermions that $\int d^M\psi \langle \widetilde{\psi}|O|\psi\rangle$ is proportional to $\mathrm{Tr}\left((-1)^F O\right)$ and $\int d^M\psi \langle \widetilde{\psi}|O|-\psi\rangle$ is proportional to $\mathrm{Tr}\,(O)$.
5. (Extra credit x2) In this problem we will study $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{T}$ symmetries for Dirac fermions.

   Let's first take $\mathcal{R}$ to be the spacetime transformation

   $$\begin{aligned} x^{0\prime} &= x^0 \\ x^{1\prime} &= -x^1 \\ x^{i\prime} &= x^i \qquad\qquad (i > 1), \end{aligned} \tag{6.105}$$

   and let's look for a transformation

   $$\Psi'(x) = D_{\mathcal{R}}\, \Psi(\mathcal{R}x) \tag{6.106}$$

   which leaves the Dirac Lagrangian invariant. Here $D_{\mathcal{R}}$ is some matrix which it is our job to find. Show that the invariance of the Lagrangian (with $m > 0$) is equivalent to

   $$\begin{aligned} D^\dagger_{\mathcal{R}} D_{\mathcal{R}} &= I \\ D^\dagger_{\mathcal{R}} \gamma^0 D_R &= \gamma^0 \\ D^\dagger_{\mathcal{R}} \gamma^0\gamma^1 D_{\mathcal{R}} &= -\gamma^0\gamma^1 \\ D^\dagger_{\mathcal{R}} \gamma^0\gamma^i D_{\mathcal{R}} &= \gamma^0\gamma^i \qquad\qquad (i > 1), \end{aligned} \tag{6.107}$$

   and also that these equations hold for $d$ even provided that we take $D_{\mathcal{R}} = e^{i\theta_{\mathcal{R}}}\gamma\gamma^1$ with $\theta_{\mathcal{R}}$ an arbitrary phase. Argue that for odd $d$ there is no matrix $D_{\mathcal{R}}$ which works.

   Now let's define $\mathcal{C}$ to act as

   $$\Psi' = D_{\mathcal{C}}\, \Psi^*. \tag{6.108}$$

   Show that invariance of the Lagrangian requires

   $$\begin{aligned} D^\dagger_{\mathcal{C}} \gamma^0 D_{\mathcal{C}} &= -\gamma^{0T} = \gamma^{0*} \\ D^\dagger_{\mathcal{C}} \gamma^0\gamma^\mu D_{\mathcal{C}} &= \gamma^{\mu T}\gamma^{0T}, \end{aligned} \tag{6.109}$$

and also show that for $d = 0, 2, 3 \bmod 4$ we can solve these by

$$D_{\mathcal{C}} = \begin{cases} B_2^\dagger & d = 0 \bmod 4 \\ B_1^\dagger & d = 2 \bmod 4 \\ B^\dagger & d = 3 \bmod 4 \end{cases}. \tag{6.110}$$

Also show that there is no solution for $d = 1 \bmod 4$.

Finally we'll define the spacetime transformation $\mathcal{T}$ to act as

$$\begin{aligned} x^{0\prime} &= -x^0 \\ x^{i\prime} &= x^i, \end{aligned} \tag{6.111}$$

and look for a field transformation

$$\Psi'(x) = D_{\mathcal{T}} \Psi(\mathcal{T}x) \tag{6.112}$$

which leaves the Dirac Lagrangian invariant. Noting that this transformation must be induced by an antiunitary operator, which therefore will take the complex conjugate of the $\gamma$-matrices and the $i$ appearing in the Lagrangian, show that the invariance of the Lagrangian is equivalent to

$$\begin{aligned} D_{\mathcal{T}}^\dagger D_{\mathcal{T}} &= I \\ D_{\mathcal{T}}^\dagger \gamma^{0*} D_{\mathcal{T}} &= -\gamma^0 \\ D_{\mathcal{T}}^\dagger \gamma^{0*} \Gamma^{i*} D_{\mathcal{T}} &= -\gamma^0 \gamma^i. \end{aligned} \tag{6.113}$$

Moreover show that for $d = 0, 1, 2 \bmod 4$ these conditions are satisfied for

$$D_{\mathcal{T}} = \begin{cases} e^{i\theta_{\mathcal{T}}} \Gamma^0 B_1 & d = 0 \bmod 4 \\ e^{i\theta_{\mathcal{T}}} \Gamma^0 B & d = 1 \bmod 4 \\ e^{i\theta_{\mathcal{T}}} \Gamma^0 B_2 & d = 2 \bmod 4 \end{cases}, \tag{6.114}$$

and also that there is no solution for $d = 3 \bmod 4$.

Finally show that the combination of these three transformations is equivalent up to a phase to the $\mathcal{CRT}$ transformation we defined in the previous section.

# 7 Perturbative calculations in Yukawa theory

In this section we will compute some perturbative correlation functions and S-matrix elements in the interacting **Yukawa theory**[20]

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m_\phi^2}{2}\phi^2 - i\overline{\psi}\left(\not{\partial} + m_\psi\right)\psi - ig\phi\overline{\psi}\psi. \tag{7.1}$$

As we mentioned before this theory is a model of a nucleon (i.e. a proton or a neutron) $\psi$ interacting with a pion $\phi$. As also mentioned before, a better model of this is what I'll call **true Yukawa theory**:

$$\mathcal{L}_{true} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m_\phi^2}{2}\phi^2 - i\overline{\psi}\left(\not{\partial} + m_\psi\right)\psi - g\phi\overline{\psi}\gamma\psi. \tag{7.2}$$

The difference between the two has to do with the spatial reflection symmetry $\mathcal{R}$. You saw in the homework last week that this acts on a Dirac fermion as

$$\psi'(x) = e^{i\theta_{\mathcal{R}}} \gamma\gamma^1 \psi(\mathcal{R}x), \tag{7.3}$$

[20] We should really write $m_{\phi,0}$, $m_{\psi,0}$, and $g_0$ to emphasize that these are "bare" couplings, but for now I'll suppress this to make the equations look nicer.

which acts on $\overline{\psi}\psi$ and $\overline{\psi}\gamma\psi$ as

$$\begin{aligned}\left(\overline{\psi}\psi\right)' &= \psi^\dagger\gamma^{1\dagger}\gamma^\dagger\gamma^0\gamma\gamma^1\psi = \overline{\psi}\gamma^0\gamma^{1\dagger}\gamma^0\gamma\gamma\gamma^1\psi = \overline{\psi}\psi \\ \left(\overline{\psi}\gamma\psi\right)' &= \psi^\dagger\gamma^{1\dagger}\gamma^\dagger\gamma^0\gamma^1\psi = \overline{\psi}\gamma^0\gamma^{1\dagger}\gamma^0\gamma\gamma^1\psi = -\overline{\psi}\gamma\psi.\end{aligned} \tag{7.4}$$

In other words $\overline{\psi}\psi$ transforms as a scalar under spatial reflection while $\overline{\psi}\gamma\psi$ transforms as a pseudoscalar. Thus in order for spatial reflection to be a symmetry, in the theory with Lagrangian (7.1) the $\mathcal{R}$-transformation of $\phi$ should be the scalar transformation

$$\phi'(x) = \phi(\mathcal{R}x), \tag{7.5}$$

while in the theory with Lagrangian (7.2) $\phi$ must be a pseudoscalar:

$$\phi'(x) = -\phi(\mathcal{R}x). \tag{7.6}$$

In the real world pions are pseudoscalars, so it is the second choice which is correct. Nonetheless the true Yukawa theory is a bit more annoying to calculate in, and also has Feynman rules which are a bit less similar to quantum electrodynamics, so in this section we'll mostly stick with the Lagrangian (7.1).

Before studying Yukawa theory in detail I must first confess that there is a sense in which it is rather unnatural. A potential

$$V(\phi) = g_1\phi + \frac{m_\phi^2}{2}\phi^2 + \frac{g_3}{6}\phi^3 + \frac{g_4}{4!}\phi^4 \tag{7.7}$$

for $\phi$ would be perfectly allowed by all of the symmetries of the Lagrangian (7.1) (the Yukawa term breaks the $\phi' = -\phi$ symmetry of the free scalar theory), and each of these terms is relevant or marginal for $d \leq 4$. Why then have we only included the $\phi^2$ term? In fact the other terms are generated by loop diagrams via renormalization group flow even if we set them to zero in the bare Lagrangian, and so we haven't really gotten rid of them anyways. The right way to deal with this systematically is to just include the full potential in the Lagrangian and allow its couplings to be renormalized as usual. Our main purpose in studying this theory however is to get some practice with fermions before we move on to studying quantum electrodynamics, which does not have this problem, and so we will take the more lazy approach of simply assuming that the bare $g_1$, $g_3$, and $g_4$ couplings have been tuned to cancel whatever contributions to these terms are generated by loop diagrams. We can thus ignore both these terms in the action and those diagrams.[21]

## 7.1 Correlation functions in Yukawa theory

With these preliminaries out of the way, our starting point for studying Yukawa theory will be our path integral expression

$$\langle\Omega|T\Phi(x_1)\dots\Phi(x_m)\Psi^{a_\ell}(y_\ell)\dots\Psi^{a_1}(y_1)\overline{\Psi}^{b_1}(z_1)\dots\overline{\Psi}^{b_\ell}(z_\ell)|\Omega\rangle = \frac{\int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\phi\dots\phi\psi\dots\psi\overline{\psi}\dots\overline{\psi}e^{iS_\epsilon}}{\int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_\epsilon}}. \tag{7.8}$$

Here I suppressed indices and locations on the right-hand side to save space and $S_\epsilon$ is the full interacting action with the time integral evaluated according to the $i\epsilon$ prescription. As we did last semester, we will approximate the numerator and denominator of this integral perturbatively by Taylor expanding in the interaction term. Let's first study the denominator:

$$\int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_\epsilon} \sim \sum_{n=0}^{\infty}\frac{g^n}{n!}\int\mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\left(\int d^dw\phi\overline{\psi}\psi(w)\right)^n e^{iS_{\epsilon,0}}, \tag{7.9}$$

where

$$S_{\epsilon,0} = \int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)}dt\int d^{d-1}x\left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m_\phi^2}{2}\phi^2 - i\overline{\psi}\left(\not{\partial} + m\right)\psi\right) \tag{7.10}$$

[21] The situation is somewhat better in the true Yukawa theory (7.2), as spatial reflection symmetry now rules out the linear and cubic terms in the potential since $\phi$ is a pseudoscalar. The quartic term however still must be included.

is the free action. I've written $\sim$ to indicate that the perturbation series is divergent and only gives an asymptotic approximation to the exact answer, as we discussed last semester. We saw in the previous section that moments in the Gaussian distribution for a fermionic theory are computed by

$$\frac{1}{\det A}\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\,\psi^{a_m}\dots\psi^{a_1}\overline{\psi}^{b_1}\dots\overline{\psi}^{b_m}e^{-\overline{\psi}^T A\psi} = \sum_{\pi\in S_m}(-1)^{p(\pi)}(A^{-1})^{a_1}{}_{b_{\pi(1)}}\dots(A^{-1})^{a_1}{}_{b_{\pi(m)}}, \tag{7.11}$$

where I've taken the liberty of changing variables from $\xi$ to $\overline{\psi}$ so $A^{-1}$ is now just the spinor Feynman propagator $S_F^{ab}$. Similarly we saw last semester that for a free scalar field we have

$$\sqrt{\det\left(\frac{A_\phi}{2\pi}\right)}\int \mathcal{D}\phi\,\phi_{a_1}\dots\phi_{a_m}\,e^{-\frac{1}{2}\phi^T A_\phi\phi} = \sum_{P_m}\prod_{(j,k)\in P}(A_\phi^{-1})_{a_j a_k}, \tag{7.12}$$

where $P_m$ is the set of unordered pairings of $m$ objects and $A_\phi^{-1}$ is the scalar Feynman propagator $G_F$. Note that the bosonic expression has no sign factor, while the fermion expression does. For expectation values that involve both bosons and fermions we simply have the product of these two expressions, switching to field theory notation we have:

$$\begin{aligned}&\frac{1}{\det A}\sqrt{\det\left(\frac{A_\phi}{2\pi}\right)}\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\,\phi(x_1)\dots\phi(x_m)\psi^{a_\ell}(y_\ell)\dots\psi^{a_1}(y_1)\overline{\psi}^{b_1}(z_1)\dots\overline{\psi}^{b_\ell}(z_\ell)e^{iS_{\epsilon,0}}\\ &=\sum_{P_m}\sum_{\pi\in S_\ell}(-1)^{p(\pi)}\left(\prod_{p=1}^{\ell}S_F^{a_p b_{\pi(p)}}(y_p-z_{\pi(p)})\right)\left(\prod_{(j,k)\in P_m}G_F(x_j-x_k)\right).\end{aligned} \tag{7.13}$$

As in our discussion of the free scalar theory, we can interpret the terms in this sum using Feynman diagrams. We now have two kinds of lines: directed lines for spinor propagators, which we'll draw so that the arrow points from $\overline{\Psi}$ to $\Psi$, and undirected lines for scalar propagators. For example in the free fermion+scalar theory we have the six-point function

$$\langle T\phi(x_1)\phi(x_2)\Psi^{a_2}(y_2)\Psi^{a_1}(y_1)\overline{\Psi}^{b_1}(z_1)\overline{\Psi}^{b_2}(z_2)\rangle = G_F(x_2-x_1)\left[S_F^{a_1 b_1}(y_1-z_1)S_F^{a_2 b_2}(y_2-z_2) - S_F^{a_1 b_2}(y_1-z_2)S_F^{a_2 b_1}(y_2-z_1)\right], \tag{7.14}$$

which comes from the two Feynman diagrams shown in figure 4. In working out the signs such diagrams it is convenient to use something called Wick contraction notation, which indicate how the operators in the correlation function are paired. For these two diagrams the contractions are

$$\langle\phi\phi\psi\psi\overline{\psi}\,\overline{\psi}\rangle \tag{7.15}$$

and

$$\langle\phi\phi\psi\psi\overline{\psi}\,\overline{\psi}\rangle, \tag{7.16}$$

where you can see that to work out the first contraction no fermion exchanges are needed, while to work out the second one fermion exchange is needed so there is a minus sign.

We can now evaluate each term on the right-hand side of equation (7.9) using Feynman diagrams. Each interaction vertex contributes a power of $g\phi\overline{\psi}\psi$, and then we sum over pairings and permutations and integrate over the interaction vertex locations. In principle that is all there is to it, but we can streamline things a bit by thinking some more about the structure of the diagrams we get. First a pleasant surprise: the Yukawa interaction vertex has no symmetry under permuting the incoming propagators, so it is almost always the case that each permutation of the interaction vertices in a Feynman diagram leads to a distinct choice of a pairing $P_m$ and a permutation $\pi$. In other words in Yukawa theory there are almost never symmetry factors! The only exceptions are disconnected diagrams and vacuum bubble diagrams with no external legs, and as we will see we rarely actually need to compute these since they cancel in computations of connected

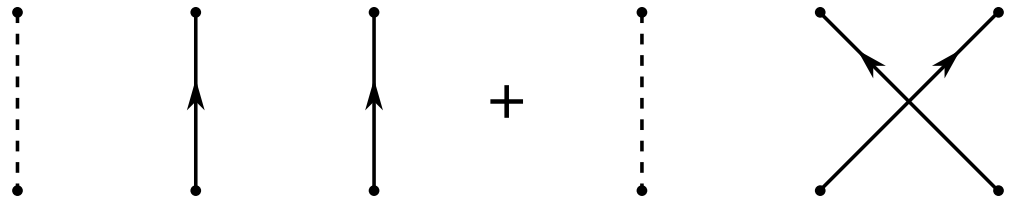

Figure 4: The two free contributions to a six point function $\langle T\Phi\Phi\Psi\Psi\overline{\Psi}\overline{\Psi}\rangle$ in Yukawa theory. The second diagram contributes with a minus sign since we need to exchange two fermions before contracting them.

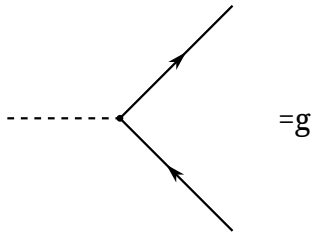


Figure 5: The interaction vertex for Yukawa theory.

correlation functions. Feel free to take a moment to go and celebrate. On the other hand we now need to work out the sign of the permutation $\pi$ for each diagram by permuting the fermions among the interaction vertices until each $\psi$ is directly to the left of the $\overline{\psi}$ it will pair up with. Moreover each interaction vertex has one fermion line going in and one fermion line going out (see figure 5), so all the fermion propagators will be combined together into lines which either go through the diagram from an external $\psi$ to an external $\overline{\psi}$ (neither of which we have for the denominator (7.9)), or else close off into loops. The structure of the Yukawa interaction is such as we go along one of the lines, we simply multiply the spinor propagator matrices $S_F^{ab}$ together from right to left as we go in the direction of the arrow. In particular for the denominator (7.9) we only have loops, and in that case the overall sign is easy to work out: each fermion loop introduces a term of the form

$$\overline{\psi}\psi(x_n)\dots\overline{\psi}\psi(x_1), \tag{7.17}$$

where the positions are labeled in the order they are encountered going around the loop. Each $\psi$ is thus contracted with the $\overline{\psi}$ directly to its right, contributing no sign, except for the first $\overline{\psi}$ which must be exchanged with the last $\psi$ before being contracted. Therefore each fermion loop simply contributes a minus sign to the diagram. We thus arrive at a general formula for the sum of vacuum bubble diagrams:

$$\frac{1}{\det A}\sqrt{\det\left(\frac{A_\phi}{2\pi}\right)}\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_\epsilon} \sim \sum_D g^{n_D}\frac{(-1)^{p_D}}{s_D}\int dx_1\dots dx_{n_D}\left(\prod_{(i,j)\in L_D^\phi} G_F(x_i-x_j)\right)\prod_{L_D^\psi}\left(\prod_{(i,j)\in L_D^\psi} S_F(x_i-x_j)\right). \tag{7.18}$$

Here $n_D$ is the number of vertices in the diagram, $(-1)^{p_D}$ is the overall sign obtained by exchanging the fermions before you contract neighbors (in this case $p_D$ is just the number of fermion loops), $s_D$ is the symmetry factor, $L_D^\phi$ is the set of (unoriented) scalar links in the diagram, $L_D^\psi$ is the set of fermion lines through the diagram, and the product of fermion propagators is taken as matrix multiplication from right to left as you go along each fermion line. As in the scalar case we can simplify this formula by noting that the sum of all diagrams is the exponential of the sum of connected diagrams:

$$\frac{1}{\det A}\sqrt{\det\left(\frac{A_\phi}{2\pi}\right)}\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{iS_\epsilon} \sim \exp\left[\sum_C g^{n_C}\frac{(-1)^{p_C}}{s_C}\int dx_1\dots dx_{n_C}\left(\prod_{(i,j)\in L_C^\phi} G_F(x_i-x_j)\right)\prod_{L_C^\psi}\left(\prod_{(i,j)\in L_C^\psi} S_F(x_i-x_j)\right)\right]. \tag{7.19}$$

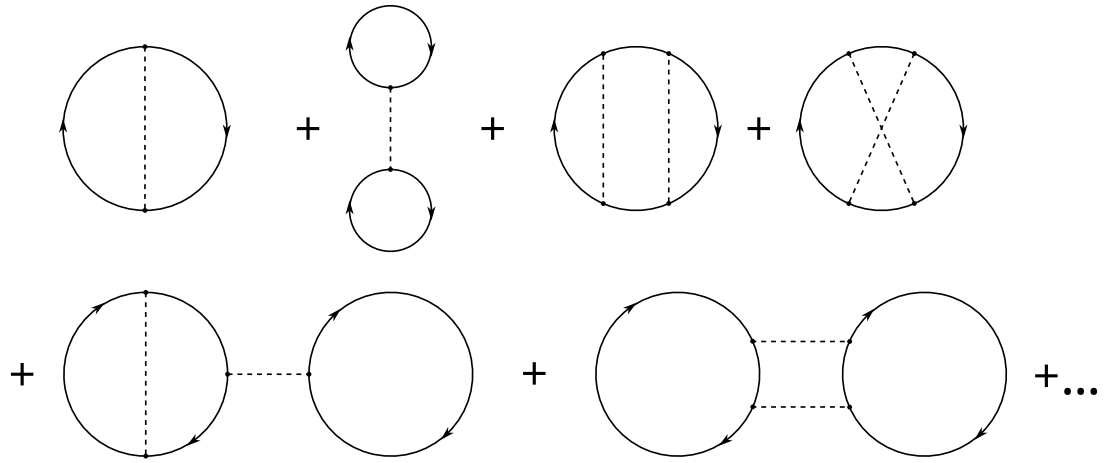

Figure 6: Leading vacuum bubble diagrams for Yukawa theory. The second and fourth diagrams involve "tadpoles" where a single scalar propagator ends on a subdiagram which is not connected to anything else.

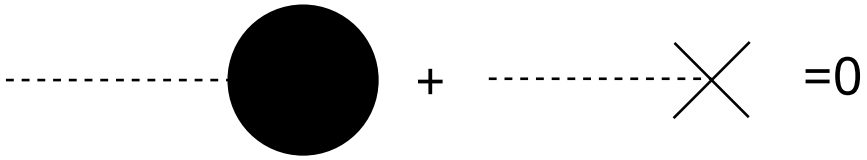


Figure 7: Canceling the sum of tadpole diagrams with a counterterm linear in $\phi$.

The connected diagrams up through three loops are shown in figure 6. The first two diagrams evaluate to

$$\frac{g^2}{2}\int d^dx d^dy G_F(x-y)\Big[\frac{1}{2}\mathrm{Tr}\Big(S_F(x-y)S_F(y-x)\Big)-\mathrm{Tr}\Big(S_F(0)\Big)\mathrm{Tr}\Big(S_F(0)\Big)\Big], \tag{7.20}$$

where the factor of 1/2 is one of the rare symmetry factors in Yukawa theory that arise when you have no external legs. You will work out the next four diagrams on the homework.

These diagrams already illustrate the fact that loops in Yukawa theory generate a linear potential term $g_1\phi$ even if we set it to zero in the bare action. This is because the Yukawa theory allows a scalar propagator to end on some subdiagram that isn't attached to anything else, for example in the second and fourth diagrams in figure 6. Such contributions to a diagram are called "tadpoles". If we add a bare $g_1\phi$ term to the action, we can tune it to cancel all tadpole contributions (see figure 7). This cancellation has nice physical interpretation: the linear term generated in the potential causes the expectation value of $\phi$ to shift away from zero, so we need to define a new field

$$\Phi'(x) = \Phi(x) - \langle\Omega|\Phi|\Omega\rangle, \tag{7.21}$$

which by construction has zero expectation value. The Feynman diagrams for $\Phi'$ are the same as those for $\Phi$, except that we no longer include tadpoles. From now on we will not draw any diagrams with tadpoles.

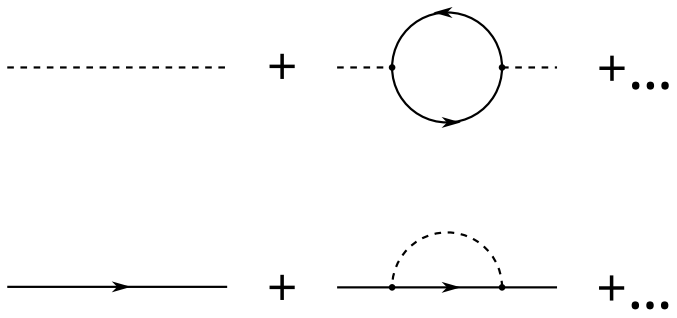

Figure 8: Leading contributions to the two-point function for a scalar and a spinor in Yukawa theory.

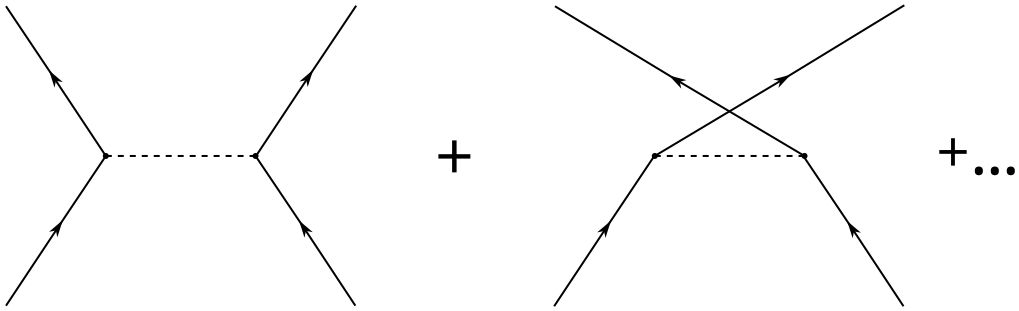

Figure 9: Leading contributions to the connected fermion four-point function in Yukawa theory.

Let's now consider the numerator in equation (7.8), which we can write as

$$\begin{aligned}&\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\,\phi(x_1)\dots\phi(x_m)\psi^{a_\ell}(y_\ell)\dots\psi^{a_1}(y_1)\overline{\psi}^{b_1}(z_1)\dots\overline{\psi}^{b_\ell}(z_\ell)e^{iS_\epsilon}\\&=\sum_{n=0}^{\infty}\frac{g^n}{n!}\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\overline{\psi}\,\phi(x_1)\dots\phi(x_m)\psi^{a_\ell}(y_\ell)\dots\psi^{a_1}(y_1)\overline{\psi}^{b_1}(z_1)\dots\overline{\psi}^{b_\ell}(z_\ell)\left(\int d^dw\phi\overline{\psi}\psi(w)\right)^n e^{iS_{\epsilon,0}}.\end{aligned} \tag{7.22}$$

The Feynman diagrams for this expression work in the same way as for the denominator, except that now there are external legs that the fermion lines can begin and end on. The disconnected vacuum bubbles exponentiate and cancel with the denominator in (7.8), so relabeling the external positions to uniformize the notation we have

$$\begin{aligned}&\langle\Omega|T\Phi(x_1)\dots\Phi(x_m)\Psi^{a_\ell}(x_{m+\ell})\dots\Psi^{a_1}(x_{m+1})\overline{\Psi}^{b_1}(x_{m+\ell+1})\dots\overline{\Psi}^{b_\ell}(x_{m+2\ell})|\Omega\rangle\\&=\sum_{\hat{C}}g^{n_{\hat{C}}}(-1)^{p_{\hat{C}}}\int dx_{m+2\ell+1}\dots dx_{m+2\ell+n_{\hat{C}}}\left(\prod_{(i,j)\in L^{\phi}_{\hat{C}}}G_F(x_i-x_j)\right)\prod_{L^{\psi}_{\hat{C}}}\left(\prod_{(i,j)\in L^{\psi}_{\hat{C}}}S_F(x_i-x_j)\right).\end{aligned} \tag{7.23}$$

Here $\hat{C}$ is the set of tadpole-free diagrams where each interaction vertex is part of a connected component which contains at least two external legs. We no longer need to write the symmetry factor since connected diagrams with external legs don't have them. The sign $(-1)^{P_{\hat{C}}}$ is worked out by exchanging the internal and external fermions so that each $\psi$ is to the left of the $\overline{\psi}$ it is contracted with. Fermion loops still each just contribute $-1$, so we only really need to work this out for fermion lines which involve the external legs.

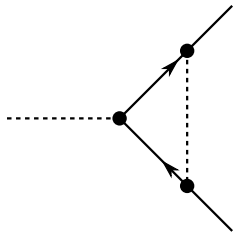

Figure 10: A one-loop correction to the Yukawa interaction vertex.

We can simplify this expression a bit more by instead computing the *connected* correlation function, also known as the cumulant. For example

$$\begin{aligned}\langle T\Psi^{a_1}(x_2)\Psi^{a_1}(x_1)\overline{\Psi}^{b_1}(y_1)\overline{\Psi}^{b_2}(y_2)\rangle_c =&\langle T\Psi^{a_1}(x_2)\Psi^{a_1}(x_1)\overline{\Psi}^{b_1}(y_1)\overline{\Psi}^{b_2}(y_2)\rangle - \langle T\Psi^{a_1}(x_1)\overline{\Psi}^{b_1}(y_1)\rangle\langle T\Psi^{a_2}(x_2)\overline{\Psi}^{b_2}(y_2)\rangle \\ &+ \langle T\Psi^{a_1}(x_1)\overline{\Psi}^{b_2}(y_2)\rangle\langle T\Psi^{a_2}(x_2)\overline{\Psi}^{b_1}(y_1)\rangle. \end{aligned} \tag{7.24}$$

For the connected correlation function we then have

$$\begin{aligned}&\langle\Omega|T\Phi(x_1)\dots\Phi(x_m)\Psi^{a_\ell}(x_{m+\ell})\dots\Psi^{a_1}(x_{m+1})\overline{\Psi}^{b_1}(x_{m+\ell+1})\dots\overline{\Psi}^{b_\ell}(x_{m+2\ell})|\Omega\rangle_c \\ &= \sum_C g^{n_C}(-1)^{p_C}\int dx_{m+2\ell+1}\dots dx_{m+2\ell+n_C}\left(\prod_{(i,j)\in L_C^\phi} G_F(x_i-x_j)\right)\prod_{L_C^\psi}\left(\prod_{(i,j)\in L_C^\psi} S_F(x_i-x_j)\right), \end{aligned} \tag{7.25}$$

where now $C$ is the set of connected tadpole-free diagrams. The tree and one-loop diagrams for the fermion and scalar two-point functions are shown in figure 8, and the tree-level contributions to the fermion four-point function in figure 9. Evaluating the two-point functions through one-loop gives

$$\begin{aligned}\langle\Omega|T\Phi(x_2)\Phi(x_1)|\Omega\rangle &= G_F(x_1-x_1) - g^2\int d^dy d^dz G_F(x_2-y)G_F(z-x_1)\mathrm{Tr}\left(S_F(z-y)S_F(y-z)\right) + O(g^4) \\ \langle\Omega|T\Psi(x_2)\overline{\Psi}(x_1)|\Omega\rangle &= S_F(x_2-x_1) + g^2\int d^dy d^dz G_F(y-z)S_F(x_2-y)S_F(y-z)S_F(z-x_1) + O(g^4), \end{aligned} \tag{7.26}$$

while the two tree-level contributions to the connected fermion four-point function give

$$\begin{aligned}\langle\Omega|\Psi^{a_2}(x_2)\Psi^{a_1}(x_1)\overline{\Psi}^{b_1}(y_1)\overline{\Psi}^{b_2}(y_2)|\Omega\rangle_c =& g^2\int d^dz_1 d^dz_2 G_F(z_2-z_1)S_F^{a_1c}(x_1-z_1)S_F^{cb_1}(z_1-y_1)S_F^{a_2d}(x_2-z_2)S_F^{db_2}(z_2-y_2) \\ &- g^2\int d^dz_1 d^dz_2 G_F(z_2-z_1)S_F^{a_2c}(x_2-z_1)S_F^{cb_1}(z_1-y_1)S_F^{a_1d}(x_1-z_2)S_F^{db_2}(z_2-y_2) \\ &+ O(g^4) \end{aligned} \tag{7.27}$$

At one-loop there are many diagrams contributing to this four-point function. Ten of them involve replacing one of the propagators in the tree-level diagrams with its one-loop correction as in figure 8. Four more arise from replacing one of the interaction vertices with a one-loop correction, as shown in figure 10. Finally there are four "ladder" diagrams shown in figure 11. As you can see the amount of work to compute higher-loop correlation functions grows quite quickly with the number of loops.

## 7.2 Correlation functions in momentum space

We are eventually interested in computing scattering amplitudes in Yukawa theory. Last semester we saw that in quantum field theory we can do this by way of the LSZ reduction formula, which I'll review in the

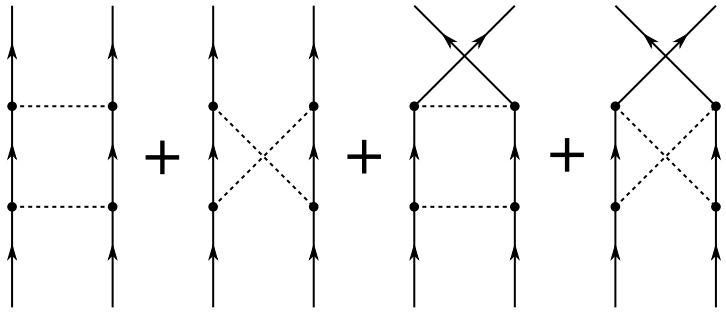

Figure 11: Ladder contributions to the connected four-fermion correlation function.

following subsection, and the key input into the LSZ formula is the Fourier transform of the time-ordered connected correlation functions:

$$\langle\Omega|T\Phi(k_1)\dots\Phi(k_m)\Psi^{a_\ell}(k_{m+\ell})\dots\Psi^{a_1}(k_{m+1})\overline{\Psi}^{b_1}(k_{m+\ell+1})\dots\overline{\Psi}^{b_\ell}(k_{m+2\ell})|\Omega\rangle_c$$
$$= \int d^dx_1\dots d^dx_{m+2\ell}e^{-ik_1\cdot x_1-\dots-ik_{m+2\ell}\cdot x_{m+2\ell}}\langle\Omega|T\Phi(x_1)\dots\Phi(x_m)\Psi^{a_\ell}(x_{m+\ell})\dots\Psi^{a_1}(x_{m+1})\overline{\Psi}^{b_1}(x_{m+\ell+1})\dots\overline{\Psi}^{b_\ell}(x_{m+2\ell})|\Omega\rangle_c. \tag{7.28}$$

In perturbation theory these position integrals, and also the integrals over the interaction vertices, can be evaluated explicitly just as in our scalar theory by using our integral representations

$$G_F(x-y) = \int\frac{d^dp}{(2\pi)^d}\frac{-i}{p^2+m_\phi^2-i\epsilon}e^{ip\cdot(x-y)}$$
$$S_F(x-y) = \int\frac{d^dp}{(2\pi)^d}\frac{i(\not{p}+im)}{p^2+m_\psi^2-i\epsilon}e^{ip\cdot(x-y)} \tag{7.29}$$

for the scalar and vector propagators. As in the scalar case the rules for evaluating a connected Feynman diagram $C$ contributing to this Fourier transform are quite simple to remember:

1. Write a factor of $g^{n_C}$, wnere $n_C$ is the number of interaction vertices in $C$.

2. Draw a momentum direction for each internal and external line of the diagram, with the internal fermion momenta chosen to point in the direction of the arrow on the propagator, the internal scalar momenta chosen to point in arbitrary directions, and the external momenta chosen to point outwards for both scalars and fermions.

3. Label external momenta by the momenta in the Fourier transform, and label the internal momenta imposing momentum conservation at each interaction vertex.

4. Multiply by an overall momentum-conserving $\delta$-function $(2\pi)^d\delta^d(k_1+\dots+k_{m+2\ell})$.

5. Write a momentum-space scalar propagator

$$\hat{G}_F(p) = \frac{-i}{p^2+m_\phi^2-i\epsilon} \tag{7.30}$$

for each scalar line, both internal and external, write a momentum-space fermion propagator

$$\hat{S}_F(p) = \frac{i(\not{p}+im)}{p^2+m_\psi^2-i\epsilon} \tag{7.31}$$

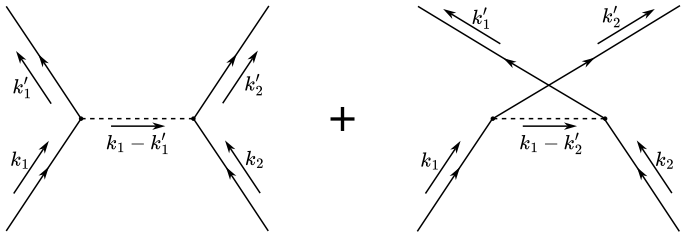


Figure 12: Momentum labels for the tree-level contribution to the fermion four-point function.

for each internal fermion line and each external fermion line whose fermion and momentum arrows are aligned, and write a momentum space fermion propagator

$$\hat{S}_F(-p) = \frac{i(-\not{p} + im)}{p^2 + m_\psi^2 - i\epsilon} \tag{7.32}$$

for each external fermion line whose fermion and momentum arrows are opposite.

6. Fermion propagators are multiplied as matrices along fermion lines just as in the position-space correlator.

7. Work out the overall sign of the diagram by counting fermion exchanges as in the position space correlator.

In the LSZ formula we will want to reverse the signs of the external momenta in the Fourier transform corresponding to incoming particles, in general the rule continues to be that we should write $\hat{S}_F(p)$ for external lines whose momentum and fermion directions are aligned while we should write $\hat{S}_F(-p)$ for external lines whose momentum and fermion directions are opposite.

For example let's apply these rules to the tree-level fermion four-point function from figure 9. Assigning momentum labels as in figure 12, we have

$$\begin{aligned}\langle\Omega|T\Psi^{a_2}(k_2')\Psi^{a_1}(k_1')\overline{\Psi}^{b_1}(-k_1)\overline{\Psi}^{b_2}(-k_2)|\Omega\rangle_c = (2\pi)^d\delta^d(k_1'+k_2'-k_1-k_2)g^2\Big[&\hat{S}_F^{a_1c}(k_1')\hat{S}_F^{cb_1}(k_1)\hat{G}_F(k_1-k_1')\hat{S}_F^{a_2d}(k_2')\hat{S}_F^{db_2}(k_2)\\ &- \hat{S}_F^{a_2c}(k_2')\hat{S}_F^{cb_1}(k_1)\hat{G}_F(k_1-k_2')\hat{S}_F^{a_1d}(k_1')\hat{S}_F^{db_2}(k_2)\Big].\end{aligned} \tag{7.33}$$

## 7.3 Scattering in Yukawa theory

We'll now work out how to use the LSZ reduction formula to compute perturbative scattering amplitudes in Yukawa theory. Let's first recall the general version of this formula:

$$\begin{aligned}\langle\Omega|TO_N'^{a_N}(k_N')\dots O_1'^{a_1}(k_1')O_1^{b_1\dagger}(-k_1)\dots O_M^{b_M\dagger}(-k_M)|\Omega\rangle_c \xrightarrow[\substack{k_i^0\to\omega_{n_i,\vec{k}_i}\\ k_i'^0\to\omega_{n_i',\vec{k}_i'}}]{} &\prod_{j=1}^N\left(Z_{n_j'}\sum_{\sigma_j'}u^{a_j}(\vec{k}_j',\sigma_j',n_j')\times\frac{-i\sqrt{2\omega_{n_j',\vec{k}_j'}}}{k_j'^2+m_{n_j'}^2-i\epsilon}\right)\\ &\times\prod_{i=1}^M\left(Z_{n_i}^*\sum_{\sigma_i}u^{b_i*}(\vec{k}_i,\sigma_i,n_i)\times\frac{-i\sqrt{2\omega_{n_i,\vec{k}_i}}}{k_i^2+m_{n_i}^2-i\epsilon}\right)\\ &\times\langle k_1',\sigma_1',n_1';\dots;k_N',\sigma_N',n_N',-|k_1,\sigma_1,n_1;\dots;k_M,\sigma_M,n_M,+\rangle_c.\end{aligned} \tag{7.34}$$

On the left hand side we have the Fourier transform

$$\langle\Omega|TO_N'^{a_N}(k_N')\dots O_1'^{a_1}(k_1')O_1^{b_1\dagger}(-k_1)\dots O_M^{b_M\dagger}(-k_M)|\Omega\rangle_c = \int d^dx_1\dots d^dx_N d^dx_1'\dots d^dx_N' e^{ik_1\cdot x_1+\dots+ik_N\cdot x_N-ik_1'\cdot x_1'-\dots-ik_N'\cdot x_N'} \times \langle\Omega|TO_N'^{a_N}(x_N')\dots O_1'^{a_1}(x_1')O_1^{b_1\dagger}(x_1)\dots O_M^{b_M\dagger}(x_M)|\Omega\rangle_c \tag{7.35}$$

of a time-ordered connected correlation function in an interacting quantum field theory, where you can see that we have indeed flipped the signs of the momenta for operators associated to particles in the initial state. On the right-hand side the $Z_n$ factors are defined by the matrix elements

$$\begin{aligned}\langle\Omega|O^a(0)|\vec{k},\sigma,n\rangle &= \frac{Z_n u^a(\vec{k},\sigma,n)}{\sqrt{2\omega_{n,\vec{k}}}}\\ \langle\vec{k},\sigma,n^c|O^a(0)|\Omega\rangle &= \frac{Z_n v^a(\vec{k},\sigma,n^c)}{\sqrt{2\omega_{n^c,\vec{k}}}},\end{aligned} \tag{7.36}$$

where $|\vec{k},\sigma,n\rangle$ is a one-particle state of particle type $n$, momentum $\vec{k}$, and spin/helicity $\sigma$, $n^c$ is the antiparticle of $n$, and $u^a(\vec{k},\sigma,n)$ and $v^a(\vec{k},\sigma,n^c)$ are the objects appearing in the mode decomposition of a free field $\Phi^a$ with the same quantum numbers as $O^a$:

$$\Phi^a(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[u^a(\vec{p},\sigma,n)a_{\vec{p},\sigma,n}e^{ip\cdot x} + v^a(\vec{p},\sigma,n^c)a^\dagger_{\vec{p},\sigma,n^c}e^{-ip\cdot x}\right]. \tag{7.37}$$

We saw last semester that $u^a$ and $v^a$ are completely determined by Lorentz invariance up to an overall factor, which is the reason they appear both in the free field and in the interacting matrix elements (7.36) (that it is the same $Z_n$ in both lines of (7.36) is a consequence of $\mathcal{CRT}$ symmetry). The arrow in (7.34) means that the external momenta $k$ and $k'$ are all taken to be close to on-shell, meaning to each be close to obeying $k^2 = -m_n^2$ with $m_n$ the mass of a particle of type $n$. This formula says that as long as the $O_i$ operators have a nonzero amplitude to annihilate particles of type $n_i$ (this is measured by the $Z$ factors), and the $O_i^\dagger$ operators have a nonzero amplitude to create them (this is measured by the $Z^*$ factors), then the Fourier transform of the time-ordered correlation function has a multidimensional simple pole in this limit (the factors of $\frac{1}{k^2+m^2-i\epsilon}$) and the residue of this pole is the connected S-matrix element

$$\langle k_1',\sigma_1',n_1';\dots;k_N',\sigma_N',n_N',-|k_1,\sigma_1,n_1;\dots;k_M,\sigma_M,n_M,+\rangle_c$$

up to polarization factors involving $u$ and $u^*$ and the kinematic factors of $\sqrt{2\omega}$ (recall that "+" here indicates an "in" state and "−" indicates an "out" state). The LSZ formula is completely non-perturbative, and holds for any stable particles in the theory (e.g. hydrogen atoms in QED or protons in QCD).

