Robustness of Finite State Automata

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Abstract. The classical robust control deals with systems which can be approximated by finite order linear time-invariant (LTI) models, uses integral constraints, such as induced gain bounds, to assess robustness with respect to the error of such approximation, and employs H-Infinity optimization to design robust linear controllers. In this paper[1], a parallel approach is developed, in which finite state stochastic automata play the role of LTI models. Analogs of the Kalman-Yakubovich-Popov Lemma, the S-procedure losslessness theorem, and H-Infinity design are derived.

Introduction

Robustness analysis and optimization is a major source of efficient design tools for the modern control engineer. The classical robust control deals with systems which can be approximated by LTI models. The differnece between such approximations and the true system dynamics is described by integral constraints, such as induced gain bounds or Integral Quadratic Constraints [1]. Constructively verifiable conditions of stability and performance, such as the small gain theorem, are used to assess stability and performance of systems defined by nominal LTI dynamics and integral constraints. Ultimately, the task of robust LTI feedback design is reduced to induced gain minimization, such as H-Infinity optimization, which employs extensively quadratic Lyapunov functions.

While being the dominant tool for computer-aided design and analysis of systems modeled by near-linear differential equations, this framework fails to provide adequate treatment in the case of hybrid systems, i.e. systems which combine continuous and discrete state dynamics.

A major objective of the paper is creation of an alternative robust control framework in which finite state stochastic automata serve as a basic system model. Systems under consideration are represented as interconnections of "nominal" controlled finite state automata and the "uncertain feedback" systems described by integral constraints representing modeling error. Lyapunov functions are used for analysis and design.

The theorems presented in this paper are quite elementary, and can be viewed as simplified versions of the standard results of dynamic programming [2]. However, they highlight a potentially powerful framework for nonlinear

[1] to appear in the Proceedings of Mohammed Dahleh memorial symphosium. Email: ameg@mit.edu. This work was supported by NSF, AFOSR, and DARPA

feedback design. In this framework, one has to start with finding a *reduced model* of the original system.

1 System Models

In this section, basic principles of system modeling and design using finite state stochastic automata and integral constraints are introduced.

1.1 Finite Alphabet Feedback Design

This subsection contains motivation for using the uncertain finite state automata models as defined later in the paper.

Observer-Based Feedback. Our ultimate goal is to develop tools for optimizing the controller K in the feedback loop shown on Figure 1, where P

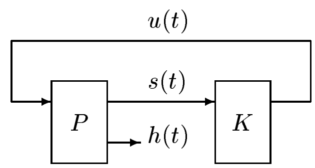

Fig. 1. General Feedback Design Setup

is a strictly causal discrete time system (possibly uncertain, infinite dimensional, and randomized) with control input $u(t) \in U$, sensor output $s(t) \in S$, and cost output $h(t) \in \mathbf{R}$. We consider the case of discrete decision-making, which means that the set U (the control alphabet) is finite. The objective is to design a causal system K (the *feedback controller*) with input $s(t)$ and output $u(t)$ such that $h(t)$ is non-negative "on average" on the trajectories of the closed loop system, which is expressed by the inequality

$$\inf_{T>0} \sum_{t=0}^{T} \mathbf{E}h(t) > -\infty. \tag{1}$$

The general task of designing and optimizing K is very difficult. However, an important simplifying assumption will be made throughout the paper, that K must be found in the observer-based form shown on Figure 2, where D is the one step delay block, $m(t) \in M$ is the observer state, $x(t) \in X$ is the observer output (the set X is *finite*), function $E: \ U \times S \times M \to X \times M$ defines the observer dynamics, and function $g : X \times \Xi \to U$ defines the control decision randomized by an independent random number generator $\theta(t) \in \Theta$. The challenging task of designing the observer function E is not discussed

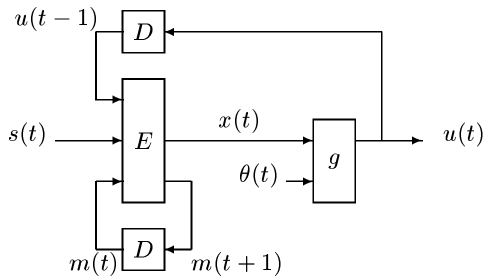

Fig. 2. Observer-Based Feedback

systematically in this paper. It is assumed that some preliminary effort (for example, a model reduction and quantization algorithm) has already produced E. Our objective is to develope algorithms for design and analysis of the randomized memoryless feedback part of the observer-based controller represented by function g and by the probability distribution p_θ of θ.

