MIT 6.S890 — Topics in Multiagent Learning | Thu, Nov 21st 2024

Lecture 18

PPAD-completeness of Nash equilibria (Part II)

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We conclude the journey we started in Lecture 17 by giving a glimpse of how the PPAD-hardness of finding ϵ-approximate Nash equilibria was shown by Daskalakis, C., Goldberg, P. W., & Papadimitriou, C. H. [DGP09].

The proof can be broken down into two main steps:
- Reduction from the End-of-the-line problem to (approximate) Brouwer.
- Reduction from (approximate) Brouwer to (approximate) Nash equilibria.

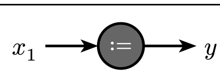
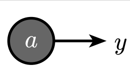
The first step is relatively easy, and we will not cover it here. The second step is more involved and requires a careful construction of a reduction from Brouwer to Nash equilibria. This is the part we will focus on in this lecture.

The key idea is the following: in the reduction from END-OF-THE-LINE to Brouwer, we have defined somewhere a function continuous f for which we need to find an approximate fixed point. We now need that we can construct a game such that a Nash equilibrium of the game is the same as a fixed point of f (up to approximations). The issue is that it is not clear how we can have games "compute" functions. Can we construct games in such a way that their behavior at Nash equilibria can be seen as "computing something"? The answer is positive, as we see next.

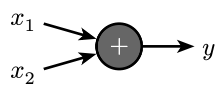
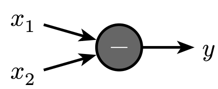
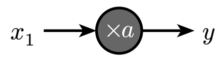
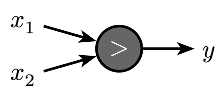
1 Arithmetic Circuit SAT

We show that given a function represented as an *arithmetic circuit*, it is possible to construct a game whose Nash equilibria correspond to computing a fixed point of the function. This is the key idea behind the reduction from Brouwer to Nash equilibria.

In particular, we will restrict our attention to functions constructed through circuits that are composed of the following:
- Variable nodes $v_1, ..., v_n$;
- Gate nodes $g_1, ..., g_m$ of six possible types:

Gate	**Symbol**	**Input-output relationship**
Assignment		$y = x_1$
Constant		$y = a$

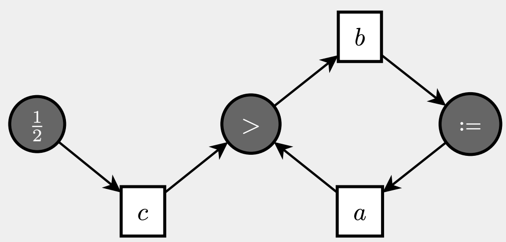
*These notes are class material that has not undergone formal peer review. The TA and I are grateful for any reports of typos. Some of the content of the lecture was adapted from material from Costis Daskalakis.

Gate	Symbol	Input-output relationship
Addition		$y = \min\{1, x_1 + x_2\}$
Subtraction		$y = \max\{0, x_1 - x_2\}$
Multiplication		$y = \max\{0, \min\{1, a \cdot x_1\}\}$
Comparison		$y = \begin{cases} 1, & \text{if } x_1 > x_2 \\ 0, & \text{if } x_1 < x_2 \\ \text{any}, & \text{if } x_1 = x_2. \end{cases}$ This gate does not restrict the output when $x_1 = x_2$.

- Directed edges connecting variables to gates and gates to variables (loops are allowed);
- Variable nodes have in-degree 1; gates have 0, 1, or 2 inputs depending on type as above; gates & nodes have arbitrary fanout.

Definition 1.1 (Arithmetic Circuit SAT problem). Given an arithmetic circuit satisfying the description above, output an assignment of values $v_1, ..., v_n \in [0, 1]$ that satisfies all the gates.

Example 1.1. Consider the following diagram.



The only satisfying assignment is $a = b = c = 1/2$.

It is easy to see that this problem has the flavor of a Brouwer fixed point.

Theorem 1.1 ([DGP09]). The Arithmetic Circuit SAT problem always admits a solution, and it is PPAD-complete to find it.

2 From gates to games

It is possible to convert an Arithmetic Circuit SAT instance into a Nash equilibrium computation problem in a *multiplayer game*. (The game can also be converted into a two-player [CDT09] or three-player game [DGP09], but we do not show how in this lecture).

The idea is to use *gadgets*: constructions that simulate the behavior of the gates in the circuit.

2.1 Addition gate

Consider any game that contains the following interaction between four players x, y, z, w, each of which has two actions, denoted $\{0, 1\}$. With a slight abuse of notation, we will call x, y, z, w the probability of playing action 1; hence, $x, y, z, w \in [0, 1]$.

Example 2.1 (Addition gadget game). Consider any game that contains as a substructure the gadget shown in the diagram below, and payoffs set as follows.

▸ *Payoffs of player w.* The payoff of player w is defined as follows. If w plays 0, her payoff does not depend on z's strategy, but only on x and y, according to the payoff table

	$y = 0$	$y = 1$
$x = 0$	0	1
$x = 1$	1	2

If w plays 1, her payoff does not depend on x and y's strategy and depends on z's according to the table

	$z = 0$	$z = 1$
	0	1

▸ *Payoffs of player z.* The payoff of player z is defined according to the table

	$z = 0$	$z = 1$
$w = 0$	1/2	1
$w = 1$	1/2	0

▸ *Other payoffs and considerations.* Players x and y utilities are independent on the strategies of w and z. Player w does not affect other players in the game.

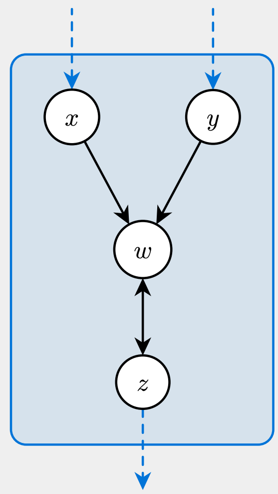


Figure 2: Addition gadget game. The dashed blue edges denote possible edges in the games, which do not affect the result in Theorem 2.1.

Theorem 2.1. In all Nash equilibria of the game, $z = \min\{x + y, 1\}$.

Proof. Suppose that $z < \min\{x + y, 1\}$. Then, $z < x + y$. But then w will deterministicaly play $w = 0$, which will force z to play $z = 1$. This is a contradiction, since by hypothesis $z < \min\{x + y, 1\}$, which implies $z < 1$.

Suppose now that $z > \min\{x + y, 1\}$. In this case, $\min\{x + y, 1\} \neq 1$, as otherwise this would imply $z > 1$ which is impossible. Thus, $z > x + y$. This implies $w = 1$ and hence $z = 0$, which is again impossible since $z > x + y$, which implies $z > 0$.

The only remaining possibility is therefore $z = \min\{x + y, 1\}$, as we wanted to show. □

Bibliography

[DGP09] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou, "The Complexity of Computing a Nash Equilibrium," *SIAM Journal on Computing*, vol. 39, no. 1, pp. 195–259, 2009, doi: 10.1137/070699652.

[CDT09] X. Chen, X. Deng, and S.-H. Teng, “Settling the complexity of computing two-player Nash equilibria,” *Journal of the ACM (JACM)*, vol. 56, no. 3, pp. 1–57, 2009.