In perturbative theories whose spectrum of light particles is close to that of a free theory, we can simply take $O$ and $O^\dagger$ to be the fundamental fields associated to those particles. In particular in Yukawa theory we can use $\Phi$ as the $O$ operator for creating and annihilating scalar particles, $\Psi$ as the $O$ operator for annihilating fermions and creating antifermions, and $\overline{\Psi}$ as the $O$ operator for annihilating antifermions and

creating fermions. The wave function renormalization constants $Z_\phi$ and $Z_\psi$ are defined by

$$\begin{aligned}
\langle\Omega|\Psi^a(0)|\vec{k},\sigma,f\rangle &= \frac{Z_\psi u^a(\vec{k},\sigma)}{\sqrt{2\omega_{f,\vec{k}}}} \\
\langle\vec{k},\sigma,\overline{f}|\Psi^a(0)|\Omega\rangle &= \frac{Z_\psi v^a(\vec{k},\sigma)}{\sqrt{2\omega_{f,\vec{k}}}} \\
\langle\vec{k},\sigma,f|\overline{\Psi}^a(0)|\Omega\rangle &= \frac{Z_\psi^* \overline{u}^a(\vec{k},\sigma)}{\sqrt{2\omega_{f,\vec{k}}}} \\
\langle\Omega|\overline{\Psi}^a(0)|\vec{k},\sigma,\overline{f}\rangle &= \frac{Z_\psi^* \overline{v}^a(\vec{k},\sigma)}{\sqrt{2\omega_{f,\vec{k}}}} \\
\langle\Omega|\Phi(0)|\vec{k},s\rangle &= \frac{Z_\phi}{\sqrt{2\omega_{s,\vec{k}}}} \\
\langle\vec{k},s|\Phi(0)|\Omega\rangle &= \frac{Z_\phi^*}{\sqrt{2\omega_{s,\vec{k}}}},
\end{aligned} \tag{7.38}$$

where $u$ and $v$ are now the ones we constructed by solving the Dirac equation. The frequencies are given by

$$\begin{aligned}
\omega_{s,\vec{k}} &= \sqrt{|k|^2 + m_\phi^2} \\
\omega_{f,\vec{k}} &= \sqrt{|k|^2 + m_\psi^2},
\end{aligned} \tag{7.39}$$

where $m_\phi$ and $m_\psi$ are the physical masses of the scalar and fermion particles (*not* the bare quantities appearing in the Lagrangian).

In the LSZ formula however we need to be careful about the fact that $\overline{\Psi}$ is not the complex conjugate of $\Psi$. This has no effect for fermions in the final state, but for fermions in the initial state we should multiply each $O^\dagger$ by $\gamma^0$ on the right, which converts each $u^*$ on the right-hand side of (7.34) to a $\overline{u}$. Similarly since we are using $\overline{\Psi}$ as the $O$ which annihilates antifermions, the "$Zu$" appearing in (7.34) for each final-state antifermion becomes a $Z_\psi^*\overline{v}$. $\overline{\Psi}^\dagger = \gamma^{0\dagger}\Psi$ would therefore supply factors of $(Z_\psi^*\overline{v})^* = Z_\psi\gamma^{0\dagger}v$ for each initial-state antifermion, but since instead want to use $\Psi$ to create initial state antifermions we should multiply by $\gamma^0$ on the right and so we should replace $Z^*u^*$ in the LSZ formula by $Z_\psi v$ for each initial state antifermion. To summarize, in the LSZ formula for Yukawa theory we have:

- **Final state scalar $\langle\vec{k}',s|$:** $\Phi(k')$ on the left-hand side, $Z_\phi \frac{-i\sqrt{2\omega_{s,\vec{k}'}}}{(k')^2+m_\phi^2-i\epsilon}$ on the right-hand side.
- **Initial state scalar $|\vec{k},s\rangle$:** $\Phi(-k)$ on the left-hand side, $Z_\phi^* \frac{-i\sqrt{2\omega_{s,\vec{k}}}}{k^2+m_\phi^2-i\epsilon}$ on the right-hand side.
- **Final state fermion $\langle\vec{k}',\sigma',f|$:** $\Psi^a(k')$ on the left-hand side, $Z_\psi u^a(\vec{k}',\sigma')\frac{-i\sqrt{2\omega_{f,\vec{k}'}}}{(k')^2+m_\psi^2-i\epsilon}$ on the right-hand side.
- **Initial state fermion $|\vec{k},\sigma,f\rangle$:** $\overline{\Psi}^b(-k)$ on the left-hand side, $Z_\psi^*\overline{u}^b(\vec{k},\sigma)\frac{-i\sqrt{2\omega_{f,\vec{k}}}}{k^2+m_\psi^2-i\epsilon}$ on the right-hand side.
- **Final state antifermion $\langle\vec{k}',\sigma',\overline{f}|$:** $\overline{\Psi}^a(k')$ on the left-hand side, $Z_\psi^*\overline{v}^a(\vec{k}',\sigma')\frac{-i\sqrt{2\omega_{f,\vec{k}'}}}{(k')^2+m_\psi^2-i\epsilon}$ on the right-hand side.

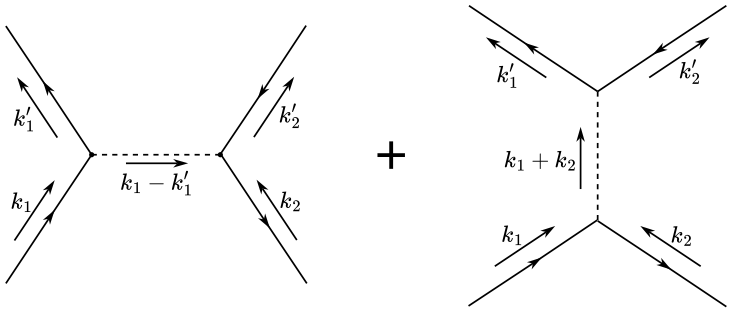


Figure 13: Feynman diagrams for $2 \to 2$ scattering of a fermion and an antifermion.

- **Initial state antifermion $|\vec{k}, \sigma, \overline{f}\rangle$:** $\Psi^b(-k)$ on the left-hand side, $Z_\psi v^b(\vec{k}, \sigma) \frac{-i\sqrt{2\omega_{f,\vec{k}}}}{k^2 + m_\psi^2 - i\epsilon}$ on the right-hand side.

Let's now see what the LSZ formula says about how to compute the S-matrix of Yukawa theory in perturbation theory. Let's first consider the example of fermion-fermion scattering at tree-level, for which we computed the relevant four-point function in equation (7.33). We are supposed to take the external momenta on-shell, in which case each of the fermion propagators in (7.33) can be approximated as

$$\hat{S}_F(p) = \frac{i(\not{p} + i m_\psi)}{p^2 + m_\psi^2 - i\epsilon} \approx \frac{-i \sum_\sigma u(p,\sigma)\overline{u}(p,\sigma)}{p^2 + m_\psi^2 - i\epsilon}. \tag{7.40}$$

I emphasize that the sum rule replacing the numerator by a sum over $u\overline{u}$ only works when the momentum is on-shell. Comparing this to the LSZ rules just stated, we see that to extract the S-matrix we should simply replace each final state fermion propagator by a $\frac{1}{\sqrt{2\omega_{\vec{k}'}}}\overline{u}(\vec{k}', \sigma')$ and each initial state fermion propagator by a $\frac{1}{\sqrt{2\omega_{\vec{k}}}} u(\vec{k}, \sigma)$ (here we are using that $Z_\psi = Z_\phi = 1$ and the physical mass equals the bare mass at tree-level). The tree-level $2 \to 2$ fermion connected scattering amplitude is thus given by

$$\begin{aligned}\langle k_2', \sigma_2', f; k_1', \sigma_1', f | p_1, \sigma_1, f; p_2, \sigma_2, f\rangle_c = &\frac{1}{\sqrt{2\omega_{\vec{k_1}}}} \frac{1}{\sqrt{2\omega_{\vec{k_1}}}} \frac{1}{\sqrt{2\omega_{\vec{k_1'}}}} \frac{1}{\sqrt{2\omega_{\vec{k_2'}}}} g^2 (2\pi)^d \delta^d(k_1' + k_2' - k_1 - k_2) \\ &\times \Big[ \frac{-i\overline{u}(\vec{k}_1', \sigma_1') u(\vec{k_1}, \sigma_1) \times \overline{u}(\vec{k}_2', \sigma_2') u(\vec{k_2}, \sigma_2)}{(k_1 - k_1')^2 + m_\phi^2 - i\epsilon} \\ &\quad - \frac{-i\overline{u}(\vec{k}_2', \sigma_2') u(\vec{k_1}, \sigma_1) \times \overline{u}(\vec{k}_1', \sigma_1') u(\vec{k_2}, \sigma_2)}{(k_1 - k_2')^2 + m_\phi^2 - i\epsilon} \Big].\end{aligned} \tag{7.41}$$

Back in our discussion of scattering theory we introduced a covariant amplitude $i\widetilde{\mathcal{M}}_c$ which is related to the connected $S$-matrix by stripping off the momentum conserving $\delta$-function and the annoying non-covariant factors of $\sqrt{2\omega_{\vec{k}}}$. Here we thus have the nice tree-level covariant amplitude

$$i\widetilde{\mathcal{M}}_c(ff \to ff) = -ig^2 \Big[ \frac{\overline{u}(\vec{k}_1', \sigma_1') u(\vec{k_1}, \sigma_1) \times \overline{u}(\vec{k}_2', \sigma_2') u(\vec{k_2}, \sigma_2)}{(k_1 - k_1')^2 + m_\phi^2 - i\epsilon} - \frac{\overline{u}(\vec{k}_2', \sigma_2') u(\vec{k_1}, \sigma_1) \times \overline{u}(\vec{k}_1', \sigma_1') u(\vec{k_2}, \sigma_2)}{(k_1 - k_2')^2 + m_\phi^2 - i\epsilon} \Big]. \tag{7.42}$$

It is also important to understand the rules for scattering which involves antifermions in the initial/final states. For example we can consider $2 \to 2$ fermion/antifermion scattering, as shown in figure 13. The

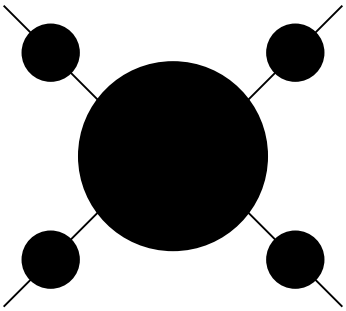

Figure 14: Decomposing a four-point function into a central contribution from the sum of pruned diagrams contracted with the exact propagators on its external lines.

contribution of these diagrams to the four-point function is

$$\langle\Omega|T\overline{\Psi}^{a_2}(k_2')\Psi^{a_1}(k_1')\overline{\Psi}^{b_1}(-k_1)\Psi^{b_2}(-k_2)|\Omega\rangle = (2\pi)^d\delta^d(k_1'+k_2'-k_1-k_2)g^2\Big[-\hat{S}_F^{a_1c}(k_1')\hat{S}_F^{cb_1}(k_1)\hat{G}_F(k_1-k_1')\hat{S}^{b_2d}(-k_2)\hat{S}^{da_1}(-k_2') + \hat{S}_F^{b_1c}(-k_2)\hat{S}_F^{cb_1}(k_1)\hat{G}_F(k_1+k_2)\hat{S}_F^{a_1d}(k_1')\hat{S}_F^{da_2}(-k_2')\Big]. \tag{7.43}$$

The signs come from working out the contractions

$$\langle\overline{\psi}\psi\overline{\psi}\psi\overline{\psi}\psi\overline{\psi}\psi\rangle \tag{7.44}$$

and

$$\langle\overline{\psi}\psi\overline{\psi}\psi\overline{\psi}\psi\overline{\psi}\psi\rangle, \tag{7.45}$$

and the negative momenta in fermion propagators are for the lines where the fermion and momentum arrows are pointing in opposite directions. Taking these external propagators on-shell, we can use that close to on-shell we have

$$\hat{S}_F(-p) = \frac{i\left(-\not{p}+im_\psi\right)}{p^2+m_\psi^2-i\epsilon} \approx \frac{i\sum_\sigma v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma)}{p^2+m_\psi^2-i\epsilon}. \tag{7.46}$$

Comparing this to our LSZ rules we see that to extract the S-matrix we should replace each final state antifermion propagator by $\frac{-v(k',\sigma')}{\sqrt{2\omega_{\vec{k}'}}}$ and each initial state propagator with $\frac{-\overline{v}(k,\sigma)}{\sqrt{2\omega_{\vec{k}}}}$ (here we are again using that $Z_\psi = 1$ at this order in $g$). We therefore have the covariant connected amplitude

$$i\widetilde{\mathcal{M}}_c(f\overline{f}\to f\overline{f}) = ig^2\left[\frac{\overline{u}(\vec{k}_1',\sigma_1')u(\vec{k}_1,\sigma_1)\times\overline{v}(\vec{k}_2,\sigma_2)v(\vec{k}_2',\sigma_2')}{(k_1-k_1')^2+m_\phi^2-i\epsilon} - \frac{\overline{v}(\vec{k}_2,\sigma_2)u(\vec{k}_1,\sigma_1)\times\overline{u}(\vec{k}_1',\sigma_1')v(\vec{k}_2',\sigma_2')}{(k_1+k_2)^2+m_\phi^2-i\epsilon}\right]. \tag{7.47}$$

Let's now generalize what we learned in this example to the case of an arbitrary scattering amplitude in Yukawa theory. The first point to make is that we showed last semester that the full non-perturbative two-point function becomes proportional to the free two point function in the limit that its momentum goes

on-shell:

$$\begin{aligned}\langle\Omega|T\Psi^a(k_2)\overline{\Psi}^b(k_1)|\Omega\rangle \to &(2\pi)^d\delta(k_2+k_1)\frac{|Z_\psi|^2 i(\not{k}_2+im_\psi)}{k_2^2+m_\psi^2-i\epsilon}\\ &= (2\pi)^d\delta(k_2+k_1)\frac{-i|Z_\psi|^2\sum_\sigma u(\vec{k}_2,\sigma)\overline{u}(\vec{k}_2,\sigma)}{k_2^2+m_\psi^2-i\epsilon}\\ &= (2\pi)^d\delta(k_2+k_1)\frac{i|Z_\psi|^2\sum_\sigma v(-\vec{k}_2,\sigma)\overline{v}(-\vec{k}_2,\sigma)}{k_2^2+m_\psi^2-i\epsilon}\end{aligned} \tag{7.48}$$

with the very important caveat that the mass $m_\psi$ appearing here is the physical mass of the fermion, while in the propagator we've been using above it is really the bare mass $m_{\psi,0}$. As before the exact scalar two-point function also has this property:

$$\langle\Omega|T\Phi(k_2)\Phi(k_1)|\Omega\rangle = (2\pi)^d\delta(k_2+k_1)\frac{-i|Z_\phi|^2}{k_2^2+m_\phi^2-i\epsilon}. \tag{7.49}$$

A general diagram contributing to the correlation function on the left-hand side of the LSZ formula (7.34) can be written as a "pruned" diagram with the property that there is no internal line can be cut which separates one external line from the rest of the diagram contracted with arbitrary diagrams contributing to the exact two-point function on each of the external legs (see figure 14). To compute the scattering amplitude we therefore simply need to strip off these external exact propagators, except that there is a small mismatch: removing an exact fermion propagator removes a factor of $|Z_\psi|^2$ and removing an exact scalar propagator removes a factor of $|Z_\phi|^2$, but in the LSZ formula we only want to remove a factor of $Z_\psi$,$Z_\psi^*$, $Z_\phi$, or $Z_\phi^*$. We therefore have a leftover factor of $Z$ or $Z^*$ in each case, leading to the following rule:

$i\widetilde{\mathcal{M}}_c$ =Sum over pruned connected tadpole-free Feynman diagrams, with overall momentum $\delta$-function removed and with external propagators replaced by the factors shown in figure 15. (7.50)

The minus signs for the external lines involving antiparticles are typically dropped since they never contribute when we compute the square of the amplitude, so you are free to drop them whenever you compute a cross section or decay rate, but they need to be there if you want to get the right expression for the S-matrix.[22]

## 7.4 Extracting the Yukawa potential

Logically the next step should be to compute a scattering cross section in Yukawa theory, but as you will see this is still a bit involved, so we will first take a slight detour to extract some physics from the scattering amplitude. The idea is to consider elastic scattering of two distinguishable nucleons, say a proton and a neutron, due to pion exchange. Since the particles are distinguishable, now only the first diagram in 12 contributes. The covariant amplitude is thus simply

$$i\widetilde{\mathcal{M}}_c(ff\to ff) = \frac{-ig^2\overline{u}(\vec{k}_1',\sigma_1')u(\vec{k_1},\sigma_1)\times\overline{u}(\vec{k}_2',\sigma_2')u(\vec{k_2},\sigma_2)}{(k_1-k_1')^2+m_\phi^2} \tag{7.51}$$

Let's study this scattering amplitude in the non-relativistic limit where both fermion momenta are small compared to their masses, in which case we can treat the momenta in $u$ and $\overline{u}$ as being at rest. From our sum rules we have

$$\overline{u}(\vec{0},\sigma)u(\vec{0},\sigma') = -2im\delta_{\sigma,\sigma'}, \tag{7.52}$$

and in the non-relativistic limit we simply have

$$(k_1-k_1')^2 = |\vec{k}_1-\vec{k}_1'|^2, \tag{7.53}$$

[22]Most QFT books (*ahem* Peskin, Schwartz) don't write these sign factors, which they can get away with since they don't actually work out the LSZ formula for spinor fields and in any case the signs drop out when you square the amplitude. Srednicki at least acknowledges they are there.

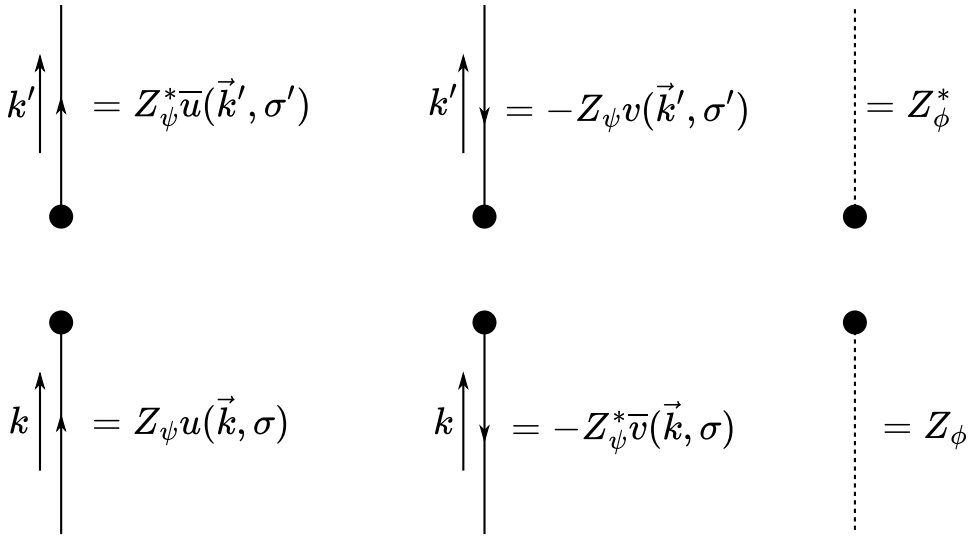


Figure 15: Factors for external lines in computing the Yukawa theory S-matrix. Here the dots indicate the rest of the diagram, so the top row are final state particles/antiparticles and the bottom row are initial state particles/antiparticles.

so in the non-relativistic limit this amplitude becomes

$$i\widetilde{\mathcal{M}}_c(p\,n \to p\,n) \approx \frac{4ig^2 m_\psi^2 \delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'}}{|\vec{k}_1 - \vec{k}_1'|^2 + m_\phi^2}. \tag{7.54}$$

We can now compare this to the usual Born approximation for nonrelativistic scattering off of a potential, which says that the scattering amplitude is[23]

$$i\widetilde{\mathcal{M}}_c \approx -4im_\psi^2 \times V(\vec{k}' - \vec{k}), \tag{7.55}$$

so apparently we have

$$V(\vec{k}) = -\frac{g^2}{|k|^2 + m_\phi^2}. \tag{7.56}$$

You will show on the homework that this is the Fourier transform of the potential

$$V_{Yukawa}(\vec{x}) = -\frac{g^2}{4\pi}\frac{1}{|x|}e^{-m_\phi|x|}. \tag{7.57}$$

Note that potential is *attractive*: the proton and neutron are pulled together by the Yukawa force. This is what holds nuclei together!

We can also consider the non-relativistic scattering of a proton and an antineutron, which has scattering amplitude

$$i\widetilde{\mathcal{M}}_c(p\,\overline{n} \to p\,\overline{n}) = ig^2 \frac{\overline{u}(\vec{k}_1', \sigma_1')u(\vec{k}_1, \sigma_1) \times \overline{v}(\vec{k}_2, \sigma_2)v(\vec{k}_2', \sigma_2')}{(k_1 - k_1')^2 + m_\phi^2} \tag{7.58}$$

In the non-relativistic limit this becomes

$$i\widetilde{\mathcal{M}}_c(p\,\overline{n} \to p\,\overline{n}) \approx \frac{4ig^2 m_\psi^2 \delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'}}{|\vec{k}_1 - \vec{k}_1'|^2 + m_\phi^2}, \tag{7.59}$$

[23]To derive this you need to translate $\widetilde{\mathcal{M}}_c$ back to the S-matrix and remember that in the non-relativistic limit we have $\sqrt{2\omega_{f,\vec{k}}} \approx \sqrt{2m_\psi}$.

with the different overall sign canceling the differing sign from the sum rule $\overline{v}(0,\sigma)v(0,\sigma') = 2m_\psi\delta_{\sigma,\sigma'}$, so the potential between a fermion and an antifermion is the same as for a fermion and a fermion. In particular it is again attractive! To get repulsive forces we need to exchange a particle of spin one, as we will see when we study quantum electrodynamics.

## 7.5 Cross section for fermion-fermion scattering

Let's now see how to turn a Yukawa scattering amplitude into a genuine differential cross section. The general formula for doing this with two particles in the initial state and an arbitrary of particles in the final state is

$$d\sigma(\alpha\to\beta) = u_\alpha^{-1}(2\pi)^d\delta^d(k_\beta - k_\alpha)\frac{|\widetilde{\mathcal{M}}_c(\alpha\to\beta)|^2}{4E_1E_2\left(\prod_{j=1}^{N_\beta}2E_{\beta,j}\right)}d\beta, \tag{7.60}$$

where $E_1$ and $E_2$ are the energies of the initial particles, $E_{\beta,j}$ is the energy of the $j$th outgoing particle, and

$$u_\alpha = \frac{\sqrt{(k_1\cdot k_2)^2 - m_1^2m_2^2}}{E_1E_2} \tag{7.61}$$

is the relativistic relative velocity of the initial particles. Last semester we showed that working in the center of mass frame $\vec{k}_1 = -\vec{k}_2$ with two particles in the final state we can simplify this to

$$\frac{d\sigma}{d\Omega_{d-2}} = \frac{1}{2^{I_{final}}}\frac{1}{(2\pi)^{d-2}}\frac{|k'|^{d-3}}{16|k|E_{tot}^2}|\widetilde{\mathcal{M}}_c|^2, \tag{7.62}$$

where $|k|$ is the magnitude of the spatial momentum of either incoming particle, $I_{final}$ is equal to one if the final state particles are identical and zero if they are distinguishable, $E_{tot}$ is the total center of mass energy

$$E_{tot} = \sqrt{|k|^2+m_1^2} + \sqrt{|k|^2+m_2^2} \tag{7.63}$$

of the collision, and

$$|k'| = \frac{\sqrt{(E_{tot}^2 - (m_1')^2 - (m_2')^2)^2 - 4(m_1')^2(m_2')^2}}{2E_{tot}} \tag{7.64}$$

is the magnitude of the spatial momenta for either outgoing particle. These formulas simplify when the external masses are all equal, as we then have $|k| = |k'|$ and $E_{tot} = 2\sqrt{|k|^2+m^2}$.

Let's apply this to our $ff\to ff$ amplitude (7.42) in Yukawa theory. This is the fortunate situation where all external masses are equal, so we simply have

$$\frac{d\sigma}{d\Omega_{d-2}} = \frac{1}{128(2\pi)^{d-2}}\frac{|k|^{d-4}}{|k|^2+m_\psi^2}|\widetilde{\mathcal{M}}_c|^2. \tag{7.65}$$

Our remaining job is thus to compute the square of the covariant matrix element (7.42). Writing the square isn't so illuminating if we don't do anything else, but as we discussed last semester the situation improves if we sum over the spins/helicities in the initial and final states. This isn't purely a matter of laziness: in a typical collider experiment it is hard to measure the spins/helicities of the outgoing particles, and the initial state particle beams typically are not spin/helicity polarized. To capture this situation we should therefore *sum* over spin/helicity in the final state and *average* over spin/helicity in the initial state.[24]

$$\frac{d\sigma_{ave}}{d\Omega_{d-2}} \equiv \frac{1}{2^{2\lfloor\frac{d-2}{2}\rfloor}}\sum_{\sigma,\sigma'}\frac{d\sigma}{d\Omega_{d-2}} = \frac{1}{128(2\pi)^{d-2}2^{2\lfloor\frac{d-2}{2}\rfloor}}\frac{|k|^{d-4}}{|k|^2+m_\psi^2}\sum_{\sigma,\sigma'}|\widetilde{\mathcal{M}}_c|^2. \tag{7.66}$$

[24] If you are puzzled about why we treated the initial and final states differently, think about the quantum mechanics of the experiment. A random initial spin is described by a normalized density matrix, for example $\rho = \frac{1}{2}|\downarrow\rangle\langle\downarrow| + \frac{1}{2}|\uparrow\rangle\langle\uparrow|$ for a spin-1/2 particle, while a measurement which cannot distinguish between spin up and spin down in the final state is a projection $P = |\downarrow\rangle\langle\downarrow| + |\uparrow\rangle\langle\uparrow|$.

Here we've used that the irreducible spinor representation of $SO(d-1)$ is $2^{\lfloor \frac{d-2}{2} \rfloor}$-dimensional. In $d=4$ this expression is simply

$$\frac{d\sigma_{ave}}{d\Omega_2} = \frac{1}{2048\pi^2}\frac{1}{|k|^2+m_\psi^2}\sum_{\sigma,\sigma'}|\widetilde{\mathcal{M}}_c|^2. \tag{7.67}$$

We can compute the spin sum explicitly using a standard sequence of tricks. We first note that if $f$ and $g$ are commuting spinors (for us they will always be $u$ or $v$), then

$$(\overline{f}g)^* = f^T\gamma^{0*}g^* = g^\dagger\gamma^{0\dagger}f = -\overline{g}f. \tag{7.68}$$

We are interested in computing

$$\sum_{\sigma,\sigma'}|\widetilde{\mathcal{M}}_c|^2 = g^4\sum_{\sigma,\sigma'}\left|\frac{\overline{u}(\vec{k}_1',\sigma_1')u(\vec{k_1},\sigma_1)\times\overline{u}(\vec{k}_2',\sigma_2')u(\vec{k_2},\sigma_2)}{(k_1-k_1')^2+m_\phi^2-i\epsilon} - \frac{\overline{u}(\vec{k}_2',\sigma_2')u(\vec{k_1},\sigma_1)\times\overline{u}(\vec{k}_1',\sigma_1')u(\vec{k_2},\sigma_2)}{(k_1-k_2')^2+m_\phi^2-i\epsilon}\right|^2. \tag{7.69}$$

Each of the four terms which arises can be simplified using (7.68) together with our spin-sum identities[25]

$$\begin{aligned}\sum_\sigma u(\vec{p},\sigma)\overline{u}(\vec{p},\sigma) &= -(\not{p}+im_\psi)\\ \sum_\sigma v(\vec{p},\sigma)\overline{v}(\vec{p},\sigma) &= -(\not{p}-im_\psi).\end{aligned} \tag{7.70}$$

Indeed for the square of the first term we have (using an abbreviated notation)

$$\begin{aligned}\sum_{\sigma,\sigma'}|\overline{u}_{1'}u_1\overline{u}_{2'}u_2|^2 &= \sum_{\sigma,\sigma'}\overline{u}_{1'}u_1\overline{u}_1u_{1'}\times\overline{u}_{2'}u_2\overline{u}_2u_{2'}\\ &= \sum_{\sigma,\sigma'}\mathrm{Tr}\left(u_1\overline{u}_1u_{1'}\overline{u}_{1'}\right)\times\mathrm{Tr}\left(u_2\overline{u}_2u_{2'}\overline{u}_{2'}\right)\\ &= \mathrm{Tr}\left[(\not{k}_1+im_\psi)\left(\not{k}_1'+im_\psi\right)\right]\mathrm{Tr}\left[(\not{k}_2+im_\psi)\left(\not{k}_2'+im_\psi\right)\right],\end{aligned} \tag{7.71}$$

while for the square of the second we similarly have

$$\sum_{\sigma,\sigma'}|\overline{u}_{2'}u_1\overline{u}_{1'}u_2|^2 = \mathrm{Tr}\left[(\not{k}_1+im_\psi)\left(\not{k}_2'+im_\psi\right)\right]\mathrm{Tr}\left[(\not{k}_2+im_\psi)\left(\not{k}_1'+im_\psi\right)\right]. \tag{7.72}$$

The cross terms have a different index structure, indeed we have

$$\sum_{\sigma,\sigma'}\overline{u}_{1'}u_1\overline{u}_{2'}u_2\overline{u}_1u_{2'}\overline{u}_2u_{1'} = \mathrm{Tr}\left[\left(\not{k}_1'+im_\psi\right)(\not{k}_1+im_\psi)\left(\not{k}_2'+im_\psi\right)(\not{k}_2+im_\psi)\right] \tag{7.73}$$

and

$$\sum_{\sigma,\sigma'}\overline{u}_{2'}u_1\overline{u}_{1'}u_2\overline{u}_2u_{2'}\overline{u}_1u_{1'} = \mathrm{Tr}\left[\left(\not{k}_1'+im_\psi\right)(\not{k}_2+im_\psi)\left(\not{k}_2'+im_\psi\right)(\not{k}_1+im_\psi)\right]. \tag{7.74}$$

To further evaluate these traces we need to learn how to take the trace of a product of $\gamma$-matrices. This is sometimes called "trace technology". Let's start simple: from our product-Pauli representation of the $\gamma$-matrices we clearly have

$$\mathrm{Tr}(\gamma^\mu) = 0. \tag{7.75}$$

From the Dirac algebra we also have

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = \frac{1}{2}\mathrm{Tr}\left(\{\gamma^\mu,\gamma^\nu\}\right) = 2^{\lfloor\frac{d}{2}\rfloor}\eta^{\mu\nu}. \tag{7.76}$$

[25] Actually here we only need the first of these, but when you compute amplitudes involving antiparticles in the homework you'll need the second.

Things get trickier as we go to more $\gamma$-matrices, so let's first take $d$ to be even. We can then show that the trace of the product of any odd number of $\gamma$-matrices is zero:

$$\begin{aligned}\mathrm{Tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_n}\right) &= \mathrm{Tr}\left(\gamma\gamma\gamma^{\mu_1}\dots\gamma^{\mu_n}\right)\\ &= \mathrm{Tr}\left(\gamma\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma\right)\\ &= (-1)^n\mathrm{Tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma\gamma\right)\\ &= (-1)^n\mathrm{Tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_n}\right).\end{aligned} \tag{7.77}$$

We can also show that the trace of the product of any odd number of $\gamma$-matrices together with $\gamma$ is zero as well:

$$\mathrm{Tr}\left(\gamma\gamma^{\mu_1}\dots\gamma^{\mu_n}\right) = (-1)^n\mathrm{Tr}\left(\gamma^{\mu_1}\dots\gamma^{\mu_n}\gamma\right) = (-1)^n\mathrm{Tr}\left(\gamma\gamma^{\mu_1}\dots\gamma^{\mu_n}\right). \tag{7.78}$$

For the above calculation what we need the trace of the product of four $\gamma$ matrices: we can compute this as follows:

$$\begin{aligned}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) &= 2\eta^{\mu\nu}\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) - \mathrm{Tr}\left(\gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta\right)\\ &= 2\eta^{\mu\nu}\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) - 2\eta^{\mu\alpha}\mathrm{Tr}\left(\gamma^\nu\gamma^\beta\right) + \mathrm{Tr}\left(\gamma^\nu\gamma^\alpha\gamma^\mu\gamma^\beta\right)\\ &= 2\eta^{\mu\nu}\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) - 2\eta^{\mu\alpha}\mathrm{Tr}\left(\gamma^\nu\gamma^\beta\right) + 2\eta^{\mu\beta}\mathrm{Tr}\left(\gamma^\nu\gamma^\alpha\right) - \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right),\end{aligned} \tag{7.79}$$

and thus

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 2^{\lfloor\frac{d}{2}\rfloor}\left(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}\right). \tag{7.80}$$

It is useful to re-package these results as statements about the traces of slashed vectors:

$$\begin{aligned}\mathrm{Tr}\left(\not{a}\not{b}\right) &= 2^{\lfloor\frac{d}{2}\rfloor}a\cdot b\\ \mathrm{Tr}\left(\not{a}\not{b}\not{c}\not{d}\right) &= 2^{\lfloor\frac{d}{2}\rfloor}\Big((a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\Big).\end{aligned} \tag{7.81}$$

Applying these to our spin sums, we have

$$\begin{aligned}\sum_{\sigma,\sigma'}|\overline{u}_{1'}u_1\overline{u}_{2'}u_2|^2 &= 2^{2\lfloor\frac{d}{2}\rfloor}\left(k_1\cdot k_1' - m_\psi^2\right)\left(k_2\cdot k_2' - m_\psi^2\right)\\ \sum_{\sigma,\sigma'}|\overline{u}_{2'}u_1\overline{u}_{1'}u_2|^2 &= 2^{2\lfloor\frac{d}{2}\rfloor}\left(k_1\cdot k_2' - m_\psi^2\right)\left(k_2\cdot k_1' - m_\psi^2\right)\\ \sum_{\sigma,\sigma'}\overline{u}_{1'}u_1\overline{u}_{2'}u_2\overline{u}_1u_{2'}\overline{u}_2u_{1'} &= \sum_{\sigma,\sigma'}\overline{u}_{2'}u_1\overline{u}_{1'}u_2\overline{u}_2u_{2'}\overline{u}_1u_{1'}\\ &= 2^{\lfloor\frac{d}{2}\rfloor}\Big(m_\psi^4 - m_\psi^2\left(k_1\cdot k_2 + k_1\cdot k_1' + k_1\cdot k_2' + k_1'\cdot k_2 + k_1'\cdot k_2' + k_2\cdot k_2'\right)\\ &\qquad + (k_1\cdot k_1')(k_2\cdot k_2') - (k_1\cdot k_2)(k_1'\cdot k_2') + (k_1\cdot k_2')(k_2\cdot k_1')\Big).\end{aligned} \tag{7.82}$$

The spin-summed matrix element squared is therefore given by

$$\begin{aligned}\sum_{\sigma,\sigma'}|\widetilde{\mathcal{M}}_c|^2 = g^4 2^{2\lfloor\frac{d}{2}\rfloor}\Bigg[&\frac{\left(k_1\cdot k_1' - m_\psi^2\right)\left(k_2\cdot k_2' - m_\psi^2\right)}{((k_1-k_1')^2+m_\phi^2)^2} + \frac{\left(k_1\cdot k_2' - m_\psi^2\right)\left(k_2\cdot k_1' - m_\psi^2\right)}{((k_1-k_2')^2+m_\phi^2)^2}\\ &- \frac{1}{2^{\lfloor\frac{d-2}{2}\rfloor}}\frac{1}{((k_1-k_1')^2+m_\phi^2)((k_1-k_2')^2+m_\phi^2)}\Big(m_\psi^4 - m_\psi^2\left(k_1\cdot k_2 + k_1\cdot k_1' + k_1\cdot k_2' + k_1'\cdot k_2 + k_1'\cdot k_2' + k_2\cdot k_2'\right)\\ &+ (k_1\cdot k_1')(k_2\cdot k_2') - (k_1\cdot k_2)(k_1'\cdot k_2') + (k_1\cdot k_2')(k_2\cdot k_1')\Big)\Bigg].\end{aligned} \tag{7.83}$$

We can evaluate these expressions in the center of mass frame, where

$$\begin{aligned} k_1 &= (\omega, \vec{k}) \\ k_2 &= (\omega, -\vec{k}) \\ k_1' &= (\omega, \vec{k}\,') \\ k_2' &= (\omega, -\vec{k}\,'), \end{aligned} \tag{7.84}$$

in terms of which we have

$$\begin{aligned} k_1 \cdot k_2 &= -\omega^2 - |k|^2 \\ k_1 \cdot k_1' &= -\omega^2 + |k|^2 \cos\theta \\ k_1 \cdot k_2' &= -\omega^2 - |k|^2 \cos\theta \\ k_1' \cdot k_2 &= -\omega^2 - |k|^2 \cos\theta \\ k_1' \cdot k_2' &= -\omega^2 - |k|^2 \\ k_2 \cdot k_2' &= -\omega^2 + |k|^2 \cos\theta \\ (k_1 - k_1')^2 &= 2|k|^2(1 - \cos\theta) \\ (k_1 - k_2')^2 &= 2|k|^2(1 + \cos\theta) \end{aligned} \tag{7.85}$$

Here is $\theta$ is the angle between $\vec{k}$ and $\vec{k}\,'$, and we see that the squared amplitude has an interesting angular dependence. This is quite different from our tree-level $2 \to 2$ squared scattering amplitude in $\phi^4$ theory, which was just $\lambda^2$. In limit where $m_\psi = 0$ the expression for the differential cross section isn't too bad, we have

$$\begin{aligned} \sum_{\sigma,\sigma'} |\widetilde{\mathcal{M}}_c|^2 = g^4 2^{2\lfloor \frac{d}{2} \rfloor} |k|^4 \Bigg[ &\frac{(1-\cos\theta)^2}{(2|k|^2(1-\cos\theta) + m_\phi^2)^2} + \frac{(1+\cos\theta)^2}{(2|k|^2(1+\cos\theta) + m_\phi^2)^2} \\ &+ \frac{1}{2^{\lfloor \frac{d-4}{2} \rfloor}} \frac{1+\cos\theta}{(2|k|^2(1-\cos\theta) + m_\phi^2)(2|k|^2(1+\cos\theta) + m_\phi^2)} \Bigg]. \end{aligned} \tag{7.86}$$

Life is even simpler if we take the fermions to be distinguishable, in which case only the first term contributes so we have the full differential cross section

$$\frac{d\sigma_{ave}}{d\Omega_{d-2}} = \frac{g^4 |k|^{d-2}}{32(2\pi)^{d-2} m_\phi^4} \times \frac{(1-\cos\theta)^2}{(1 + 2\frac{|k|^2}{m_\phi^2}(1-\cos\theta))^2}. \tag{7.87}$$

This has several interesting features, for example when $|k| \ll m_\phi$ the cross section is zero at $\theta = 0$ and peaked at $\theta = \pi$, while for $|k| \gg m_\phi$ it is roughly independent of angle.

**Problems:**

1. Write out the values of the three-loop bubble diagrams in figure 6. You don't need to evaluate the integrals over positions, and I'll forgive you if you ignore the symmetry factors.

2. Write out the contribution to the connected fermion four-point function from the four ladder diagrams shown in figure 11. You don't need evaluate the position integrals.

3. Write out the tree-level contributions to the connected fermion four-point function in true Yukawa theory with interaction $-g\phi\overline{\psi}\gamma\psi$.

4. Show that (7.56) is the Fourier transform of (7.57).

5. Compute the tree-level spin-summed/averaged differential cross section for $f\,\overline{f} \to f\,\overline{f}$ scattering (you can start from the covariant amplitude (7.47)). If you insist you can work in $d = 4$, and it is ok to leave it in a form similar to (7.83).

6. Compute the tree-level spin-summed differential cross section for $s\,s \to f\,\overline{f}$ scattering. Make sure to include both diagrams which contribute. If you insist you can work in $d = 4$, and it is ok to leave it in a form similar to (7.83).

7. Extra credit: compute the tree-level spin-summed/averaged differential cross section for $f\,f \to f\,f$ scattering in the true Yukawa theory with interaction $-g\phi\overline{\psi}\gamma\psi$. You can work in $d = 4$.

# 8 Lattice fermions and the 2D Ising model

In this section we'll learn a bit about some novel phenomena that arise when we try to put spinor fields on a spatial lattice. This is not merely a technicality, the issues we encounter are the gateway to a number of deep theoretical ideas related to anomalies and the renormalization group. Moreover lattice field theory is a big business, especially here at MIT, and if you aren't careful you can get in serious trouble with lattice fermions. Beyond this, we'll also see that as an added bonus these methods give us an easy approach to solving the classical Ising model in two spatial dimensions. The original solution due to Onsager is famously impenetrable, so having an easy approach is quite nice! For this section we will work entirely in $1+1$ dimensions, as the essential points already arise there and it is the Euclidean version of this theory that is related to the 2D Ising model.[26]

## 8.1 Scalar field on the lattice in $1+1$ dimensions

As a warmup, let's first consider the lattice formulation of a $1+1$ dimensional massless scalar field with continuum Lagrangian

$$L = -\frac{1}{2}\int dx\, \partial_\mu \Phi_{cont} \partial^\mu \Phi_{cont} \tag{8.1}$$

and continuum Hamiltonian

$$H = \frac{1}{2}\int dx \left(\Pi_{cont}^2 + (\partial_1 \Phi_{cont})^2\right). \tag{8.2}$$

The canonical commutation relation is[27]

$$[\Phi_{cont}(0,x), \Pi_{cont}(0,y)] = i\delta(x-y). \tag{8.3}$$

We'd like to define a version of this theory where the field lives only on a spatial lattice, with site locations

$$x_i = an \tag{8.4}$$

with $n \in \mathbb{Z}$. The length parameter $a$ is called the **lattice spacing**. Since the fields are now labeled by a discrete parameter, we should rescale them so that the canonical commutation relation has a discrete Kronecker $\delta$ instead of a continuous $\delta$-function:

$$\Phi(n) = \sqrt{a}\Phi_{cont}(an) \tag{8.5}$$

$$\Pi(n) = \sqrt{a}\Pi_{cont}(an), \tag{8.6}$$

[26] Many of the topics in this section are nicely reviewed in two papers by my adviser Leonard Susskind from 1977-1978, I put them on the course website.

[27] In this section we will temporarily depart from our convention that $x$ is a spacetime vector with components $x^\mu$ and $\vec{x}$ is a spatial vector with components $x^i$, as it seems a bit much to write a one-dimensional vector with an arrow. Thus $x$ and $y$ in this equation are spatial positions, not spacetime vectors.

in terms of which we have

$$[\Phi(n), \Pi(m)] = i\delta_{m,n}. \tag{8.7}$$

These results follow from the fact that in the continuum limit we have the replacements

$$\sum_n a \to \int dx$$
$$\frac{1}{a}\delta_{n,m} \to \delta(x-y). \tag{8.8}$$

The spatial derivative in the Hamiltonian should be replaced by a finite difference

$$\partial_1 \Phi_{cont}(x) \to \frac{\Phi_{cont}(x+a) - \Phi_{cont}(x)}{a}, \tag{8.9}$$

so in terms of the lattice field the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2}\sum_n a\left(\frac{\Pi(n)^2}{a} + \frac{\big(\Phi(n+1)-\Phi(n)\big)^2}{a^3}\right) \\ &= \frac{1}{2}\sum_n \left(\Pi(n)^2 + \frac{\big(\Phi(n+1)-\Phi(n)\big)^2}{a^2}\right). \end{aligned} \tag{8.10}$$

We thus have the lattice equations of motion

$$\begin{aligned} \dot{\Phi}(t,n) &= i[H, \Phi(t,n)] \\ &= \Pi(t,n) \\ \dot{\Pi}(t,n) &= i[H, \Pi(t,n)] \\ &= \frac{i}{2a^2}\sum_m [(\Phi(t,m+1) - \Phi(t,m)), \Pi(t,n)] \\ &= \frac{\Phi(t,n+1) + \Phi(t,n-1) - 2\Phi(t,n)}{a^2}, \end{aligned} \tag{8.11}$$

which we can combine to get a lattice version of the massless Klein-Gordon equation:

$$\ddot{\Phi}(t,n) = \frac{\Phi(t,n+1) + \Phi(t,n-1) - 2\Phi(t,n)}{a^2}. \tag{8.12}$$

We can understand the physics of this equation by looking for solutions of the form

$$\Phi(t,n) = Ae^{ikan - i\omega t}. \tag{8.13}$$

Substituting this expression into (8.12), we find the dispersion relation

$$\omega^2 = \frac{2 - e^{ika} - e^{-ika}}{a^2} = \frac{2}{a^2}\left(1 - \cos(ka)\right). \tag{8.14}$$

This dispersion relation has the periodicity

$$k \sim k + \frac{2\pi}{a}; \tag{8.15}$$

this is because $k$ and $k + \frac{2\pi}{a}$ give the same solution $\Phi(t,n)$. We should therefore restrict to $k \in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right]$ (this range is sometimes called the **Brillouin zone**). The dispersion relation is plotted in figure 16. There are two key points to make about this dispersion relation:

(1) For $k \ll \frac{1}{a}$ we have $\omega^2 \approx k^2$, so the low-energy modes near $k = 0$ match onto the solutions of the continuum Klein-Gordon equation.