Uncertain Finite State Automata Models. For the purpose of designing the randomized memoryless feedback defined by g and p_θ, the observer based feedback system can be described by the diagram on Figure 3, where H

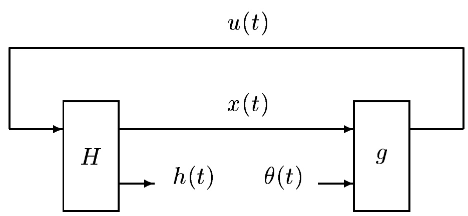

Fig. 3. Closed Loop System

represents the combination of plant and observer. While the set of possible values of $v(t)$ and $x(t)$ at given time is finite, the exact dynamics of H can be extremely complicated. We propose the use of *uncertain stochastic finite state automata* (FSA) with *integral constraints* (IC) as a tool for simplified representation (abstraction) of H, as shown on Figure 4, i.e.

$$x(t+1) = f(x(t), u(t), w(t), \xi(t)), \tag{2}$$

where $w(t) \in W$ is the *uncertain input*, representing possible mismatch between H and the FSA model, $\xi(t) \in \Xi$ is the output of an independent random number generator with a given probability distribution p_ξ, $f: X \times U \times W \times \Xi \to X$ is the function defining the FSA. The behav-

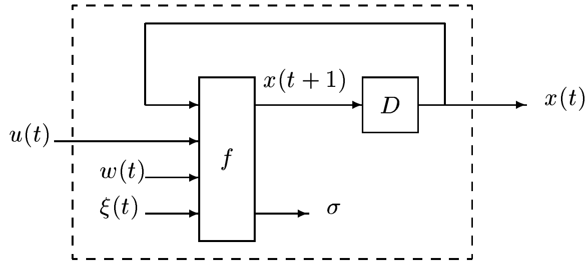

Fig. 4. Uncertain FSA Model

ior of the uncertain input $w(t)$ is constrained by a set of *integral constraints*

$$\inf_{T>0} \sum_{t=0}^{T} \mathbf{E}\sigma(x(t), u(t), w(t)) > -\infty, \tag{3}$$

which are assumed to hold for *all* functions $\sigma : X \times U \times W \to \mathbf{R}$ from a given convex compact set $\tilde{\sigma} = \{\sigma\}$. In addition, a given function $\sigma_0 : X \times U \times W \to \mathbf{R}$ is assumed to provide a lower bound for the averages of $h(t)$, in the sense that the performance inequality in (1) holds for every set of input/output signals $(u(t), x(t), h(t))$ produced by H, as long as there exist $w(t), \xi(t)$ satisfying (2),(3), and

$$\inf_{T>0} \sum_{t=0}^{T} \mathbf{E}\sigma_0(x(t), u(t), w(t)) > -\infty. \tag{4}$$

Under these assumption, the design problem under consideration can be formulated as that of finding a function g and a random number generator distribution p_θ such that the inequality (4) is satisfied for all solutions of (2) satisfying $u(t) = g(x(t), \theta(t))$ and (3).

1.2 FSA Models

In this paper, finite state stochastic automata (FSA) are used to define *nominal*, i.e. precisely known, system models. Thus, they play the role of finite order LTI models of the classical robust control.

Random Variables. The set of all random variables ξ with values in a given *finite* set Ξ (without loss of generality, ξ can be viewed as a measureable function $\xi : \Omega \mapsto \Xi$, where $\Omega = [0,1]$ is the "set of elementary events") is denoted by $\mathcal{R}(\Xi)$. $\mathcal{D}(\Xi)$ is the set of all *probability distributions* on Ξ, i.e. functions $p : \Xi \to [0,1]$ such that

$$\sum_{\bar{\xi} \in \Xi} p(\bar{\xi}) = 1.$$

The function $\pi = \pi_{\Xi}$ maps every random variable $\xi \in \mathcal{R}(\Xi)$ into its *distribution* $p_\xi = \pi(\xi) \in \mathcal{D}(\Xi)$. In other words, $p_\xi = \pi(\xi)$ means that $p_\xi(\bar{\xi}) = \mathbf{P}(\xi = \bar{\xi})$ is the probability that the random variable ξ takes the value $\bar{\xi}$. In particular, if $h: Z \to \mathbf{R}$ is a given function, and $z \in \mathcal{R}(Z)$, then the *expected value* $\mathbf{E}h(z)$ is given by

