6.S979 Lecture 7

Reminder: Pset 1 due 10/2

Last time:
- Bell-KS theorem
- "Contextuality"
- Magic square
  → 2-player game

Today: Generalizations of MS
⋮ a hidden variable theory
⋮

1) Linear system game:

a_1	a_2	a_3	$\cdot\ +1$
a_4	a_5	a_6	
a_7	a_8	a_9	
$+1$	$+1$	-1	

Linear equations over $\mathbb{Z}_2$

$a_1 + a_2 + a_3 = 0 \mod 2$
⋮

$a_1 + a_4 + a_7 = 0 \mod 2$

$\vdots$

$a_3 + a_6 + a_9 = 1 \mod 2$

row   col

Alice   Bob

$x_1 x_2 x_3$ $y_1 y_2 y_3$

LS game:

$$\left.\begin{array}{l} a_{i_1} + a_{i_2} = b_1 \\ \vdots \end{array}\right\} Ma = b \implies \text{2-player game}$$

equation $\vec{m}$ $j \in \vec{m}$ var in equation

Alice G_M Bob

assignment to all vars in eq. $\vec{m}$ assigned to a_j

$\forall i$ $\vec{m}_i \neq 0$, returns a_i

If $Ma = b$ is satisfiable, then $\omega(G_M) = 1$

If $Ma = b$ has a "quantum soln."

then $\omega^*(G_M) = 1$ ↺

<u>Def:</u> A quantum soln. is a collection of operators $\{A_1, \ldots, A_n\}$ s.t.

1) $A_i^2 = I \quad \forall i$

2) $\forall$ row i, $\forall j, k$ s.t. M_{ij} & M_{ik} are both nonzero

$$[A_j, A_k] = 0$$

(all vars in an equation should be simultaneously measurable)

3) $\forall$ row i, $\prod_j A_j^{M_{ij}} = (-1)^{b_i} I$

Can generalize to $\mathbb{Z}_d$ (mod d)

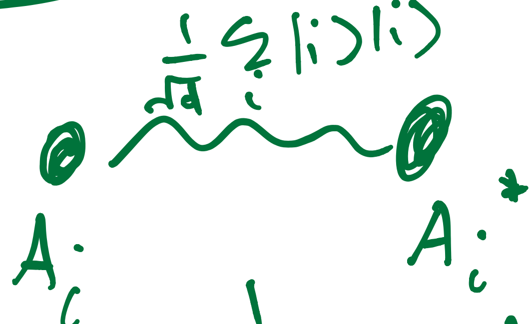
1) $A_i^d = I$, 3) $\prod_j A_j^{M_{ij}} = \omega^{b_i} I$

$\Rightarrow$ eigenvalues are d-th roots of unity

"generalized observables"

$\omega = e^{2\pi i/d}$

<u>Q. sol $\Rightarrow$ val. 1 Q. strat</u>



consistent w/ prob 1

$\{A_i\}$ q. sol. $\Rightarrow$ satisfy eq. w/ prob 1

val. 1 Q. strat $\Rightarrow$ Alice's observables are a q. sol

Def: Solution group

Generators: $\{a_1, \ldots, a_n, J\}$

1. $a_i^d = J^d = e \leftarrow$ identity
2. $a_i J = J a_i \quad \forall i$
3. $\forall\, i,j$ in same equation,
$a_i a_j = a_j a_i$
4. $J^{-b_i} \prod_j a_j^{M_{ij}} = e \quad \forall i$

A quantum sol is a unitary rep of sol. group where
$$J \longmapsto \omega \cdot I$$

Consequence:
If from sol group relations, can prove that $J = e$, then $\nexists$ quantum sol.

[Cleve, Liu, Slofstra]
If d prime and $J \neq e$
then $\exists$ q. sol.

MS mod d (d odd prime)

$$a_1 \ a_2 \ a_3 = 0 \mod d$$
$$a_4 \ a_5 \ a_6 = 0 \mod d$$
$$a_7 \ a_8 \ a_9 = 0 \mod d$$
$$\begin{array}{ccc} \| & \| & \| \\ 0 & 0 & I \mod d \end{array}$$