(2) Away from $k = 0$ there are no other low-energy modes, so the continuum limit works nicely.

Our main point for this section will be that for fermions we aren't so lucky with point (2).

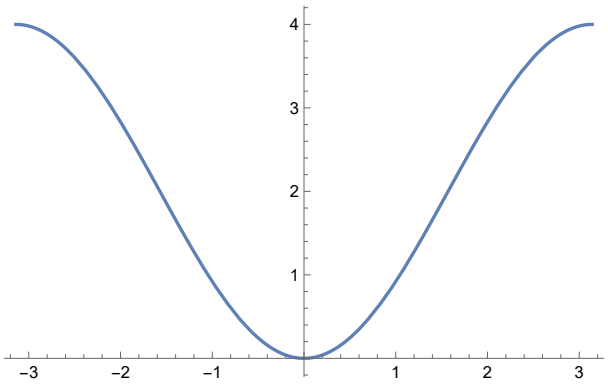


Figure 16: The dispersion relation $\omega^2(k)$ for a lattice scalar field, in units where $a = 1$. Note that the only low-energy states are near $k = 0$.

## 8.2 Lattice fermions in $1+1$ dimensions

Turning now to spinors, I'll remind you of our two-dimensional $\gamma$-matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{8.16}$$

Thus a massless complex Dirac fermion

$$\Psi^{cont} = \begin{pmatrix} \Psi_L^{cont} \\ \Psi_R^{cont} \end{pmatrix} \tag{8.17}$$

has continuum Lagrangian

$$L = i \int dx \big( \Psi_L^{cont*} (\partial_0 - \partial_1) \Psi_L^{cont} + \Psi_R^{cont*} (\partial_0 + \partial_1) \Psi_R^{cont} \big) \tag{8.18}$$

and continuum Hamiltonian

$$H = i \int dx \big( \Psi_L^{cont*} \partial_1 \Psi_L^{cont} - \Psi_R^{cont*} \partial_1 \Psi_R^{cont} \big). \tag{8.19}$$

The canonical anticommutation relation is

$$\{ \Psi_i^{cont}(0,x), \Psi_j^{cont*}(0,y) \} = \delta_{ij} \delta(x-y), \tag{8.20}$$

with $i = L, R$. The equations of motion are

$$\begin{aligned} (\partial_0 - \partial_1) \Psi_L^{cont} &= 0 \\ (\partial_0 + \partial_1) \Psi_L^{cont} &= 0, \end{aligned} \tag{8.21}$$

so $\Psi_L^{cont}$ is a function only of $t+x$, i.e. it is "left-moving", and $\Psi_R^{cont}$ is a function only of $t-x$, i.e. it is "right-moving". I emphasize that $\Psi_L^{cont}$ and $\Psi_R^{cont}$ are decoupled, so we can have a sensible theory which has only one or the other - field theories in $d = 2$ with inequivalent numbers of left-moving and right-moving massless fermions are called **chiral theories**.[28]

[28] In $d = 4$ chiral theories are those with inequivalent numbers of left-handed and right-handed massless fermions. The term is also sometimes used when there are equal numbers but the left-handed and right-handed massless fermions have different interactions.

In this section we will be particularly interested in Majorana fermions, which obey the constraint $\Psi^{cont*} = B_1\Psi^{cont}$. Here this just means that $\Psi_L^{cont}$ and $\Psi_R^{cont}$ are real. The Lagrangian and Hamiltonian for a Majorana spinor in $1+1$ dimensions are

$$\begin{aligned} L &= \frac{i}{2}\int dx\big(\Psi_L^{cont}(\partial_0 - \partial_1)\Psi_L^{cont} + \Psi_R^{cont}(\partial_0+\partial_1)\Psi_R^{cont}\big) \\ H &= \frac{i}{2}\int dx\big(\Psi_L^{cont}\partial_1\Psi_L^{cont} - \Psi_R^{cont}\partial_1\Psi_R^{cont}\big), \end{aligned} \tag{8.22}$$

and its canonical anticommutation relation is

$$\{\Psi_i^{cont}(0,x), \Psi_j^{cont}(0,y)\} = \delta_{ij}\delta(x-y). \tag{8.23}$$

As in the scalar case we can introduce a latticized version

$$\Psi_{L,R}(n) = \sqrt{a}\Psi_{L,R}^{cont}(na) \tag{8.24}$$

of our spinor field, which in the Dirac case obeys the anticommutation relation

$$\{\Psi_i(n), \Psi_j^*(m)\} = \delta_{ij}\delta_{nm} \tag{8.25}$$

and in the Majorana case obeys

$$\{\Psi_i(n), \Psi_j(m)\} = \delta_{ij}\delta_{nm}. \tag{8.26}$$

What about the Hamiltonian? Perhaps the most obvious choice of Hamiltonian for a single right-moving Majorana-Weyl fermion $\Psi_R$ is

$$-\frac{i}{2}\sum_n \Psi_R(n)\frac{\Psi_R(n+1)-\Psi_R(n)}{a}, \tag{8.27}$$

but this isn't actually hermitian since the second term is an (infinite) imaginary constant (remember that $\Psi_R(n)^2 = \frac{1}{2}$ by the anticommutation relations). On the other hand we don't care about an additive constant in the Hamiltonian, so we can instead take the Hamiltonian to be

$$H = -\frac{i}{2a}\sum_n \Psi_R(n)\Psi_R(n+1). \tag{8.28}$$

We can make this look more like the continuum Hamiltonian by writing it as

$$H = -\frac{i}{2}\sum_n \Psi_R(n)\frac{\Psi_R(n+1)-\Psi_R(n-1)}{2a}, \tag{8.29}$$

which is now manifestly hermitian since the dagger just exchanges the two sums. The quantity

$$\frac{\Psi_R(n+1)-\Psi_R(n-1)}{2a} \tag{8.30}$$

is called the **symmetric derivative**. It is actually a better approximation to the continuum derivative than the naive discrete derivative (8.9), since if $f$ is a smooth function we have

$$\frac{f(x+\epsilon)-f(x-\epsilon)}{2\epsilon} = f'(x) + O(\epsilon^2). \tag{8.31}$$

The equation of motion for this Hamiltonian is

$$\begin{aligned} \dot{\Psi}_R(t,n) &= i[H, \Psi_R(t,n)] \\ &= \frac{1}{2a}\sum_m [\Psi_R(t,m)\Psi_R(t,m+1), \Psi_R(t,n)] \\ &\quad - \frac{\Psi_R(t,n+1)-\Psi_R(t,n-1)}{2a}, \end{aligned} \tag{8.32}$$

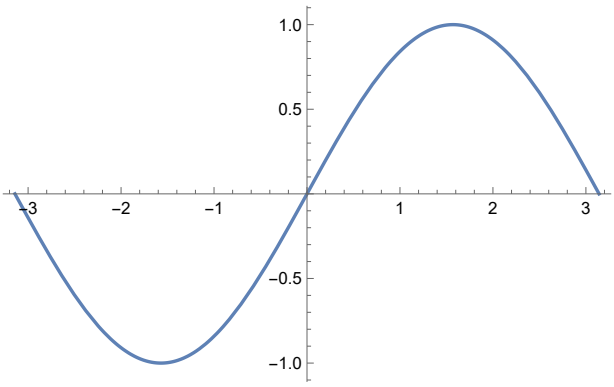


Figure 17: The dispersion relation $\omega(k)$ for a right-moving lattice Majorana-Weyl spinor $\Psi_R$ in $1+1$ dimensions, in units where $a = 1$. Note in particular the extra low-energy modes near $k \approx \pi$, these are the left-moving doublers.

which is a discrete version of the Dirac equation (8.21) with the spatial derivative replaced by the symmetric derivative.

We can find the dispersion relation as in the scalar case, by looking for solutions of the form

$$\Psi_R(t, n) = Ae^{ikan - i\omega t}. \tag{8.33}$$

The lattice Dirac equation (8.32) gives

$$-i\omega = -\frac{e^{ika} - e^{-ika}}{2a}, \tag{8.34}$$

or in other words

$$\omega = \frac{\sin(ka)}{a}. \tag{8.35}$$

This dispersion relation is plotted in figure 17. Note that for a spinor it is $\omega$ which is an analytic function of $k$, while for the scalar it was $\omega^2$. In particular to get $\omega > 0$ we should take $k > 0$, which reflects the right-moving nature of the spinor field. For small $k$ we have

$$\omega \approx k, \tag{8.36}$$

which is precisely the continuum dispersion relation for a right-moving continuum fermion. On the other hand we now have a surprise: there are more light excitations near $k = \frac{\pi}{a}$! We can get a sense of what these excitations look like by defining $\psi_R(n) = e^{i\pi n}\chi(n)$ and $k = \frac{\pi}{a} - \kappa$ with $0 < \kappa a \ll 1$, in terms of which have

$$\chi(n) = Ae^{-i\kappa x - i\omega t} \tag{8.37}$$

with

$$\omega \approx \kappa. \tag{8.38}$$

In other words these are precisely the low-energy excitations of a *left-moving fermion*; our attempt to construct a lattice version of a one-component right-moving Majorana-Weyl spinor has instead resulted in a theory whose low-energy description is a two-component Majorana spinor with both left-moving and right-moving fields described by the continuum Hamiltonian (8.22)! This phenomenon is called **fermion doubling**, and the extra left-moving mode we generated is called a **fermion doubler**.

## 8.3 Nielsen-Ninomiya theorem

You may be wondering if the fermion doubling phenomenon we just discovered can be avoided by some more clever choice of the lattice Hamiltonian, or more generally if it is only a problem in $1+1$ dimensions. In fact fermion doubling is a quite general phenomenon, which is present in any spacetime dimension. It is particularly robust in even spacetime dimensions, where it is formalized in the **Nielsen-Ninomiya theorem**. This says that any weakly-interacting lattice fermion theory in even spacetime dimensions which respects locality, lattice translation symmetry, and chiral symmetry necessarily has an equal number of upper and lower component massless Weyl fermions at low energies. In other words you can't put chiral theories on a lattice. We won't get into the general proof of the theorem here, but I'll give you the essence of the argument for the $1+1$ dimensional case we just discussed.[29]

Let's consider a rather general lattice Hamiltonian

$$H = \frac{1}{2}\sum_{n,m} \Psi(n) K(n-m) \Psi(m), \tag{8.39}$$

where $\Psi$ obeys the canonical anticommutation relation

$$\{\Psi(n), \Psi(m)\} = \delta_{n,m} \tag{8.40}$$

and we can take $K(n-m) = -K(m-n)$. Hermiticity requires $K$ to be pure imaginary, and the fact that $K$ depends only on $n-m$ is the a consequence of lattice translation invariance. You will show on the homework that the equation of motion for this system is

$$\dot{\Psi}(t,n) = i\sum_m K(m-n)\Psi(t,m), \tag{8.41}$$

and that solutions of the form

$$\Psi(t,n) = Ae^{ikan - i\omega t} \tag{8.42}$$

obey a dispersion relation

$$\omega(k) = \sum_m K(m) e^{-ikam}. \tag{8.43}$$

The right-hand side of this equation is just the discrete Fourier transform of $K$. Note in particular that we have the periodicity

$$\omega\left(k + \frac{2\pi}{a}\right) = \omega(k), \tag{8.44}$$

which is again a consequence of lattice translation symmetry.

To implement the idea of locality, we should require $K(n)$ to vanish at large $n$. In simple lattice models such as the one we considered in the previous section $K$ has compact support, which certainly implies this vanishing, but we can relax that assumption quite a bit and still prove the theorem. The most convenient assumption to make is that $K$ falls off fast enough that the sum $\sum_m K(m)m$ is absolutely convergent, which implies that the Fourier transform of $K$ has a continuous first derivative. From (8.43), we therefore see that $\omega(k)$ is a continuously differentiable map from $\mathbb{S}^1$ to $\mathbb{R}$.

The Nielsen-Ninomiya theorem now follows immediately from the observation that any such function must cross the surface $\omega = 0$ an even number of times to make sure you get back to where you started as you go around the circle. Moreover half of those crossings must go from negative to positive as you increase $k$, while the other half must go from positive to negative (see figure 18 for an illustration). Crossings of the former type give rise to right-movers, while crossings of the latter type give rise to left-movers. Thus the number of left-movers and right-movers must be equal!

[29] There is one sense in which $1+1$ is special: the antiparticle of a left/right mover is also a left/right mover, while in $3+1$ dimensions the antiparticle of a left/right handed particle is right/left handed. This means that in the argument we now give we won't need to assume anything about chiral symmetry, while in $3+1$ dimensions one needs to.

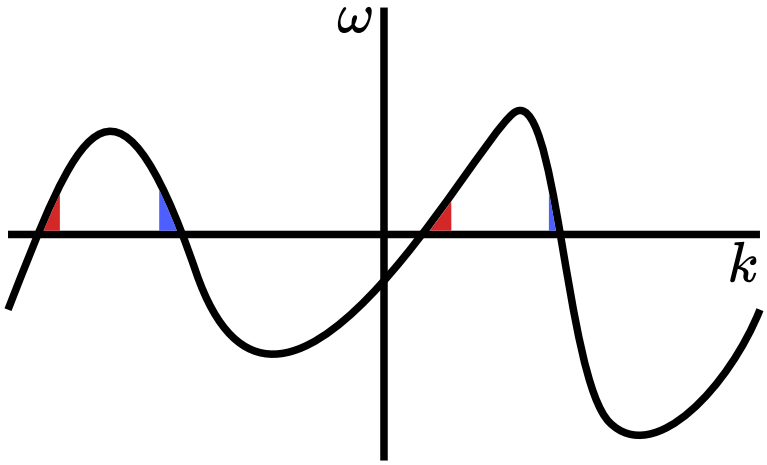


Figure 18: The Nielsen-Ninomiya theorem in $1+1$ dimensions: a fermion dispersion relation $\omega(k)$ must be periodic and have continuous first derivative, so it crosses $\omega = 0$ an equal number of times in either direction. Low-energy right-movers are shaded red and low-energy left-movers are shaded blue.

Fermion doubling has a rather embarrassing consequence: the standard model of particle physics is a chiral theory, so here in 2024 we still don't know how to write down a lattice model which regulates the standard model at short distances. This means that we don't actually have a non-perturbatively good definition of the standard model, which is a bit unsettling since it is supposed to be part of our current understanding of the fundamental laws of nature.

There is much more that could be said about lattice fermions, in particular there are many proposals for how to minimize the number of doublers that all have various advantages and disadvantages, but we will have to leave it here. Will Detmold is teaching a class on lattice field theory next semester, so that is one place to go if you want more!

## 8.4 Classical 2D Ising model

The 2D Ising model is of great importance to the history of statistical mechanics, as it was the first solvable model of statistical physics with a genuine phase transition. The basic idea is simple: we have a single classical spin $\sigma = \pm 1$ at each site $\vec{x}$ of a square two-dimensional spatial lattice, with an energy functional that wants to align neighboring spins. More concretely we can parametrize the lattice sites by

$$\vec{x} = (an_x, an_y), \tag{8.45}$$

with $n_x \in n_y \in \mathbb{Z}$, and the energy functional is given by

$$E[\sigma] = -J \sum_{n_x, n_y} \Big( \sigma(n_x, n_y)\sigma(n_x+1, n_y) + \sigma(n_x, n_y)\sigma(n_x, n_y+1) \Big). \tag{8.46}$$

This gives a model of a two-dimensional classical ferromagnet with an "easy" direction of magnetization, which is to be distinguished from the Heisenberg model which allows the spin to point in any direction.

At zero temperature the Ising model has two obvious ground states: we can have all of the spins to be up (i.e. $\sigma = 1$ everywhere), or we can have all of the spins to be down (i.e. $\sigma = -1$ everywhere). Either of these configurations is called "ordered". On the other hand at infinite temperature all states are equally likely, so each spin is independently random and we say the system is "disordered". At finite temperature things are more interesting. In the partition function

$$Z(\beta) = \sum_{\sigma} e^{-\beta E[\sigma]} \tag{8.47}$$

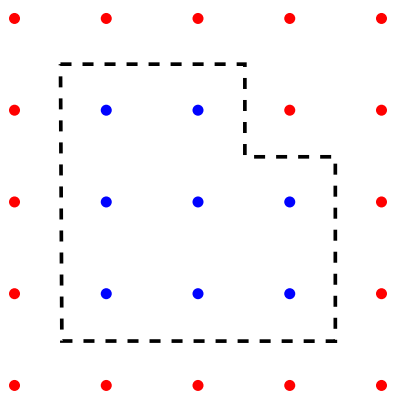

Figure 19: Flipping a cluster in the Ising model with perimeter $L = 12$. At high temperature it is entropically favorable to do this even though it increases the energy at the perimeter.

there is a battle between energy and entropy: misaligning some of the spins costs energy, and is thus punished by the Boltzmann factor $e^{-\beta E}$, but the more spins are misaligned the more ways there are of misaligning them and this can compete with the Boltzmann suppression at high enough temperature (remember that $\beta = 1/T$). There is a beautiful argument due to Peierls that there is some nonzero temperature below which the energy wins, ordering the system (almost all spins up or down), and some other finite temperature above which the entropy wins and the system is disordered. The idea is to look at the energy cost of flipping a connected cluster of spins compared to the gain in entropy of doing so. Say that the spins are currently all aligned. The cost in energy of flipping a cluster whose boundary perimeter (meaning the number of bonds which are misaligned) is $L$ is

$$\Delta E = 2JL, \tag{8.48}$$

while we can upper bound the number $N(L)$ of clusters of perimeter $L$ containing some fixed lattice site $(n_x, n_y)$ by saying that as we go around the perimeter for each link we need to choose one of three directions to go in, which gives

$$N(L) < 3^L. \tag{8.49}$$

Therefore when

$$2\beta J > \log 3 \tag{8.50}$$

the gain in entropy from flipping a cluster can't possibly balance the energy cost of doing so, so it is better not to flip and the system is ordered. See figure 19 for an illustration. On the other hand, although it is hard to compute $N(L)$ exactly it isn't too hard to argue that it is *lower* bounded by

$$N(L) < C^L \tag{8.51}$$

for some $1 < C < 3$ (basically the constraints that the perimeter needs to close and not self-intersect aren't strong enough to completely kill the exponential from choosing which direction to go at each step). This means that in the partition function the gain in entropy will certainly beat the energy cost if we have

$$2\beta J < \log C, \tag{8.52}$$

in which case many clusters will form and the system will disorder.

It is harder to say what happens in the intermediate temperature window

$$\frac{2J}{\log 3} < T < \frac{2J}{\log C} \tag{8.53}$$

between these two bounds, but in fact what happens is that there is a single critical temperature

$$T_c = \frac{2J}{\log(1+\sqrt{2})} \tag{8.54}$$

above which the system is disordered and below which it is ordered. Above the critical temperature the spin-spin correlation function decays exponentially with distance,

$$\langle \sigma(0)\sigma(x)\rangle \sim e^{-|x|/\xi}, \tag{8.55}$$

where $\xi$ is called the **correlation length**. At the critical temperature the correlation length goes to infinity and the system becomes scale-invariant, so the spin-spin correlation function can only decay as a power:

$$\langle \sigma(0)\sigma(x)\rangle \sim \frac{1}{|x|^\eta}. \tag{8.56}$$

The dimensionless number $\eta$ is another example of a **critical exponent**, similar to the critical exponent $\nu$ that we computed approximately in the three-dimensional Ising model using the $\epsilon$-expansion at the end of the previous semester. As in that case, we can rewrite $\eta$ in terms of the **anomalous dimension** of the spin operator $\sigma$:

$$\eta = 2\Delta_\sigma. \tag{8.57}$$

Computing these critical exponents is the great achievement of Onsager's solution, and in particular there is the famous formula

$$\Delta_\sigma = \frac{1}{8}. \tag{8.58}$$

In the remainder of this section we will see how the Ising model can be rewritten in terms of a free lattice fermion, which makes it clear why it is soluble.[30]

As a side comment, it is great fun to produce samples of the 2D Ising model using the **Metropolis algorithm**. Starting from an arbitrary initial configuration this works by going through the sample and flipping each spin either if doing so decreases the total energy or with probability $e^{-\beta\Delta E}$ even if it doesn't. This algorithm converges towards a typical sample from the thermal distribution with inverse temperature $\beta$ rather quickly, giving a powerful way to study statistical systems numerically. See figure 20 for some Ising samples generated this way.

## 8.5 Transfer matrix and the Hamiltonian formulation for the 1D Ising model

One of the most powerful ideas in statistical mechanics is the transfer matrix, which in many cases allows us to convert a classical statistical system at finite temperature in $D$ spatial dimensions to a quantum system at zero temperature in $D$ Euclidean spacetime dimensions. Roughly speaking this limit replaces thermal fluctuations by quantum fluctuations, with $\hbar$ being the effective "temperature". As a warmup let's first see how this works for the one-dimensional classical Ising model, which has energy functional

$$E[\sigma] = \frac{J}{2}\sum_n (\sigma(n+1)-\sigma(n))^2. \tag{8.59}$$

Here I've taken the liberty of shifting the ground state energy to be zero, which simplifies life. The thermal partition function of this system is

$$Z(\beta) = \sum_\sigma e^{-\beta E[\sigma]}, \tag{8.60}$$

[30] The realization that the model can be rewritten in terms of free fermions goes back to Schultz, Mattis, and Lieb in 1964. Elliott Lieb is a legendary mathematical physicist, who also proved the strong subadditivity of von Neumann entropy and important results on the quantum stability of matter. He is still around (he's 92), I had fun talking to him when I was a postdoc at Princeton and I just saw him last month when I visited!

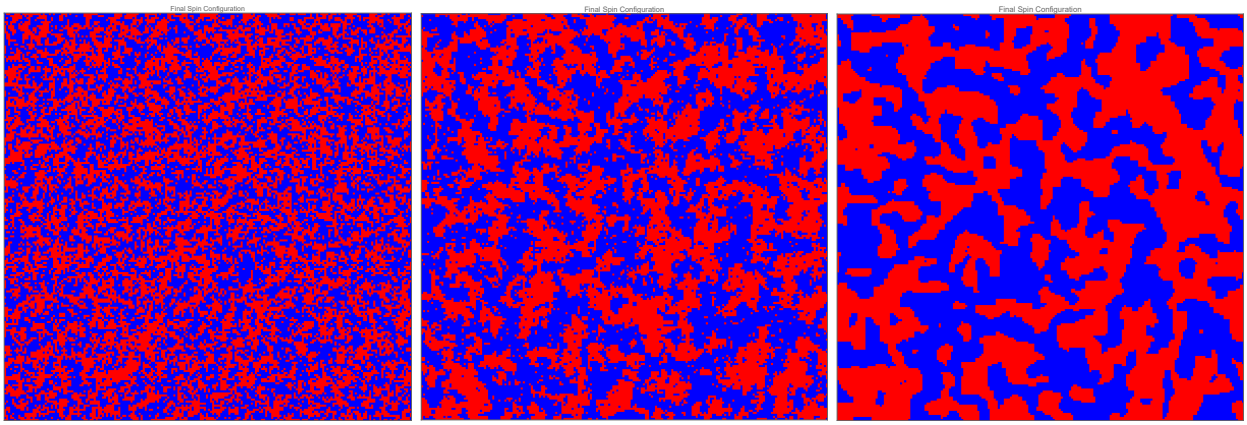


Figure 20: Samples of the 2D Ising model on a $200 \times 200$ lattice generated by the Metropolis algorithm starting from random spins. On the left we have $T = 4J$, in the center we are at the critical temperature $T = \frac{2J}{\log(1+\sqrt{2})} \approx 2.269J$, and on the right we have $T = .5J$. At high temperature you can see the spins stay disordered, at the critical temperature blocks of arbitrary sizes are forming and disappearing, while at low temperature the spins are coalescing into large aligned regions which will eventually merge and order the magnet.

and if we study this with periodic boundary conditions (i.e. so that $\sigma(N+1) = \sigma(1)$) then we can write this partition function as

$$Z(\beta) = \mathrm{Tr}\left(T^N\right) \tag{8.61}$$

where

$$T_{\sigma',\sigma} = e^{-\frac{\beta J}{2}(\sigma'-\sigma)^2} \tag{8.62}$$

is called the **transfer matrix**. We can think of $T$ as the contribution to the partition function from the link connecting spin $\sigma$ to spin $\sigma'$, and the trace arises because of our periodic boundary conditions. In matrix notation we have

$$T = \begin{pmatrix} 1 & e^{-2\beta J} \\ e^{-2\beta J} & 1 \end{pmatrix}. \tag{8.63}$$

The connection to quantum mechanics arises because when $\beta J \gg 1$ we can view $T$ as the infinitesimal Euclidean evolution by a Hamiltonian:

$$T \approx 1 - \epsilon H + O(\epsilon^2), \tag{8.64}$$

with

$$\begin{aligned} \epsilon &= e^{-2\beta J} \\ H &= -\sigma_x. \end{aligned} \tag{8.65}$$

We can then take the continuum limit $\epsilon \to 0$, in which case a correlation function involving Euclidean time separation $\tau$ comes from a lattice separation $\Delta n$ given by

$$\Delta n = \frac{\tau}{\epsilon}. \tag{8.66}$$

That the "bare coupling" $\beta J$ goes to infinity in this limit is an example of renormalization group flow. By construction correlation functions computed in the continuum limit will match those computed at distances which are large compared to the lattice scale in the original model. Indeed we can compute the spin-spin correlation function: the ground state is simply

$$|\Omega\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right), \tag{8.67}$$

and the spin-spin correlation function is

$$\langle\Omega|e^{-\tau\sigma_x}\sigma_z e^{\tau\sigma_x}\sigma_z|\Omega\rangle = e^{-2\tau}, \tag{8.68}$$

which indeed decays exponentially at large $\tau$ as we would expect for a disordered phase (the 1D Ising model is always disordered).

## 8.6 Hamiltonian formulation of the 2D Ising model

Let's now repeat this analysis for the 2D Ising model, which we will generalize to allow the coupling $J$ to be different in the two different directions:

$$E[\sigma] = \frac{J_x}{2}\sum_{n_x,n_y}(\sigma(n_x+1,n_y)-\sigma(n_x,n_y))^2 + \frac{J_y}{2}\sum_{n_x,n_y}(\sigma(n_x,n_y+1)-\sigma(n_x,n_y))^2. \tag{8.69}$$

We can again write the partition function with periodic boundary conditions in the $y$ direction as

$$Z(\beta) = \mathrm{Tr}\left(T^{N_y}\right), \tag{8.70}$$

where now $T$ is a $2^{N_x}\times 2^{N_x}$ matrix whose entries are given by

$$T_{\sigma',\sigma} = \exp\left[-\frac{\beta J_y}{2}\sum_{n_x}(\sigma'(n_x)-\sigma(n_x))^2 - \frac{\beta J_x}{2}\sum_{n_x}(\sigma(n_x+1)-\sigma(n_x))^2\right]. \tag{8.71}$$

Here you should think of $\sigma(n_x)$ as the spins in row $n_y$ and $\sigma'(n_x)$ as the spins in the row $n_y+1$. The transfer matrix includes the Ising interactions for the links connecting these two rows (the first term) and also the Ising interactions within the $n_y$ row. Multiplying the transfer matrix together $N_y$ times and then taking the trace thus accounts for each interaction once.

To get a quantum interpretation of the transfer matrix we again want to take the limit of continuous time, which we will arbitrarily interpret to be the $y$ direction. We can guess the right way to take this limit by noting that if $\sigma$ and $\sigma'$ differ at $m$ sites then we have

$$T_{\sigma',\sigma} = e^{-2m\beta J_y}\exp\left[-\frac{\beta J_x}{2}\sum_{n_x}(\sigma(n_x+1)-\sigma(n_x))^2\right]. \tag{8.72}$$

We want a limit where

$$T \approx 1 - i\epsilon H, \tag{8.73}$$

so in particular we want the $m=0$ components to be close to one. In other words we need $\beta J_x$ to be small. Moreover we would like first corrections to the identity for $m=0$ and $m=1$ to have comparable size, as otherwise we will get a trivial theory where there is no relation between the different rows. The right way to do this is to take

$$\begin{aligned}\beta J_y &\to \infty\\ \beta J_x &\to \lambda e^{-2\beta J_y}\end{aligned} \tag{8.74}$$

with $\lambda$ fixed, as (8.73) then holds with

$$\begin{aligned}\epsilon &= e^{-2\beta J_y}\\ H &= -\sum_n \sigma_x(n) + \frac{\lambda}{2}\sum_n(\sigma_z(n+1)-\sigma_z(n))^2\\ &= -\sum_n \sigma_x(n) - \lambda\sum_n \sigma_z(n)\sigma_z(n+1) + \text{constant}.\end{aligned} \tag{8.75}$$

This Hamiltonian is traditionally called the **1D Ising model in a transverse field**, or sometimes the **quantum Ising chain**. The two terms in the Hamiltonian do not commute, so the ground state is a tug of war between them. When $\lambda$ is small the ground state is just the product of the $\sigma_x = 1$ state for each spin, with no long-range correlation, while when $\lambda$ is large then the spins all want to be aligned in the $\sigma_z$ basis. This of course is just the phase structure we found for the original 2D Ising model, which apparently has survived the quantum limit! In particular we should expect a phase transition at some critical value of $\lambda$; we will now see that the transition happens at $\lambda = 1$ and is governed by a free Majorana fermion.

## 8.7 2D Ising as a free Majorana fermion

Now we come to the moment of truth. Let's return to our lattice Majorana fermion $\Psi(n)$ with Hamiltonian

$$H = \frac{-i}{2a} \sum_n \Psi(n)\Psi(n+1). \tag{8.76}$$

We've already seen that at low energies this theory is described by the continuum free massless Majorana fermion theory (8.22) with both left and right moving fields $\Psi^{cont}_{L/R}$ due to fermion doubling. This theory is scale-invariant, so its correlation functions decay as powers just as we saw for the 2D Ising model at the critical point. In fact they are the same powers, as we will now see.

The key idea is to use our old Jordan-Wigner transformation to rewrite these Majorana fermions in terms of qubits. We saw back in the second section that we can represent the algebra of $2N$ Majorana fermions on the Hilbert space of $N$ qubits, with the fermions represented by products of Pauli operators. More concretely we have a qubit for each even $n$, and we have the following representations:

$$\begin{aligned} \psi(2m) &= \ldots \sigma_z \otimes \sigma_z \otimes \frac{\sigma_x}{\sqrt{2}} \otimes I \otimes I \otimes \ldots \\ \psi(2m+1) &= \ldots \sigma_z \otimes \sigma_z \otimes \frac{\sigma_y}{\sqrt{2}} \otimes I \otimes I \otimes \ldots, \end{aligned} \tag{8.77}$$

where the $\sigma_x$ and $\sigma_y$ are acting on the $m$th qubit. We can therefore rewrite the Hamiltonian (8.76) as

$$\begin{aligned} H &= -\frac{i}{2a} \sum_m \Big( \Psi(2m)\Psi(2m+1) + \Psi(2m-1)\Psi(2m) \Big) \\ &= \frac{1}{4a} \sum_m \Big( \sigma_z(m) + \sigma_x(m-1)\sigma_x(m) \Big). \end{aligned} \tag{8.78}$$

We can act with a rotation by $\pi/2$ in the $zx$ plane and shift the sum in the second term to rewrite this as

$$H = \frac{1}{4a} \Big( -\sigma_x(m) + \sigma_z(m)\sigma_z(m+1) \Big) \tag{8.79}$$

which (up to an overall rescaling) is precisely the transverse field Ising Hamiltonian (8.75)! More precisely, it is the transverse field Ising Hamiltonian with $\lambda = 1$, which we can thus identify as the critical point since we've now shown it has a massless dispersion relation at low energy. Therefore all the critical exponents of the 2D Ising model can be computed using the free Majorana fermion theory (8.22), and indeed they agree with the critical exponents from Onsager's solution and also with experiment! There is much more that could be said about this, but we need to get on to electromagnetism so I'll content myself with a few comments:

- You might be wondering what happens in the fermion description if we set $\lambda \neq 1$ in the transverse field Ising description. Working this out shows that it introduces a mass in the fermion theory, so the correlation length indeed becomes finite away from criticality.

- You may be concerned that the anisotropic continuous-time limit we used to derive the transverse-field Ising Hamiltonian is quite different from the square lattice isotropic Ising model we started with. The reason we ended up with the same critical exponents is the renormalization group: the details of the lattice structure (such as whether or not it is isotropic) do not matter in the continuum since by Polchinski's theorem they are irrelevant. On the other hand questions which specifically refer to bare parameters, such as the transition temperature $T_c$ (or the transition value of $\lambda$) do depend on the lattice details and therefore cannot be computed from the continuum free field theory.

- The next thing we'd do if we had more time is the free field computation of the spin-spin correlation function to recover Onsager's famous $\Delta_\sigma = 1/8$. This is trickier than you might expect however, as it isn't so obvious how to represent the spin operator $\sigma_z(m)$ in terms of the continuum Majorana fermion. The correct thing to do is to remove a small disk in the vicinity of the operator from the path integral and then impose boundary conditions at the edge of the disk where the fermion changes sign as you go around it.[31] To compute the two-point function the best thing to do is make use of the conformal symmetry of the problem, which relates the scaling dimension of this operator to the energy of the ground state on a spatial circle with periodic boundary conditions for the fermion. This ground state energy is UV-divergent, but can be renormalized to give a nice finite answer called the **Casimir energy**. When the dust settles you indeed find $\Delta_\sigma = 1/8$.

**Problems:**

1. Confirm the equation of motion (8.41) and the dispersion relation (8.43) for the general lattice Hamiltonian (8.39), and also check that the frequency $\omega(k)$ is real.

# 9 Quantum electrodynamics I: quantization of the electromagnetic field

In this section we now turn to **quantum electrodynamics**, usually called QED, which is the quantum field theory describing the interactions of photons with charged particles. QED is responsible for most daily phenomena we encounter, from atoms to chemistry to solids, and its predictions have been confirmed experimentally to remarkable accuracy. It is the quantum field theory for which quantum field theory was invented, and it is the still the prototypical example. It has several novel features that we have not yet encountered:

- **Helicity one:** So far we have only studied field theories whose particles have spin/helicity zero or 1/2. The photon has helicity one, which brings new complications. The fundamental issue is that massive and massless particles of spin/helicity zero or 1/2 have the same number of states, but for spin/helicity one and higher a massless particle has fewer helicity states than a massive particle has spin states. For example in $d = 4$ a spin one particle has three spin states $(-1, 0, 1)$ but a helicity one particle only has two $(\pm 1)$. So our formalism needs to somehow be discontinuous in the massless limit.

- **Gauge symmetry:** In order to solve the previous problem, it turns out that we need to introduce a new kind of symmetry. This is a symmetry which can be implemented by a different amount at different points in spacetime, and we will see that such symmetries amount to redundancies rather than physical transformations. This is the mechanism that removes the extra spin state of massless photon. Gauge symmetries can be quite confusing, and in particular they imply that some of the natural observables in a QFT with a gauge symmetry are nonlocal.

[31] It may not be clear that this really defines a valid local operator in the theory, but it does. To understand this properly it is important to realize that the fermion parity symmetry $\Psi'_{L/R} = -\Psi_{L/R}$ must be treated as a gauge symmetry (see the next section) in order for the match between the critical Ising and massless Majorana Hamiltonians to work. Essentially this is because of the Jordan-Wigner strings - they should really be thought of as "Wilson lines" for fermion parity, and in particular if we work with periodic boundary conditions in space then the transverse Ising Hilbert space only allows the fermions to act in pairs.

- **Infrared divergences:** The scattering theory we have presented so far, based on the LSZ formula, works best when all particles are massive. In QED the photon is massless, which leads to new complications in defining asymptotic states. In particular an electron can never be found in pure isolation, it always needs to carry around its Coulomb field, and this field is made out of a large number of photons. We will see that this implies that the S-matrix element from some fixed number of photons and electrons to some other fixed number is always zero, since the probability is one that an infinite number of "soft" photons are radiated in any scattering process. We therefore need to learn how to do scattering with infinite numbers of particles in the initial/final state.

We clearly have our work cut out for us, so let's begin.

## 9.1 Maxwell theory

You at some point learned about Maxwell's equations, which in SI units are written as

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon_0}\\ \vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{E} &= -\dot{\vec{B}}\\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{J}+\epsilon_0\mu_0\dot{\vec{E}}.\end{aligned} \tag{9.1}$$

Together with the Lorentz force law

$$\vec{F} = q\vec{E}+q\vec{v}\times\vec{B} \tag{9.2}$$

these give a complete description of the classical electromagnetic force. Of course SI units are not for real scientists; a better choice is **Heaviside-Lorentz units**, in which these equations read

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \rho\\ \vec{\nabla}\cdot\vec{B} &= 0\\ \vec{\nabla}\times\vec{E} &= -\frac{1}{c}\dot{\vec{B}}\\ \vec{\nabla}\times\vec{B} &= \frac{1}{c}\left(\vec{J}+\dot{\vec{E}}\right)\\ \vec{F} &= q\vec{E}+q\frac{\vec{v}}{c}\times\vec{B}.\end{aligned} \tag{9.3}$$

We will of course also set $c = 1$, in which case you can remember that to get to Heaviside-Lorentz units (with $c = 1$) you can just set $\epsilon_0 = \mu_0 = 1$ - every physics undergraduate's dream come true!

You hopefully also learned that you can automatically solve two of Maxwell's equations by writing $\vec{E}$ and $\vec{B}$ in terms of a scalar potential $\phi$ and a vector potential $\vec{A}$, via

$$\begin{aligned}\vec{B} &= \vec{\nabla}\times\vec{A}\\ \vec{E} &= -\vec{\nabla}\phi-\dot{\vec{A}}.\end{aligned} \tag{9.4}$$

This parametrization is not unique, as we can shift both $\vec{A}$ and $\phi$ by a **gauge transformation**

$$\begin{aligned}\vec{A}' &= \vec{A}+\vec{\nabla}\Omega\\ \phi' &= \phi-\dot{\Omega},\end{aligned} \tag{9.5}$$

where $\Omega$ is an arbitrary function of space and time, without changing $\vec{E}$ and $\vec{B}$.

This all looks much nicer if we adopt more relativistic notation. We can combine $\phi$ and $\vec{A}$ into a one-form **gauge field** as

$$A_\mu = (-\phi, \vec{A}), \tag{9.6}$$

in terms of which the gauge transformation is simply

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Omega. \tag{9.7}$$

The electromagnetic current becomes a spacetime vector

$$J^\mu = (\rho, \vec{J}). \tag{9.8}$$

We can combine $\vec{E}$ and $\vec{B}$ into the antisymmetric **field strength tensor**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{9.9}$$

in terms of which we have

$$E_i = F_{i0} \tag{9.10}$$

and

$$B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}. \tag{9.11}$$

Maxwell's equations in this notation are simply

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= J^\mu \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0, \end{aligned} \tag{9.12}$$

with the second line following automatically from the expression for $F$ in terms of $A$. You showed on the homework last semester that these equations follow from the Maxwell Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu, \tag{9.13}$$

where $A_\mu$ is treated as the fundamental dynamical variable. In this section we will treat $J^\mu$ as a background source for $A_\mu$, while in the next section we will build $J^\mu$ out of dynamical charged fields. Equations (9.12) and (9.13) are valid in any dimension, so from now on we will consider Maxwell theory in $d$ spacetime dimensions. We will soon see that the theory is inconsistent unless $J^\mu$ obeys the conservation equation $\partial_\mu J^\mu = 0$, which is a reasonable requirement for a background current. One indication of this is that with this requirement the Lagrangian density is invariant under gauge transformations up to a total derivative:

$$A'_\mu J^\mu - A_\mu J^\mu = \partial_\mu \Omega J^\mu = \partial_\mu(\Omega J^\mu) - \Omega \partial_\mu J^\mu. \tag{9.14}$$

## 9.2 Towards a Hamiltonian formulation

Life gets more challenging when we try to convert the Maxwell Lagrangian (9.13) to a Hamiltonian. The reason this is complicated is the following. Naively we might expect to view $A_\mu$ and $\dot{A}_\mu$ at $t = 0$ as "initial data", which can then be evolved forward using Maxwell's equations to predict the future. This is an example of what is called the initial value problem, which is the essence of the Hamiltonian formulation of classical mechanics: given a point in phase space specified by $q$ and $p$, we uniquely evolve forward (or backward) in time by solving Hamilton's equation. In Maxwell theory however the evolution isn't unique: we can always do a gauge transformation $A'_\mu = A_\mu + \partial_\mu \Omega$, with $\Omega$ vanishing in the vicinity of $t = 0$ but doing something arbitrary elsewhere, and this gives a distinct but equally valid solution of the equations of motion with the same "initial data". The naive initial value problem therefore isn't well-posed in Maxwell theory. This has two more concrete manifestations:

(1) The time derivative of $A_0$ doesn't appear in the Lagrangian, so its canonical momentum $\Pi^0$ is zero. This isn't consistent with the canonical commutation relation $[A_0(\vec{x}, \Pi^0(\vec{y})] = i\delta^{d-1}(\vec{x} - \vec{y})$.

(2) The canonical conjugate of $A_i$ is

$$\Pi^i = \frac{\partial \mathcal{L}}{\partial \partial_0 A_i} = F^{i0} = -E^i, \tag{9.15}$$

so we can write the 0th component of the equation of motion as

$$\partial_i F^{0i} = -\partial_i \Pi^i = \rho. \tag{9.16}$$

This of course is just Gauss's law, but the the point here is that it doesn't involve the time derivative of $\Pi^i$ and thus gives a **constraint** on the canonical variables $A_i$ and $\Pi^i$ instead of a genuinely dynamical equation of motion. This constraint again is not compatible with the canonical commutation relations

$$[A_i(\vec{x}, \Pi^j(\vec{y})] = i\delta_i^j \delta^{d-1}(\vec{x} - \vec{y}), \tag{9.17}$$

as these would allow us to adjust each component of $\vec{\Pi}$ independently by conjugating by $e^{i\int d^{d-1}x A_i(\vec{x}) f^i(\vec{x})}$ with $f^i$ arbitrary.