$$\mathbf{E}h(z) = \sum_{\bar{z} \in Z} h(\bar{z}) p_z(\bar{z}).$$

For two random variables $x \in \mathcal{R}(X)$ and $y \in \mathcal{R}(Y)$, their *direct product* $\xi = (x, y)$ is the random variable from $\mathcal{R}(X \times Y)$ defined by $\xi(\tau) = (x(\tau), y(\tau))$ for all $\tau \in \Omega = [0,1]$. x and y are called *independent* if

$$p_\xi(\bar{x}, \bar{y}) = p_x(\bar{x}) p_y(\bar{y}) \;\; \forall \, \bar{x} \in X, \; \bar{y} \in Y,$$

where p_ξ, p_x, p_y are the distributions of ξ, x, y respectively. We will write $x \perp y$ when x and y are independent.

Finite State Automata. Let X, V, Z be three finite sets. A function $f: X \times V \times Z \mapsto X$ and a probability distribution $p_z \in \mathcal{D}(Z)$ define a *finite state automata* $\mathcal{A} = \mathcal{A}(f, p_z)$ as a relation between sequences of random variables $x(t) \in \mathcal{R}(X)$ and $v(t) \in \mathcal{R}(V)$ expressed by

$$x(t+1) = f(x(t), v(t), z(t)), \quad z(t) \perp (x(t), v(t)), \quad \pi_Z(z(t)) = p_z. \qquad (5)$$

Here $v(t)$ is the input of a discrete time dynamical system with state $x(t)$ and independent identically distributed random number generator $z(t)$.

An equivalent expression of the FSA relation (5) is given by

$$p_x^{t+1}(\tilde{x}) = \sum_{\tilde{x} = f(\bar{x}, \bar{v}, \bar{z})} p_{(x,v)}^t(\bar{x}, \bar{v}) p_z(\bar{z}),$$

where $p_x^{t+1}, p_{(x,v)}^t$ are the distributions of $x(t+1)$ and $(x(t), v(t))$ respectively.

In this paper, FSA are used as simplified models of systems.

Memoryless Automata. Let U, X, Θ be three finite sets. A function $g: X \times \Theta \to U$ and a probability distribution $p_\theta \in \mathcal{D}(\Theta)$ define a *memoryless automata* $\mathcal{M} = \mathcal{M}(g, p_\theta)$ as a relation between sequences of random variables $x(t) \in \mathcal{R}(X)$ and $u(t) \in \mathcal{R}(U)$ expressed by

$$u(t) = g(x(t), \theta(t)), \quad \theta(t) \perp x(t), \quad \pi_\Theta(\theta(t)) = p_\theta. \qquad (6)$$

Here $x(t)$ is the input of a memoryless system with output $u(t)$ and an independent identically distributed random number generator $\theta(t)$. An equivalent way of defining the memoryless automata is by specifying a function

$p_{u|x}: X \to \mathcal{D}(U)$, which defines the conditional distribution of $u(t)$ for every given value $x(t) = \bar{x}$. The relation between g, p_θ, and $p_{u|x}$ is given by

$$p_{u|x}(\bar{x}, \bar{u}) = \sum_{\bar{\theta}:\ g(\bar{x},\bar{\theta})=\bar{u}} p_\theta(\bar{\theta}).$$

An equivalent expression of (6) is given by

$$p_u^t(\bar{u}) = \sum_{\bar{x},\bar{\theta}:\ \bar{u}=g(\bar{x},\bar{\theta})} p_x^t(\bar{x}) p_\theta(\bar{\theta}) = \sum_{\bar{x}\in X} p_{u|x}(\bar{x}, \bar{u}) p_x^t(\bar{x}),$$

where p_u^t, p_x^t are the distributions of $u(t)$ and $x(t)$ respectively.

In this paper, memoryless automata are used as feedback laws.

1.3 Integral Constraints

Most systems of practical interest cannot be represented *exactly* as finite state automata. However, they can frequently be *approximated* to a reasonable degree of accuracy by FSA, just as many mildly nonlinear systems can be approximated by LTI models. Integral constraints (IC) serve as indicators of accuracy of such approximations, playing the role of L2 gain bounds or Integral Quadratic Constraints in the classical robust control. In addition, IC can express the objectives for feedback design, in the same way as L2 gain bounds serve as design objectives of the standard H-Infinity optimization.