No classical solution

Sol. group relations

$\Rightarrow J^2 = e = J^d$

$\Rightarrow J = e$

$\Rightarrow$ No q. sol

Generalized Paulis

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$X^2 = Z^2 = I$

$$XZ = -ZX$$

$$X = \sum_i |i+1\rangle\langle i|$$

$$Z = \sum_i \omega^i |i\rangle\langle i|$$

$$X^d = Z^d = I$$

$$ZX = \sum_i \omega^{i+1} |i+1\rangle\langle i|$$

$$= \omega \left(\sum_i \omega^i |i+1\rangle\langle i| \right)$$

$$= \underline{\omega} \cdot XZ$$

$MS \longrightarrow$ "tests anticommutation"

$$A_1 A_5 = - A_5 A_1$$

$MS_d \rightarrow$ tests phase commutation

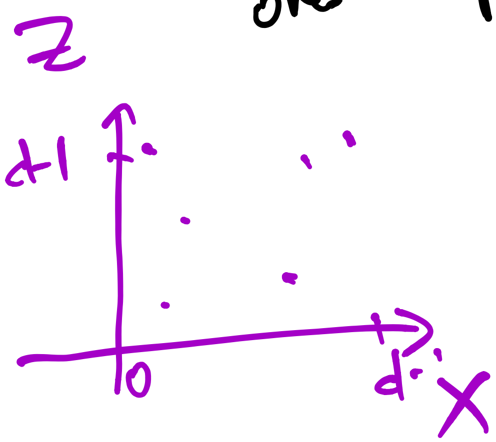
$$a_1 a_5 = J a_5 a_1$$

$$\Longrightarrow J^2 = e$$

There is a hidden variable model for mod d X & Z operators acting on a big class of states

"Quasiprobability distribution on phase space"

x & z values



$$W_\psi(x,z) \not\geq 0$$

$$\sum_z W_\psi(x,z) = \Pr_\psi[x]$$

$$\sum_x W_\psi(x,z) = \Pr_\psi[z]$$

Wigner function W

for states & measurements

$$W_\psi(x,z) = \frac{1}{d}\langle\psi|A(x,z)|\psi\rangle$$

$$W_{M_k}(x,z) = \operatorname{tr}\left(A(x,z)\,M_k\right)$$

$$\Pr[k] = \langle\psi|M_k|\psi\rangle$$

$$= \sum_{x,z} W_\psi(x,z)\cdot W_{M_k}(x,z)$$

$$= \mathop{\mathbb{E}}_{(x,z)\sim W_\psi} W_{M_k}(x,z)$$

hidden state (x,z) sampled from W_ψ

Fact: 1) For any mod d X or Z measurement

$$W_{M_k} \geq 0$$

2) $\exists$ states ψ for which $W_\psi \not\geq 0$

Thm: [Qassim & Wallman]

Can map any q. sol. built out of mod d X, Z, & products to a classical sol.

Pf: Fix the hidden state to $(x, z) = (0, 0)$

$$W_\psi(x, z) = \delta_{x=0, z=0}$$

$$W_{M_k}(x,z) = \mathrm{tr}\left(A(x,z)M_k\right)$$

$$A(0,0) = \sum_i |-i\rangle\langle i|$$

$$= \sum_i |i + 2^{-1}d\rangle\langle i|$$

for any observable $\underline{\omega^k Z^p X^q}$

Wigner function gives det. outcome $\nu(\omega^k Z^p X^q)$

$$= k + 2^{-1}pq$$

2^{-1} ↑ multiplicative inverse mod d

$\{A_1, \ldots, A_n\}$ q. sol.

$\Rightarrow \{\nu(A_1), \ldots, \nu(A_n)\}$ d. sol.

$$\nu(AB) = \nu(A)\nu(B)$$

if A & B commute

$$A = Z^p X^q, \quad B = Z^r X^s$$

$ZX = \omega XZ$ $\qquad [A, B] = 0$ if

$$ps - qr = 0$$

$$\nu(AB) = \nu(Z^p X^q Z^r X^s)$$

$$= \nu(\omega^{-rq} Z^{p+r} X^{q+s})$$

$$= -rq + 2^{-1} \cdot (p+r)(q+s)$$

$$= -\overset{2\cdot 2^{-1}}{1}\, rq + 2^{-1}(p \cdot q + r \cdot s)$$

$$+ 2^{-1}(p \cdot s + r \cdot q)$$

$$= \underbrace{2^{-1}(p \cdot q)} + 2^{-1}(r \cdot s)$$

$$+ 2^{-1}\underbrace{(p \cdot s - r \cdot q)}_{= 0}$$

$$= r(A) + r(B) + 0$$

Takeaway:

A HVM can describe a subtheory of QM with

— Non-compatible measurements

— Entanglement

"Stabilizer states & Clifford circuits"

Caveat:

- mod 2 vs mod d
- stabilizers & cliffords are computationally simulable mod 2, but no HVM

Linear systems games and self-testing

Thm [Coladangelo Stark]:

Suppose soln. group has a <u>unique</u> irrep σ where

$$J \overset{\sigma}{\longmapsto} \omega I$$

Then you have a robust self-test for σ

$$(V_A \otimes V_B)(A_i \otimes I |\psi\rangle)$$

$$\approx (\sigma(a_i) \otimes I |\hat{\psi}\rangle) |junk\rangle$$

"Self-test for measurements"