We can view the condition $\Pi^0 = 0$ as a constraint as well, so the basic problem we need to solve in order to come up with a sensible Hamiltonian formulation of Maxwell theory is to figure out how to handle the constraints

$$\begin{aligned} \Pi^0 &= 0 \\ \partial_i \Pi^i + \rho &= 0. \end{aligned} \tag{9.18}$$

The pair of constraints (9.18) are an example of what are called **first-class constraints**, which means that if we compute the commutators of the quantities on the left-hand side of the constraint equations using the naive canonical commutation relation

$$[A_\mu(\vec{x}), \Pi^\nu(\vec{y})] = i\delta_\mu^\nu \delta^{d-1}(\vec{x} - \vec{y}) \tag{9.19}$$

we get something which vanishes after imposing the constraints. That is clearly true here, as the constraints only involve $\Pi^\mu$ and not $A_\mu$. First class constraints are always related to gauge symmetries, essentially because their commutators form a Lie algebra that we can exponentiate to construct the gauge transformations and they always commute with the Hamiltonian since it is gauge-invariant.[32] We can check explicitly here that they generate the gauge symmetry:[33]

$$\begin{aligned} i\left[\int d^{d-1}x' \dot{\Omega}(\vec{x}')\Pi^0(\vec{x}'), A_0(\vec{x})\right] &= \dot{\Omega}(\vec{x}) \\ i\left[-\int d^{d-1}x' \Omega(\vec{x}')\left(\partial_j \Pi^j(\vec{x}') + \rho(\vec{x}')\right), A_i(\vec{x})\right] &= \partial_i \Omega(\vec{x}). \end{aligned} \tag{9.20}$$

In the second of these we neglected a boundary term; this is justified for gauge transformations which vanish at infinity. We will consider the case of gauge transformations which don't vanish at infinity later in the section. In the meantime we have learned the following: in order to have a well-posed initial value problem in a system with first-class constraints, *it is necessary to view gauge transformations which vanish at infinity as redundancies of description rather than physical transformations.* There are two standard ways to do this:

- **Gauge-fixing:** Impose some kind of additional requirement on the dynamical variables which removes the gauge symmetry. For example in electrodynamics we can impose the Coulomb gauge condition $\vec{\nabla} \cdot \vec{A} = 0$, which we will shortly argue removes the gauge freedom. We can then solve the constraints explicitly (including the gauge-fixing condition) to get an unconstrained system.

[32] There is a converse to this statement called Noether's second theorem, which in this language says that if the Lagrangian has a gauge symmetry then it is generated by first-class constraints.

[33] In the second of these we neglected a boundary term, we will be more careful about it later in the section.

- **Quotient by gauge transformations:** Apply an equivalence relation $A_\mu \sim A_\mu + \partial_\mu \Omega$ to the set of solutions, so that they are only defined modulo gauge transformations which vanish at infinity. Physical observables are then required to be gauge-invariant, in which case there is a good initial value problem for all observables. Quantum mechanically we start with a larger Hilbert space that doesn't obey the constraints and then we restrict to the set of gauge-invariant states which are annihilated by them.

The first method is more standard in practice, but it has two unfortunate aspects. The first is that fixing the gauge breaks manifest Lorentz invariance, leading to unwieldy expressions which magically end up being Lorentz-invariant at the end of the calculation. The second is that gauge-fixing introduces apparent non-locality in the Hamiltonian, which again magically cancels at the end of the calculation. For these reasons gauge-fixing is often a source of confusion in quantum field theory, for students and researchers alike. The second method is more elegant, as it preserves manifest Lorentz invariance and locality at every step, but it is also more abstract and in particular it requires us to introduce a larger Hilbert space that includes "unphysical" states. We then need to confirm that our evolution does not mix unphysical states with physical states. Moreover some calculations in the Hamiltonian formalism (which anyways breaks covariance) are easier once we fix the gauge. Our approach here will be to take the second approach as fundamental and derive the first approach from within it, which I think is the most enlightening way to proceed: we never need to worry about whether the theory we are defining is Lorentz-invariant and local, but we are free to fix the gauge whenever doing so is convenient.

## 9.3 Gauge-invariant quantization

The idea of gauge-invariant quantization is that we start with a large set $\mathcal{H}_{big}$ of wave functionals $\Psi[A]$ that are arbitrary functions of all $d$ components of the gauge field $A_\mu(\vec{x})$. The canonical momenta are defined by

$$\Pi^\mu(\vec{x}) = -i\frac{\delta}{\delta A_\mu(\vec{x})}, \tag{9.21}$$

which by construction obeys the naive canonical commutation relation (9.19). We then construct the true Hilbert space $\mathcal{H}$ by restricting to states which are annihilated by the two constraints (9.18). Since these constraints generate the gauge symmetry, this is the same as restricting to the set of gauge-invariant states. People thus often use the terms "physical Hilbert space" or "gauge-invariant Hilbert space" to describe $\mathcal{H}$. The set of states in $\mathcal{H}_{big}$ which are annihilated by $\Pi^0$ is quite simple: it is the functionals which are independent of $A_0$. We therefore can just write the wave functional as $\Psi[\vec{A}]$. The physical states are then those wave functionals obeying

$$i\partial_j \frac{\delta}{\delta A_j(\vec{x})}\Psi[\vec{A}] = -\rho(\vec{x})\Psi[\vec{A}]. \tag{9.22}$$

We also need to define a Hamiltonian on $\mathcal{H}$. We can first try to construct a Hamiltonian on $\mathcal{H}_{big}$ in the usual way:

$$\begin{aligned} H &= \int d^{d-1}x\big[\dot{A}_0\Pi^0 + \dot{\vec{A}}\cdot\vec{\Pi} + \frac{1}{2}F^{0i}F_{0i} + \frac{1}{4}F_{ij}F^{ij} - A_\mu J^\mu\big] \\ &= \int d^{d-1}x\left[\frac{1}{2}\vec{\Pi}\cdot\vec{\Pi} + \frac{1}{4}F_{ij}F^{ij} - \vec{A}\cdot\vec{J} + \vec{\nabla}\cdot(A_0\vec{\Pi}) - A_0\left(J^0 + \vec{\nabla}\cdot\vec{\Pi}\right) + \dot{A}_0\Pi^0\right]. \end{aligned} \tag{9.23}$$

We can drop the total derivative term by assuming boundary conditions where $A_0 \to 0$ at $\infty$. This Hamiltonian however is not well-defined on $\mathcal{H}_{big}$, because $\dot{A}_0$ is not something we know how to build out of $A_\mu$ and $\Pi^\mu$. Indeed if we compute the commutator of $H$ with $A_0$ we just get

$$\dot{A}_0 = \dot{A}_0, \tag{9.24}$$

which although true is not very helpful. This is a concrete illustration of our inability to define a good time evolution on the full Hilbert space $\mathcal{H}_{big}$. On the other hand there is no such obstruction to defining the

Hamiltonian on the physical Hilbert space $\mathcal{H}$: since the last two terms in (9.23) are both proportional to the constraints, they vanish when $H$ acts on states in $\mathcal{H}$. We therefore can take the Hamiltonian on $\mathcal{H}$ to simply be

$$H = \int d^{d-1}x \left[ \frac{1}{2}\vec{\Pi} \cdot \vec{\Pi} + \frac{1}{4} F_{ij} F^{ij} - \vec{A} \cdot \vec{J} \right]. \tag{9.25}$$

For the theory to be consistent, we need to check that this Hamiltonian evolves gauge-invariant states to gauge-invariant states. This is a bit subtle in general because if $\rho$ is time-dependent then the Gauss constraint is also time-dependent, so in the Schrodinger picture the gauge-invariant subspace changes with time. The condition we need is that

$$\left( \vec{\nabla} \cdot \vec{\Pi}(\vec{x}) + \rho(t+\epsilon, \vec{x}) \right) e^{-i\epsilon H(t)} = e^{-i\epsilon H(t)} \left( \vec{\nabla} \cdot \vec{\Pi}(\vec{x}) + \rho(t, \vec{x}) \right) \tag{9.26}$$

for small $\epsilon$, or equivalently

$$i[H(t), \vec{\nabla} \cdot \vec{\Pi}(\vec{x})] = -\dot{\rho}(t, \vec{x}). \tag{9.27}$$

This is indeed true:

$$\begin{aligned} i[H(t), \vec{\nabla} \cdot \vec{\Pi}(\vec{x})] &= -i \int d^{d-1}x' [\vec{A} \cdot \vec{J}(t, \vec{x}'), \vec{\nabla} \cdot \vec{\Pi}(\vec{x})] \\ &= \int d^{d-1}x' \partial_i J^i(t, \vec{x}') \delta^{d-1}(\vec{x}' - \vec{x}) \\ &= \partial_i J^i(t, \vec{x}) \\ &= -\dot{\rho}(t, \vec{x}). \end{aligned} \tag{9.28}$$

Note that when $\dot{\rho} \neq 0$ this means that the Hamiltonian isn't gauge-invariant. This is an artifact of treating $J^\mu$ as a background field; once we build $J^\mu$ out of dynamical fields then this gauge transformation for the Hamiltonian will be canceled by a compensating transformation from the matter Lagrangian so $H$ will be gauge-invariant. It is important to emphasize that this argument *only works if $J^\mu$ is conserved*: it is inconsistent to try to quantize a Maxwell field coupled to a current which isn't conserved.

## 9.4 Maxwell theory in $1+1$ dimensions

As a simple example of this approach, we can consider electrodynamics in $1+1$ dimensions with $J = 0$ on a spatial circle of circumference $L$. The equations of motion say that

$$\partial_0 F^{01} = \partial_1 F^{01} = 0, \tag{9.29}$$

so the electric field

$$E = F^{01} \tag{9.30}$$

is constant in spacetime. What are the quantum states? The Gauss constraint (9.22) says that the wave functional depends on $A_1$ at different points in exactly the same way, or in other words that it only depends on $A_1$ through the combination

$$h = \int_0^L dx A_1(x), \tag{9.31}$$

which is called the **holonomy** around the loop. It is a gauge-invariant variable since

$$h' = \int_0^L dx \left( A_1(x) + \partial_1 \Omega(x) \right) = h + \Omega(L) - \Omega(0) = h, \tag{9.32}$$

since we must have $\Omega(0) = \Omega(L)$ because we are on a circle. The classical phase space of this system is two-dimensional, and in fact $h$ and $-E$ are canonical conjugates:

$$[h, -E] = i. \tag{9.33}$$

The Hamiltonian (9.25) is simply

$$H = \frac{L}{2}E^2, \tag{9.34}$$

so Maxwell theory in $1+1$ dimensions is the same as the problem of the quantum mechanics of a non-relativistic particle on an infinite line! There are few interesting things about this example. First of all there are no propagating photons - naively $A_\mu$ has two independent components which could support propagating waves, but the gauge constraints removed both of them. There is however still a single remaining degree of freedom, the holonomy $h$ and its canonical conjugate $-E$. $h$ is particularly interesting because it is *nonlocal*; in gauge theories nonlocal observables are fairly often of interest. Another worthwhile observation is that the energy grows linearly with the system size $L$; we will eventually see that this is the mechanism behind the confinement of quarks and gluons into hadrons in quantum chromodynamics (QCD), which is the theory of the strong nuclear force.

The holonomy $h$ brings up an interesting question about the foundations of quantum electrodynamics. In classical electromagnetism we learn that the fundamental degrees of freedom are the electric and magnetic fields, with $A_\mu$ something of a mathematical afterthought. The holonomy $h$ is novel in this regard because *there is no way to express it in terms of electric and magnetic fields.* Indeed in $1+1$ dimensions there is no magnetic field, and $h$ is clearly independent of $E$ since they don't commute. Thus there apparently is more to electromagnetism than just $\vec{E}$ and $\vec{B}$! On the other hand, is the holonomy really measureable? In fact it is, as is beautifully illustrated by the **Aharanov-Bohm effect**. Say we have a non-relativistic charged particle of mass $m$ moving on a circle of circumference $L$, about which the holonomy $h$ is not zero. Without loss of generality we can take $A_1$ to be constant, in which case $h = LA_1$. The Hamiltonian for a non-relativistic particle moving in an electromagnetic field is

$$H = \frac{|p - qA|^2}{2m} - qA_0, \tag{9.35}$$

which in this case is just

$$H = \frac{(p - qh/L)^2}{2m}. \tag{9.36}$$

The periodic boundary conditions require $p$ to be quantized so that the wave function is single-valued, with quantization

$$p = \frac{2\pi n}{L}, \tag{9.37}$$

so the energy levels of this system are

$$E_n = \frac{(2\pi n - qh)^2}{2mL^2}. \tag{9.38}$$

Note in particular that they depend on $h$, which thus can indeed be measured by looking at the time evolution of the relative phase of the wave function in a superposition of two energy eigenstates.

## 9.5 Higher dimensions

We can also apply this method to quantize Maxwell theory in higher dimensions, returning now to $d$-dimensional Minkowski spacetime. Following our usual practice we should begin by solving the field equations in the Heisenberg picture. We need to be careful however: the Hamiltonian is only defined on the gauge-invariant Hilbert space $\mathcal{H}$, so we cannot construct a Heisenberg evolution for $A_\mu$ since it doesn't act within $\mathcal{H}$. We thus need to apply our quantization algorithm to the field strength $F_{\mu\nu}$ instead of $A_\mu$. We won't pursue this all the way to the end, as the last part of the calculation is a bit tedious, but we will set it up since this will teach us some important things. The last step we will then complete using the gauge-fixing method in the subsection after next.

We'll nonetheless begin by constructing *classical* solutions of the $A_\mu$ equations of motion, which we can then use construct the field strength tensor as a Heisenberg operator on $\mathcal{H}$. As usual we look for solutions of the form

$$A_\mu(x) = e_\mu e^{ik\cdot x}, \tag{9.39}$$

where $e_\mu$ is called the **polarization one-form**. Substituting this into Maxwell's equation (with $J = 0$) we find

$$k^\mu(k_\mu e_\nu - k_\nu e_\mu) = k^2 e_\nu - (e \cdot k)k_\nu = 0. \tag{9.40}$$

Let's first consider the case where $k^2 \neq 0$. This equation then says that $e_\nu \propto k_\nu$. Solutions of this type are "pure gauge", since if $e_\mu = \alpha k_\mu$ then we have

$$A_\mu = -i\partial_\mu \left(\alpha e^{ik\cdot x}\right). \tag{9.41}$$

Pure gauge solutions do not contribute to the field strength tensor $F_{\mu\nu}$, as we can confirm from the expression

$$F_{\mu\nu} = i(k_\mu e_\nu - k_\nu e_\mu)e^{ik\cdot x} \tag{9.42}$$

which vanishes if $e_\mu \propto k_\mu$. What about solutions with $k^2 = 0$? The equation of motion (9.40) then tells us that we must either have $k_\nu = 0$ or else $k$ null with $e \cdot k = 0$. The holonomy in the previous section is an example of the former kind of solution, but in infinite volume we will adopt boundary conditions where $A \to 0$ at infinity so $k = 0$ solutions can't be excited. We are thus left with the solutions with null $k$ obeying

$$e \cdot k = 0. \tag{9.43}$$

To parametrize the set of $e$ which obey this equation we can find a basis of $d-1$ linearly independent vectors which are orthogonal to $k$, but these don't all describe distinct physical polarization states. Indeed since $k$ is null the polarization vectors $e_\mu$ and $e_\mu + \alpha k_\mu$ are both orthogonal to $k$ but they lead to the same field strength tensor $F_{\mu\nu}$. We therefore should impose an *equivalence relation*

$$e_\mu \sim e_\mu + \alpha k_\mu \tag{9.44}$$

on the set of polarization one-forms, which leads to a basis of $d-2$ distinct polarizations. These of course are the $d-2$ polarization states of a helicity-one particle, the photon, in $d$ spacetime dimensions.

To understand photon polarization more explicitly, we can recall our usual rule for the Lorentz transformations of polarization tensors. This is that for each little group element $\Lambda$ obeying $\Lambda k = k$, with $k$ the reference momentum, we have

$$D^a{}_{a'}(\Lambda)u^{a'}(\vec{k},\sigma) = u^a(\vec{k},\sigma')\hat{D}_{\sigma',\sigma}(\Lambda). \tag{9.45}$$

The little group of a massless particle is generated by the subset $SO(d-2)$ of the rotation group that preserves the direction of the reference momentum, together with a set of "null rotations" generated by the combinations of a rotation that changes the direction of $k$ together with a boost the changes it back. Specializing to $d = 4$ and taking the reference momentum to point in the $z$ direction,

$$k^\mu = \omega(1,0,0,1), \tag{9.46}$$

the generators of the little group are

$$\mathcal{J}^{12} = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{J}^{13}+\mathcal{J}^{01} = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathcal{J}^{23}+\mathcal{J}^{02} = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{9.47}$$

Raising the index on the polarization one-form to get a four vector $e^\mu$, Maxwell's equations tell us that we need $k \cdot e = 0$ and thus

$$e^\mu = \begin{pmatrix} \alpha\omega \\ \beta \\ \gamma \\ \alpha\omega \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \\ \gamma \\ 0 \end{pmatrix} + \alpha k^\mu. \tag{9.48}$$

We can figure out the helicity basis from the infinitesimal version of (9.45) (replacing $u \to e$),

$$\left(\mathcal{J}^{12}\right)^\mu{}_\nu \, e^\nu(\vec{k},\pm) = \pm e^\mu(\vec{k},\pm), \tag{9.49}$$

which tells us that polarization vectors of definite helicity are

$$e^\mu(\vec{k},\pm) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix} + \alpha_\pm k^\mu \tag{9.50}$$

with $\alpha_\pm$ arbitrary. You've hopefully seen these polarization vectors before, $\sigma = +1/-1$ describes right/ left handed circular polarization. As usual the polarization vectors for general momenta are then obtained by acting on these with an arbitrary Lorentz transformation $L_p$ that maps $k$ to $p$:

$$e^\mu(\vec{p},\sigma) = (L_p)^\mu{}_\nu e^\nu(\vec{k},\sigma). \tag{9.51}$$

The action of the rest of the little group on $e^\mu(\vec{k},\sigma)$ is more surprising. When we discussed the little group for massless particles last semester, we took the the generators $\mathcal{J}^{13}+\mathcal{J}^{01}$ and $\mathcal{J}^{23}+\mathcal{J}^{02}$ to act trivially in the representation $\hat{D}$, since otherwise a massless particle would need to have an infinite number of helicity states. This however is not what happens with the helicity vectors (9.50); instead we have

$$\left(\beta\left(\mathcal{J}^{13}+\mathcal{J}^{01}\right)+\gamma\left(\mathcal{J}^{23}+\mathcal{J}^{02}\right)\right)^\mu{}_\nu e^\nu(\vec{k},\pm) = \frac{i(\beta\pm i\gamma)}{\sqrt{2}}k^\mu. \tag{9.52}$$

So in other words (9.45) is actually impossible to satisfy for a massless particle of helicity one created/annihilated by a one-form gauge field! We can rephrase this by noting that if we try to define a "physical" Heisenberg gauge field

$$A^{phys}_\mu(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[e_\mu(\vec{p},\sigma)a_{\vec{p},\sigma}e^{ip\cdot x} + e^*_\mu(\vec{p},\sigma)a^\dagger_{\vec{p},\sigma}e^{-ip\cdot x}\right] \tag{9.53}$$

out of creation and annihilation operators on the gauge-invariant Fock space of helicity-one photons with some particular choice of $e_\mu$, under Lorentz transformations we have

$$U(\Lambda)^\dagger A^{phys}_\mu(x)U(\Lambda) = \Lambda_\mu{}^\nu A^{phys}_\nu(\Lambda^{-1}x) + \partial_\mu\Omega(x) \tag{9.54}$$

for some function $\Omega$. In other words $A^{phys}_\mu$ doesn't transform as a one-form under Lorentz transformations. On the other hand $A^{phys}_\mu$ is "almost" a one-form: it is a one-form up to a gauge transformation. Since we are viewing gauge symmetries as redundancies, we can hope that this is still enough for us to have a Lorentz-invariant theory and in fact it is. This is clear from our more general formalism: we started with a genuine one-form gauge field $A_\mu$ on $\mathcal{H}_{big}$, and our two constraints (9.18) hold in any Lorentz frame. The quantity $A^{phys}_\mu$ is really an example of a "gauge-fixed" gauge field, as we will discuss in the subsection after next, and once we gauge-fix we lose manifest Lorentz invariance.

The next step is now to introduce the Heisenberg field strength operator

$$F_{\mu\nu}(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[e_{\mu\nu}(\vec{p},\sigma)a_{\vec{p},\sigma}e^{ip\cdot x} + e^*_{\mu\nu}(\vec{p},\sigma)a^\dagger_{\vec{p},\sigma}e^{-ip\cdot x}\right], \tag{9.55}$$

with

$$e_{\mu\nu}(\vec{p},\sigma) = i\left(p_\mu e_\nu(\vec{p},\sigma) - p_\nu e_\mu(\vec{p},\sigma)\right). \tag{9.56}$$

These polarization tensors do obey equation (9.45), since the extra gauge transformation we found for $e_\mu$ cancels between the two terms. From here the remaning steps are the same as for the scalar and the spinor field: we take the Fourier transform of $F_{\mu\nu}$ to extract $a_{\vec{p},\sigma}$ and $a^\dagger_{\vec{p},\sigma}$, use the canonical commutation relations (9.19) to show that these obey

$$\begin{aligned}[a_{\vec{p},\sigma}, a_{\vec{p}',\sigma'}] &= [a^\dagger_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}] = 0\\ [a_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}] &= \delta_{\sigma,\sigma'}(2\pi)^{d-1}\delta^{d-1}(\vec{p}-\vec{p}'),\end{aligned} \tag{9.57}$$

and then substitute into the Hamiltonian to see that

$$H = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \omega_{\vec{p}} a^\dagger_{\vec{p}\sigma} a_{\vec{p}\sigma} + \text{constant}. \tag{9.58}$$

These calculations are not hard in principle, but in practice they are somewhat tedious due to the proliferation of indices. We'll instead do this calculation below using the gauge-fixing method, which makes it more manageable.

## 9.6 Boundary conditions and asymptotic symmetries

Before proceeding to gauge fixing however, we'll now make an aside to deal with the possibility of gauge transformations which don't vanish at infinity. Let's recall the variation of the Maxwell Lagrangian density:

$$\delta\mathcal{L} = -\partial_\mu \delta A_\nu F^{\mu\nu} + \delta A_\mu J^\mu = (\partial_\mu F^{\mu\nu} + J^\mu)\delta A_\nu + \partial_\mu \left(F^{\mu\nu}\delta A_\nu\right). \tag{9.59}$$

To simplify our discussion we'll first take the spatial boundary to be at some finite location $\Gamma$, which we can take to infinity later. For example we can take $\Gamma$ to be the set of spacetime points $(t, \vec{x})$ obeying $|x| = R$ for some large but finite $R$. The variation of the action has a boundary term

$$\delta S \supset \int_\Gamma d^{d-1}x n_\mu F^{\mu\nu}\delta A_\nu, \tag{9.60}$$

where $n^\mu$ is the outward-pointing normal vector at the boundary $\Gamma$. In order for the action to be stationary (up to future and past terms) about solutions of the equations of motion, we therefore need to have

$$n_\mu F^{\mu\nu}\delta A_\nu|_\Gamma = 0 \tag{9.61}$$

for all variations $\delta A$ which obey the boundary conditions. One way to satisfy this is **Neumann boundary conditions**:

$$n_\mu F^{\mu\nu}|_\Gamma = 0. \tag{9.62}$$

In old-fashioned language this sets the the normal electric field and the tangential magnetic field to zero at the boundary. The alternative way is **Dirichlet boundary conditions**:

$$t^\mu A_\mu|_\Gamma = 0, \tag{9.63}$$

where $t^\mu$ is any vector which is tangent to $\Gamma$. In more geometric language the pullback of the one-form $A$ to $\Gamma$ vanishes. In old-fashioned language these boundary conditions set the normal magnetic field and the tangential electric field to zero - they are the boundary conditions one would find at the edge of a perfect conductor.[34] In quantum electrodynamics Dirichlet boundary conditions are more natural since we will want to have $A_\mu$ vanish at infinity, so we will adopt them from now on. Dirichlet boundary conditions constrain the set of allowed gauge transformations; in order to prevent a gauge transformation from generating a nonzero tangential $A$ at the boundary, we must require it to be constant there:

$$\Omega|_\Gamma = \text{constant}. \tag{9.64}$$

This constant does not need to vanish, and the right thing to do with it is a bit subtle so we will now discuss it in some detail.

Let's first remember that our earlier argument that the Gauss constraint generates gauge transformations neglected a boundary term. This was ok for gauge transformations that vanish at $\Gamma$, but we are now interested in gauge transformations which approach a nonzero constant at $\Gamma$ so we need to be more careful. To avoid

[34] Neumann boundary conditions correspond to a "perfect magnetic conductor", but given our lack of magnetic monpoles we don't know how to build one.

the boundary term, we should instead take the generator of an infinitesimal gauge transformation $\omega(x)$ on $\mathcal{H}_{big}$ to be

$$Q_\omega = \int_{\Sigma_t} d^{d-1}x \left(\omega\rho - \partial_i\omega\Pi^i\right), \tag{9.65}$$

where $\Sigma_t$ is the time slice at time $t$. You can check that this has the right commutator with $A_i$ to generate an infinitesimal gauge transformation $\Omega$ without dropping any boundary term:

$$-i[Q_\omega, A_i(t,\vec{x})] = \partial_i\omega(t,\vec{x}). \tag{9.66}$$

To relate this to the Gauss constraint we can integrate by parts, now being careful about the boundary term:

$$Q_\omega = \int_{\Sigma_t} d^{d-1}x \left(\omega(\rho + \partial_i\Pi^i)\right) - \int_{\partial\Sigma_t} d^{d-2}x\omega n_i\Pi^i. \tag{9.67}$$

On the physical Hilbert space this vanishes if $\omega|_\Gamma = 0$, reflecting the fact that such gauge transformations are redundancies. On the other hand for gauge transformations where $\omega|_\Gamma = \omega_0$ with $\omega_0$ constant, on the physical Hilbert space $\mathcal{H}$ we instead have

$$Q_\omega = -\omega_0 \int d^{d-2}x\, n_i\Pi^i = \omega_0 \int d^{d-1}x\, \rho = \omega_0 Q_{el}, \tag{9.68}$$

where $Q_{el}$ is the total electric charge. This need not vanish, which tells us something interesting: gauge transformations which approach a nonzero constant at infinity are *not* redundancies - they act nontrivially on $\mathcal{H}$ as long as the net electric charge isn't zero.[35]

Constant gauge transformations in Maxwell theory give us a new kind of internal symmetry that we haven't encountered before. Last semester we defined an internal symmetry in quantum field theory to be a unitary operator $U$ which preserves the local algebras $\mathcal{A}[R]$ and also leaves the energy-momentum tensor invariant. These conditions are both satisfied here for

$$U(\theta) = e^{i\theta Q_{el}}, \tag{9.69}$$

so this is a valid internal symmetry. We further said that an internal symmetry is a **global symmetry** if there is a local operator which transforms nontrivially under conjugation by $U$. By Gauss's law however we can express $Q_{el}$ as the electric flux through infinity, so it must commute with all local operators. It therefore isn't a global symmetry. How can it act nontrivially on the Hilbert space? We will see next time that the operators which create charged particles are necessarily non-local operators which extend out to spatial infinity, so these operators can and do have a nonzero commutator with $Q_{el}$. Symmetries of this type are often called **asymptotic symmetries**, especially in the limit where we take $\Gamma$ to infinity.

## 9.7 Quantization of the electromagnetic field in Coulomb gauge

Turning now to the gauge-fixing approach to quantizing Maxwell theory, the idea is quite simple: we pick a single representative of each gauge equivalence class for $A_\mu$, and then just use these representatives to do all calculations. This choice of representative is typically made by imposing one or more **gauge-fixing conditions**. Here are some popular gauge-fixing conditions:

- **Coulomb gauge:** $\vec{\nabla}\cdot\vec{A} = 0$.
- **Temporal gauge:** $A_0 = 0$.

[35] There is a more systematic approach to deciding which gauge transformations are redundancies using the covariant phase space approach to Hamiltonian mechanics: a continuous gauge transformation is a redundancy if and only if it is a zero mode of the pre-symplectic form. See my first paper with Jie-qiang Wu for an explanation of this formalism and also how to apply it to Maxwell theory.

- **Lorenz gauge:** $\partial^\mu A_\mu = 0$.
- **Axial gauge:** $A_1 = 0$.

For a gauge-fixing condition to be successful, we need to show that every gauge field configuration $A_\mu$ differs by a gauge transformation from one which obeys the gauge-fixing condition, and we also need to show that the representative which obeys the gauge-fixing condition is unique. The above conditions are not always strong enough to fulfil both requirements, for example we can stay in temporal gauge while doing a gauge transformation obeying $\dot{\Omega} = 0$. Similarly we can stay in Lorenz gauge while doing a gauge transformation obeying $\partial^2\Omega = 0$. On the other hand with the Dirichlet boundary conditions we discussed in the last section, axial gauge and Coulomb gauge do satisfy both requirements and thus give valid gauge-fixings of Maxwell theory. It is more convenient to use Coulomb gauge, as it preserves rotational symmetry, so we will proceed with that.

The procedure for going to Coulomb gauge is straightforward; given a gauge field $A_\mu$, we construct a gauge transformation $\Omega$ such that

$$A^c_\mu = A_\mu + \partial_\mu \Omega \tag{9.70}$$

obeys

$$\partial_i A^c_i = 0. \tag{9.71}$$

We can find $\Omega$ by solving the equation

$$\nabla^2 \Omega = -\vec{\nabla} \cdot \vec{A}, \tag{9.72}$$

which is of course just a version of the Poisson equation $\nabla^2\phi = -\rho$ that we solve to find the electrostatic potential given a charge distribution. We can solve it by introducing a spatial Green's function $K(\vec{x})$ obeying

$$\nabla^2 K = -\delta^{d-1}(\vec{x}). \tag{9.73}$$

This is the same equation that determines the Euclidean propagator of a massless scalar field in $d-1$ dimensions, so we already know the solution:

$$K(\vec{x}) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2} = \frac{1}{(d-3)\Omega_{d-2}|x|^{d-3}}. \tag{9.74}$$

We thus have

$$\Omega(t,\vec{x}) = \int d^{d-1}x' K(\vec{x}-\vec{x}')\vec{\nabla}\cdot\vec{A}(t,\vec{x}'). \tag{9.75}$$

This indeed vanishes at infinity as long as $d \geq 4$. For $d = 2, 3$ the boundary conditions are more important since the Coulomb potential grows with distance, so in discussing Coulomb gauge we'll restrict to $d \geq 4$ (we anyways already solved the theory in $d = 2$). The Coulomb gauge field thus is given by

$$A^c_i(t,\vec{x}) = A_i(t,\vec{x}) + \int d^{d-1}x' \partial_i K(\vec{x}-\vec{x}')\partial'_j A_j(t,\vec{x}'). \tag{9.76}$$

and

$$\begin{aligned} A^c_0(t,\vec{x}) &= A_0(t,\vec{x}) + \int d^{d-1}x' K(\vec{x}-\vec{x}')\partial_j \dot{A}_j(t,\vec{x}') \\ &= A_0(t,\vec{x}) + \int d^{d-1}x' K(\vec{x}-\vec{x}')\left(\nabla^2 A_0(t,\vec{x}') - \rho(t,\vec{x}')\right) \\ &= -\int d^{d-1}x' K(\vec{x}-\vec{x}')\rho(t,\vec{x}'). \end{aligned} \tag{9.77}$$

In going to the second line for the $A_0$ expression we used Gauss's law, while in going to the third we used (9.73). These expressions tell us something quite interesting: the gauge field in Coulomb gauge is a

*nonlocal* function of the fundamental gauge field $A_\mu$ and the background current $J^\mu$. This is reflected in the commutation relations it obeys:[36]

$$\begin{aligned}
&[A_i^c(\vec{x}), A_j^c(\vec{y})] = 0 \\
&[\Pi^i(\vec{x}), \Pi^j(\vec{y})] = 0 \\
&[A_i^c(\vec{x}), \Pi^j(\vec{y})] = i\delta_i^j \delta^{d-1}(\vec{x}-\vec{y}) + i\frac{\partial}{\partial x^i}\frac{\partial}{\partial x_j}K(\vec{x}-\vec{y}).
\end{aligned} \tag{9.78}$$

Using the fact that

$$\vec{\Pi} = \dot{A}^c - \vec{\nabla}A_0^c, \tag{9.79}$$

and also our expression (9.77) for $A_0^c$, we can rewrite the Hamiltonian in Coulomb gauge as

$$H = \int d^{d-1}x \left[\frac{1}{2}\dot{\vec{A}}^c \cdot \dot{\vec{A}}^c + \frac{1}{4}F_{ij}F^{ij} - \vec{A}\cdot\vec{J}\right] + \frac{1}{2}\int d^{d-1}x d^{d-1}y K(\vec{x}-\vec{y})\rho(\vec{x})\rho(\vec{y}). \tag{9.80}$$

Note in particular the non-local Coulomb interaction term, which is a consequence of the non-locality of the Coulomb gauge field. It is sometimes presented as a potential problem for the theory that needs to be surmounted, but here we know things will be fine since we started with a local and Lorentz-invariant presentation of the theory. The quantity $\dot{\vec{A}}^c$ has the same algebra with $\vec{A}^c$ as $\vec{\Pi}$ does, since they differ only by $\vec{\nabla}A_0^c$ and this depends only on $\rho$ and thus commutes with $\vec{A}^c$.

Why bother introducing the Coulomb gauge gauge field $A_\mu^c$? The reason is that, despite its inherent non-locality, it gives us a way to find a gauge-invariant operator on the physical Hilbert space $\mathcal{H}$ which obeys Maxwell's equations as its Heisenberg equations of motion - precisely the thing we couldn't accomplish using the gauge-invariant formalism. Non-locality is just the price we pay for doing this. Indeed since $A_\mu^c(x)$ obeys Maxwell's equations, when $J^\mu = 0$ we can expand it in a basis of the plane-wave solutions we constructed before:

$$A_\mu^c(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[e_\mu(\vec{p},\sigma)a_{\vec{p},\sigma}e^{ip\cdot x} + e_\mu^*(\vec{p},\sigma)a_{\vec{p},\sigma}^\dagger e^{-ip\cdot x}\right]. \tag{9.81}$$

We need to require however that the polarization vectors $e_\mu$ are consistent with the Coulomb gauge condition (9.71). Since we've now set $\rho = 0$ we will also have $A_0^c = 0$ , so we must also have $e^0(\vec{p},\sigma) = 0$. We therefore can focus on the spatial components of the gauge field:

$$\vec{A}^c(x) = \sum_\sigma \int \frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left[\vec{e}(\vec{p},\sigma)a_{\vec{p},\sigma}e^{ip\cdot x} + \vec{e}^*(\vec{p},\sigma)a_{\vec{p},\sigma}^\dagger e^{-ip\cdot x}\right]. \tag{9.82}$$

We saw before that the equations of motion require $p_\mu e^\mu(\vec{p},\sigma) = 0$, so in Coulomb gauge the polarization vectors must obey

$$\vec{p}\cdot\vec{e}(\vec{p},\sigma) = 0. \tag{9.83}$$

In $d = 4$ we have already constructed these polarization vectors, in particular for the reference momentum

$$k^\mu = \omega\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \tag{9.84}$$

we have

$$\vec{e}(\vec{k},\pm) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm i\\0\end{pmatrix}. \tag{9.85}$$

[36]There are a number of approaches to deriving this algebra. In particular Weinberg uses a general method called "Dirac brackets" to combine the gauge fixing condition with the electromagnetic constraints. This is a good thing to know about in general, but it is unnecessary for us here since we defined $A_\mu^c$ as a nonlocal functional of $A_\mu$ rather than a fundamental object in its own right. We can therefore derive the algebra of $A_\mu^c$ directly from that of $A_\mu$.

Thus Coulomb gauge simply sets $\alpha_\pm = 0$ in our earlier discussion of the polarization. For more general momenta the polarization vector is given by

$$\vec{e}(\vec{p},\pm) = R_{\vec{p}}\,\vec{e}(\vec{k},\pm), \tag{9.86}$$

where $R_{\vec{p}}$ is some fixed rotation which turns $\vec{k}$ to point in the direction of $\vec{p}$. These polarization vectors obey

$$\begin{aligned}\vec{e}^{\,*}(\vec{p},\sigma)\cdot\vec{e}(\vec{p},\sigma') &= \delta_{\sigma,\sigma'}\\ \sum_\sigma e_i^*(\vec{p},\sigma)e_j(\vec{p},\sigma) &= \delta_{ij} - \frac{p_i p_j}{|p|^2},\end{aligned} \tag{9.87}$$

as you can easily check for the reference momentum. You can fairly easily convince yourself that these equations hold in general dimensions, as they are simply expressing that we have adopted an orthonormal basis of polarization vectors which is complete in the subspace orthogonal to $\vec{p}$.

To study the algebra of the $a_{\vec{p},\sigma}$ we can extract them the usual way:

$$a_{\vec{p},\sigma} = \sqrt{\frac{\omega_{\vec{p}}}{2}}\int d^{d-1}x e^{-i\vec{p}\cdot\vec{x}}\vec{e}^{\,*}(\vec{p},\sigma)\cdot\left(\vec{A}^c(0,\vec{x}) + \frac{i}{\omega_{\vec{p}}}\dot{\vec{A}}^c(0,\vec{x})\right). \tag{9.88}$$

We can then extract the algebra using the Coulomb gauge commutation relations (9.78), for example we have

$$\begin{aligned}[a_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}] &= \sqrt{\frac{\omega_{\vec{p}}\omega_{\vec{p}'}}{4}}\int d^{d-1}x d^{d-1}x' e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}e_i^*(\vec{p},\sigma)e_{i'}(\vec{p}',\sigma')\times\left(\frac{i}{\omega_{\vec{p}}}[\dot{A}_i(\vec{x}),A_{i'}(\vec{x}')] - \frac{i}{\omega_{\vec{p}'}}[A_i(\vec{x}),\dot{A}_{i'}(\vec{x}')]\right)\\ &= \sqrt{\frac{\omega_{\vec{p}}\omega_{\vec{p}'}}{4}}\int d^{d-1}x d^{d-1}x' e^{-i\vec{p}\cdot\vec{x}+i\vec{p}'\cdot\vec{x}'}e_i^*(\vec{p},\sigma)e_{i'}(\vec{p}',\sigma')\times\left(\frac{2}{\omega_{\vec{p}}}\delta_{i,i'}\delta^{d-1}(\vec{x}-\vec{x}') + \frac{2}{\omega_{\vec{p}'}}\partial_i\partial_{i'}K(\vec{x}-\vec{x}')\right)\\ &= \delta_{\sigma,\sigma'}(2\pi)^{d-1}\delta^{d-1}(\vec{p}-\vec{p}').\end{aligned} \tag{9.89}$$

In going to the last line we have used that the derivatives acting on $K$ will become powers of $\vec{p}$ or $\vec{p}'$ in the Fourier transform, and the Fourier transform will also set $\vec{p}=\vec{p}'$. When these powers are contracted with $\vec{e}$ or $\vec{e}^{\,*}$ they vanish due to the Coulomb gauge condition (9.83). Similarly we have

$$[a_{\vec{p},\sigma}, a_{\vec{p}',\sigma'}] = [a^\dagger_{\vec{p},\sigma}, a^\dagger_{\vec{p}',\sigma'}] = 0. \tag{9.90}$$

Finally one can compute the Hamiltonian, on the homework you will show that this is given (still with $J^\mu = 0$) by

$$\begin{aligned}H &= \frac{1}{2}\sum_\sigma\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\omega_{\vec{p}}\left(a_{\vec{p},\sigma}a^\dagger_{\vec{p},\sigma} + a^\dagger_{\vec{p},\sigma}a_{\vec{p},\sigma}\right)\\ &= \sum_\sigma\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\omega_{\vec{p}}\,a^\dagger_{\vec{p},\sigma}a_{\vec{p},\sigma} + \text{constant}.\end{aligned} \tag{9.91}$$

Thus the quantization of the Maxwell field in Coulomb gauge (and thus in any gauge) indeed produces the Fock space of non-interacting photons.

We can also compute the two-point function of the Coulomb gauge field in the usual way:

$$\begin{aligned}\langle\Omega|A^c_\mu(x_2)A^c_\nu(x_1)|\Omega\rangle &= \sum_{\sigma,\sigma'}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{d^{d-1}p'}{(2\pi)^{d-1}}\frac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}e_\mu(\vec{p},\sigma)e^*_\nu(\vec{p},\sigma')e^{ip\cdot x_2 - ip'\cdot x_1}\langle\Omega|a_{\vec{p},\sigma}a^\dagger_{\vec{p}',\sigma'}|\Omega\rangle\\ &= \int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{\sum_\sigma e_\mu(\vec{p},\sigma)e^*_\nu(\vec{p},\sigma)}{2\omega_{\vec{p}}}e^{ip\cdot(x_2-x_1)}.\end{aligned} \tag{9.92}$$

In fact this equation is true in any gauge for the appropriate choice of $e_\mu$; imposing Coulomb gauge we have $A_0 = 0$ and the two-point function of the spatial components of $A^c$ is

$$\langle \Omega | A_i^c(x_2) A_j^c(x_1) | \Omega \rangle = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{\delta_{ij} - \frac{p_i p_j}{|p|^2}}{2\omega_{\vec{p}}} e^{ip\cdot(x_2 - x_1)}. \tag{9.93}$$

This is a rather unpleasant and non-covariant expression, which we landed on to the complicated Lorentz transformation properties of $A_\mu^c$. We can also consider the time-ordered correlation function, which has the nicer (but still not covariant) expression

$$\langle \Omega | T A_\mu^c(x_2) A_\nu^c(x_1) | \Omega \rangle = \int \frac{d^d p}{(2\pi)^d} \frac{-i \sum_\sigma e_\mu(\vec{p}, \sigma) e_\nu^*(\vec{p}, \sigma)}{p^2 - i\epsilon} e^{ip\cdot(x_2 - x_1)}, \tag{9.94}$$

which again is valid also in other gauges. Next time we will see using the path integral that if we are sufficiently clever we can justify replacing the helicity sum in the numerator with $\eta_{\mu\nu}$ in Feynman diagram calculations, which certainly improves our quality of life going forward.[37]

**Problems:**

1. Work out the relationship between SI and Heaviside-Lorentz units (not setting $c = 1$). Hint: you can rescale the definition of charge by $(q_{SI}, \rho_{SI}, \vec{J}_{SI}) = \alpha \times (q_{HL}, \rho_{HL}, \vec{J}_{HL})$. How should you rescale $\vec{E}_{SI}$ and $\vec{B}_{SI}$ and what is the value of $\alpha$?

2. In $d = 3$ there is a variant of the Maxwell action we can write down called the *Chern-Simons action*, given by

$$S_{CS} = \frac{k}{4\pi} \int d^3x \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma}. \tag{9.95}$$

   The factor of $4\pi$ is there because for subtle topological reasons if we define our Maxwell field to allow for magnetic monopoles (i.e. we take the gauge group to be $U(1)$ instead of $\mathbb{R}$), then we need to take $k$ to be an integer. What are the equations of motion of this theory and what is the conjugate momentum to $\vec{A}$? What is the Hamiltonian?