Definition of Integral Constraints. A sequence $\{y(t)\}$ of real-valued integrable random variables $y(t)$ such that

$$\inf_{T>0} \sum_{t=0}^{T} \mathbf{E} y(t) > -\infty$$

is said to *satisfy the integral constraint* $\mathcal{I}[y(t)] \geq 0$.

Let X, V be two finite sets, and let $\sigma: X \times V \to \mathbf{R}$ be a function. Two sequences of random variables $x(t) \in \mathcal{R}(X)$ and $v(t) \in \mathcal{R}(V)$ are said to *satisfy the integral constraint (IC) defined by* σ if

$$\inf_{T>0} \sum_{t=0}^{T} \mathbf{E} \sigma(x(t), v(t)) > -\infty, \tag{7}$$

in which case we will write $\mathcal{I}[\sigma(x(t), v(t))] \geq 0$.

An equivalent way to express the IC in (7) is given by

$$\inf_{T>0} \sum_{t=0}^{T} \sum_{\bar{x},\bar{v}} \sigma(\bar{x}, \bar{v}) p_{(x,v)}^t > -\infty,$$

where $p_{(x,v)}^t$ is the distribution of $(x(t), v(t))$.

In this paper, integral constraints are used for defining performance criteria and for constraining undermodeled behavior of uncertain models.

Integral Constraints as Performance Specifications. In this case the IC in (7) is a hypothesis to be verified by a system analysis procedure, or a design criterion to be satisfied by a design choice.

For example, assume that certain elements $x_0 \in X, v_0 \in V$ are designated as *zero* values. If the input $v = v(t)$ of FSA (5) represents a *control decision*, the informal performance criterion may require that $x(t)$, $v(t)$ take zero values $x(t) = x_0$, $v(t) = v_0$ with probability 1 as $t \to \infty$. If σ is defined in such way that $\sigma(x_0, v_0) = 0$ and $\sigma(\bar{x}, \bar{v}) < 0$ for $(\bar{x}, \bar{v}) \neq (x_0, v_0)$, then (6) implies that $\mathbf{P}(x(t) = x_0, v(t) = v_0) \to 1$ as $t \to \infty$. In this case (6) plays a role similar to that of a quadratic performance criterion in the classical linear-quadratic optimization.

Another example of a performance specified by an IC is as follows. Assume the input v models a disturbing noise, and one has to verify that $x(t)$ is not very sensitive to $v(t)$, which means that, on average, $x(t)$ will take a "non-zero" value $x(t) \neq x_0$ with a frequency not exceeding γ times the frequency of $v(t)$ taking non-zero values $v(t) \neq v_0$, where $\gamma > 0$ is a given number quantifying the degree of sensitivity. In this case, σ can be defined by

$$\sigma(\bar{x}, \bar{v}) = \begin{cases} -1, & \bar{x} \neq x_0, \ \bar{v} = v_0, \\ \gamma, & \bar{x} = x_0, \ \bar{v} \neq v_0, \\ \gamma - 1, & \bar{x} \neq x_0, \ \bar{v} \neq v_0, \\ 0, & \bar{x} = x_0, \ \bar{v} = v_0. \end{cases} \tag{8}$$

and then (7) plays a role similar to that of a L2 gain bound in H-Infinity optimization.

Integral Constraints as Uncertainty Bounds. In this case the IC in (7) is a constraint limiting the behavior of uncertain signals within the system.

For example, $v(t)$ may represent the output of an undermodeled subsystem Δ with input $x(t)$. Assume that Δ satisfies a "low sensitivity" condition which means that on average its output $v(t)$ takes non-zero values $v(t) \neq v_0$ with a frequency smaller than $1/\gamma$ times the frequency at which its input $x(t)$ takes non-zero values $x(t) \neq x_0$. This condition can be represented by the IC $\mathcal{I}[-\sigma(x, v)] \geq 0$, where σ is the function defined in (8).

2 Analysis and Design of FSA/IC Models

This section presents general results on analysis and design of systems defined as finite state automata with integral constraints.