3. Confirm equations (9.88) and (9.90).

4. Derive the Hamiltonian (9.91) starting from (9.80).

5. Our first theory with charged dynamical fields coupled to electromagnetism is **scalar electrodynamics**, which is a theory of a complex scalar field $\Phi$ coupled to a Maxwell gauge field $A_\mu$ with Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (\partial_\mu \Phi^* + iqA_\mu \Phi^*)(\partial^\mu \Phi - iqA^\mu \Phi) - m^2 |\Phi|^2. \tag{9.96}$$

   What is the current density appearing in Maxwell's equations? What is the equation of motion for $\Phi$? What are the canonical conjugates of $\Phi$ and $\Phi^*$, and what is the current density expressed in terms of the canonical coordinates and momenta? Show that this action is invariant under the gauge transformation

$$\begin{aligned} A'_\mu &= A_\mu + \partial_\mu \Omega \\ \Phi' &= e^{iq\Omega} \Phi. \end{aligned} \tag{9.97}$$

[37] If you want to try to justify this replacement now, there is a somewhat convoluted explanation in section 8.5 of Weinberg. The rough idea is that the non-covariance of the propagator in Coulomb gauge is precisely that which is needed to conspire to remove the non-locality in the Coulomb gauge Hamiltonian (9.80) which arises once dynamical charges are present. From the point view of our covariant formulation no such gymnastics are needed.

# 10 Quantum electrodynamics II: charged matter and the path integral

In the previous section we quantized the electromagnetic field in the presence of a conserved background current $J^\mu$. In the real world currents are of course made out of charges, so we now need to introduced charged matter fields. We will then use the path integral approach to derive the Feynman rules for photons coupled to spinor and scalar charged fields.

## 10.1 Charged matter

The obvious guess for how to make a conserved current $J^\mu$ out of matter fields is to take it to be the Noether current for a global symmetry. The integral of $\rho = J^0$ over space is the charge which generates the global symmetry for which $J^\mu$ is the Noether current, so apparently

$$\left[\int d^{d-1}x J^0(t,\vec{x}), H\right] = 0. \tag{10.1}$$

We saw in the previous section however that in order to construct a sensible Hilbert space and Hamiltonian for quantum electrodynamics it is necessary to impose the Gauss constraint

$$\vec{\nabla}\cdot\vec{\Pi} + \rho = 0 \tag{10.2}$$

independently at each point in spacetime. In order for the Hamiltonian to preserve the set of states which are annihilated by this constraint, it must be invariant under a combined operation where we simultaneously perform an arbitrary gauge transformation on $A_\mu$ and also a transformation of the matter fields where the "amount" of the symmetry transformation at each point in spacetime and is set by the gauge transformation. In other words the gauge symmetry must act as a local symmetry both on $A_\mu$ and on any charged matter fields.

One simple example of such a theory is **scalar electrodynamics**, which takes a complex scalar field of charge $q$ and couples it to a Maxwell field $A_\mu$ through the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \left(\partial_\mu\Phi^\dagger + iqA_\mu\Phi^\dagger\right)\left(\partial^\mu\Phi - iqA^\mu\Phi\right) - m^2\Phi^\dagger\Phi. \tag{10.3}$$

You showed on the previous homework that this Lagrangian density is invariant under the combined gauge transformation

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \partial_\mu\Omega(x) \\ \Phi'(x) &= e^{iq\Omega(x)}\Phi(x). \end{aligned} \tag{10.4}$$

Note in particular that the quantity

$$D_\mu\Phi = \left(\partial_\mu - iqA_\mu\right)\Phi, \tag{10.5}$$

usually called the **gauge covariant derivative** of $\Phi$, has the simple gauge transformation

$$(D_\mu\Phi)' = e^{iq\Omega}D_\mu\Phi. \tag{10.6}$$

This example is actually a bit more complicated than one might like however, since the quadratic term in $A$ suggests that perhaps the current $J^\mu$ has some nontrivial dependence on $A_\mu$. This concern is not actually realized once we express the current in terms of the canonical momenta, but anyways we can avoid it by instead coupling to a charged spinor field $\Psi$. This brings us to our next example of a gauge theory coupled to matter, **spinor electrodynamics**:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\overline{\Psi}\left(\not{D} + m\right)\Psi, \tag{10.7}$$

where

$$D_\mu = \partial_\mu - iqA_\mu \tag{10.8}$$

is again the gauge covariant derivative and $q$ is the electric charge of the spinor field. The current which couples to $A_\mu$ is simply

$$J^\mu = -q\overline{\Psi}\gamma^\mu\Psi, \tag{10.9}$$

which is indeed the Noether current for the charge rotation symmetry $\Psi' = e^{iq\theta}\Psi$. The spinor electrodynamics Lagrangian is invariant under the gauge symmetry

$$\begin{aligned} A'_\mu(x) &= A_\mu(x) + \partial_\mu\Omega(x) \\ \Psi'(x) &= e^{iq\Omega(x)}\Psi(x). \end{aligned} \tag{10.10}$$

Spinor electrodynamics is arguably the most successful scientific theory we possess, explaining most of atomic physics, chemistry, radiation, etc, to extraordinary precision. We will spend this section and the next two studying it in some detail.

## 10.2 Some comments on charge quantization and topology

We'll now make an aside on the possible values of the charge $q$. The scalar or spinor Lagrangians we've written above make sense for any real value of $q$. On the other hand there is extraordinarily good experimental evidence for the principal of **charge quantization**, which says that the electric charge of any object in nature is an integer multiple of the proton charge. In SI units the proton charge is

$$e = 1.602176634 \times 10^{-19}\,\mathrm{C}, \tag{10.11}$$

while in Heaviside-Lorentz units it is is dimensionless,

$$e = .302822\ldots. \tag{10.12}$$

This dimensionless number is usually reported in terms of the **fine structure constant**

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = .0072973525643(11) \approx \frac{1}{137}. \tag{10.13}$$

Why should charge be quantized? We don't know for sure of course, but there is a very plausible potential explanation in terms of the topology of the gauge group. So far we have been assuming that $\Omega$ is a function of spacetime that takes values in $\mathbb{R}$. On the other hand let's imagine that there is some constant $e$ such that we should identify

$$\Omega \sim \Omega + \frac{2\pi}{e}. \tag{10.14}$$

Looking at the matter field transformation

$$\Psi' = e^{iq\Omega}\Psi, \tag{10.15}$$

this will only be consistent with the periodicity of $\Omega$ if we have

$$\frac{q}{e} \in \mathbb{Z}, \tag{10.16}$$

i.e. if charge quantization holds. Since there is extraordinarily good evidence for charge quantization, most physicists view this as strong evidence that the gauge group of electromagnetism is indeed $U(1)$ instead of $\mathbb{R}$.

Why didn't we have to discuss the topology of the gauge group earlier? For the most part we were studying Maxwell theory in Minkowski space, in which case the spectrum of the Hamiltonian (i.e. the set of photon states) is independent of whether the gauge group topology is $U(1)$ or $\mathbb{R}$. The only immediate effect of having the topology be $U(1)$ is that we can only couple the gauge field to charged fields with $q \in \mathbb{Z}e$. On

the other hand if the spacetime topology is nontrivial then we can detect the topology of the gauge group even in the absence of charged matter. For example in $1+1$ dimensions we found the holonomy degree of freedom

$$h = \int_0^L dx^1 A_1, \tag{10.17}$$

which we argued was gauge-invariant via the manipulation

$$h' = h + \Omega(L) - \Omega(0). \tag{10.18}$$

If $\Omega$ is not periodic, i.e. the gauge group is $\mathbb{R}$, then this manipulation was correct. If $\Omega$ has periodicity $\frac{2\pi}{e}$ however, then it is possible to have gauge transformations which have nontrivial winding as we go around the circle:[38]

$$\Omega(L) = \Omega(0) + \frac{2\pi n}{e} \qquad n \in \mathbb{Z}. \tag{10.19}$$

We then can only conclude that

$$h' = h + \frac{2\pi n}{e} \qquad n \in \mathbb{Z}. \tag{10.20}$$

In order to get something gauge-invariant we thus need to promote the holonomy to a **Wilson loop**

$$W = e^{ieh}. \tag{10.21}$$

Another way to think about this is that the quantum mechanics of $U(1)$ Maxwell theory on a circle in $1+1$ dimensions is that of a particle moving on a circle of circumference $\frac{2\pi}{e}$ (not to be confused with the spatial circle, whose circumference is $L$).

There is another important feature of $U(1)$ gauge theory: if the gauge group is $U(1)$ then the theory allows for **magnetic monopoles**, which you will show on the homework have possible magnetic charges

$$Q_{magnetic} = \int_{\mathbb{S}^2} \vec{B} \cdot d\vec{A} = \frac{2\pi n}{e} \qquad n \in \mathbb{Z}. \tag{10.22}$$

So far these have not been observed in nature, which one might use as an argument against the possibility of $U(1)$ topology for the gauge group, but on the other hand they are expected to be heavy and so hard to produce and the quantization of charge is quite compelling.

## 10.3 Gauge-invariant charged operators

In quantum electrodynamics charged local operators such as $\Phi$ or $\Psi$ are not gauge-invariant, so they do not act within the physical Hilbert space $\mathcal{H}$. We can nonetheless use them to build gauge-invariant operators that do act on $\mathcal{H}$. There are a number of ways to do this, here are some examples:

- Combine $\Psi$ and $\overline{\Psi}$ or $\Phi$ and $\Phi^\dagger$ at the same point into a gauge-invariant local operator such as the current $J^\mu = -q\overline{\Psi}\gamma^\mu\Psi$ or the mass operator $\overline{\Psi}\Psi$.
- Combine $\Psi$ and $\overline{\Psi}$ or $\Phi$ and $\Phi^\dagger$ at different points using a **Wilson Line**:

  $$\overline{\Psi}(y) e^{iq\int_C A\cdot dx}\Psi(x), \tag{10.23}$$

  where $C$ is any curve in spacetime which goes from $x$ to $y$ and $q$ is the charge of $\Psi$. This is gauge-invariant due to the Wilson line transformation law:

  $$e^{iq\int_C A'\cdot dx} = e^{iq\int_C A\cdot dx + iq\int_C \partial\Omega\cdot dx} = e^{iq\Omega(y)-iq\Omega(x)}. \tag{10.24}$$

  The operator (10.23) is non-local, acting with it on the vacuum creates an fermion and antifermion connected by a line of elecric flux.

[38] If you like topology this has to do with the fact that $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

- Connect a local charged operator to spatial infinity using an infinitely long Wilson line:

$$\widetilde{\Psi}_C(x) = e^{iq\int_C A\cdot dx}\Psi(x), \tag{10.25}$$

  where now $C$ runs from $x$ to an arbitrary point $y$ at spatial infinity. Acting on the vacuum this operator creates an antifermion connected to infinity by a line of electric flux. It is invariant under any gauge transformation which vanishes at infinity, which you'll recall are the ones we quotient by in defining the physical Hilbert space $\mathcal{H}$. $\widetilde{\Psi}_C(x)$ is sometimes called a "dressed" field, with the Wilson line being the "dressing" of the "bare" charged field $\Psi(x)$.

The last operator is interesting because it transforms nontrivially under gauge transformations which approach a nonzero constant at infinity. We saw in the last section that these are generated by the total electric charge operator $Q_{electric}$, and indeed we have

$$e^{-i\theta Q_{electric}}\widetilde{\Psi}_C(x)e^{i\theta Q_{electric}} = e^{i\theta q}\widetilde{\Psi}_C(x) \tag{10.26}$$

This is essentially just Gauss's law: the electric field created by $\widetilde{\Psi}_C(x)$ is detected by

$$Q_{electric} = \int_{\mathbb{S}^{d-2}_\infty} \vec{E}\cdot d\vec{A}. \tag{10.27}$$

Dressed operators such as $\widetilde{\Psi}_C(x)$ have an interesting interpretation in terms of gauge-fixing. For example let's take $C$ to be a straight line in the $x^1$ direction from $(t, x, \vec{x}^\perp)$ to $(t, \infty, \vec{x}^\perp)$. Now let's go to axial gauge $A_1 = 0$. We then simply have

$$\widetilde{\Psi}_C(x) = \Psi(x), \tag{10.28}$$

so in other words we can think of $\widetilde{\Psi}_C(x)$ as the gauge-invariant description of the charged field in axial gauge! This naturally suggests the question of how to come up with a gauge-invariant description of the charged field in other gauges, for example Coulomb gauge. To answer this question it is useful to first introduce a rather general set of dressings, defined by

$$\widetilde{\Psi}(t,\vec{x}) = e^{i\int_{\Sigma_t} d^{d-1}y\vec{E}_{cl}\cdot\vec{A}}\Psi(x), \tag{10.29}$$

where $\vec{E}_{cl}$ is any solution of Gauss's law with charge distribution

$$\rho(\vec{y}) = q\delta^{d-1}(\vec{x}-\vec{y}) \tag{10.30}$$

at time $t$. The axial gauge dressing just mentioned has this form with

$$\vec{E}_{cl}(y^1,\vec{y}^\perp) = q\theta(y^1-x^1)\delta^{d-2}(\vec{y}^\perp-\vec{x}^\perp)\hat{x}_1, \tag{10.31}$$

which you can check indeed obeys Gauss's law. In general we have the gauge transformation

$$\begin{aligned} e^{i\int_{\Sigma_t} d^{d-1}y\vec{E}_{cl}\cdot\vec{A}'} &= e^{i\int_{\Sigma_t} d^{d-1}y\vec{E}_{cl}\cdot\vec{\nabla}\Omega}e^{i\int_{\Sigma_t} d^{d-1}y\vec{E}_{cl}\cdot\vec{A}} \\ &= e^{i\Omega_\infty Q_{electric} - iq\Omega(x)}e^{i\int_{\Sigma_t} d^{d-1}y\vec{E}_{cl}\cdot\vec{A}}, \end{aligned} \tag{10.32}$$

so $\widetilde{\Psi}(t,\vec{x})$ has the same gauge transformation as $\widetilde{\Psi}_C(t,\vec{x})$ and in particular is invariant under gauge transformations which vanish at infinity. As you might guess, to get the Coulomb gauge dressing we should take

$$\vec{E}_{cl}(\vec{y}) = \vec{\nabla}\phi_{cl} \tag{10.33}$$

with

$$\phi_{cl}(\vec{y}) = \frac{q}{(d-3)\Omega_{d-2}|y-x|^{d-3}}, \tag{10.34}$$

since then we have

$$\int_{\Sigma_t} d^{d-1}y \vec{E}_{cl} \cdot \vec{A} = -\int_{\Sigma_t} \phi_{cl} \vec{\nabla} \cdot \vec{A} + \int_{\partial\Sigma_t} \phi_{cl} \vec{A} \cdot d\vec{A} \tag{10.35}$$

which vanishes in Coulomb gauge since $\vec{\nabla} \cdot \vec{A} = 0$ and $\phi_{cl}\vec{A}$ goes to zero at infinity fast enough to make the boundary term vanish.[39]

## 10.4 QED path integral I: gauge-invariant formalism

To prepare for perturbative calculations in quantum electrodynamics, we'll now introduce the path integral formalism. Let's first recall that the physical Hilbert space $\mathcal{H}$ is constructed from $\mathcal{H}_{big}$ by imposing the constraints

$$\begin{aligned} \Pi^0 &= 0 \\ \vec{\nabla} \cdot \vec{\Pi} + J^0 &= 0. \end{aligned} \tag{10.36}$$

The first constraint is easy enough to impose by restricting to wave functionals which depend only on $\vec{A}$, but the second is more complicated. Let's therefore define an "intermediate" Hilbert space $\mathcal{H}_{int}$ consisting of wave functionals which are independent of $A_0$, but on which we haven't yet imposed the Gauss constraint. We'll then define a projection $P_{GI}$ on $\mathcal{H}_{int}$ which projects to the physical Hilbert space $\mathcal{H}$ that is annihilated by the Gauss constraint.

We next recall that the key step in deriving the path integral is computing the matrix elements in the position basis of the infinitesimal time-evolution operator $e^{-i\epsilon H}$, for example in the bosonic case we had

$$\langle q'|e^{-i\epsilon H(Q,P)}|q\rangle \approx \int \frac{dp}{2\pi} e^{ip(q'-q)-i\epsilon H(q',p)}. \tag{10.37}$$

In electrodynamics however the Hamiltonian $H$ is only well-defined on the physical Hilbert space, so this equation cannot hold in $\mathcal{H}_{int}$. In order to compute something sensible, we need to include the projection $P_{GI}$ in some way. To this end we'll note that we can write $P_{GI}$ in terms of a path integral over a quantity we will prophetically call $a_0$:

$$P_{GI} = \frac{\int \mathcal{D}a_0 e^{i\epsilon \int d^{d-1}x a_0 (J^0 + \vec{\nabla}\cdot\vec{\Pi})}}{\int \mathcal{D}a_0}, \tag{10.38}$$

where $a_0$ is taken to go to zero at spatial infinity and the infinite quantity in the denominator is there to ensure that acting on a gauge-invariant state we get one.[40] We then have the following manipulation:[41]

$$\begin{aligned} \langle \vec{A}'|e^{-i\epsilon H(\vec{A},\vec{\Pi},\vec{J})} P_{GI}|\vec{A}\rangle &\approx \int \mathcal{D}\pi \langle \vec{A}'|(1 - i\epsilon H(\vec{A},\vec{\Pi},\vec{J})|\pi\rangle\langle\pi|P_{GI}|\vec{A}\rangle \\ &\approx \frac{1}{\int \mathcal{D}a_0} \int \mathcal{D}a_0 \int \mathcal{D}\pi \langle \vec{A}'|\pi\rangle\langle\pi|\vec{A}\rangle (1 - i\epsilon H(\vec{a}', \vec{\pi}, \vec{j}))(1 + i\epsilon \int d^{d-1}x a_0 (j^0 + \vec{\nabla}\cdot\vec{\pi})) \\ &\approx \frac{1}{\int \mathcal{D}a_0} \int \mathcal{D}a_0 \int \mathcal{D}\pi \exp\left[ i \left( (\vec{a}' - \vec{a})\cdot\vec{\pi} - i\epsilon H(\vec{a},\vec{\pi},\vec{j}) + i\epsilon \int d^{d-1}x a_0 (j^0 + \vec{\nabla}\cdot\vec{\pi}) \right) \right] \\ &\approx \frac{1}{\int \mathcal{D}a_0} \int \mathcal{D}a_0 \int \mathcal{D}\pi e^{i\epsilon \int d^{d-1}x \mathcal{L}(a,\vec{\pi})}, \end{aligned} \tag{10.39}$$

[39] The vanishing of the boundary term is clear for $d > 4$, as $\vec{A}$ and $\phi_{cl}$ should both fall off like $1/r^{d-3}$ so we can naively estimate the size of the boundary term as $r^{d-2}/r^{2d-6} = 1/r^{d-4}$. For $d = 4$ one needs to think more the structure of $\vec{A}$ in Coulomb gauge to see that the boundary term still vanishes, I'll leave the details to you.

[40] If we take the gauge group to be $U(1)$ then we should integrate $a_0$ at each point in space over the range $(-\frac{\pi}{\epsilon e}, \frac{\pi}{\epsilon e})$, which becomes $(-\infty, \infty)$ in the continuum limit $\epsilon \to 0$. We can therefore interpret the factor in the denominator as the volume of the gauge group on a given time slice. This statement becomes precise if we introduce a lattice regulator, as we may do later in the semester.

[41] Here I don't write the matter fields explicitly except through their appearance in the current $J$, but of course these need to be included in the Hamiltonian and integrated over as well.

where $\mathcal{L}$ is the covariant Maxwell Lagrangian written in terms of $a$ and $\pi$ and as usual we have ordered $H$ (and the matter fields in $J^0$) so that canonical momenta appear to the right. Thus the covariant path integral automatically imposes a projection onto gauge-invariant states in addition to time evolution by the gauge-invariant Hamiltonian! Following our usual procedures we then end up with a fully gauge-invariant and covariant expression for time-ordered correlation functions of gauge-invariant operators in the vacuum:

$$\langle\Omega|TO_N[A,\Phi]\dots O_1[A,\Phi]|\Omega\rangle = \frac{\int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, O_N[a,\phi]\dots O_1[a,\phi]e^{iS_\epsilon[a,\phi,\phi^*]}}{\int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, e^{iS_\epsilon[a,\phi,\phi^*]}}, \tag{10.40}$$

where as usual the $i\epsilon$ prescription in the action is used to project onto the vacuum at early and late times and $\Phi/\phi$ represent the matter fields. $O_1$ and $O_N$ are taken to be gauge-invariant, otherwise on the left-hand side we should sandwich them between projection operators $P_{GI}$.

Unfortunately in gauge theories the gauge-invariant path integral (10.40) is in some sense "too pure" for practical calculations. The basic problem is that the integrals in the numerator and the denominator both generate infinite factors due to the fact that directions in field space which correspond to to gauge transformations are "flat directions" of the integral. We therefore should divide both the numerator and the denominator by the (infinite) volume of the gauge group in order for them to be (reasonably) well-defined. In fact our above argument essentially generated these factors, in the guise of the denominator $\int \mathcal{D}A_0$. This leads to some inconvenient infinities when we try to do perturbative calculations. For example we can try to compute a two-point function of the gauge field $A_\mu$. $A_\mu$ itself isn't gauge-invariant of course, but its two-point function will regularly show up as in intermediate step in calculations of gauge-invariant quantities so we had best say a little about it. We can write the kinetic term in the exponent as

$$-\frac{i}{4}\int d^dx f_{\mu\nu}f^{\mu\nu} = -\frac{1}{2}\int d^dx d^dy a_\mu(x)a_\nu(y)\mathcal{A}^{\mu\nu}(x,y), \tag{10.41}$$

with[42]

$$\mathcal{A}^{\mu\nu}(x,y) = i\left(\frac{\partial^2}{\partial x^\mu \partial x^\nu} - \eta^{\mu\nu}\frac{\partial^2}{\partial x_\alpha \partial x^\alpha}\right)\delta^d(x-y). \tag{10.42}$$

The photon propagator

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^dp}{(2\pi)^d}\hat{\Delta}_{\mu\nu}(p)e^{ip\cdot(x-y)} \tag{10.43}$$

should be the inverse of $\mathcal{A}^{\mu\nu}$ in the sense that

$$\int d^dy \mathcal{A}^{\mu\nu}(x,y)\Delta_{\nu\lambda}(y-z) = \delta^\mu_\lambda\delta^d(x-z). \tag{10.44}$$

In momentum space this is equivalent to the requirement that

$$i(p^2\eta^{\mu\nu} - p^\mu p^\nu)\hat{\Delta}_{\nu\lambda}(p) = \delta^\mu_\lambda. \tag{10.45}$$

We now meet the essential problem: the matrix $(p^2\eta^{\mu\nu} - p^\mu p^\nu)$ is not invertible! Indeed it has $p_\nu$ as a zero eigenvector, and so there is no quantity $\hat{\Delta}_{\nu\lambda}(p)$ obeying (10.45). This shouldn't be a surprise of course, as $A_\mu$ isn't gauge-invariant and we saw before that to give it a well-defined two-point function we need to gauge fix. This isn't to say that (10.40) is wrong by the way, it isn't; the ambiguity in defining the inverse of $\mathcal{A}^{\mu\nu}$ all cancels when we use use the propagator in gauge-invariant quantities. Indeed the Euclidean lattice version of (10.40) is the starting point for rigorous lattice calculations in quantum electrodynamics, so it had better not have any fundamental problems. Nonetheless for perturbative calculations it is useful to have a nice photon propagator, we'll now see how to do this.

[42] We should replace the time derivatives here by $(1+i\epsilon)\partial_t$ to implement the $i\epsilon$ prescription, I'll leave this implicit to preserve covariant notation.

## 10.5 QED path integral II: fixing the gauge

The most obvious way fix the gauge in the QED path integral is simply to only integrate over configurations which respect the gauge-fixing condition. This approach to the path integral is described in Weinberg's book, where he derives it starting from the canonical quantization in Coulomb gauge that we discussed in the previous section. It leads to path integrals which are somewhat difficult to evaluate in practice however, as one ends up needing to insert a factor

$$\prod_x \delta(\vec{\nabla}\cdot\vec{A}(x)) \tag{10.46}$$

in the Lagrangian path integral (10.40) to impose the gauge condition (and thus to get rid of the infinite factors of the gauge group volume in the numerator and denominator in (10.40)). It was realized by Faddeev and Popov however that we can get a more manageable path integral if we instead impose the gauge constraint more softly, via a Gaussian suppression of configurations that don't obey the gauge fixing condition rather than a delta function suppression. We'll now see how to do this starting directly from our gauge-invariant path integral (10.40). Working just with the numerator, we have the following manipulation

$$\begin{aligned}
\int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, O_N[a,\phi]\dots O_1[a,\phi]e^{iS_\epsilon[a,\phi,\phi^*]} &\propto \int \mathcal{D}f e^{-\frac{i}{2\xi}\int d^dx f^2}\int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, O_N[a,\phi]\dots O_1[a,\phi]e^{iS_\epsilon[a,\phi,\phi^*]}\\
&\propto \int \mathcal{D}\Omega\mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, O_N[a,\phi]\dots O_1[a,\phi]e^{iS_\epsilon[a,\phi,\phi^*]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu+\partial^2\Omega)^2}\\
&= \int \mathcal{D}\Omega\mathcal{D}a_\Omega\mathcal{D}\phi_\Omega\mathcal{D}\phi^*_\Omega\, O_N[a_\Omega,\phi_\Omega]\dots O_1[a_\Omega,\phi]e^{iS_\epsilon[a_\Omega,\phi_\Omega,\phi^*_\Omega]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu_\Omega)^2}\\
&\propto \int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, O_N[a,\phi]\dots O_1[a,\phi]e^{iS_\epsilon[a,\phi,\phi^*]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu)^2}.
\end{aligned} \tag{10.47}$$

In the first line we simply multiplied by a constant, and then in the second line we changed variables in the $f$ integral from $f$ to $\Omega$ via the field-dependent expression

$$f = \partial_\mu A^\mu + \partial^2\Omega. \tag{10.48}$$

This change of variables generates a field-independent determinant $\det(\partial^2)$ which we discarded (hence the $\propto$). Each value of $f$ is attained in this integral since we can always solve (10.48) for $\Omega$ using the Green's function for the scalar wave operator $\partial^2$. The solution is not unique since we can add to $\Omega$ any function which solves the massless wave equation to $\Omega$, so the integral over $\Omega$ generates an additional constant factor reflecting this redundancy (which is the same redundancy which prevented the Lorenz gauge from being a complete gauge-fixing). In the third line we then used the gauge invariance of the path integral measure and action, as well as the operator insertions $O_1\dots O_N$, using the abbreviated notation

$$\begin{aligned}
(a_\Omega)_\mu &= a_\mu + \partial_\mu\Omega\\
\phi_\Omega &= e^{iq\Omega}\phi,
\end{aligned} \tag{10.49}$$

and then in the final line we changed the integration variables from $(a_\Omega,\phi_\Omega)$ to $(a,\phi)$ and then performed and discarded the integral over $\Omega$ which now gives a field-independent constant. You may worry about the convergence of the $f$ integral, but implicitly here we are using the $i\epsilon$ prescription $t=\tau(1-i\epsilon)$ so the measure in the $d^dx$ integral includes a factor of $(1-i\epsilon)$ which makes the $f$-integral convergent for $\xi>0$. The final upshot of this is that we can replace equation (10.40) by[43]

$$\langle\Omega|TO_N[A,\Phi]\dots O_1[A,\Phi]|\Omega\rangle = \frac{\int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, O_N[a,\phi]\dots O_1[a,\phi]e^{iS_\epsilon[a,\phi,\phi^*]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu)^2}}{\int \mathcal{D}a\mathcal{D}\phi\mathcal{D}\phi^*\, e^{iS_\epsilon[a,\phi,\phi^*]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu)^2}}, \tag{10.50}$$

[43]It must be acknowledged that the interpretation of this procedure in the operator formalism isn't so clear: does canonical quantization make sense if we only suppress configurations which violate the gauge-fixing condition rather than forbidding them entirely? The answer to this question ends up being "yes", but we need to introduce a rather involved formalism called **BRST quantization** to see it. We won't pursue this further here, as anyways we already have a gauge-invariant operator interpretation of the path integral (10.40) through our construction of the physical subspace of $\mathcal{H}_{big}$.

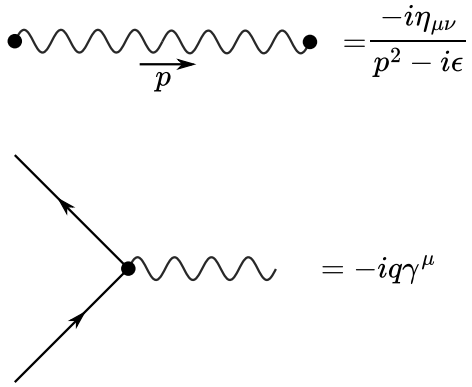


Figure 21: Feynman rules for spinor electrodynamics. Photon propagators are indicated by wavy lines.

for arbitrary $\xi > 0$. This is a substantial improvement on (10.40), as at least those gauge transformations which have $\partial^2\Omega \neq 0$ are now suppressed in the integral.

Let's see what this says about the photon propagator. Including our new "gauge-fixing" term in the action, equation (10.42) is now modified to

$$\mathcal{A}^{\mu\nu}(x,y) = i\left(\left(1-\frac{1}{\xi}\right)\frac{\partial^2}{\partial x^\mu \partial x^\nu} - \eta^{\mu\nu}\frac{\partial^2}{\partial x_\alpha \partial x^\alpha}\right)\delta^d(x-y). \tag{10.51}$$

To get the propagator we thus now want to solve

$$i\left(p^2\eta^{\mu\nu} - \left(1-\frac{1}{\xi}\right)p^\mu p^\nu\right)\hat{\Delta}_{\nu\lambda}(p) = \delta^\mu_\lambda, \tag{10.52}$$

whose solution you will show on the homework is

$$\hat{\Delta}_{\mu\nu}(p) = -i\left(\frac{\eta_{\mu\nu}}{p^2 - i\epsilon} - (1-\xi)\frac{p_\mu p_\nu}{(p^2-i\epsilon)^2}\right). \tag{10.53}$$

The position-space photon propagator is thus given by

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^dp}{(2\pi)^d}\left(\frac{-i\eta_{\mu\nu}}{p^2-i\epsilon} + i(1-\xi)\frac{p_\mu p_\nu}{(p^2-i\epsilon)^2}\right)e^{ip(x-y)}. \tag{10.54}$$

Anyone sane would look at these expressions and immediately set $\xi = 1$, which is called **Feynman gauge**. There are however a few other options which are considered, most commonly the **Lorenz gauge** $\xi = 0$, in which case the Gaussian in the path integral becomes a $\delta$-function imposing the Lorenz gauge condition. True masochists can work with $\xi$ as a free parameter, as it gives a useful check that any physical observable must be independent of $\xi$. We however will stick with Feynman gauge, so from now on our photon propagator is

$$\Delta_{\mu\nu}(x-y) = \int \frac{d^dp}{(2\pi)^d}\frac{-i\eta_{\mu\nu}}{p^2-i\epsilon}e^{ip(x-y)} = \eta_{\mu\nu}G_F(x-y). \tag{10.55}$$

## 10.6 Feynman rules for spinor electrodynamics

We are now in a position to state the Feynman rules for computing gauge-invariant correlation functions in spinor electrodynamics. These are exactly the same as for Yukawa theory, except for three updates:

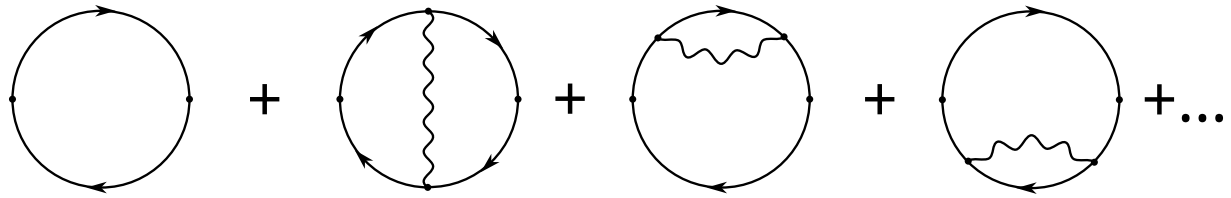

Figure 22: Leading diagrams for the two-point function of the gauge-invariant operator $\overline{\psi}\psi$ (self-contractions are removed by normal ordering).

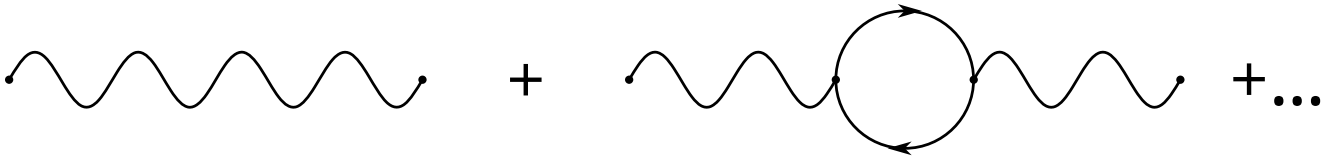

Figure 23: Leading diagrams for the photon propagator.

- Instead of the Yukawa interaction $-ig\phi\overline{\psi}\psi$, which supplies a factor of $g$ for each vertex, we now have the QED interaction $-qa_\mu\overline{\psi}\gamma^\mu\psi$, which supplies a factor of $-iq\gamma^\mu$ for each vertex. The $\gamma^\mu$ matrix is multiplied by the fermion propagators which are attached to the vertex, with the multiplication going from right to left in the direction of the fermion arrows as in Yukawa theory.
- Instead of the scalar propagator
$$\hat{G}_F(p) = \frac{-i}{p^2 + m_\phi^2 - i\epsilon} \tag{10.56}$$
we now have the photon propagator
$$\hat{\Delta}_{\mu\nu}(p) = \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon}. \tag{10.57}$$
- In order to use these expressions the external operators must be gauge-invariant.

The new ingredients are shown in diagrammatic form in figure 21.

To illustrate these rules the leading diagrams contributing to the two-point function of the gauge-invariant operator $\overline{\Psi}\Psi$ are shown in figure 22. They give the two-point function

$$\begin{aligned}\langle\Omega|T\overline{\Psi}\Psi(x)\overline{\Psi}\Psi(y)|\Omega\rangle = &- q^2\mathrm{Tr}\Big(S_F(x-y)S_F(y-x)\Big) - q^2(-iq)^2\int d^dz_1\int d^dz_2\Delta_{\mu\nu}(z_2-z_1)\\ &\times\mathrm{Tr}\Bigg(S_F(x-z_2)\gamma^\nu S_F(z_20y)S_F(y-z_1)\gamma^\mu S_F(z_1-x)\\ &\quad + S_F(x-y)S_F(y-z)2)\gamma^\nu S_F(z_2-z_1)\gamma^\mu S_F(z_1-x)\\ &\quad + S_F(x-z_2)\gamma^\nu S_F(z_2-z_1)\gamma^\mu S_F(z_1-y)S_F(y-x)\Bigg).\end{aligned} \tag{10.58}$$

In the next section we will learn how to compute perturbative scattering amplitudes in QED.

**Problems:**

1. Show that the Maxwell kinetic term indeed can be written as (10.41) with $\mathcal{A}^{\mu\nu}$ given by (10.42), and also that (10.53) solves (10.52).

2. Starting from (10.53), write an expression for the free time-ordered two-point function of $F_{\mu\nu}$ in momentum space. Make sure that your answer is independent of $\xi$.

3. Write out position and momentum space expressions for the one-loop correction to the $F_{\mu\nu}$ two-point function from the diagram shown in figure 23. You don't need to evaluate the trace or the loop integrals.

4. In this problem we'll pursue one approach to the Dirac quantization of magnetic monopoles. One way to think about magnetic monopoles is that on a two-sphere surrounding the monopole the gauge field is given in spherical coordinates by[44]

$$A_\phi = -\frac{n}{2e}(1+\cos\theta). \tag{10.59}$$

   Show that this gauge potential describes a constant magnetic flux through the sphere, whose integral is

$$\int \vec{B}\cdot d\vec{A} = \frac{2\pi n}{e}. \tag{10.60}$$

   Compute the holonomy

$$h = \int_0^{2\pi} d\phi A_\phi \tag{10.61}$$

   around a circle of constant $\theta$. Does this vanish at $\theta = \pi$? How about at $\theta = 0$? In $U(1)$ gauge theory the Wilson line

$$W = e^{ieh} \tag{10.62}$$

   should become the identity for a circle of vanishing size. Applying this to the circle at $\theta = 0$, what do we learn about $n$?

5. Extra credit: show that the boundary term in (10.35) indeed vanishes for $d = 4$. You'll need to remember how to solve Maxwell's equations in Coulomb gauge given a fixed background current, and you'll also need to be a bit clever.

# 11 Quantum electrodynamics III: scattering

In this section we will learn how to compute perturbative scattering amplitudes in spinor electrodynamics (scalar electrodynamics works in a similar way). We will follow our usual practice of using the LSZ formula to extract the Feynman rules, being (somewhat) careful to address the new subtleties that arise from gauge invariance.

## 11.1 Asymptotic states in quantum electrodynamics

In the last section we arrived at a simple covariant formula for computing correlation functions of gauge-invariant operators in quantum electrodynamics using the path integral (now switching to spinor notation

[44] Note this is using the physics convention where $\theta \in (0,\pi)$ is the polar angle and $\phi \in (0, 2\pi)$ is the azimuthal angle. Also beware that here $A_\phi$ is the $\phi$ component of the one-form $A$, i.e. the one-form is $A_\phi d\phi$. $A_\phi$ is *not* the component of a vector $A$ in the direction of a unit vector $\hat{\phi}$, so if you want to use formulas for the spherical curl e.g. in Griffiths then you need to figure out how to rescale $A_\phi$. Your life will be easier if you instead just integrate the two-form $F_{\mu\nu}$ on the two-sphere using the intrinsic definition of the integral of a two-form on a two-manifold.

for the charged fields):

$$\langle\Omega|TO_N[A,\Psi,\overline{\Psi}]\dots O_1[A,\Psi,\overline{\Psi}]|\Omega\rangle = \frac{\int \mathcal{D}a\mathcal{D}\psi\mathcal{D}\overline{\psi}\, O_N[a,\psi,\overline{\psi}]\dots O_1[a,\psi,\overline{\psi}]e^{iS_\epsilon[a,\psi,\overline{\psi}]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu)^2}}{\int \mathcal{D}a\mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{iS_\epsilon[a,\psi,\psi^*]-\frac{i}{2\xi}\int d^dx(\partial_\mu a^\mu)^2}}, \tag{11.1}$$

where $O_1 \dots O_N$ are all gauge-invariant. We also saw how to evaluate this path integral perturbatively using Feynman diagrams, in which we chose to set $\xi = 1$ and use the photon propagator

$$\hat{\Delta}_{\mu\nu}(p) = \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon}. \tag{11.2}$$

As usual the spinor propagator is

$$\hat{S}_F(p) = \frac{i(\not{p} + im)}{p^2 + m^2 - i\epsilon}, \tag{11.3}$$

where $m$ is the (bare) mass of the electron, and the interaction vertex factor is $-iq\gamma^\mu$. The next step in most QFT textbooks is to take each $O_i$ in (11.1) to be $A_\mu$, $\Psi$, or $\widetilde{\Psi}$, and then apply the LSZ formula to extract the S-matrix by taking the external momenta to be on shell and stripping off the resulting poles. Unfortunately however these operators are not gauge-invariant, so this manipulation is not justified without some further explanation.

For $A_\mu$ the idea is simple: what the LSZ procedure really requires us to do is take the Fourier transform of $A_\mu$, take the external momentum to be on-shell, and then contract with $\epsilon^{\mu *}(p,\sigma)$ for a photon in the final state or $\epsilon^\mu(p,\sigma)$ for a photon in the initial state. In the path integral this means that the operator we are really inserting is

$$e^{\mu *}(\vec{p},\sigma)\int d^dx e^{-ip\cdot x}a_\mu(x) \tag{11.4}$$

with $p^0 = |p|$ for final state photons and

$$e^{\mu}(\vec{p},\sigma)\int d^dx e^{ip\cdot x}a_\mu(x) \tag{11.5}$$

with $p^0 = |p|$ for initial state photons. The polarization vectors here are chosen in some arbitrary gauge, for example we can use the Coulomb gauge $e^\mu$. Regardless of the choice we make the operators (11.4) and (11.5) are actually gauge-invariant, since under a gauge transformation

$$a'_\mu = a_\mu + \partial_\mu\Omega \tag{11.6}$$

the Fourier transform of $a_\mu$ changes by $ip_\mu\hat{\Omega}(p)$ and on-shell we have $e\cdot p = e^*\cdot p = 0$. Thus we can use (11.4) and (11.5) in (11.1) with no problem.[45]

The situation is substantially more complicated for the electron/positron fields $\Psi$ and $\overline{\Psi}$. As explained in the previous section, to make these gauge-invariant we need to dress them using a classical solution of Gauss's law:

$$\widetilde{\Psi}(x) = e^{iq\int_{\Sigma_t} d^{d-1}x\vec{E}_{cl}\cdot\vec{A}}\Psi(x). \tag{11.7}$$

We can think of the dressing as creating a **coherent state** of the electromagnetic field, and in a proper treatment of the asymptotic states of QED this coherent state must be included in order to get sensible transition amplitudes. The theory of these asymptotic states goes well beyond what we can reasonably cover in this class, there is a nice four-paper series "coherent soft-photon states and infrared divergences" by Kibble from 1968 which I encourage you to look at if you want to learn more about it. There is also a famous paper by Faddeev and Kulish a few years later which is more commonly cited, it simplifies somewhat Kibble's formalism but the connection to LSZ is less clear. I will content myself with just making a few comments:

[45] Of course we could also just use the gauge-invariant operator $F_{\mu\nu}$ in the LSZ formula, which would remove the need for this discussion, but that introduces an unnecessary extra index and for the non-abelian gauge theories we'll discuss next semester the analogue of $F_{\mu\nu}$ isn't gauge-invariant so we might as well stick with $A_\mu$.

- Due to the nonlocality of $\widetilde{\Psi}$ and the masslessness of the photon, when we Fourier transform correlation functions involving $\widetilde{\Psi}$ and go on the mass shell the singularity we find is modified from a simple pole to a branch cut. Extracting the scattering amplitude now requires isolating the coefficient of this cut.

- The details of the solution $\vec{E}_{cl}$ end up not contributing to the coefficient of the branch cut as the external momentum goes on shell, it is only the small-momentum part of its Fourier transform which matters and this serves primarily to cancel certain "soft photon infrared divergences" that arise at higher orders in perturbation theory applied to the "bare" correlation function.

- Indeed in the standard textbook treatment of QED the dressing is neglected entirely: one simply inserts bare charged fields into the path integral (11.1) and then tries extract an S-matrix. A price is paid however for doing this: the dressing factors no longer cancel the soft photon infrared divergences and in fact the latter set the scattering amplitude to zero![46] The textbook approach to this (see chapter 13 of Weinberg for a nice presentation) is to introduce a small photon mass to regulate the divergence, square the regularized amplitude to get a differential cross section, and then sum over the number of low-energy photons in the final state. One can then argue that this "inclusive" differential cross section is finite in the limit that the mass of the photon goes to zero.

I must confess that I was never happy with the standard approach described in this third bullet point - why should we need to introduce a nonzero photon mass when the correlation functions of gauge-invariant operators are all finite? Is it really true that it is only the squares of transition amplitudes which make sense in QED, and if so is this a modification of quantum mechanics? The approach based on dressing and coherent states described by Kibble/Fadeev/Kulish is much more satisfying to me, since charged particles really do come together with the Coulomb fields and neglecting this is a crime for which one should be and is punished. This does not mean that the differential cross sections computed by the standard method are wrong of course, but to properly interpret them it is better to remember the dressing.

Fortunately for us the QED calculations we will actually do are either at lowest order in perturbation theory, in which case we can neglect the dressing part of $\widetilde{\Psi}$, or they involve one-loop diagrams which can be interpreted entirely in terms of gauge-invariant correlation functions where these infrared subtleties are not important. We thus will for the most part leave the infrared problem here.

## 11.2 External leg factors

In our discussions of $\phi^4$ theory and Yukawa theory, we saw that to compute the connected covariant scattering amplitude $i\widetilde{M_c}$ one sums over pruned connected Feynman diagrams, with the external fermion factors shown in figure 15. Up to the infrared issues just mentioned we can use the same factors for external fermions in spinor electrodynamics, so the final ingredient we need are the external factors for photons in the initial and final states. We can define a photon renormalization factor $Z_\gamma$ by

$$\langle \Omega | F_{\mu\nu}(0) | \vec{k}, \sigma, \gamma \rangle = Z_\gamma \frac{i\big(k_\mu e_\nu(\vec{k},\sigma) - k_\nu e_\mu(\vec{k},\sigma)\big)}{\sqrt{2\omega_{\vec{k}}}}, \tag{11.8}$$

where $|\vec{k}, \sigma, \gamma\rangle$ is a one-photon state (one-photon states are well-defined in QED, unlike one-electron states since the latter need to be dressed). To get an element of $i\widetilde{M_c}$ we want to remove the LSZ factors of

$$Z_\gamma \sum_{\sigma'} e_\mu(\vec{k}', \sigma') \frac{-i\sqrt{2\omega_{\vec{k}'}}}{(k')^2 - i\epsilon} \langle \vec{k}', \sigma', \gamma | \tag{11.9}$$

for each final-state photon and

$$Z_\gamma \sum_{\sigma} e^*_\mu(\vec{k}, \sigma) \frac{-i\sqrt{2\omega_{\vec{k}}}}{(k)^2 - i\epsilon} |\vec{k}, \sigma, \gamma\rangle \tag{11.10}$$

[46] This zero is quite intuitive, what it is saying is that you are never going to produce an electron without its Coulomb field.

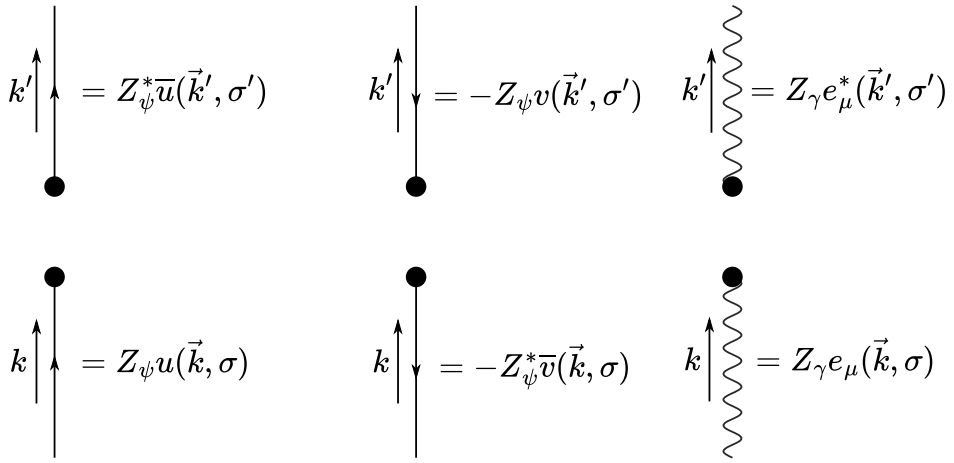


Figure 24: External leg factors for computing the connected covariant amplitude $\widetilde{M_c}$ in quantum electrodynamics. As in Yukawa theory we should sum over all pruned connected diagrams, with the external propagators replaced by these factors. The dot indicates the interaction vertex which is connected to the rest of the diagram, so the top row are final state factors and the bottom row are initial state factors.

for each initial-state photon in correlation functions involving $A_\mu$. The exact external propagator legs in the correlation function near the mass shell each have the form

$$Z_\gamma^2 \frac{-i\eta_{\mu\nu}}{k^2 - i\epsilon} J^\nu, \tag{11.11}$$

where $J^\nu$ indicates the current that the propagator is contracted with at the interaction vertex, and so to extract the scattering states we should strip off the exact external propagator by only drawing pruned diagrams and then contracting each interaction current $J^\mu$ which is attached to an external line with a factor of $Z_\gamma e_\mu(\vec{k}', \sigma')$ for each photon in the final state and a factor of $Z_\gamma e_\mu(\vec{k}, \sigma)$ for each photon in the initial state (the factors of $\sqrt{2\omega_{\vec{k}}}$ are absorbed into the definition of $i\widetilde{M_c}$, which removes these factors and also the overall momentum-conserving $\delta$-function. These rules are shown in figure 24.

In the traditional approach to the renormalization of QED the phase of $\Psi$ and the sign of $A_\mu$ are adjusted so that both $Z_\psi$ and $Z_\gamma$ are real and positive, and one then defines the conventional factors

$$\begin{aligned} Z_\psi &= \sqrt{Z_2} \\ Z_\gamma &= \sqrt{Z_3}. \end{aligned} \tag{11.12}$$

We will see in the next section that the electron charge $q$ must be renormalized, and this renormalization is conventionally parametrized by a constant $Z_1 > 0$ via

$$q = \sqrt{Z_3} \frac{Z_2}{Z_1} q_0, \tag{11.13}$$

where $q_0$ is the bare electron charge. In fact however this constant is not free: if we define the renormalized fields

$$\begin{aligned} \Psi_R &= Z_2^{-1/2} \Psi \\ A_{R,\mu} &= Z_3^{-1/2} A_\mu, \end{aligned} \tag{11.14}$$

then to preserve the gauge invariance of the gauge covariant derivative we must have

$$\partial_\mu - i q_R A_{R,\mu} = \partial_\mu - i q_0 A_\mu, \tag{11.15}$$

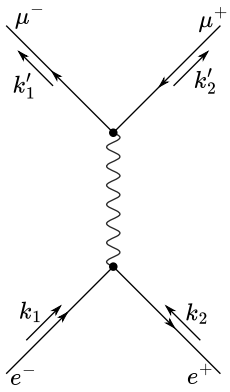


Figure 25: The tree-level contribution to $e^+\,e^- \to \mu^+\,\mu^-$.

and thus

$$Z_1 = Z_2. \tag{11.16}$$

The electron mass $m$ is also renormalized as

$$m_R = m + \delta m, \tag{11.17}$$

so to work out the renormalization of QED we need to compute the three quantities $Z_2$, $Z_3$, and $\delta m$. We will return to this in the next section when we consider loop diagrams.

## 11.3 $e^+\,e^- \to \mu^+\,\mu^-$ at tree level

Let's now consider our first QED scattering amplitude, the scattering of an electron-positron pair to a muon-antimuon pair. This process only has one diagram which contributes, which is shown in figure 25 (had we had another electron-positron pair in the final state there would have been another diagram). According to the Feynman rules, the connected covariant matrix element is

$$\begin{aligned} i\widetilde{M}_c &= (-iq)^2 \overline{u}(\vec{k}_1', \sigma_1')\gamma^\mu v(\vec{k}_2', \sigma_2')\overline{v}(\vec{k}_2, \sigma_2)\gamma^\nu u(\vec{k}_1, \sigma_1)\frac{-i\eta_{\mu\nu}}{(k_1+k_2)^2 - i\epsilon} \\ &= iq^2 \frac{\overline{u}_{1'}\gamma^\mu v_{2'}\overline{v}_2\gamma_\mu u_1}{(k_1+k_2)^2}, \end{aligned} \tag{11.18}$$

where in the second line we've adopted the same abbreviated notation for $u$ and $v$ that we used in Yukawa theory and also set $\epsilon \to 0$ since there are no integrals over momentum since we are at tree level. To compute the spin-averaged differential cross section we need to square this and sum/average over the external spins. Noting that

$$(\overline{f}\gamma^\mu g)^* = g^\dagger \gamma^0 \gamma^\mu \gamma^0 \gamma^{0\dagger} f = \overline{g}\gamma^\mu f \tag{11.19}$$

for $f$, $g$ equal to any $u$ or $v$, we have (setting $d = 4$)

$$\begin{aligned}\sum_{\sigma,\sigma'} |\widetilde{M}_c|^2 &= \frac{q^4}{(k_1+k_2)^4} \sum_{\sigma,\sigma'} \overline{u}_{1'}\gamma^\mu v_{2'}\overline{v}_{2'}\gamma^\nu u_{1'}\overline{v}_2\gamma_\mu u_1\overline{u}_1\gamma_\nu v_2 \\ &= \frac{q^4}{(k_1+k_2)^4} \mathrm{Tr}\Big[(\not{k}'_1 + im_\mu)\gamma^\mu(\not{k}'_2 - im_\mu)\gamma^\nu\Big]\mathrm{Tr}\Big[(\not{k}_1 + im_e)\gamma_\nu(\not{k}_2 - im_e)\gamma_\mu\Big] \\ &= \frac{16q^4}{(k_1+k_2)^4}\Big((k'_1)^\mu(k'_2)^\nu + (k'_1)^\nu(k'_2)^\mu - (k'_1\cdot k'_2)\eta^{\mu\nu} + m_\mu^2\eta^{\mu\nu}\Big)\Big((k_1)_\mu(k_2)_\nu + (k_1)_\nu(k_2)_\mu - (k_1\cdot k_2)\eta_{\mu\nu} + m_e^2\eta_{\mu\nu}\Big) \\ &= \frac{32q^4}{(k_1+k_2)^4}\Big((k_1\cdot k'_1)(k_2\cdot k'_2) + (k_1\cdot k'_2)(k_2\cdot k'_1) - m_\mu^2(k_1\cdot k_2) - m_e^2(k'_1\cdot k'_2) + 2m_\mu^2 m_e^2\Big).\end{aligned} \tag{11.20}$$

As usual we can convert this into a spin summed/averaged differential cross section in the center of mass frame using the formula

$$\frac{d\sigma_{ave}}{d\Omega} = \frac{|k'|}{64\pi^2 E_{tot}^2|k|} \times \frac{1}{4}\sum_{\sigma,\sigma'} |\widetilde{M}_c|^2. \tag{11.21}$$

In the center of mass frame the particles all have the same energy $\omega = E_{tot}/2$, with

$$\begin{aligned}|k| &= \sqrt{\omega^2 - m_e^2} \\ |k'| &= \sqrt{\omega^2 - m_\mu^2},\end{aligned} \tag{11.22}$$

and the various inner products are given by

$$\begin{aligned}k_1\cdot k'_1 = k_2\cdot k'_2 &= -\omega^2 + \sqrt{\omega^2 - m_e^2}\sqrt{\omega^2 - m_\mu^2}\cos\theta \\ k_1\cdot k'_2 = k_2\cdot k'_1 &= -\omega^2 - \sqrt{\omega^2 - m_e^2}\sqrt{\omega^2 - m_\mu^2}\cos\theta \\ k_1\cdot k_2 &= -2\omega^2 + m_e^2 \\ k'_1\cdot k'_2 &= -2\omega^2 + m_\mu^2 \\ (k_1+k_2)^2 &= 4\omega^2\end{aligned} \tag{11.23}$$

We thus have

$$\begin{aligned}\frac{1}{4}\sum_{\sigma,\sigma'} |\widetilde{M}_c|^2 &= \frac{q^4}{2}\left(\left(1 - \sqrt{1-\frac{m_\mu^2}{\omega^2}}\sqrt{1-\frac{m_\mu^2}{\omega^2}}\cos\theta\right)^2 + \left(1 + \sqrt{1-\frac{m_\mu^2}{\omega^2}}\sqrt{1-\frac{m_\mu^2}{\omega^2}}\cos\theta\right)^2 + \frac{2}{\omega^2}(m_\mu^2 + m_e^2)\right) \\ &= q^4\left(1 + \frac{m_\mu^2 + m_e^2}{\omega^2} + \left(1 - \frac{m_\mu^2}{\omega^2}\right)\left(1 - \frac{m_e^2}{\omega^2}\right)\cos^2\theta\right),\end{aligned} \tag{11.24}$$

and therefore

$$\frac{d\sigma_{ave}}{d\Omega} = \frac{q^4}{64\pi^2 E_{tot}^2}\frac{\sqrt{1-\frac{m_\mu^2}{\omega^2}}}{\sqrt{1-\frac{m_e^2}{\omega^2}}}\left(1 + \frac{m_\mu^2 + m_e^2}{\omega^2} + \left(1 - \frac{m_\mu^2}{\omega^2}\right)\left(1 - \frac{m_e^2}{\omega^2}\right)\cos^2\theta\right). \tag{11.25}$$

The spin summed/averaged total cross section is

$$\begin{aligned}\sigma_{ave} &= \int d\Omega \frac{d\sigma_{ave}}{d\Omega} \\ &= \frac{q^4}{16\pi E_{tot}^2} \frac{\sqrt{1-\frac{m_\mu^2}{\omega^2}}}{\sqrt{1-\frac{m_e^2}{\omega^2}}} \left(1 + \frac{m_\mu^2 + m_e^2}{\omega^2} + \frac{1}{3}\left(1 - \frac{m_\mu^2}{\omega^2}\right)\left(1 - \frac{m_e^2}{\omega^2}\right)\right) \\ &= \frac{q^4}{12\pi E_{tot}^2} \frac{\sqrt{1-\frac{m_\mu^2}{\omega^2}}}{\sqrt{1-\frac{m_e^2}{\omega^2}}} \left(1 + \frac{m_\mu^2 + m_e^2}{2\omega^2} + \frac{m_e^2 m_\mu^2}{4\omega^4}\right). \end{aligned} \tag{11.26}$$

Note that the cross section vanishes when $\omega = m_\mu$ and must be taken to zero below this, since for $\omega < m_\mu$ the electron and positron do not have enough energy to produce the muon and antimuon. This kind of sudden turning on of a production cross section is called a **threshold**.

In the real world we have

$$\begin{aligned} m_e &\approx 5.1 \times 10^5 \,\mathrm{eV} \\ m_\mu &\approx 1.1 \times 10^8 \,\mathrm{eV}, \end{aligned} \tag{11.27}$$

so in situations where $\omega \gg m_e$ it is sensible to set $m_e = 0$ in these formulas, leading to

$$\begin{aligned} \frac{d\sigma_{ave}}{d\Omega} &= \frac{q^4}{64\pi^2 E_{tot}^2} \sqrt{1-\frac{m_\mu^2}{\omega^2}} \left(1 + \frac{m_\mu^2}{\omega^2} + \left(1 - \frac{m_\mu^2}{\omega^2}\right)\cos^2\theta\right) \\ \sigma_{ave} &= \frac{q^4}{12\pi E_{tot}^2} \sqrt{1-\frac{m_\mu^2}{\omega^2}} \left(1 + \frac{m_\mu^2}{2\omega^2}\right), \end{aligned} \tag{11.28}$$

which is a bit simpler but still has clear threshold behavior. In the vicinity of the threshold the angular dependence of the differential cross section is suppressed, so when the muon/antimuon pair is produced almost at rest their momenta are essentially equally likely to be pointing in any direction. On the other hand in the ultra high energy limit $\omega \gg m_\mu$ we can lmost take $m_\mu$ to zero, leading to the nice expressions

$$\begin{aligned} \frac{d\sigma_{ave}}{d\Omega} &= \frac{q^4}{64\pi^2 E_{tot}^2} \left(1 + \cos^2\theta\right) \\ \sigma_{ave} &= \frac{q^4}{12\pi E_{tot}^2}. \end{aligned} \tag{11.29}$$

In particular there is now strong angular dependence of the differential cross section, with the cross section being peaked at $\theta = 0$ ("forward scattering") and $\theta = \pi$ ("backward scattering"), with the differential cross section being minimized at $\theta = \pi/2$. In the total cross section the power of $q$ can be immediately seen from the diagram and the power of $E_{tot}$ is set by dimensional analysis, so we only need to evaluate the diagram to get the $12\pi$.

## 11.4 $e^+ e^- \to$ hadrons, $R$ ratio

There is a nice application to quark physics of our formula (11.28) for the total cross section of $e^+ e^- \to \mu^+ \mu^-$. We have not yet discussed quantum chromodynamics (QCD), which is the fundamental theory of the strong interactions, but it consists of a set of spinor "quark" fields interacting with a generalized version of the gauge field $A_\mu$ called the "gluon" field. There are six kinds of quarks, each coming in one of three "colors", and the masses and electric charges of these quarks are given by:

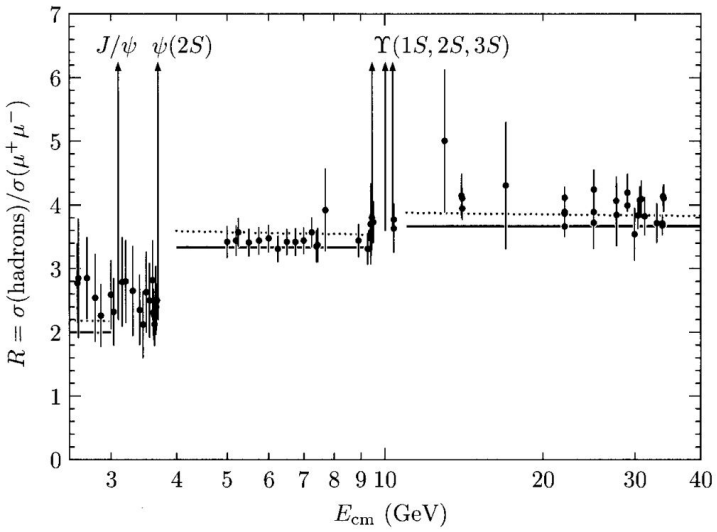


Figure 26: Experimental measurement of the R ratio of $\sigma_{ave}(e^+\,e^- \to \text{hadrons})$ to the high-energy limit of $\sigma_{ave}(e^+\,e^- \to \mu^+\,\mu^-)$. The jumps around 3 GeV and 8 GeV correspond to the thresholds for the charm and bottom quarks respectively, with the various other peaks corresponding to hadronic resonances not captured by our simple tree-level analysis. The solid black line is the tree-level result described in the text. Image shamelessly stolen from Peskin and Schroeder.

| Quark type | mass | charge |
|---|---|---|
| down | $4.7 \times 10^6$ eV | $-\frac{1}{3}e$ |
| up | $2.2 \times 10^6$ eV | $\frac{2}{3}e$ |
| strange | $9.5 \times 10^7$ eV | $-\frac{1}{3}e$ |
| charm | $1.3 \times 10^9$ eV | $\frac{2}{3}e$ |
| bottom | $4.2 \times 10^9$ eV | $-\frac{1}{3}e$ |
| top | $172 \times 10^9$ eV | $\frac{2}{3}e$ |

Since quarks are electrically charged, we can pair-produce them by colliding electron and positron pairs just as we saw for muons in the previous subsection. At tree level the diagram is the same as in figure 25, except that we should label the final state quarks as $q$ and $\overline{q}$ instead of $\mu^-$ and $\mu^+$. Moreover the electron charge $q$ should be replaced by the quark electric charge the upper interaction vertex (but not the lower one), the muon mass should be replaced by the quark mass, and we should multiply by a factor of 3 since there are three colors of quark. Roughly speaking we thus should expect the total cross section for $e^+\,e^- \to q\,\overline{q}$ to be given by

$$\sigma_{ave}(e^+\,e^- \to q\,\overline{q}) = R\sigma(e^+\,e^- \to \mu^+\,\mu^-), \tag{11.30}$$

with

$$R = 3\sum_i q_i^2, \tag{11.31}$$

with the sum being over all quark types whose mass is less than $E_{com}/2$ and $q_i$ being the electric charge of the $i$th type of quark. There is a major problem with this expectation however: in QCD the potential energy between a quark and antiquark grows linearly with distance, so if we try to separate them eventually there is enough energy to pull more quarks and antiquarks into existence. These additional quarks/antiquarks combine with the original quark and antiquark to form **hadrons**, which are bound states of quarks and gluons such as pions, protons, and neutrons that are uncharged under the strong force. This dynamical process is called **hadronization**, and it is the mechanism which enforces **confinement** - one can never

find quarks, antiquarks, or gluons in isolation. You will learn more about confinement next semester. The amazing thing however is that hadronization happens at energy scales which are of order

$$\Lambda_{QCD} \approx 2 \times 10^8 \,\mathrm{eV}, \tag{11.32}$$

and as long as we are colliding our electron-positron pair at energies which are large compared to this then we can cleanly separate the perturbative scattering process shown in figure 25 (with the muon/antimuon replaced by a quark/antiquark) from the complicated hadronization dynamics. Indeed the only effect of hadronization is to replace the outgoing quarks by **jets**, which are collimated streams of hadrons that are moving roughly in the direction the initial quarks were moving in. Therefore a better version of (11.30) is

$$\sigma_{ave}(e^+ e^- \to \text{hadrons}) = R\sigma(e^+ e^- \to \mu^+ \mu^-), \tag{11.33}$$

with $R$ again given at first pass by (11.31). On the right-hand side we are now just looking at the total cross section where the electron-positron pair produces some kind of hadronic final state - we can hope that away from special energies this will be dominated by back-to-back jets coming from a $q\bar{q}$ pair. Let's see what to expect:

- For center of mass energies above $\Lambda_{QCD}$ but below $2m_c \approx 2.6\,\mathrm{GeV}$ the R ratio is

$$3\left(\frac{2}{3}\right)^2 + 6\left(\frac{1}{3}\right)^2 = 2. \tag{11.34}$$

- For center of mass energies above $2m_c$ but below $2m_b \approx 8.4\,\mathrm{GeV}$ the R ratio is

$$2 + 3\left(\frac{2}{3}\right)^2 = \frac{10}{3} \approx 3.3. \tag{11.35}$$

- For center of mass energies above $2m_b$ and below $2m_t$, the latter of which has not been reached in any $e^+ e^-$ collider, the R ratio is

$$\frac{10}{3} + 3\left(\frac{1}{3}\right)^2 = 11/3 \approx 3.7. \tag{11.36}$$

The experimental data (as of 1995, sorry) is shown in figure 26, as you can see this rather naive prediction actually works quite well! It also is quite sensitive to the details of QCD, to get these numbers we needed to know about how many quarks there are of each charge and also their masses.

## 11.5 Compton scattering

Another important tree-level process in quantum electrodynamics is **Compton scattering**, which is the process $e^- \gamma \to e^- \gamma$. The two diagrams which contribute at tree-level are shown in figure 27. These two diagrams sum to

$$i\widetilde{M}_c(e^- \gamma \to e^- \gamma) = (-iq)^2 \overline{u}_{1'} \left( \frac{\not{\epsilon}^*_{2'} i(\not{k}_1 + \not{k}_2 + im)\not{\epsilon}_2}{(k_1+k_2)^2 + m^2 - i\epsilon} + \frac{\not{\epsilon}_2 i(\not{k}_1 - \not{k}'_2 + im)\not{\epsilon}^*_{2'}}{(k_1-k'_2)^2 + m^2 - i\epsilon} \right) u_1. \tag{11.37}$$

Before squaring this we can simplify it a bit by noting that

$$\begin{aligned} (k_1+k_2)^2 + m^2 &= 2k_1 \cdot k_2 \\ (k_1-k'_2)^2 + m^2 &= -2k_1 \cdot k'_2 \\ (\not{p} + im)\gamma^\mu u(\vec{p},\sigma) &= \left(\gamma^\mu(-\not{p} + im) + 2p^\mu\right) u(\vec{p},\sigma) = 2p^\mu u(\vec{p},\sigma), \end{aligned} \tag{11.38}$$

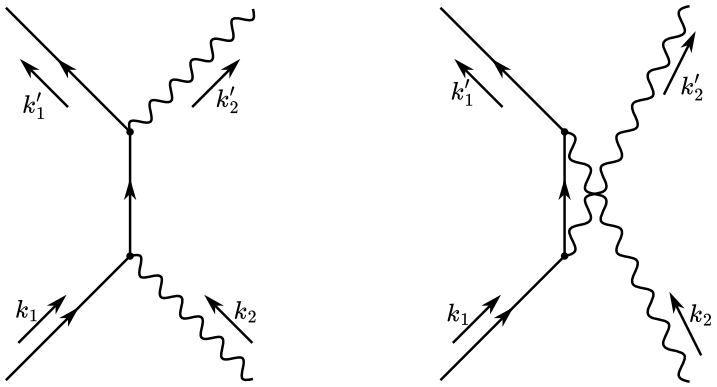


Figure 27: Compton scattering at tree level.

which give

$$i\widetilde{M}_c(e^-\,\gamma \to e^-\,\gamma) = \frac{-iq^2}{2}\overline{u}_{1'}\left(\frac{\not\epsilon^*_{2'}\not k_2\not\epsilon_2 + 2(k_1\cdot e_2)\not\epsilon^*_{2'}}{k_1\cdot k_2} + \frac{\not\epsilon_2\not k_2'\not\epsilon^*_{2'} - 2(k_1\cdot e^*_{2'})\not\epsilon_2}{k_1\cdot k_2'}\right)u_1. \tag{11.39}$$

Squaring this amplitude and summing over spins we find

$$\sum_{\sigma,\sigma'}|\widetilde{M}_c|^2 = \frac{q^4}{4}\left(\sum_\sigma e_{2,\mu}e^*_{2,\alpha}\right)\left(\sum_{\sigma'} e_{2',\beta}e^*_{2',\nu}\right)\mathrm{Tr}\Bigg[(\not k_1' + im)\left(\frac{\gamma^\nu\not k_2\gamma^\mu + 2k_1^\mu\gamma^\nu}{k_1\cdot k_2} + \frac{\gamma^\mu\not k_2'\gamma^\nu - 2k_1^\nu\gamma^\mu}{k_1\cdot k_2'}\right)$$
$$\times(\not k_1 + im)\left(\frac{\gamma^\alpha\not k_2\gamma^\beta + 2k_1^\alpha\gamma^\beta}{k_1\cdot k_2} + \frac{\gamma^\beta\not k_2'\gamma^\alpha - 2k_1^\beta\gamma^\alpha}{k_1\cdot k_2'}\right)\Bigg]. \tag{11.40}$$

Note that on the right-hand side we have already summed over the electron spins, but we have left the photon polarization sums to be evaluated.

In evaluating the polarization sum here there is a quite useful trick. In Coulomb gauge we saw that the polarization sum is given by

$$\sum_\sigma e_i(\vec p,\sigma)e^*_j(\vec p,\sigma) = \delta_{ij} - \frac{p_ip_j}{|p|^2}, \tag{11.41}$$

while any sum involving $e_0$ vanishes. We can write this in a more covariant way by introducing a null **rigging vector** $\ell^\mu$, which is defined to obey

$$\begin{aligned}\ell^\mu p_\mu &= -1\\ \ell^\mu e_\mu(\vec p,\sigma) &= 0.\end{aligned} \tag{11.42}$$

More concretely in $d=4$ if $p^\mu = (\omega,0,0,\omega)$ then we have $\ell^\mu = \frac{1}{2\omega}(1,0,0,-1)$. We then have

$$\sum_\sigma e_\mu(\vec p,\sigma)e^*_\nu(\vec p,\sigma) = \eta_{\mu\nu} + \ell_\mu p_\nu + p_\mu\ell_\nu, \tag{11.43}$$

as you can check by contracting this tensor with $p^\mu$, $\ell^\mu$, and $e^\mu$ in either index and seeing that it agrees with (11.41). This expression actually holds in any gauge provided that after a gauge transformation $e'_\mu = e_\mu + \alpha p_\mu$ we choose a new rigging vector so that (11.42) continues to hold. The nice thing about this expression is that in Feynman amplitudes the external polarization one-forms are always contracted with an insertion of

the electric current $j^\mu = -q\overline{\psi}\gamma^\mu\psi$. Since this current is conserved, if we replace any external factor $e_\mu(\vec{p},\sigma)$ or $e^*_\mu(\vec{p},\sigma)$ with $p_\mu$ in a squared scattering amplitude then we must get zero by the current conservation equation $p_\mu j^\mu = 0$. This statement is called the **Ward-Takehashi identity**, as it is a version of the Ward identity we discussed last semester for correlation functions involving conserved currents.[47] This means that in expressions such as (11.40), where both indices are contracted with a scattering amplitude, we can simply make the replacement

$$\sum_\sigma e_\mu(\vec{p},\sigma)e^*_\nu(\vec{p},\sigma) \to \eta_{\mu\nu}. \tag{11.45}$$

This is essentially the same reason that we could replace the complicated Coulomb gauge propagator by the Feynman gauge propagator.

Returning now to our Compton scattering calculation, we have

$$\sum_{\sigma,\sigma'}|\widetilde{M}_c|^2 = \frac{q^4}{4}\eta_{\mu\alpha}\eta_{\nu\beta}\mathrm{Tr}\Bigg[(\not{k}'_1 + im)\left(\frac{\gamma^\nu \not{k}_2 \gamma^\mu + 2k_1^\mu\gamma^\nu}{k_1\cdot k_2} + \frac{\gamma^\mu \not{k}'_2\gamma^\nu - 2k_1^\nu\gamma^\mu}{k_1\cdot k'_2}\right)$$
$$\times(\not{k}_1 + im)\left(\frac{\gamma^\alpha \not{k}_2 \gamma^\beta + 2k_1^\alpha\gamma^\beta}{k_1\cdot k_2} + \frac{\gamma^\beta \not{k}'_2\gamma^\alpha - 2k_1^\beta\gamma^\alpha}{k_1\cdot k'_2}\right)\Bigg]. \tag{11.46}$$

From here it is "just" a slog in computing traces of $\gamma$-matrices to evaluate this, I've left the details to you in the homework and the result is:

$$\frac{1}{4}\sum_{\sigma,\sigma'}|\widetilde{M}_c|^2 = 2q^4\left[\frac{k_1\cdot k'_2}{k_1\cdot k_2} + \frac{k_1\cdot k_2}{k_1\cdot k'_2} + 2m^2\left(\frac{1}{k_1\cdot k'_2} - \frac{1}{k_1\cdot k_2}\right) + m^4\left(\frac{1}{k_1\cdot k'_2} - \frac{1}{k_1\cdot k_2}\right)^2\right]. \tag{11.47}$$

The next step is to convert this expression to a differential cross section. So far we have been doing this in the center of mass frame, but Compton scattering is usually studied in the "lab" frame where the initial electron is at rest. We therefore need to revisit the relationship between the covariant matrix element and the differential cross section. In the lab frame we have

$$\begin{aligned} k_1 &= (m,\vec{0}) \\ k_2 &= (\omega,\omega\hat{n}) \\ k'_1 &= (E'_1,\vec{k}'_1) \\ k'_2 &= (\omega',\omega'\hat{n}'), \end{aligned} \tag{11.48}$$

where $\hat{n}$ and $\hat{n}'$ are unit vectors, and with a little effort one can show that the energy of the outgoing photon is

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)} \tag{11.49}$$

where $\theta$ is the angle between $\hat{n}$ and $\hat{n}'$. An analysis similar to the one we did in the center of mass frame (see Peskin and Schroeder for help) then shows that the relationship between squared matrix element and

[47] This argument is a bit too quick, as the Ward identity we derived last semester had some contact terms on the right-hand side:

$$\partial_\mu\langle TJ^\mu(x)O_1(y_1)\dots O_n(y_n)\rangle = i\sum_{m=1}^n \delta^d(x-y_m)\langle TO_1(y_1)\dots\delta_S O(y_m)\dots O_n(y_n)\rangle + \dots, \tag{11.44}$$

where $\delta_S O$ is the symmetry transformation of $O$ and the $\dots$ indicates possible non-universal terms proportional to derivatives of $\delta^d(x-y_m)$ which are called **Schwinger terms**. To get from this to a statement about scattering amplitudes we can observe that QED scattering amplitudes have the form $e_{\mu_1 *}(\vec{k}'_1,\sigma'_1)\dots e_{\nu_1}(\vec{k_1},\sigma_1)\dots\langle\beta|TJ^{\mu_1}(k'_1)\dots J^{\nu_1}(k_1)\dots|\alpha\rangle$, where $|\alpha\rangle$ and $|\beta\rangle$ are the initial and final states of the electrons/positrons. If we replace one of the $e$ or $e^*$ by its associated $k^\mu$ we can replace the current correlator by the Fourier transform of the right-hand side of (11.44). The universal contact term does not contribute since the currents are all neutral under the gauge symmetry. It is more subtle to argue that there are no Schwinger term contributions, for spinor electrodynamics the argument is essentially that the current $J^\mu = -q\overline{\Psi}\gamma^\mu\Psi$ involves no derivatives so when we compute $[J^0, J^\nu]$ there is no place for a derivative of a $\delta$-function to come from. A diagrammatic argument leading to the same conclusion is given in Peskin and Schroeder.

the differential cross section in the lab frame for the elastic scattering of a massless particle off of a massive particle is given by

$$\frac{d\sigma}{d\Omega} = \frac{(\omega')^{d-2}}{16m^2\omega^2}\frac{1}{(2\pi)^{d-2}}|\widetilde{M_c}|^2. \tag{11.50}$$

Specializing to $d = 4$ the spin summed/averaged differential cross section is thus

$$\frac{d\sigma_{ave}}{d\Omega} = \frac{(\omega')^2}{64\pi^2m^2\omega^2}\frac{1}{4}\sum_{\sigma,\sigma'}|\widetilde{M_c}|^2. \tag{11.51}$$

Returning to Compton scattering, in the lab frame we have

$$\begin{aligned} k_1 \cdot k_2 &= -m\omega \\ k_1 \cdot k_2' &= -m\omega', \end{aligned} \tag{11.52}$$

so combining everything we at last arrive at the famous **Klein-Nishina formula** for the differential cross section of Compton scattering:

$$\begin{aligned} \frac{d\sigma_{ave}}{d\Omega} &= \frac{q^4}{32\pi^2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + m^2\left(\frac{1}{\omega'} - \frac{1}{\omega}\right)^2\right] \\ &= \frac{q^4}{32\pi^2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right], \end{aligned} \tag{11.53}$$

where $\omega'$ is given in terms of $\omega$ and the scattering angle $\theta$ by equation (11.49). In particular in the non-relativistic limit $\omega \ll m$ we have $\omega' \approx \omega$ and thus

$$\frac{d\sigma_{ave}}{d\Omega} \approx \frac{q^4}{32\pi^2m^2}\left(1 + \cos^2\theta\right), \tag{11.54}$$

which a glance at an electromagnetism textbook (or wikipedia) will convince you is indeed the differential cross section for the Thomson scattering of light off of a charged particle in classical electromagnetism. It is peaked at forward and backward scattering just as we found for the high-energy limit of $e^+\,e^- \to \mu^+\,\mu^-$. The total cross section in this limit is

$$\sigma_{ave} = \frac{q^4}{6\pi m^2}, \tag{11.55}$$

so as far as low-energy photons are concerned the "size" of an electron is

$$r_{thomson} = \sqrt{\sigma_{ave}/\pi} = \frac{q^2}{\sqrt{6}\pi m}. \tag{11.56}$$

Note that this is smaller than the Compton wavelength $1/m$ of the electron by a factor of $\frac{4}{\sqrt{6}}\alpha \approx .01$, with $\alpha$ being the fine-structure constant. On the other hand in the high-energy limit we instead have

$$\frac{d\sigma_{ave}}{d\Omega} \approx \frac{q^4}{32\pi^2m\omega(1 - \cos\theta)}, \tag{11.57}$$

which is peaked at forward scattering as you might expect for high energy scattering off of a fixed target.

**Problems:**

1. Compute the differential cross section for electron-muon scattering $e^-\mu^- \to e^-\,\mu^-$. You should find that you can re-use some results from our analysis of $e^+\,e^- \to \mu^+\,\mu^-$.

2. Derive the $\gamma$-matrix contraction identities

$$\begin{aligned}\gamma^\mu\gamma_\mu &= d\\ \gamma^\mu\gamma^\alpha\gamma_\mu &= -(d-2)\gamma^\alpha\\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu &= (d-4)\gamma^\alpha\gamma^\beta + 4\eta^{\alpha\beta}\\ \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\mu &= -2\gamma^\gamma\gamma^\beta\gamma^\alpha - (d-4)\gamma^\alpha\gamma^\beta\gamma^\gamma.\end{aligned}$$

3. Evaluate the trace from the spin sum in the Compton scattering expression (11.46), showing that the result is (11.47). Hint: you would be wise to use the contraction identities from the previous problem before computing the trace, as this way you never need to compute a trace of more than four $\gamma$-matrices. If necessary you can look at Peskin and Schroeder for help, but beware that they use different conventions for the metric and the $\gamma$-matrices.

4. Write out the connected covariant amplitude $i\widetilde{M}_c$ for Moller scattering $e^-\,e^-\to e^-\,e^-$. Make sure to include both diagrams.

5. Extra credit: Compute the spin summed/averaged differential cross section for Moller scattering.

# 12 Quantum electrodynamics IV: one loop renormalization and $g-2$

In this section we discuss the classic one-loop calculations of quantum electrodynamics. The primary experimental result will be Schwinger's famous calculation of the one-loop contribution to the magnetic moment of the electron, but we will first discuss the renormalization of the electron and photon propagators and the renormalization group flow equation for the charge $q$.

## 12.1 Self-energy of the electron (theory)

We begin with a discussion of the renormalization of the electron propagator in quantum electrodynamics. Let's first recall a convenient alternative expression for the free momentum space propagator:

$$\hat{S}_F(p) = \frac{i}{\not{\widetilde{p}} - im_0}, \tag{12.1}$$

where $\widetilde{p}$ is $p$ except that its 0 component is $p_0(1+i\epsilon)$ and we are now distinguishing between the bare mass $m_0$ and the physical mass $m$ that we will introduce in a moment. We can parametrize the exact propagator as

$$\hat{S}_F^{exact}(p) = \frac{i}{\not{\widetilde{p}} - i(m_0 + \Sigma(\not{p}))}, \tag{12.2}$$

where $\Sigma(\not{\widetilde{p}})$ is a matrix called the **self-energy** of the electron. By Lorentz invariance it must have the form

$$\Sigma(\not{p}) = A(\widetilde{p}^2) + i\not{\widetilde{p}}B(\widetilde{p}^2), \tag{12.3}$$

so we can write the exact propagator as

$$\hat{S}_F^{exact}(p) = \frac{i}{(1+B(\widetilde{p}^2))\not{\widetilde{p}} - i(m_0 + A(\widetilde{p}^2))} \tag{12.4}$$

Multiplying the top and bottom by $(1+B(\widetilde{p}^2))\not{\widetilde{p}} + i(m_0 + A(\widetilde{p}^2))$ and extracting the $i\epsilon$ we have

$$\hat{S}_F^{exact}(p) = \frac{i\left((1+B)\not{p} + i(m_0 + A(p^2))\right)}{(1+B(p^2))^2p^2 + (m_0 + A(p^2))^2 - i\epsilon} \tag{12.5}$$

The physical mass is found by the location of the pole in $p^2$, so we have

$$m = \frac{m_0 + A(-m^2)}{1 + B(-m^2)}. \tag{12.6}$$

There is an easy way to remember this formula: we can rewrite it as

$$\delta m \equiv m - m_0 = \Sigma(\not{p})|_{\not{p}=im}. \tag{12.7}$$

It is tempting to derive this "by inspection" from equation (12.2) (as most textbooks do), but it is not actually possible to have $\not{p} = im$ for some choice of momentum $p^\mu$ (none of the $\gamma$-matrices are proportional to the identity and indeed in the standard $d = 4$ representation they have only off diagonal components) so that argument should be viewed as heuristic. We can also extract the field renormalization constant $Z_2$; to do this it is convenient to rewrite (12.5) as

$$\hat{S}_F^{exact}(p) = \frac{i\left(\not{p} + i\frac{m_0 + A(p^2)}{1 + B(p^2)}\right)}{(1 + B(p^2))\left(p^2 + \left(\frac{m_0 + A(p^2)}{1 + B(p^2)}\right)^2\right) - i\epsilon} \tag{12.8}$$

Near the pole $p^2 = -m^2$ can write the numerator as

$$i\left(\not{p} + i\frac{m_0 + A(p^2)}{1 + B(p^2)}\right) = i\left(\not{p} + im + O(p^2 + m^2)\right) \tag{12.9}$$

and the denominator as

$$\begin{aligned}(1 + B(p^2))\left(p^2 + \left(\frac{m_0 + A(p^2)}{1 + B(p^2)}\right)^2\right) &= (1 + B(-m^2))\left(p^2 + m^2 + \frac{2m(A'(-m^2) - B'(-m^2)m)}{1 + B(-m^2)}(p^2 + m^2)\right) \\ &= \Big(1 + B(-m^2) + 2mA'(-m^2) - 2m^2B'(-m^2)\Big)(p^2 + m^2), \end{aligned} \tag{12.10}$$

so the residue is given by

$$Z_2 = \frac{1}{1 + B(-m^2) + 2mA'(-m^2) - 2m^2B'(-m^2)}. \tag{12.11}$$

As for the mass there is an easier way to remember this: we can write it as

$$Z_2 = \frac{1}{1 - i\Sigma'(\not{p})|_{\not{p}=im}}, \tag{12.12}$$

which we can naively "derive" from (12.5) with the same caveat as before.

## 12.2 Self-energy of the electron (calculation)

How can we compute $\Sigma(\not{p})$? As in our discussion of $\phi^4$ theory, there is a trick using the idea of one-particle irreducible diagrams. Guessing the answer, let's denote the sum of 1PI electron two-point function diagrams (with the two external propagators removed) by $\Sigma$. The full propagator is then given by

$$\hat{S}_F^{exact}(p) = \hat{S}_F(p) + \hat{S}_F(p)\Sigma(\not{p})\hat{S}_F(p) + \hat{S}_F(p)\Sigma(\not{p})\hat{S}_F(p)\Sigma(\not{p})\hat{S}_F(p) + \ldots, \tag{12.13}$$

which is a geometric series we can evaluate as

$$\begin{aligned}\hat{S}_F^{exact}(p) &= \hat{S}_F(p)\left(1 - \Sigma(\not{p})\hat{S}_F(p)\right)^{-1} \\ &= \left(\hat{S}_F(p)^{-1} - \Sigma(\not{p})\right)^{-1} \\ &= \frac{i}{\not{p} - i(m_0 + \Sigma(\not{p}))}, \end{aligned} \tag{12.14}$$

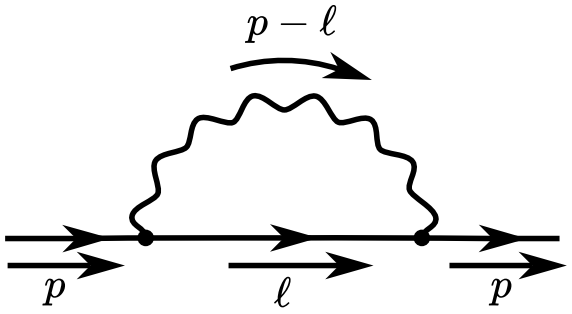


Figure 28: Leading contribution to the electron self-energy in spinor electrodynamics.

so indeed our guess was correct: $\Sigma(\not{p})$ is the sum of all 1PI electron two-point function diagrams.