2.1 Analysis of FSA/IC Models

Analogs of the Kalman-Yakubovich-Popov Lemma and the S-procedure losslessness theorem will be formulated and proven here for FSA/IC models.

First, existence of a *storage function* is shown to be a necessary and sufficient condition for an integral constraint to be satisfied for *all* trajectories of a given finite state automata model. Second, it is shown that a finite set of IC $\mathcal{I}[\sigma_k(x,v)] \geq 0$, $k = 1, 2, ..., n$, imposed on the set of all trajectories of a given FSA implies IC $\mathcal{I}[\sigma_0(x,v)] \geq 0$ if and only if there exist nonnegative coefficients $c_k \geq 0$ such that, for

$$\sigma(\bar{x}, \bar{v}) = \sigma_0(\bar{x}, \bar{v}) - \sum_{k=1}^{n} c_k \sigma_k(\bar{x}, \bar{v}),$$

the IQC $\mathcal{I}[\sigma(x,v)] \geq 0$ is satisfied for *all* trajectories of the FSA.

Integral Constraints and Storage Functions. The following statement can be used to check whether a given integral constraint is satisfied for *all* trajectories of a given finite state automata.

Theorem 1. *Let X, V, Z be finite sets, $p_z \in \mathcal{D}(Z)$. Let $f: X \times V \times Z \to X$ and $\sigma: X \times V \to \mathbf{R}$ be two functions. Then the following two conditions are equivalent.*

(a) The IC $\mathcal{I}[\sigma(x,v)] \geq 0$ from (7) holds for all sequences of random variables $x(t), v(t), z(t)$ satisfying (5).
(b) There exists a function $H: X \to \mathbf{R}$ such that

$$\sigma(\bar{x}, \bar{v}) + H(\bar{x}) \geq \sum_{\bar{z} \in Z} H(f(\bar{x}, \bar{v}, \bar{z})) p_z(\bar{z}) \quad \forall \quad \bar{x} \in X, \ \bar{v} \in V. \tag{9}$$

Essentially, (9) means that

$$\mathbf{E}\sigma(x(t), v(t)) \geq \mathbf{E}H(x(t+1)) - \mathbf{E}H(x(t)) \tag{10}$$

for all solutions of the FSA equation (7), i.e., using the terminology by J.C.Willems, that the function $H = H(x)$ can serve as a *storage function* for (7) with *supply rate* $\sigma = \sigma(x,v)$.

Proof of Theorem 1. Taking into account (10), the implication (b)⇒(a) is straightforward: taking a sum of such inequalities with $t = 0, 1, \ldots, T$ yields

$$\sum_{t=0}^{T} \mathbf{E}\sigma(x(t), v(t)) \geq \mathbf{E}H(x(T+1)) - \mathbf{E}H(x(0)).$$

Since X is finite, $\mathbf{E}H(x(T+1))$ is uniformly bounded, and hence

$$\inf_{T \geq 0} \sum_{t=0}^{T} \mathbf{E}\sigma(x(t), v(t)) > -\infty.$$

To prove that (a) implies (b), consider the case when there exists no function $H: X \to \mathbf{R}$ satisfying (9). This means that the minimum of the convex function

$$f(H) = \max_{\bar{x}\in X, \bar{v}\in V} \left\{ -H(\bar{x}) - \sigma(\bar{x},\bar{v}) + \sum_{\bar{z}\in Z} H(f(\bar{x},\bar{v},\bar{z}))p_z(\bar{z}) \right\} \tag{11}$$

over the vector space $\mathcal{H} = \{H\}$ of all functions $H: X \to \mathbf{R}$ is positive. In terms of the dual linear program, this means existence of a probability distriution $r \in \mathcal{D}(X \times V)$ such that

$$\sum_{\bar{x}\in X, \bar{v}\in V} r(\bar{x},\bar{v})\sigma(\bar{x},\bar{v}) = c < 0, \tag{12}$$

$$\sum_{\bar{v}\in V} r(\tilde{x},\bar{w}) = \sum_{\bar{x}\in X, \bar{v}\in V, \bar{z}\in Z:\ f(\bar{x},\bar{v},\bar{z})=\tilde{x}} r(\bar{x},\bar{v})p_z(\bar{z}) \ \ \forall\, \tilde{x} \in X. \tag{13}$$

Let

$$\bar{r}(\bar{x}) = \sum_{\bar{w}\in W} r(\bar{x},\bar{w}).$$

We will use the following (obvious) observation.