Let's now compute this at one-loop. The only diagram is shown in figure 28, its value is

$$\Sigma(\not{p}) = (-iq)^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{i\gamma^\mu(\not{\ell} + im_0)\gamma_\mu}{\ell^2 + m_0^2 - i\epsilon} \frac{-i}{(\ell - p)^2 + \mu^2 - i\epsilon}. \tag{12.15}$$

Anticipating the possibility of an infrared divergence I've included a regulating mass $\mu$ in the photon propagator, this is eventually taken to zero once any infrared divergences are dealt with. The charge $q$ here should really be the bare charge $q_0$, but at this order the difference doesn't matter so I'll stick with $q$. The first thing we should always do with such an integral is analytically continue to Euclidean signature via $\ell^0 = i\ell_E^0$ and $p^0 = ip_E^0$, and also using the first two $\gamma$-contraction identities you derived on the previous homework, we have

$$\Sigma(\not{p}) = iq^2 \int \frac{d^d\ell}{(2\pi)^d} \frac{(d-2)\not{\ell} - idm_0}{(\ell^2 + m_0^2)((\ell - p)^2 + \mu^2)}. \tag{12.16}$$

We can then combine the denominators using the Feynman parameter identity

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}, \tag{12.17}$$

which gives

$$\begin{aligned}\Sigma(\not{p}) &= iq^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(d-2)\not{\ell} - idm_0}{\Big((\ell - xp)^2 + x(1-x)p^2 + (1-x)m_0^2 + x\mu^2\Big)^2} \\ &= iq^2 \int_0^1 dx \int \frac{d^d\ell}{(2\pi)^d} \frac{(d-2)\left(\not{\ell} + x\not{p}\right) - idm_0}{\Big(\ell^2 + x(1-x)p^2 + (1-x)m_0^2 + x\mu^2\Big)^2},\end{aligned} \tag{12.18}$$

where in the second line we shifted the integration variable. The form (12.3) of $\Sigma$ is now clear. The term involving $\not{\ell}$ integrates to zero by rotational invariance. The remaining terms go like $1/\ell^4$ at large $\ell$, so this integral has a logarithmic UV divergence. We therefore need to regulate it, for example using Pauli-Villars or dimensional regularization. In the former we subtract the same integrand with $\mu$ replaced by $\Lambda$, after which the momentum integral can be evaluated in $d = 4$, e.g. using mathematica, to give

$$\Sigma(\not{p}) = \frac{q^2}{8\pi^2} \int_0^1 dx \left(2m_0 + ix\not{p}\right) \log\left(\frac{x\Lambda^2}{x(1-x)p^2 + (1-x)m_0^2 + x\mu^2}\right). \tag{12.19}$$

This integral is finite in the limit that $\mu \to 0$ so the self-energy does not have an infrared divergence. We can compute one-loop renormalization of the electron mass using (12.7), which at this order we can rewrite as

$$\delta m \approx \Sigma(\not{p})|_{\not{p}=im_0} \tag{12.20}$$

since $\Sigma$ already has a factor of $q^2$ so the difference between $m$ and $m_0$ on the right-hand side is higher order in $q^2$. We thus have

$$\begin{aligned}\delta m &= \frac{q^2 m_0}{8\pi^2}\int_0^1 dx\,(2-x)\log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2}\right)\\ &= \frac{3q^2 m_0}{32\pi^2}\left(1+2\log\left(\frac{\Lambda^2}{m_0^2}\right)\right).\end{aligned} \tag{12.21}$$

To compute the field renormalization similarly we need

$$i\Sigma'(\not{p})|_{\not{p}=im_0} = \frac{q^2}{8\pi^2}\int_0^1 dx\left[-x\log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right) + \frac{2(2-x)(1-x)x}{(1-x)^2 + x\frac{\mu^2}{m_0^2}}\right]. \tag{12.22}$$

In the first term we can take $\mu \to 0$, but doing so in the second gives a logarithmic divergence in the integral at $x = 1$. Evaluating the integral and expanding at small $\mu$ one finds

$$i\Sigma'(\not{p})|_{\not{p}=im_0} = -\frac{q^2}{8\pi^2}\left(\frac{9}{4}+\log\left(\frac{\Lambda}{m_0}\right)-\log\left(\frac{m_0^2}{\mu^2}\right)\right), \tag{12.23}$$

so at one loop we have

$$Z_2 = 1 - \frac{q^2}{8\pi^2}\left(\frac{9}{4}+\log\left(\frac{\Lambda}{m_0}\right)-\log\left(\frac{m_0^2}{\mu^2}\right)\right). \tag{12.24}$$

Before moving on I'll mention that one can also compute the electron self-energy using dimensional regularization. Evaluating the momentum integral using our general formula

$$\int_0^\infty d\ell \frac{\ell^{a-1}}{(\ell^2+\sigma^2)^b} = \frac{\sigma^{a-2b}}{2}\frac{\Gamma(a/2)\Gamma(b-a/2)}{\Gamma(b)} \tag{12.25}$$

from last semester gives

$$\begin{aligned}\Sigma(\not{p}) &= \frac{q_d^2\Omega_{d-1}\Gamma(d/2)\Gamma(2-d/2)}{2(2\pi)^d}\int_0^1 dx\Big(dm_0 + i(d-2)x\not{p}\Big)\Big(x(1-x)p^2+(1-x)m_0^2+x\mu^2\Big)^{\frac{d-4}{2}}\\ &= \frac{q_d^2\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 dx\Big(dm_0 + i(d-2)x\not{p}\Big)\Big(x(1-x)p^2+(1-x)m_0^2+x\mu^2\Big)^{\frac{d-4}{2}},\end{aligned} \tag{12.26}$$

where we have introduced a renormalization scale $\widetilde{\mu}$ by

$$q_d = \widetilde{\mu}^{\frac{4-d}{2}} q. \tag{12.27}$$

Expanding near $d = 4$ as $d = 4 - 2\epsilon$, we have

$$\Sigma(\not{p}) = \frac{q^2}{8\pi^2}\int_0^1 dx\left[\big(2m_0 + ix\not{p}\big)\left(\frac{1}{\epsilon}+\log(4\pi)-\gamma-1+\log\left(\frac{\widetilde{\mu}^2}{x(1-x)p^2+(1-x)m_0^2+x\mu^2}\right)\right)+m_0\right], \tag{12.28}$$

which differs from (12.19) by various finite shifts. We can make them look more similar by defining

$$\log\widetilde{\mu}^2 + \frac{1}{\epsilon} + \log(4\pi) - \gamma - 1 = \log\Lambda^2, \tag{12.29}$$

in which case we now have

$$\Sigma(\not{p}) = \frac{q^2}{8\pi^2}\int_0^1 dx\left[\big(2m_0+ix\not{p}\big)\log\left(\frac{x\Lambda^2}{x(1-x)p^2+(1-x)m_0^2+x\mu^2}\right) - \log x\,\big(2m_0+ix\not{p}\big)+m_0\right]. \tag{12.30}$$

The first term in the parentheses is now the same as (12.19), while the last two are easily integrated to give

$$\Sigma_{DR}(\not{p}) = \Sigma_{PV}(\not{p}) + \frac{q^2}{8\pi^2}\left(3m_0 + \frac{i\not{p}}{4}\right). \tag{12.31}$$

Therefore the mass renormalizations in the two schemes are related as

$$\delta m_{DR} = \delta m_{PV} + \frac{11q^2 m_0}{32\pi^2} \tag{12.32}$$

and the field renormalizations are related as

$$Z_{2,DR} = Z_{2,PV} - \frac{q^2}{32\pi^2}. \tag{12.33}$$

## 12.3 Photon self-energy (theory)

As for the electron, we can write the exact photon propagator (in some gauge) in terms of a 1PI self-energy $\Pi^{\mu\nu}(p)$ via

$$\begin{aligned}\hat{\Delta}^{exact}(p) &= \hat{\Delta} + \hat{\Delta}\Pi\hat{\Delta} + \hat{\Delta}\Pi\hat{\Delta}\Pi\hat{\Delta} + \dots \\ &= \hat{\Delta}(1 - \Pi\hat{\Delta})^{-1} \\ &= (\hat{\Delta}^{-1} - \Pi)^{-1}.\end{aligned} \tag{12.34}$$

In our general set of $\xi$-gauges we have

$$(\hat{\Delta}^{-1})^{\mu\nu} = i\left(p^2\eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right)p^\mu p^\nu\right). \tag{12.35}$$

By Lorentz invariance we must have

$$\Pi^{\mu\nu}(p) = -i\left(p^2\eta^{\mu\nu}\Pi_1(p^2) - p^\mu p^\nu \Pi_2(p^2)\right) \tag{12.36}$$

for some functions $\Pi_1$ and $\Pi_2$. It will soon be clear why I extracted the factor of $p^2$ from $\Pi_1$. In the meantime the matrix we want to invert to find the propagator is

$$(\hat{\Delta}^{-1} - \Pi)^{\mu\nu} = i\left(p^2\eta^{\mu\nu}(1 + \Pi_1) - \left(1 - \frac{1}{\xi} + \Pi_2\right)p^\mu p^\nu\right), \tag{12.37}$$

whose inverse is

$$\hat{\Delta}^{exact}_{\mu\nu} = \frac{-i}{(p^2 - i\epsilon)(1 + \Pi_1)}\left[\eta_{\mu\nu} - (1 - \xi(1 + \Pi_2))\frac{p_\mu p_\nu}{p^2 - i\epsilon}\right]. \tag{12.38}$$

This propagator is simpler in the Landau gauge $\xi = 0$, which is part of its appeal, but we will stick with Feynman gauge $\xi = 1$.

We can further constrain the photon self-energy by noting that it has a simple relationship to the Fourier transform of the two-point function of the electromagnetic current: the latter is given by

$$\begin{aligned}\langle TJ(p)J(-p)\rangle &= -\left(\Pi + \Pi\hat{\Delta}\Pi + \Pi\hat{\Delta}\Pi\hat{\Delta}\Pi + \dots\right) \\ &= -\Pi\left(1 - \hat{\Delta}\Pi\right)^{-1} \\ &= -(1 - \Pi\hat{\Delta})^{-1}\Pi.\end{aligned} \tag{12.39}$$

As we discussed in the previous section, correlation functions of currents such as this one must have the property that if we dot either current into its momentum $p_\mu$ we get zero by the Ward-Takehashi identity. We therefore see that the self-energy must obey

$$p_\mu \Pi^{\mu\nu}(p) = p_\mu \Pi^{\nu\mu}(p) = 0. \tag{12.40}$$

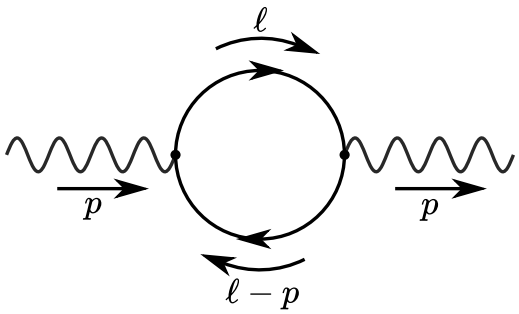


Figure 29: The one-loop contribution to the photon self-energy.

We therefore must have

$$\Pi_1(p^2) = \Pi_2(p^2) \equiv \Pi(p^2), \tag{12.41}$$

and thus

$$\Pi^{\mu\nu}(p) = -i\left(p^2\eta^{\mu\nu} - p^\mu p^\nu\right)\Pi(p^2). \tag{12.42}$$

Now we see the reason for extracting the factor of $p^2$ from $\Pi_1$: otherwise this relation would require the coefficient of $p^\mu p^\nu$ to be singular in the current two-point function, but this shouldn't have any pole since there no one-particle state in quantum electrodynamics which can be created by acting with $J^\mu$ on the vacuum. We can therefore write the exact propagator as

$$\hat{\Delta}^{exact}_{\mu\nu} = \frac{-i}{(p^2 - i\epsilon)(1+\Pi)}\left[\eta_{\mu\nu} - (1 - \xi(1+\Pi))\,\frac{p_\mu p_\nu}{p^2 - i\epsilon}\right]. \tag{12.43}$$

There are two comments to make about this result:

(1) The pole location is still at $p^2 = 0$, so there is no renormalization of the photon mass away from zero. This is reassuring, and also expected since gauge invariance requires the photon to be massless and were it massive we'd need to get its third spin state from somewhere.

(2) We can extract the field renormalization constant $Z_3$ for the photon from this formula, the residue of the pole is

$$Z_3 = \frac{1}{1+\Pi(0)}. \tag{12.44}$$

## 12.4 Photon self-energy (calculation)

Let's now compute the photon self-energy at one loop. There is one Feynman diagram, shown in figure 29, and its contribution to the self-energy is

$$\Pi^{\mu\nu}(p) = -(-iq)^2 \int \frac{d^d\ell}{(2\pi)^d}\,\frac{\mathrm{Tr}\big(i(\not{\ell} + im)\gamma^\mu i(\not{\ell} - \not{p} + im)\gamma^\nu\big)}{\big(\ell^2 + m^2 - i\epsilon\big)\big((\ell - p)^2 + m^2 - i\epsilon\big)}. \tag{12.45}$$

As usual we can simplify this integral using a Feynman parameter and a shift of the integration variable:

$$\begin{aligned}\Pi^{\mu\nu}(p) &= -q^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\mathrm{Tr}\big((\ell\!\!\!/+im)\gamma^\mu(\ell\!\!\!/-p\!\!\!/+im)\gamma^\nu\big)}{\Big(x\big(\ell^2+m^2-i\epsilon\big)+(1-x)\big((\ell-p)^2+m^2-i\epsilon\big)\Big)^2}\\ &= -q^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\mathrm{Tr}\big((\ell\!\!\!/+im)\gamma^\mu(\ell\!\!\!/-p\!\!\!/+im)\gamma^\nu\big)}{\Big((\ell-(1-x)p)^2+x(1-x)p^2+m^2-i\epsilon\Big)^2}\\ &= -q^2\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\mathrm{Tr}\big((\ell\!\!\!/+(1-x)p\!\!\!/+im)\gamma^\mu(\ell\!\!\!/-xp\!\!\!/+im)\gamma^\nu\big)}{\Big(\ell^2+x(1-x)p^2+m^2-i\epsilon\Big)^2}.\end{aligned}\tag{12.46}$$

We can simplify the numerator using our trace technology:

$$\begin{aligned}\mathrm{Tr}\big((\ell\!\!\!/+(1-x)p\!\!\!/+im)\gamma^\mu(\ell\!\!\!/-xp\!\!\!/+im)\gamma^\nu\big) =&(\ell+(1-x)p)_\alpha(\ell-xp)_\beta\mathrm{Tr}\left(\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu\right)-m^2\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)\\ =&2^{\lfloor\frac{d}{2}\rfloor}\Big((\ell+(1-x)p)^\mu(\ell-xp)^\nu+(\ell+(1-x)p)^\nu(\ell-xp)^\mu\\ &-(\ell+(1-x)p)\cdot(\ell-xp)\eta^{\mu\nu}-m^2\eta^{\mu\nu}\Big),\end{aligned}\tag{12.47}$$

and by symmetry the terms which are linear in $\ell$ will integrate to zero and thus can be dropped. Moreover the quantity $\ell^\mu\ell\nu$ must integrate to something proportional to $\eta^{\mu\nu}$, and by contracting with $\eta_{\mu\nu}$ we can see that the correct replacement is

$$\ell^\mu\ell^\nu\to\frac{\ell^2}{d}\eta^{\mu\nu}.\tag{12.48}$$

Wick rotating the integral we thus have

$$\Pi^{\mu\nu}(p) = -iq^2 2^{\lfloor\frac{d}{2}\rfloor}\int_0^1 dx\int\frac{d^d\ell}{(2\pi)^d}\frac{\left(\frac{2}{d}-1\right)\ell^2\eta^{\mu\nu}+x(1-x)\left(p^2\eta^{\mu\nu}-2p^\mu p^\nu\right)-m^2\eta^{\mu\nu}}{\left(\ell^2+x(1-x)p^2+m^2\right)^2}.\tag{12.49}$$

This integral has a quadratic UV divergence, and also a subleading logarithmic divergence. As usual it is up to us how to regulate the integral, but now we can get into trouble if we aren't careful. For example say we use a hard momentum cutoff $\ell^2<\Lambda^2$. Then the leading divergence has the form

$$\Lambda^2\eta^{\mu\nu},\tag{12.50}$$

which does *not* match the form (12.42) that we argued for from gauge invariance in the previous subsection. A hard momentum cutoff violates gauge invariance! This doesn't mean that it is absolutely wrong, but it means that we need to include a gauge non-invariant counterterm (in this case a UV-divergent photon mass) to restore gauge invariance. Including gauge non-invariant terms in the action to fix the gauge non-invariance of the cutoff is rather inconvenient in practice, so essentially nobody does it. For example in lattice gauge theory the cutoff always preserves gauge invariance. Here we will use dimensional regularization, which also preserves gauge invariance as we will now see (essentially because the current conservation equation $\partial_\mu J^\mu=0$ is dimension-independent.[48]

[48]There is an important caveat to this statement, which is that in chiral gauge theories such as the standard model where $\Psi_L$ and $\Psi_R$ have different gauge couplings it isn't so clear how to continue these theories away from $d=4$ while preserving gauge invariance (what do we do with $\gamma$?). This leads to the possibility of **anomalies**, which can spoil the gauge invariance of a quantum field theory without the possibility of repair by a gauge non-invariant counterterm. We will study this phenomenon in detail next semester.

Evaluating (12.49) using our integration formula (12.25) gives

$$\Pi^{\mu\nu}(p) = -iq_d^2 2^{\lfloor \frac{d}{2} \rfloor} \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^1 dx \left[ \frac{2-d}{d} \eta^{\mu\nu} \frac{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(1-\frac{d}{2}\right)}{2} (x(1-x)p^2+m^2)^{\frac{d-2}{2}} \right.$$
$$\left. + \left(x(1-x)(p^2\eta^{\mu\nu} - 2p^\mu p^\nu) - m^2\eta^{\mu\nu}\right) \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(2-\frac{d}{2}\right)}{2} (x(1-x)p^2+m^2)^{\frac{d-4}{2}} \right], \tag{12.51}$$

which we can simplify using the $\Gamma$ function identity $\Gamma(x+1) = x\Gamma(x)$ to get

$$\Pi^{\mu\nu}(p) = -iq_d^2 2^{\lfloor \frac{d}{2} \rfloor} \frac{\Omega_{d-1}}{(2\pi)^d} \Gamma\left(\frac{d}{2}\right) \Gamma\left(2-\frac{d}{2}\right) \left(p^2\eta^{\mu\nu} - p^\mu p^\nu\right) \int_0^1 dx x(1-x)(x(1-x)p^2+m^2)^{\frac{d-4}{2}}. \tag{12.52}$$

The gauge-invariant form (12.42) is now manifest, so we can extract

$$\Pi(p^2) = 2^{\lfloor \frac{d}{2} \rfloor + 1} q_d^2 \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx\, x(1-x)(x(1-x)p^2+m^2)^{\frac{d-4}{2}}. \tag{12.53}$$

To expand near $d = 4$ we need to decide how to handle the quantity $\lfloor \frac{d}{2} \rfloor$, the standard convention seems to be to just set it equal to two, so we'll respect this, but one could also choose $d/2$ and get a scheme which differs slightly from the standard dimensional regularization. Expanding near $d = 4$ as $d = 4 - 2\epsilon$, we then have

$$\Pi(p^2) = \frac{q^2}{2\pi^2} \int_0^1 dx\, x(1-x) \left[ \frac{1}{\epsilon} + \log(4\pi) - \gamma + \log\left( \frac{\widetilde{\mu}^2}{x(1-x)p^2+m^2} \right) \right]. \tag{12.54}$$

Introducing an explicit UV cutoff by[49]

$$\log \Lambda^2 = \log \widetilde{\mu}^2 + \frac{1}{\epsilon} + \log(4\pi) - \gamma, \tag{12.55}$$

we can rewrite this as

$$\Pi(p^2) = \frac{q^2}{2\pi^2} \int_0^1 dx\, x(1-x) \log\left( \frac{\Lambda^2}{x(1-x)p^2+m^2} \right), \tag{12.56}$$

so in particular we have

$$\Pi(0) = \frac{q^2}{6\pi^2} \log\left(\frac{\Lambda}{m}\right) \tag{12.57}$$

and thus

$$Z_3 \approx 1 - \Pi(0) = 1 - \frac{q^2}{6\pi^2} \log\left(\frac{\Lambda}{m}\right). \tag{12.58}$$

Having now determined $\delta m$, $Z_2$, and $Z_3$ at one loop, this completes the one-loop renormalization of spinor electrodynamics.

## 12.5 Charge renormalization and vacuum polarization

Let's now use our expression for $\Pi(p^2)$ to do some physics. The first thing we'll study is the renormalization group flow of electric charge. In the previous section we argued that due to gauge invariance the relationship between the bare charge $q_0$ and the renormalized charge $q$ must be given by

$$q = \sqrt{Z_3} q_0. \tag{12.59}$$

[49] Note that this isn't quite the same cutoff we introduced for the electron self-energy, but we can absorb the difference into a small rescaling of $\Lambda$ and it is only the coefficient of the logarithm that will be meaningful for us anyways.

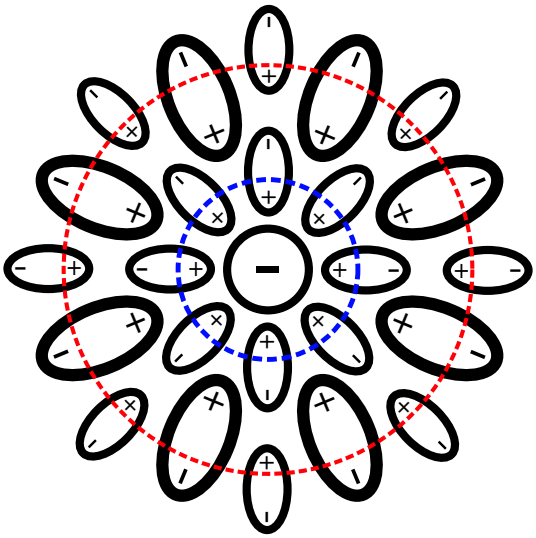

Figure 30: The scale dependence of vacuum polarization. Measuring the electric flux through a sphere which is large compared to $1/m$ feels the full screening of the vacuum pairs, while spheres which are small compared to $1/m$ are only screened by those pairs which are small enough.

In the Wilsonian approach to renormalization the way we are supposed to interpret this equation is that it tells us how to tune $q_0$ as a function of the cutoff $\Lambda$ so that $q$ stays fixed as we vary $\Lambda$. Given our calculation of $Z_3$ in the previous subsection, at one loop this tuning is apparently

$$q_0(\Lambda) = \left(1 + \frac{q^2}{12\pi^2} \log\left(\frac{\Lambda}{m}\right)\right) q. \tag{12.60}$$

In other words at this order the Wilsonian $\beta$-function of QED is given by

$$\beta(q) \equiv \frac{dq_0}{d\log\Lambda} = \frac{q_0^3}{12\pi^2}. \tag{12.61}$$

What these equations say is that the effective electromagnetic coupling grows logarithmically with increasing energy (or decreasing distance). This is sometimes described as saying quantum electrodynamics is "infrared-free", as the coupling gets weaker as we flow to lower energies.[50] If we take the cutoff $\Lambda$ to be large enough that

$$\frac{q^2}{12\pi^2} \log\left(\frac{\Lambda}{m}\right) \sim 1 \tag{12.62}$$

then the perturbative approximation we made in this calculation will break down, but this does not happen until the absurdly large energy scale

$$\Lambda = m e^{\frac{12\pi^2}{q^2}} = m e^{\frac{3\pi}{\alpha}} \approx m e^{1300}, \tag{12.63}$$

which is hardly something for us to worry about in the real world. On the other hand this does mean that spinor electrodynamics may not make sense as a continuum quantum field theory.

There is a nice physical picture of the scale-dependence of the electron charge, shown in figure 30. The idea is that the presence of a "bare" electric charge polarizes the vacuum by biasing the orientation of the

[50] It stops flowing below the mass of the electron however, and indeed we will see in the next subsection that the intrinsic definition of the renormalized charge $q$ is that it is the coupling at the scale of the electron mass.

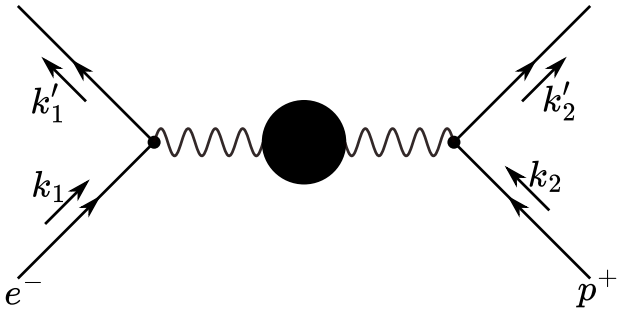


Figure 31: Electron-proton scattering by exchanging the exact photon propagator.

electron-positron pairs which are inherently present in the vacuum due to the entanglement illustrated by the nonvanishing two-point function of the electron field at spacelike separation. These pairs come in all sizes, but pairs which are larger than the Compton wavelength $1/m$ are exponentially suppressed so roughly speaking we can say they don't exist. The electric charge as measured at distances which are large compared to the Compton wavelength is screened by these pairs, since the surface where we measure the electric flux will necessarily cut some of the screening pairs and thus result in a smaller measured value of the charge than its true bare value. As we make the surface smaller than $1/m$ however, the screening pairs which are larger than the size of the surface do not contribute so the screening effect becomes less complete.

Vacuum polarization also leads to an interesting modification of the Coulomb potential at short distances. To extract this contribution we can consider the scattering of an electron off of a proton using the exact photon propagator in the exchange as in figure 31. Using our expressions (12.43) for the exact photon propagator and (12.44) for the exact photon field renormalization constant we can evaluate this diagram as

$$i\widetilde{M}_c = (iq)(-iq)\overline{u}_{1'}\gamma^\mu u_1 \overline{u}_{2'}\gamma^\nu u_2 \frac{-i\eta_{\mu\nu}(1+\Pi(0))}{((k_1-k_1')^2-i\epsilon)(1+\Pi((k_1-k_1')^2)}. \tag{12.64}$$

Taking the non-relativistic limit as in our discussion of Yukawa theory this becomes

$$i\widetilde{M} \approx \frac{iq^2 4m_e m_p \delta_{\sigma_1,\sigma_1'}\delta_{\sigma_2,\sigma_2'}}{|k_1'-k_1|^2}\frac{1+\Pi(0)}{1+\Pi(|k_1'-k_1|^2)}, \tag{12.65}$$

so comparing to the Born approximation we find that the Fourier transform of the non-relativistic potential is

$$V(\vec{k}) = \frac{-q^2(1+\Pi(0))}{|k|^2(1+\Pi(|k|^2))}. \tag{12.66}$$

Since we have computed $\Pi((p^2)$ only to leading order in $q$ we can approximate this as

$$V(\vec{k}) \approx \frac{-q^2}{|k|^2} \times \Big(1+\Pi(0)-\Pi(|k|^2)\Big). \tag{12.67}$$

From (12.56) we have

$$\Pi(0)-\Pi(p^2) = \frac{q^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(1+x(1-x)\frac{p^2}{m_e^2}\right), \tag{12.68}$$

so in the non-relativistic limit we have the approximation

$$\Pi(0)-\Pi(|k|^2) \approx \frac{q^2|k|^2}{60\pi^2 m_e^2}. \tag{12.69}$$

Therefore we have

$$V(\vec{k}) \approx -\frac{q^2}{|k|^2} - \frac{q^4}{60\pi^2 m_e^2}, \tag{12.70}$$

whose Fourier transform is

$$V(\vec{r}) = -\frac{q^2}{4\pi|r|} - \frac{q^4}{60\pi^2 m_e^2}\delta^3(\vec{r}) \tag{12.71}$$

$$= -\frac{\alpha}{|r|} - \frac{4\alpha^2}{15 m_e^2}\delta^3(\vec{r}). \tag{12.72}$$

This additional potential shows that when the electron is very close to the proton it feels an additional attraction due to the unscreening of the proton. This effect is actually measurable: it predicts a shift of the energy levels of hydrogen which at first order in perturbation theory is given by

$$\Delta E = -\frac{4\alpha^2}{15 m_e^2}|\Psi(0)|^2 \tag{12.73}$$

where $\Psi(\vec{r})$ is the unperturbed wave function. Atomic wave functions of nonzero angular momentum have wave functions which vanish at the origin, so this effect only affects $s$ orbitals. For example for the 1s and 2s states of hydrogen we have

$$\begin{aligned} |\Psi_{1s}(0)|^2 &= \frac{m_e^3\alpha^3}{\pi} \\ |\Psi_{2s}(0)|^2 &= \frac{m_e^3\alpha^3}{8\pi}, \end{aligned} \tag{12.74}$$

so the energy shifts are

$$\begin{aligned} \Delta E_{1s} &= -\frac{4 m_e \alpha^5}{15\pi} \approx -8.99 \times 10^{-7}\,\text{eV} \\ \Delta E_{2s} &= -\frac{m_e \alpha^5}{30\pi} \approx -1.12 \times 10^{-7}\,\text{eV}. \end{aligned} \tag{12.75}$$

These contributions are smaller than the usual fine structure of hydrogen by one power of $\alpha$, but they are still measurable and in fact have been measured! In ordinary hydrogen they are not the largest effects at $O(\alpha^5)$, but they become so for "muonic hydrogen" where the electron is replaced by a muon since in that case the squared wave functions are proportional to $m_\mu^3$ so we get an enhancement by a factor of $\left(\frac{m_\mu}{m_e}\right)^2 \approx 4 \times 10^4$ compared to the naive energy scale of $m_\mu$.

## 12.6 Vertex function and $g-2$ (theory)

The last one-loop effect we will consider in quantum electrodynamics is the effect of vacuum fluctuations on the motion of a charged particle in a *classical* electromagnetic field. We can implement this by the modifying the Hamiltonian of quantum electrodynamics to include a term

$$H_{back} = -\int d^{d-1}x \vec{J}\cdot\vec{A}_{cl}, \tag{12.76}$$

where $\vec{A}_{cl}$ is the vector potential for the classical gauge field. We will be particularly interested in the effects of this Hamiltonian on a single charged particle, which at first order in the background field are controlled by the matrix element

$$\langle \vec{p}', \sigma' | H_{back} | \vec{p}, \sigma \rangle = -\int d^{d-1}x \vec{A}_{cl}(x) \cdot \langle \vec{p}', \sigma' | \vec{J}(x) | \vec{p}, \sigma \rangle. \tag{12.77}$$

This motivates us to study the matrix element

$$\langle \vec{p}', \sigma' | J^\mu(x) | \vec{p}, \sigma \rangle \tag{12.78}$$

of the electromagnetic current

$$J^\mu = -q_0 \overline{\Psi} \gamma^\mu \Psi \tag{12.79}$$

between two one-particle states in quantum electrodynamics. Before trying to compute this quantity, it is good to first see how it is constrained by Lorentz invariance. To get a sense of what to expect we can first compute it in the free Dirac theory:

$$\begin{aligned} \langle \vec{p}', \sigma' | J^\mu(x) | \vec{p}, \sigma \rangle &= -q_0 \langle \Omega | a_{\vec{p}', \sigma'} \overline{\Psi}(x) \gamma^\mu \Psi(x) a^\dagger_{\vec{p}, \sigma} | \Omega \rangle \\ &= -q_0 \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{p}'}}} e^{i(p-p')\cdot x} \overline{u}(\vec{p}', \sigma') \gamma^\mu u(\vec{p}, \sigma). \end{aligned} \tag{12.80}$$

Now let's consider the interacting theory. By translation invariance we must have

$$\begin{aligned} \langle \vec{p}', \sigma' | J^\mu(x) | \vec{p}, \sigma \rangle &= \langle \vec{p}', \sigma' | e^{-ix\cdot P} J^\mu(0) e^{ix\cdot P} | \vec{p}, \sigma \rangle \\ &= e^{i(p-p')\cdot x} \langle \vec{p}', \sigma' | J^\mu(0) | \vec{p}, \sigma \rangle, \end{aligned} \tag{12.81}$$

and since the integral of $J^0$ is the total charge[51] we must have

$$\begin{aligned} q_0 (2\pi)^{d-1} \delta^{d-1}(\vec{p}' - \vec{p}) \delta_{\sigma', \sigma} &= \int d^{d-1}x \langle \vec{p}', \sigma' | J^0(t, \vec{x}) | \vec{p}, \sigma \rangle \\ &= (2\pi)^{d-1} \delta^{d-1}(\vec{p}' - \vec{p}) \langle \vec{p}', \sigma' | J^0(0) | \vec{p}, \sigma \rangle \end{aligned} \tag{12.82}$$

and thus

$$\langle \vec{p}, \sigma' | J^0(0) | \vec{p}, \sigma \rangle = q_0 \delta_{\sigma', \sigma}. \tag{12.83}$$

Moreover by current conservation we must have

$$(p - p')_\mu \langle \vec{p}, \sigma' | J^\mu(0) | \vec{p}, \sigma \rangle = 0. \tag{12.84}$$

Now let's see what we can learn from Lorentz invariance. The quantity

$$\langle \vec{p}, \sigma' | J^\mu(0) | \vec{p}, \sigma \rangle \sqrt{2\omega_{\vec{p}}} \sqrt{2\omega_{\vec{p}'}} \tag{12.85}$$

transforms as a four vector if we act with a Lorentz transformation on the one-particle states. This motivates us to define

$$\langle \vec{p}', \sigma' | J^\mu(0) | \vec{p}, \sigma \rangle = -q_0 \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{p}'}}} \overline{u}(\vec{p}', \sigma') \Gamma^\mu(p, p') u(\vec{p}, \sigma), \tag{12.86}$$

where $\Gamma^\mu$ is a spinor matrix which is also a spacetime vector, and which must be built out of $p$, $p'$, and $\gamma^\mu$ in some way. $\Gamma^\mu$ is called the **vertex function**. Here $u$ and $\overline{u}$ are serving in their role as intertwiners, converting the Lorentz transformations of the two one-particle states to the vector Lorentz transformation. In principle the matrix structure of $\Gamma^\mu$ could include $\not{p}$ or $\not{p}'$, but these can always be anticommuted through any other $\gamma$-matrices so that they are adjacent to $u$ or $\overline{u}$ respectively, in which case we can use the Dirac equation expressed as

$$\begin{aligned} \not{p} u(\vec{p}, \sigma) &= i m u(\vec{p}, \sigma) \\ \overline{u}(\vec{p}', \sigma') \not{p}' &= i m \overline{u}(\vec{p}', \sigma') \end{aligned} \tag{12.87}$$

to remove their matrix structure. The matrix structure of $\Gamma^\mu$ thus comes entirely from powers of the $\gamma$ matrices. Any term with more than one $\gamma^\mu$ can be simplified however, since only one of them can have a

[51] Note that we are using the "bare" current here, so it integrates to the bare charge.

free $\mu$ index so the rest must be contracted together (we've already removed all of their contractions with $p$ or $p'$). We can then use our $\gamma$-matrix contraction identities to remove all of these contractions. The result is that we must have

$$\overline{u}(\vec{p}',\sigma')\Gamma^\mu(p,p')u(\vec{p},\sigma) = \overline{u}(\vec{p}',\sigma')\Bigg[F\left((p-p')^2\right)\gamma^\mu - \frac{i}{2m}(p+p')^\mu G\left((p-p')^2\right) + \frac{(p-p')^\mu}{2m}H\left((p-p')^2\right)\Bigg]u(\vec{p},\sigma), \tag{12.88}$$

where $F$, $G$, and $H$ are scalar functions called **form factors**. They must be real by the hermiticity of $J^\mu$, and in fact by the current conservation condition (12.84) we must have $H = 0$. Moreover by (12.83) we must have

$$F(0) + G(0) = 1. \tag{12.89}$$

In free field theory we see from (12.80) that $F = 1$ and $G = 0$, but in interacting electrodynamics they are both nontrivial functions.

Now let's see what we can say about the interaction of our charged particle with a classical field. In the non-relativistic limit we'd like to expand the vertex function at small momenta, but at zero momentum we only recover the relation (12.89) so we need to work at first order in the momenta. In the $G$ term this is easy since we already have a factor of $p+p'$, but in the $F$ term we need to be careful since there is a linear term hiding in $\overline{u}\gamma^\mu u$. To extract it we can use the Gordon identity

$$\overline{u}(\vec{p}',\sigma')\gamma^\mu u(\vec{p},\sigma) = \frac{1}{2im}\overline{u}(\vec{p}',\sigma')\left((p+p')^\mu + 2i(p'-p)_\alpha J^{\alpha\mu}\right)u(\vec{p},\sigma) \tag{12.90}$$

that we derived back in section four, where

$$J^{\alpha\mu} = -\frac{i}{4}[\gamma^\alpha,\gamma^\mu], \tag{12.91}$$

from which we have

$$\overline{u}'\Gamma^\mu u = \overline{u}'\left[-\frac{i}{2m}(p+p')^\mu(F+G) + \frac{1}{m}(p'-p)_\alpha J^{\alpha\mu}F\right]u. \tag{12.92}$$

Both terms now have an explicit power of momenta, so in the non-relativistic limit of the spatial components we can equate the momenta in $\overline{u}'$ and $u$, which allows us to contract them to get

$$\overline{u}'\Gamma^i u \approx -(p+p')^i + 4i(p-p')_j S^{ji}_{\sigma',\sigma}F(0), \tag{12.93}$$

where $S^{ji}_{\sigma',\sigma}$ are the standard spin matrices for $SO(d-1)$. For $d=4$ we can write them as

$$S^{ji} = \frac{1}{2}\epsilon^{jik}S^k \tag{12.94}$$

with $S^k = \frac{\sigma^k}{2}$. For a time-independent background vector potential the matrix element of the interaction Hamiltonian is thus

$$\langle\vec{p}',\sigma'|H_{back}|\vec{p},\sigma\rangle \approx \frac{q_0}{2m}\int d^3x A^i_{cl}(\vec{x})e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}\Bigg(-(p+p')^i + 2i(p-p')^j\epsilon_{jik}S^k_{\sigma',\sigma}F(0)\Bigg). \tag{12.95}$$

The first term in parenthesis is just the usual $-\vec{p}\cdot\vec{A}$ coupling of a charged particle to the electromagnetic field. The second term is more interesting: it describes a **magnetic moment** of the particle. Indeed integrating by parts the second term has the form

$$-\frac{q_0F(0)}{m}\int d^3x B^k_{cl}(\vec{x})S^k_{\sigma',\sigma}e^{i(\vec{p}-\vec{p}')\cdot\vec{x}}, \tag{12.96}$$

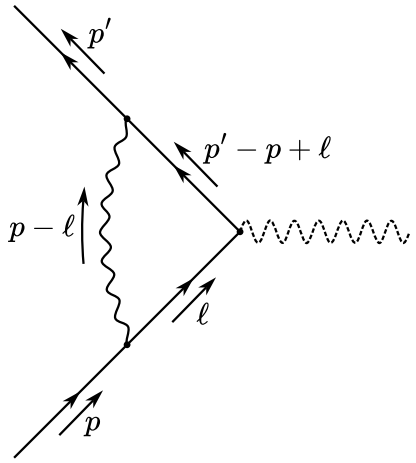


Figure 32: The one-loop correction to the vertex function. The dashed photon line indicates that it is a classical background field.

so in particular if the magnetic field is constant then we have

$$-(2\pi)^3\delta^3(\vec{p}-\vec{p}')\vec{m}_0 \cdot \vec{B}_{cl} \tag{12.97}$$

with

$$\vec{m}_0 = \frac{q_0 F(0)}{m}\vec{S}. \tag{12.98}$$

This is almost the magnetic moment of the particle, but we need to be careful to write things in terms of the renormalized magnetic field $\vec{B}_{cl,R} = Z_3^{-1/2}\vec{B}_{cl}$, as this is what we actually measure, so the true magnetic moment of the particle is

$$\vec{m} = \frac{q_0\sqrt{Z_3}F(0)}{m}\vec{S} = \frac{qF(0)}{m}\vec{S}. \tag{12.99}$$

This is usually parametrized in terms of a gyromagnetic ratio $g$ via

$$\vec{m} = \frac{gq}{2m}\vec{S}, \tag{12.100}$$

so what we have at last learned is that

$$g = 2F(0) = 2(1 - G(0)). \tag{12.101}$$

In particular in the free theory we recover Dirac's famous result that $g = 2$. We also see however that to the extent that $G(0)$ is nonzero there will be corrections to Dirac's formula; these are typically described in terms of the **anomalous magnetic moment of the electron:**

$$a_e = \frac{g-2}{2} = -G(0). \tag{12.102}$$

## 12.7 Vertex function and $g - 2$ (calculation)

It is now at last time for perhaps the most famous calculation in quantum field theory: the one-loop determination of the electron gyromagnetic ratio $g$ in quantum electrodynamics! We will compute the quantity $\overline{u}'\Gamma^\mu u$, which up through one loop is given by

$$\overline{u}'\Gamma^\mu u = Z_2\overline{u}'\gamma^\mu u + (-iq_0)^2\int \frac{d^4\ell}{(2\pi)^4}\frac{-i\overline{u}'\gamma^\nu i(\not{p}' - \not{p} + \not{\ell} + im)\gamma^\mu i(\not{\ell} + im)\gamma_\nu u}{(\ell^2 + m^2 - i\epsilon)\left((\ell - p + p')^2 + m^2 - i\epsilon\right)\left((\ell - p)^2 + \mu^2 - i\epsilon\right)} + \dots \tag{12.103}$$

The factor of $Z_2$ is there in the first term because of the on-shell external electrons, and the loop integral comes from the diagram shown in figure 32. I'll just compute the integral in $d = 4$, since in this case the numerator simplification is quite a bit more complicated if we use dimensional regularization; we'll eventually use Pauli-Villars to deal with the logarithmic UV divergence of the integral. $\mu$ is again a small photon mass to regulate an infrared divergence which arises when $\ell \approx p$, since then all three factors in the denominator are close to vanishing so there is a logarithmic infrared divergence. We can simplify the numerator using our formulas for contracting $\gamma$ matrices, and we can combine the denominators using a slightly more advanced Feynman parameter identity:

$$\frac{1}{ABC} = \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \frac{2}{(xA+yB+zC)^3}, \tag{12.104}$$

after which we can write the loop contribution as

$$\overline{u}' \Gamma^\mu u \supset 4iq_0^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{\overline{u}' \left( \not{\ell} \gamma^\mu (\not{\ell} + \not{q}) - 2im(2\ell+q)^\mu - m^2 \gamma^\mu \right) u}{\left( x(\ell^2+m^2) + y((q+\ell)^2+m^2) + z((\ell-p)^2+\mu^2) - i\epsilon \right)^3}, \tag{12.105}$$

where I've defined the **momentum transfer**

$$q = p' - p. \tag{12.106}$$

$q$ is not to be confused with the renormalized charge, which will not appear in this calculation until the very end. We can simplify the quantity in the denominator using that $p$ and $p'$ are on shell, which results in

$$x(\ell^2+m^2) + y((q+\ell)^2+m^2) + z((\ell-p)^2+\mu^2) = (\ell + yq - zp)^2 + xyq^2 + (1-z)^2 m^2 + z\mu^2, \tag{12.107}$$

so shifting the integration momentum by a constant we get

$$\begin{aligned} \overline{u}' \Gamma^\mu u \supset & 4iq_0^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \\ & \times \int \frac{d^4\ell}{(2\pi)^4} \frac{\overline{u}' \left( (\not{\ell} + z\not{p} - y\not{q}) \gamma^\mu (\not{\ell} + z\not{p} + (1-y)\not{q}) - 2im(2\ell + 2zp + (1-2y)q)^\mu - m^2\gamma^\mu \right) u}{\left( \ell^2 + xyq^2 + (1-z)^2 m^2 + z\mu^2 - i\epsilon \right)^3}. \end{aligned} \tag{12.108}$$

Simplifying the numerator further requires some work: we can drop all terms which are linear in $\ell$, we can replace $\ell^\mu \ell^\nu$ by $\frac{1}{4}\eta^{\mu\nu}\ell^2$, and we can move $\not{p}$ to the right to act on $u$ and $\not{p}'$ to the left to act on $\overline{u}'$. When the dust settles we have[52]

$$\begin{aligned} \overline{u}' \Gamma^\mu u \supset & 4iq_0^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \\ & \times \int \frac{d^4\ell}{(2\pi)^4} \frac{\overline{u}' \left( \left( -\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (z^2+2z-1)m^2 \right) \gamma^\mu + imz(z-1)(p+p')^\mu \right) u}{\left( \ell^2 + xyq^2 + (1-z)^2 m^2 + z\mu^2 - i\epsilon \right)^3}. \end{aligned} \tag{12.109}$$

This is now in a form where we can extract the form factors $F$ and $G$: Wick rotating, which supplies a factor of $i$ and removes the $i\epsilon$, we have

$$\begin{aligned} F(q^2) &= 1 + (Z_2 - 1) + 4q_0^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{\frac{1}{2}\ell^2 - (1-x)(1-y)q^2 - (z^2+2z-1)m^2}{\left( \ell^2 + xyq^2 + (1-z)^2 m^2 + z\mu^2 \right)^3} \\ G(q^2) &= -8q_0^2 m^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{z(1-z)}{\left( \ell^2 + xyq^2 + (1-z)^2 m^2 + z\mu^2 \right)^3}. \end{aligned} \tag{12.110}$$

[52]The simplification also produces a term $\overline{u}'(im(z-2)(x-y)q^\mu)u$ in the numerator, but this integrates to zero since it is antisymmetric in $x$ and $y$ while the denominator is symmetric. This is reassuring, as such a term would have given us a nonzero form factor $H$ which would have violated current conservation.

We can now (at last) evaluate these integrals using our standard formulas. The momentum integral in the expression for $F$ has logarithmic UV divergence which we can regulate using Pauli-Villars, which results in

$$F(q^2) = 1 + (Z_2 - 1) + \frac{q_0^2}{8\pi^2}\int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1)\Bigg[ - \frac{(1-x)(1-y)q^2 + (z^2+2z-1)m^2}{xyq^2 + (1-z)^2m^2 + z\mu^2} + \log\left(\frac{z\Lambda^2}{xyq^2 + (1-z)^2m^2 + z\mu^2}\right)\Bigg]. \tag{12.111}$$

Looking at our one-loop expression (12.24) for $Z_2$, we can see that the logarithmic divergence cancels so $F(q^2)$ is UV-finite. Evaluating the Feynman parameter integrals at $q^2 = 0$ we have

$$F(0) = 1 + \frac{q_0^2}{8\pi^2}. \tag{12.112}$$

Our expression for $G(q^2)$ is finite both in the IR and the UV so we can set $\mu = 0$ and $\Lambda = \infty$. We then have

$$G(q^2) = -\frac{q_0^2 m^2}{4\pi^2}\int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1)\frac{z(1-z)}{xyq^2 + (1-z)^2m^2}. \tag{12.113}$$

Evaluating this at $q^2 = 0$ we have

$$\begin{aligned} G(0) &= -\frac{q_0^2}{4\pi^2}\int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(x+y+z-1)\frac{z}{1-z} \\ &= -\frac{q_0^2}{4\pi^2}\int_0^1 dz \int_0^{1-z} dx \frac{z}{1-z} \\ &= -\frac{q_0^2}{8\pi^2}, \end{aligned} \tag{12.114}$$

which is consistent with our requirement (12.89). More importantly we thus have

$$a_e = \frac{\alpha}{2\pi} \approx .0011614, \tag{12.115}$$

which is the famous formula on Schwinger's tombstone at Mount Auburn Cemetery here in Cambridge. Higher order calculations give

$$a_e = .001159652181643(764), \tag{12.116}$$

which can be compared to the current(ish) experimental value:

$$a_e = .00115965218059(13). \tag{12.117}$$

It seems there is really something to this quantum electrodynamics business!

# 13 Spontaneous symmetry breaking and the Higgs mechanism

So far we have mostly only considered global symmetries that leave the ground state invariant. In general quantum systems this is by no means the only possibility, and a global symmetry which does not leave the ground state invariant is said to be **spontaneously broken**. There is also an analogue of spontaneous symmetry breaking for gauge symmetries, called the **Higgs mechanism**, although in that case the ground state is still unique. In this section we will learn some basics about both of these phenomena.

## 13.1 Spontaneous symmetry breaking in quantum mechanics

A rather trivial example of spontaneous symmetry breaking happens in the non-relativistic hydrogen atom: there are two degenerate 1s states, one for each spin of the electron, and they are mixed by spatial rotations. We thus can say that rotational invariance is spontaneously broken for the non-relativistic hydrogen atom.[53] This example might feel contrived, and indeed it is sometimes claimed that in quantum mechanics with a finite number of degrees of freedom obeying canonical commutation relations, with no extra label such as spin, the ground state can't be degenerate. To give you a flavor of these arguments, here is one in the context of a nonrelativistic particle moving in a potential $V(x)$ in one dimension. Say that $\psi_1$ and $\psi_2$ are normalizable wave functions obeying

$$\begin{aligned} -\psi_1'' + V(x)\psi_1(x) &= E\psi_1(x) \\ -\psi_2'' + V(x)\psi_2(x) &= E\psi_2(x). \end{aligned} \tag{13.1}$$

Multiplying the first equation by $\psi_2$ and the second equation by $\psi_1$, taking the difference, and then integrating we see that

$$\psi_1\psi_2' - \psi_1'\psi_2 = \text{constant}. \tag{13.2}$$

By normalizability $\psi_1$ and $\psi_2$ must got to zero at infinity, and for "reasonable" potentials $\psi_1'$ and $\psi_2'$ will not grow fast enough to cancel this. The constant must therefore be zero. We can then rewrite this equation as

$$(\log\psi_1)' = (\log\psi_2)', \tag{13.3}$$

which we can integrate to see that $\psi_1$ must be proportional to $\psi_2$ and thus they are really the same state.

On the other the other hand this argument involved various assumptions, and if we drop them then spontaneous symmetry breaking is possible. A simple example is the quantum mechanics of a particle on a circle $x \sim x + L$ with Lagrangian

$$L = \frac{m\dot{x}^2}{2} + \frac{\theta}{L}\dot{x}. \tag{13.4}$$

The canonical momentum is

$$p = \dot{x} + \frac{\theta}{L}, \tag{13.5}$$

and the Hamiltonian is

$$H = \frac{1}{2m}\left(p - \frac{\theta}{L}\right)^2. \tag{13.6}$$

The energy eigenstates are

$$\psi_n(x) = e^{ip_n x}, \tag{13.7}$$

with

$$p_n = \frac{2\pi n}{L}, \tag{13.8}$$

so the energy levels are

$$E_n = \frac{1}{2mL^2}\left(2\pi n - \theta\right)^2. \tag{13.9}$$

In particular if we choose $\theta = \pi$, then the $n = 0$ and $n = 1$ states are degenerate ground states. These are actually related by a symmetry, $x \to -x$. The Lagrangian may not look invariant under this symmetry, but it is invariant up to a total derivative. The symmetry is manifest in the Hamiltonian formulation, as the Hamiltonian is invariant under

$$\begin{aligned} x' &= -x \\ p' &= -p + \frac{2\theta}{L}. \end{aligned} \tag{13.10}$$

[53] In real hydrogen this degeneracy is broken by the interaction between the spin of the electron and the spin of the proton, leading to the hyperfine structure.

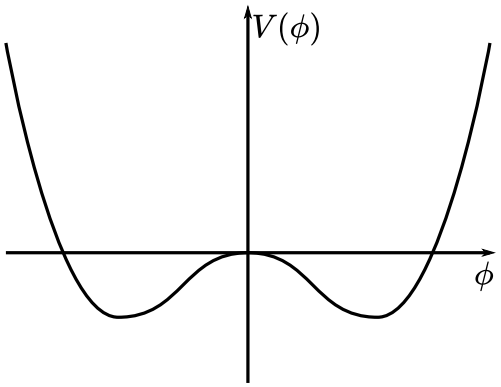


Figure 33: A scalar potential that exhibits spontaneous symmetry breaking.

Note that this symmetry only respects the periodicity of $x$ for $\theta = \pi m$ with $m \in \mathbb{Z}$, as otherwise it does not respect the quantization of $p$.[54]

## 13.2 Spontaneous symmetry breaking in quantum field theory

In the examples of spontaneous symmetry breaking in the previous subsection, the reason the ground state is not invariant is that it still has some kind of object around, an electron in the 1s orbital in the first example and a particle on a circle in the second, and these objects can transform nontrivially under the symmetry. In quantum field theory however the ground state is supposed to represent the vacuum, without any objects present, and one might expect that a state with "nothing there" should be be invariant under any global symmetry. In fact spontaneous symmetry breaking is nonetheless possible in quantum field theory, but it has a somewhat different character from the examples we just discussed.

As is often the case when meeting a new phenomenon for the first time, it is best to start with a simple example. The classic example of a quantum field theory with spontaneous symmetry breaking is $\phi^4$ theory with a negative mass squared:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{13.11}$$

The potential for this theory is shown in figure 33; $\phi = 0$ is now a local maximum rather than a global minimum, and the true minima are at

$$\phi = \pm\sqrt{\frac{6m^2}{\lambda}}. \tag{13.12}$$

We can thus guess that this theory has two degenerate ground states $|\Omega, \pm\rangle$ obeying

$$\langle\Omega, \pm|\Phi(x)|\Omega, \pm\rangle = \pm\sqrt{\frac{6m^2}{\lambda}}. \tag{13.13}$$

If this is true, then the $\phi' = -\phi$ global symmetry of the action is spontenously broken. Justifying this guess is somewhat subtle however, and in particular it is only really correct when we study the theory in infinite volume. To get a sense of where the subtlety comes from, we can go back to the quantum mechanics of a particle with Hamiltonian

$$H = \frac{p^2}{2m} - \frac{k^2}{2}x^2 + \frac{\lambda}{4!}x^4. \tag{13.14}$$

[54] This breaking of a classical symmetry when $\theta \neq \pi m$ is a simple example of an **anomaly**; we will meet other examples next semester.

This potential again has the form shown in figure 33, but we proved in the last section that theories of this type have a unique ground state. So what is going on?

We can get a better sense of what is happening by defining a pair of states $|\pm\rangle$ by the property that $|\pm\rangle$ minimizes

$$\langle \pm | H | \pm \rangle \tag{13.15}$$

subject to the constraint that

$$\langle \pm | x | \pm \rangle = \pm \sqrt{\frac{6k^2}{\lambda}}. \tag{13.16}$$

Were this theory to exhibit spontaneous symmetry breaking then $|\pm\rangle$ would be a pair of degenerate ground states. Let's study their matrix elements. The $x' = -x$ symmetry is implemented on the Hilbert space by a unitary operator $U$ that commutes with $H$, so we have

$$\langle + | H | + \rangle = \langle - | U^\dagger H U | - \rangle = \langle - | H | - \rangle \equiv a. \tag{13.17}$$

Moreover we have

$$\langle + | H | - \rangle = \langle - | U^\dagger H | - \rangle = \langle - | H U^\dagger | - \rangle = \langle - | H | + \rangle = b. \tag{13.18}$$

By hermiticity we also have

$$\langle + | H | - \rangle = \langle - | H | + \rangle^*, \tag{13.19}$$

so $b$ must be real. Within the subspace spanned by $|\pm\rangle$ we thus have

$$H = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{13.20}$$

with $a$ and $b$ both real. The real question is whether or not $b = 0$. The state $|+\rangle$ looks like a wave packet near $x = +\sqrt{\frac{6k^2}{\lambda}}$, while the state $|-\rangle$ looks like a wave packet near $x = -\sqrt{\frac{6k^2}{\lambda}}$. We thus should expect a nonzero tunnelling amplitude between them, which in the WKB approximation gives

$$b \sim e^{-\int_{-x_0}^{x_0} dx' \sqrt{2m(V(x') - E_0)}}. \tag{13.21}$$

Diagonalizing $H$ with nonzero $b$ shows that the true eigenstates are $|+\rangle \pm |-\rangle$, with energy $a \mp |b|$. Thus the ground state is

$$|\Omega\rangle = |+\rangle + |-\rangle, \tag{13.22}$$

which is invariant under the symmetry so there is no spontaneous symmetry breaking.

Let's now apply this discussion to the quantum field theory (13.11). The key difference is that in order to tunnel from $|+\rangle$ to $|-\rangle$ or vice versa, the field now needs to tunnel everywhere in space. The tunneling exponent in the WKB approximation therefore picks up an extra factor of the volume of space. More concretely if we define a constant mode

$$\begin{aligned} \phi_0 &= \frac{\int d^{d-1}x \phi}{\text{Vol}} \\ \pi_0 &= \frac{\int d^{d-1}x \dot{\phi}}{\text{Vol}}, \end{aligned} \tag{13.23}$$

the Hamiltonian derived from (13.11) becomes

$$H = \frac{\pi_0^2}{2\text{Volume}} + \text{Volume} \times V(\phi_0). \tag{13.24}$$

Thus in the tunneling exponent (13.21) we should replace $m \to \text{Volume}$ and $V \to \text{Volume} \times V$, which gives an overall factor of the spatial volume in the exponent. Thus in the infinite volume limit we have $b = 0$, so

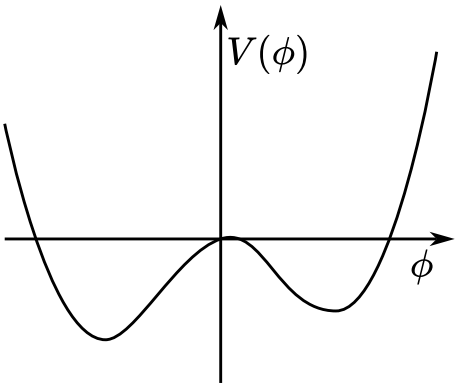


Figure 34: Turning on an explicit symmetry breaking perturbation. This forces the system to pick $|+\rangle$ or $|-\rangle$ to be the true ground state.

there are indeed two degenerate ground states exchanged by the symmetry.[55] Thus we see that spontaneous symmetry breaking is considerably more robust in quantum field theory than it is in quantum mechanics.

Let's say a bit more precisely what the difference is between this example and the quantum mechanical examples we discussed above. In the field theory model (13.11), the field $\phi$ is an example of what is called an **order parameter** - it is a dynamical variable whose expectation value tells us which vacuum we are in. In the hydrogen atom example we can take the order parameter to be the spin of the electron in the $z$-basis, while in the example with a periodic particle it is the momentum $p$. In both of the single particle cases the order parameter commutes with the Hamiltonian, as otherwise it would need to have fluctuations in the ground state. What can happen in quantum field theory however is that we can have an order parameter which has fluctuations in any finite volume, but which stops fluctuating in the limit of infinite volume (we will see this more explicitly momentarily). This allows for more interesting examples of spontaneous symmetry breaking.

## 13.3 External perturbations and cluster decomposition

Given the discussion of the last subsection you might wonder why we talk about the $|\pm\rangle$ states at all. If the Hamiltonian in finite volume is really diagonal in the $|+\rangle \pm |-\rangle$ basis, shouldn't we work in that basis instead? What is really special about the $|\pm\rangle$ states is that if we add a small perturbation to the system which explicitly breaks the symmetry, the true ground state is close to $|+\rangle$ or $|-\rangle$ rather than $|+\rangle + |-\rangle$ or $|+\rangle - |-\rangle$. We can illustrate this using our above quantum mechanics example. Say that we introduce a small linear perturbation to the potential,

$$V(x) = gx - \frac{k^2}{2}x^2 + \frac{\lambda}{4!}x^4. \tag{13.25}$$

The rule of degenerate perturbation theory is that perturbed Hamiltonian will be diagonal in the basis which diagonalizes the perturbation. If we look at the matrix elements of the perturbation we have

$$\langle \pm | gx | \pm \rangle = \pm g\sqrt{\frac{6k^2}{\lambda}} \tag{13.26}$$

by assumption, while the off-diagonal matrix elements of $gx$ are much smaller since $\langle x|+\rangle$ and $\langle x|-\rangle$ are peaked at different places. We can also see this directly from the plot of the potential as in figure 34, where

[55] If the spontaneously broken symmetry is continuous then $b$ only vanishes like a power of the volume, but it still vanishes.

the linear perturbation breaks the degeneracy and favors one minimum or the other depending on the sign of $g$. In our matrix language we are thus now diagonalizing

$$H = \begin{pmatrix} a+c & b \\ b & a-c \end{pmatrix} \tag{13.27}$$

which has eigenvalues $a \mp \sqrt{b^2+c^2}$ and eigenvectors which when $|b| \ll |c|$ are close to $|\mp\rangle$ if $c$ is positive and $|\pm\rangle$ if $c$ is negative. This idea should also be familiar in the context of the Ising model, where if we turn on a small external magnetic field this biases the system towards all up or all down depending on the sign of the field. In particular if the strength of the perturbation stays nonzero in the large volume limit then eventually we will always be in the regime $|b| \ll |c|$ no matter how small the perturbation.

For quantum field theory in infinite volume we can be somewhat more quantitative about this situation. Indeed let's say that we have a set of degenerate vacua $|v\rangle$ where $v$ is some set of parameters. We will assume translation symmetry is not broken, in which case these vacua obey

$$\vec{P}|v\rangle = 0 \tag{13.28}$$

with $\vec{P}$ the total spatial momentum, and since there are no particles in these vacua they should be discrete eigenvectors of $\vec{P}$ (i.e. they shouldn't be part of some continuous set of $\delta$-function normalized momentum eigenstates). We can thus take each $|v\rangle$ state to have norm one, and by choosing an orthonormal set of $|v\rangle$ we can assume that

$$\langle u|v\rangle = \delta_{u,v}. \tag{13.29}$$

Note that this is the usual Kronecker $\delta$, even though $v$ may include continuous parameters (in the next subsection we will see examples where it does). Our goal is now to show that we can additionally choose this basis so that for any local operator $O(x)$ we have

$$\langle u|O(x)|v\rangle = \delta_{u,v}\langle v|O(x)|v\rangle, \tag{13.30}$$

In other words local operators can never mix between the different vacua. This thus implies that any deformation of the Hamiltonian by a local operator (or a sum of local operators) will be diagonal in the $|v\rangle$ basis, and thus that these will be the states among which the ground state is selected when we add a symmetry-breaking perturbation. The idea is that for any two local operators $O_1$ and $O_2$ we have

$$\langle u|O_1(\vec{x})O_2(0)|v\rangle = \sum_w \langle u|O_1(0)|w\rangle\langle w|O_2(0)|v\rangle + \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \sum_N \langle u|O_1(0)|N,\vec{p}\rangle\langle N,\vec{p}|O_2(0)|v\rangle e^{i\vec{p}\cdot\vec{x}} \tag{13.31}$$

where $|N,\vec{p}\rangle$ are some complete set of single and multiparticle states. If we consider the limit of large $|x|$ the momentum integral should vanish by the Riemann-Lebesgue lemma (assume some mild integrability of these matrix elements), so we thus have

$$\lim_{|x|\to\infty} \langle u|O_1(\vec{x})O_2(0)|v\rangle = \sum_w \langle u|O_1(0)|w\rangle\langle w|O_2(0)|v\rangle. \tag{13.32}$$

Similarly we also have

$$\lim_{|x|\to\infty} \langle u|O_2(0)O_1(\vec{x})|v\rangle = \sum_w \langle u|O_2(0)|w\rangle\langle w|O_1(0)|v\rangle. \tag{13.33}$$

Now let's assume that $O_1$ and $O_2$ are bosonic. They therefore must commute at spacelike separation, and so we must have

$$\sum_w \langle u|O_1(0)|w\rangle\langle w|O_2(0)|v\rangle = \sum_w \langle u|O_2(0)|w\rangle\langle w|O_1(0)|v\rangle. \tag{13.34}$$

In other words the matrices $\langle u|O_1(0)|v\rangle$ and $\langle u|O_2(0)|v\rangle$ commute for any bosonic $O_1$ and $O_2$. If $O_1$ or $O_2$ is fermionic then its matrix elements between the $|v\rangle$ must vanish since each $|v\rangle$ should be invariant under

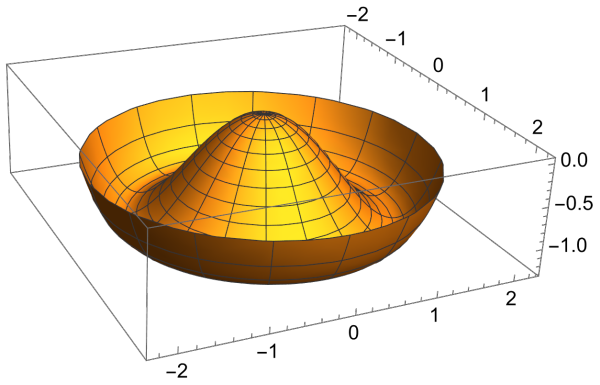


Figure 35: The sombrero potential for the $O(2)$ vector model.

fermion parity (which is built out of infinitesimal rotations). We can therefore simultaneously diagonalize all of the local operators, leading to (13.30).

We can think of these $|v\rangle$ states as being precisely the ones where the **cluster decomposition principle** is obeyed:

$$\lim_{|x|\to\infty} \langle v|O_1(\vec{x})O_2(0)|v\rangle = \langle v|O_1(0)|v\rangle\langle v|O_2(0)|v\rangle \tag{13.35}$$

for any $O_1$ and $O_2$. One consequence of this is that the fluctuations of the zero modes of operators vanish in these states:

$$\begin{aligned}\langle v|\frac{\int d^{d-1}xO(\vec{x})}{\text{Volume}}\frac{\int d^{d-1}yO(\vec{y})}{\text{Volume}}|v\rangle &= \frac{1}{\text{Volume}}\int d^{d-1}x\langle v|O(\vec{x})O(0)|v\rangle \\ &= \langle v|O(0)|v\rangle\langle v|O(0)|v\rangle \\ &= \langle v|\frac{\int d^{d-1}xO(\vec{x})}{\text{Volume}}|v\rangle\langle v|\frac{\int d^{d-1}yO(\vec{y})}{\text{Volume}}|v\rangle,\end{aligned} \tag{13.36}$$

where in the first equality I used translation invariance, in the second I used that the integral is determined by the large $|x|$ regime since the contribution from any finite region is eliminated by the infinite volume factor, and in the third I again used translation invariance. This shows that these $|v\rangle$ states are indeed those where the zero modes of the fields have definite values, and in fact usually the label $v$ is precisely just some list of these expectation values. In our $\phi^4$ theory with negative mass squared these are precisely the states $|\pm\rangle$, while in the states $|+\rangle \pm |-\rangle$ the zero mode of $\phi$ has $O(1)$ fluctuations.

## 13.4 Goldstone's theorem

Spontaneous symmetry breaking is particularly interesting when the symmetry which is broken is continuous. The classic example of this is a theory of $N$ real scalar fields $\phi_i$, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sum_i \partial^\mu\phi_i\partial_\mu\phi_i + \frac{m^2}{2}\sum_i \phi_i^2 - \frac{\lambda}{4}\left(\sum_i \phi_i^2\right)^2. \tag{13.37}$$

This theory is usually called the "$O(N)$ vector model", as it has an $O(N)$ global symmetry that rotates among the $\phi_i$ fields:

$$\phi_i' = R_{ij}\phi_j, \tag{13.38}$$

with $R \in O(N)$. For $N = 2$ this potential is called the sombrero or mexican hat potential, it is shown in figure 35. The minimum of this potential is an $\mathbb{S}^{N-1}$ in field space located at

$$|\phi| = \frac{m}{\sqrt{\lambda}}, \tag{13.39}$$

so to pick a vacuum we need to pick a unit vector in $\mathbb{R}^N$. This unit vector is preserved by an $O(N-1)$ subgroup of $O(N)$, so this situation is typically described by saying that the $O(N)$ global symmetry of the model is spontaneously broken to a $O(N-1)$ subgroup, with the latter still leaving the vacuum invariant. In particular for the case of $N=2$ the unbroken symmetry is just $O(1)=\mathbb{Z}_2$.

What are excitations of this theory about the ground state $|\hat{n}\rangle$ in which the zero mode of $\phi_i$ is pointing in the $\hat{n}$ direction? Looking at figure 35 the answer is clear: in the radial direction orthogonal to the ground state $\mathbb{S}^{N-1}$ there is a massive scalar field excitation (with positive mass squared), while for each of the $N-1$ field directions along the $\mathbb{S}^{N-1}$ there is a massless scalar field. These massless scalars are called **Goldstone bosons**, or sometimes **Nambu-Goldstone bosons**, and their existence is the hallmark of a spontaneously broken continuous symmetry. Note that there is an independent massless scalar for each broken generator of $O(N)$. In particular for $N=2$ there is one Goldstone boson. We can extract it explicitly by writing

$$\begin{aligned}\phi_1 &= \rho\cos\theta \\ \phi_2 &= \rho\sin\theta,\end{aligned} \tag{13.40}$$

in terms of which the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \frac{\rho^2}{2}\partial_\mu\theta\partial^\mu\theta + \frac{m^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4. \tag{13.41}$$

The potential causes $\rho$ to get an expectation value

$$\rho_0 = \frac{m}{\sqrt{\lambda}}, \tag{13.42}$$

so writing

$$\rho = \rho_0 + \delta \tag{13.43}$$

we have

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\delta\partial^\mu\delta - m^2\delta^2 - \frac{(\rho_0+\delta)^2}{2}\partial_\mu\theta\partial^\mu\theta + \lambda\rho_0\delta^3 + \frac{\lambda}{4}\delta^4 \tag{13.44}$$

we can see that $\delta$ is a massive scalar field of mass $\sqrt{2}m$ and $\theta$ is our massless Goldstone boson.

You may wonder if these massless bosons were an artifact of this model, or perhaps if they can acquire a mass due to loop corrections. The answer to both questions is no, as we can learn from **Goldstone's theorem:**[56]

- In a relativistic quantum field theory with an internal global symmetry group $G$ that is spontaneously broken to some subgroup $H$, for each broken direction in $G$ there is an exactly massless scalar particle.

There are various approaches to proving Goldstone's theorem, my favorite is the following. Let $J^\mu$ be the Noether current for a continuous internal global symmetry that is spontaneously broken in some vacuum state $|v\rangle$, and let $O_n$ be some set of local operators that transform nontrivially under the symmetry:

$$\delta_S O_n = \sum_m T_{nm} O_m. \tag{13.45}$$

The Ward identity that we proved last semester shows that we have

$$\frac{\partial}{\partial x^\mu}\langle v|TJ^\mu(x)O_n(0)|v\rangle = i\delta^d(x)\langle v|\delta_S O_n(0)|v\rangle + \ldots, \tag{13.46}$$

where $\ldots$ indicates possible Schwinger terms involving derivatives of the $\delta$-function. Defining the Fourier transform

$$\langle v|TJ^\mu(x)O_n(0)|v\rangle = \int \frac{d^dp}{(2\pi)^d}\langle v|TJ^\mu(p)O_n(0)|v\rangle e^{ip\cdot x}, \tag{13.47}$$

[56]This is not the most general version of the theorem. There is also a non-relativistic version of Goldstone's theorem, and the spontaneous breaking of spacetime symmetries such as translation and rotation invariance also leads to Goldstone bosons.

we can rewrite (13.46) as

$$p_\mu \langle v|TJ^\mu(p)O_n(0)|v\rangle = \langle v|\delta_S O_n(0)|v\rangle + \ldots, \tag{13.48}$$

where the ... terms come from Schwinger terms and are proportional to positive powers of momentum (from the derivatives on the $\delta$ function). Now let's consider the limit of zero momentum. As long as $\langle v|\delta_S O_n(0)|v\rangle \neq 0$, the two point function in momentum space must have a term

$$\langle v|TJ^\mu(p)O_n(0)|v\rangle \supset \langle v|\delta_S O_n(0)|v\rangle \times \frac{p^\mu}{p^2}. \tag{13.49}$$

Therefore the two-point function must have a pole at zero momentum, which I'll remind you is equivalent to saying there is a massless particle which can be created by $O(0)$ and annihilated by $J^\mu$. Moreover the momentum $p^\mu$ can be recognized as the intertwiner which allows a vector field $J^\mu$ to annihilate a particle of zero spin (think $\partial^\mu\phi$ in free field theory), so our massless particle has spin zero. This particle must have the same internal symmetry charges as $J^\mu$ since it is annihilated by it. This conclusion required that

$$\langle v|\delta_S O_n(0)|v\rangle = \sum_m T_{nm}\langle v|O_m(0)|v\rangle \neq 0, \tag{13.50}$$

but this is precisely the condition that the vacuum transforms nontrivially under symmetry generated by $J^\mu$ (assuming that we choose $O_n$ to include any fields which have a vacuum expectation value). Moreover if there is more than one $J^\mu$ for which this is true then there is an independent Goldstone boson for each spontaneously-broken direction in the internal global symmetry group. Indeed let $J^\mu_a$ be a basis for the set of broken currents and let $b$ run over the set of Goldstone bosons contributing to this two-point function. The relevant matrix elements are determined by Lorentz invariance to be

$$\begin{aligned} \langle v|J^\mu(0)|\vec{p},b\rangle &= \frac{iF_{ab}p^\mu}{\sqrt{2\omega_{\vec{p}}}} \\ \langle \vec{p},b|O_n(0)|v\rangle &= \frac{Z_{bn}}{\sqrt{2\omega_{\vec{p}}}}, \end{aligned} \tag{13.51}$$

and from the two-point function we have

$$-i\sum_b F_{ab}Z_{bn} = \sum_m T^a_{nm}\langle v|O_m(0)|v\rangle. \tag{13.52}$$

We can view the right hand side as a matrix $T_{nm}$, which by definition has a rank which is equal to the number $N_b$ of broken generators (i.e. the number of generators $T^a_{nm}$ such that no linear combination of them leaves the vacuum invariant). The left-hand side is a product of two matrices, so since their product has rank $N_b$ they must each separately have rank at least $N_b$. In particular $Z_{bn}$ has rank $N_b$, so $O_n$ creates $N_b$ linearly-independent boson states. This is the most we could possibly create, since $F_{ab}$ has rank $N_b$.

There are many examples of Goldstone bosons in nature, here are a few:

- In QCD in the limit $m_u = m_d = 0$ there is an $SU(2)$ chiral symmetry. This symmetry turns out to be spontaneously broken, leading to three Goldstone bosons: the charged pions $\pi_\pm$ and the neutral pion $\pi_0$.
- In superfluid liquid helium particle number symmetry is spontaneously broken, which leads to a massless Goldstone boson whose dynamics account for the remarkable bulk properties of the system.
- In a crystal spatial translation symmetry is spontaneously broken, which leads to Goldstone bosons called phonons.
- In the early universe it seems that there was an approximate spontaneously broken shift symmetry of some field called the *inflaton*, whose approximate Goldstone boson fluctuations eventually gave rise to the anisotropy of the cosmic microwave background and the stars and galaxies we see today.

## 13.5 The Abelian Higgs model

What happens if a gauge symmetry is spontaneously broken? This turns out to be a rather slippery question to make precise, and in particular to the extent that spontaneous breaking of a gauge symmetry makes sense it does *not* lead to any vacuum degeneracy.[57] It is thus safest not to refer to (or think of) the phenomenon as spontaneous symmetry breaking; I'll instead refer to it as the **Higgs mechanism**.[58] Rather than give a general discussion we'll instead just consider a simple example, the **Abelian Higgs model**, which is another name for scalar QED with a negative mass squared:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - (D_\mu\phi)^* D_\mu\phi + m^2|\phi|^2 - \frac{\lambda}{4}|\phi|^4. \tag{13.53}$$

Here we take $\phi$ to have charge $q$, so the covariant derivative is

$$D_\mu\phi = (\partial_\mu - iqA_\mu)\,\phi. \tag{13.54}$$

We can guess that the potential causes $\phi$ to get a nonzero expectation value, but since $\phi$ is not gauge-invariant we need to be more careful about analyzing what this theory does. It turns out the most useful approach is to introduce the field redefinition

$$\phi(x) = \rho(x)e^{iq\theta(x)}, \tag{13.55}$$

with $\rho > 0$. This field redefinition is not good near $\rho = 0$, but at least to all orders in perturbation theory we are ok as long as we end up expanding around somewhere other than $\rho = 0$. In terms of these variables the gauge transformations of this theory are

$$\begin{aligned} A'_\mu &= A_\mu + \partial_\mu\Omega \\ \theta' &= \theta + \Omega \\ \rho' &= \rho. \end{aligned} \tag{13.56}$$

Substituting this redefinition into the covariant derivative we have

$$D_\mu\phi = (\partial_\mu\rho - iq(A_\mu - \partial_\mu\theta)\rho)e^{iq\theta}, \tag{13.57}$$

so the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial_\mu\rho\partial^\mu\rho + m^2\rho^2 - \frac{\lambda}{4}\rho^4 - q^2\rho^2\left(A_\mu - \partial_\mu\theta\right)\left(A^\mu - \partial^\mu\theta\right). \tag{13.58}$$

The potential thus sets $\rho$ to

$$\rho_0 = \sqrt{\frac{2m^2}{\lambda}}. \tag{13.59}$$

I emphasize that $\rho$ is gauge-invariant, so this expectation value does *not* break the gauge symmetry; the vacuum is gauge-invariant (as it had better be since it should be a physical state). The Lagrangian (13.58) has a number of remarkable features, which we will now discuss in turn:

- The variable $\theta$ appears only in derivatives, so one might naively expect it to lead to a massless scalar particle. If we remove the photon from the theory then this is indeed the case, with $\theta$ being the Goldstone boson for the spontaneous breaking of $U(1)$ global charge rotation symmetry. In fact this is just the $O(2)$ vector model written in slightly different variables. In the theory (13.58) however, we can remove the $\theta$ field by doing the gauge transformation $\Omega = -\theta$. This is called going to **unitarity gauge**, which has gauge-fixing condition $\theta = 0$. The theory (in any gauge) thus has no physical Goldstone boson, it has been removed by the gauge redundancy. This is consistent with the general slogan that the Higgs mechanism is not the same thing as spontaneous symmetry breaking.

[57] This has been shown rigorously from a lattice point of view, where it is called "Elitzur's theorem".

[58] The history of the Higgs mechanism is a convoluted mess, with the list of people deserving some credit including Anderson, Brout, Englert, Guralnik, Hagen, Higgs, Kibble, and Nambu. I'm not enough of a historian to take sides, but it does need a name and Higgs is a fine one.

- If we look at fluctuations of the $\rho$ field, which we can parametrize as $\rho = \rho_0 + \frac{\delta}{\sqrt{2}}$, the kinetic term for $\delta$ has the form

$$\mathcal{L} \supset -\frac{1}{2}\partial_\mu \delta \partial^\mu \delta - m^2 \delta^2, \tag{13.60}$$

so this theory has a massive scalar excitation with mass $m_{Higgs} = \sqrt{2}m$. This is called the **Higgs boson**, and a close analogue of it was discovered in 2012 by the ATLAS and CMS experiments at the Large Hadron Collider in Geneva.

- The quadratic action for the photon now includes a mass term

$$\mathcal{J} \supset -\frac{\mu^2}{2} A_\mu A^\mu, \tag{13.61}$$

with the photon mass $\mu$ being given by[59]

$$\mu = \sqrt{2} q \rho_0 = \frac{2qm}{\sqrt{\lambda}}. \tag{13.64}$$

This term does not look gauge-invariant, but the full Lagrangian is; the Higgs mechanism thus gives us a gauge-invariant way to describe realize a massive photon. More concretely the gauge-invariant massive photon field is $\widetilde{A}_\mu = A_\mu - \partial_\mu \theta$. This situation is sometimes described by saying that the massive photon has "eaten the Goldstone boson". Of course in the standard model the true photon is massless, but an analogue of this mechanism leads to three heavy massive particles of spin one called the $W_\pm$ and $Z$ bosons.

- In the standard model the Higgs mechanism gives mass to matter fields in addition to gauge bosons. Here is how it works. Say we have two charged spinors $\Psi_1$ and $\Psi_2$, with charges $q_1$ and $q_2$ respectively, and moreover let's assume that $q_1 + q = q_2$. Then we can have a gauge invariant Yukawa-type term in the Lagrangian

$$\mathcal{L} \supset -ig\phi\overline{\psi}_2\psi_1 + \text{c.c.} \tag{13.65}$$

If $\phi$ gets an expectation value then this becomes a mass term that mixes $\psi_1$ and $\psi_2$, and we can rediagonalize the spinor basis to find a pair of massive spinors.

## 13.6 Superconductivity

The first physics application of the Higgs mechanism was not actually to particle physics; it was originally proposed (before the work of Higgs) by Nambu and then Anderson as a model of superconductivity. In this context it is therefore sometimes called the Anderson-Higgs mechanism, especially by condensed matter physicists, although I don't think that is fair to Nambu. In any event the key point is that a superconductor is really nothing but a material in which some low-energy field with electric charge $q$ has a radial part $\rho$ which gets an expectation value. In other words it is a material in which electromagnetism is Higgsed! We won't concern ourselves here with why this happens, for that we need the BCS theory for conventional superconductors and something fancier for unconventional superconductors, but given this definition we can easily understand the most famous superconducting phenomona without any need for complicated models.[60]

The idea is that inside a superconducting material we can describe the low-energy behavior using the photon field $A_\mu$ and the field $\theta$ corresponding to the phase of the operator whose radial part gets an expectation value. In order for the low-energy action to be gauge-invariant, it can depend on $A_\mu$ and $\theta$ only through

[59] To see that this term indeed has the interpretation of a photon mass, we can look at how it modifies Maxwell's equations. These become

$$\partial^\mu F_{\mu\nu} = \mu^2 A_\nu, \tag{13.62}$$

which in Lorenz gauge $\partial_\mu A^\mu = 0$ have the form

$$\partial^2 A_\nu = \mu^2 A_\nu. \tag{13.63}$$

Thus the solutions in momentum space obey $p^2 = -\mu^2$.

[60] We here only give a rather cursory treatment, see section 21.6 of Weinberg for more.

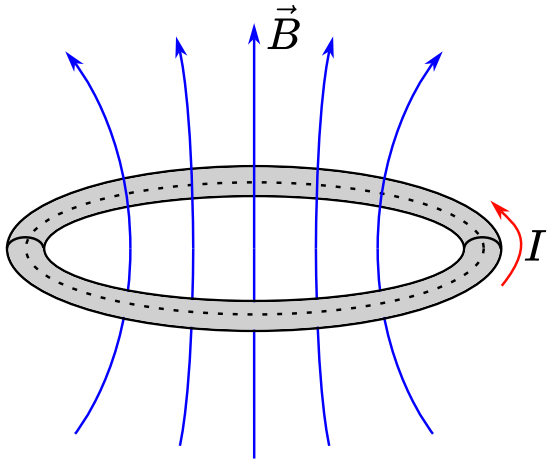


Figure 36: Magnetic flux through a superconducting loop in the presence of a persistent current $I$: the flux through the disk bounded by the dashed line is quantized in units of $2\pi/q$, so the current cannot decay. The current is flowing at the edge of the loop, inside of it both $\vec{B}$ and $\vec{J}$ vanish.

the combination $A_\mu - \partial_\mu \theta$. Moreover we'd like the system to be stable when $A$ and $\theta$ both vanish, so the energy should have a minimum when this quantity vanishes. These expectations are realized for the Abelian Higgs Lagrangian (13.58), but we do not need to to use the details of this Lagrangian (and in fact it is only a good approximation near the transition temperature, where it is referred to as the Landau-Ginzburg theory). In general if we are deep inside a superconductor which is in (or near) its ground state we should expect that

$$A_\mu = \partial_\mu \theta. \tag{13.66}$$

Since this is pure gauge, the electric and magnetic fields must both vanish. The vanishing of the electric field is familiar for a perfect conductor, the vanishing of the magnetic field inside of a superconductor is called the **Meissner effect**.[61]

Of course the most famous feature of a superconductor is electrical conductivity with zero resistance. To understand this, we first need to appreciate that since $\theta$ is the phase of an operator of charge $q$ it is actually a periodic variable obeying

$$\theta \sim \theta + \frac{2\pi n}{q}. \tag{13.67}$$

Now let's consider a closed loop of superconducting material. Consider circular path $L$ which circles the loop once within the material, which we can view as the boundary of a disk $D$ that extends outside of the superconducting material. See figure 36 for an illustration. We can compute the magnetic flux through $D$ via the following calculation:

$$\int_D \vec{B} \cdot d\vec{A} = \int_L \vec{A} \cdot d\vec{x} = \int_L \vec{\nabla}\theta \cdot d\vec{x} = \theta(2\pi) - \theta(0) = \frac{2\pi n}{q} \qquad n \in \mathbb{Z}. \tag{13.68}$$

Thus the magnetic flux through the loop is quantized, with the quantization integer being set by the number of times that $\theta$ winds around the loop. Since this flux is linearly proportional to the current circulating around the loop (see e.g. the Biot-Savart law), this means the current is also quantized. In particular this means that the current cannot decay continuously, so if it starts with $n \neq 0$ then it must remain so. In other words a superconducting loop can support a **persistent current** that lasts forever without any external

[61] Incidentally this already shows that there is more to a superconductor than just being a perfect conductor - the magnetic field inside the latter has to be constant, but it doesn't have to be zero.

potential applied, which is the hallmark of superconductivity! Experimentally the set of allowed currents can be measured and gives $q = -2e$, so apparently in a typical superconductor the operator whose radial component gets an expectation value has the charge of two electrons - the excitations it annihilates are typically called **Cooper pairs**.

It is interesting to consider where the persistent current is located within the current loop. Inside the loop we have $\vec{B} = 0$ by the Meissner effect, and so by Ampere's law we have (assuming the current is stationary) $\vec{J} = \vec{\nabla} \times \vec{B} = 0$. So there is no current flowing inside the loop. It therefore must all be flowing right at the surface! This is a typical feature of superconductors - most of the interesting physics has to do with what happens at an interface between a superconductor and a material (such as air) in which electromagnetic gauge symmetry is not Higgsed. Another example of such a phenomenon is the **Josephson effect**, where bringing two superconductors close together results in an oscillating current between them. The Josephson effect has many applications, for example in SQUIDs (which are very precise magnetometers) and as qubits for potential quantum computing architectures.