Lemma 1. *For any random variable $\hat{x} \in \mathcal{R}(X)$ such that $\pi_X(\hat{x}) = \bar{r}$ there exists a random variable $\hat{v} \in \mathcal{R}(V)$ such that $\pi_{X\times V}((\hat{x},\hat{v})) = r$.*

Lemma 1 can be used to construct sequences of random variables $x = x(t)$, $v = v(t)$ satisfying (5) and such that

$$\mathbf{E}\bar{\sigma}(x(t),v(t)) = c < 0 \ \ \forall\, t = 0,1,2,\ldots, \tag{14}$$

which contradicts assumption (a). Indeed, let $x(0)$ be a random variable with values on X and probability distribution $\bar{r}$. Using Lemma 1 with $\hat{x} = x(0)$, define $v(0)$ by $v(0) = \hat{v}$. Then, by (12), the inequality in (14) holds, and, by (13), the probability distribution of $x(1)$ is the same as that of $x(0)$. Now the process can be repeated by applying Lemma 1 with $\hat{x} = x(t)$ and using $v(t) = \hat{v}$ for $t = 1,2,\ldots$.

Calculation of Storage Functions Formally speaking, checking validity of a given IC on the trajectories of a given FSA using Theorem 1 amounts to solving a linear program with respect to H. However, the following observations may help in reducing complexity of the analysis.

Theorem 2. *Let $M > 0$ be a given number. The following conditions are equivalent:*

(a) inequality (9) has a solution H with

$$\max_{\bar{x}\in X} H(\bar{x}) - \min_{\bar{x}\in X} H(\bar{x}) \le M;$$

(b) for all $n = 1, 2, \ldots$ the functions $H_n : X \to \mathbf{R}$ defined by

$$H_{n+1} = \alpha(H_n), \quad H_0 = 0,$$

where $\bar{H} = \alpha(H)$ is given by

$$\bar{H}(\bar{x}) = \max\left\{0, \max_{\bar{v}}\left\{-\sigma(\bar{x}, \bar{v}) + \sum_{\bar{z}\in Z} H(f(\bar{x}, \bar{v}, \bar{z}))p_z(\bar{z})\right\}\right\},$$

satisfy the conditions

$$0 = \min_{\bar{x}\in X} H_n(\bar{x}), \quad \max_{\bar{x}\in X} H_n(\bar{x}) \leq M.$$

Moreover, if conditions (a),(b) are satisfied then one such H is given by

$$H(\bar{x}) = H_-(\bar{x}) = \lim_{n\to\infty} H_n(\bar{x}),$$

and $H(\bar{x}) \geq H_-(\bar{x})$ for any other non-negative solution of (9).

Proof.
(a)⇒(b) If $H : X \to \mathbf{R}$ is a solution of (9) then for every constant $c \in \mathbf{R}$ the function $H_c(\bar{x}) = H(\bar{x}) + c$ is also a solution. Therefore, it is sufficient to look only for those solutions of (9) which satisfy

$$\min_{\bar{x}\in X} H(\bar{x}) = 0. \tag{15}$$

Note that α is a *monotonically non-decreasing* transformation, in the sense that

$$\alpha_+(H_1)(\bar{x}) \geq \alpha_+(H_2)(\bar{x}) \ \forall\ \bar{x} \in X \text{ whenever } H_1(\bar{x}) \geq H_2(\bar{x}) \ \forall\ \bar{x} \in X.$$

Hence $H_* \geq 0 = H_0$ implies $H_* = \alpha(H_*) \geq \alpha(H_0) = H_1$, and, further by induction, $M \geq H_* \geq H_n$.
(a)⇐(b) Since $H_1 \geq 0 = H_0$ and α is monotonically non-decreasing, we have $H_{n+1} \geq H_n$ for all n. Hence for every $\bar{x} \in X$ the sequence $H_n(\bar{x})$ is monotonically non-decreasing and bounded, and thus converges to a limit H_- such that $H_- = \alpha(H_-)$ and $0 \leq H_-(\bar{x}) \leq M$ for all $\bar{x} \in X$. Therefore $H = H_-$ satisfies (9) as well.

S-Procedure Losslessness. The following statement can be used to check whether a given integral constraint is satisfied for all of those trajectories of a given finite state automata which satisfy integral constraints from a given set.

Theorem 3. *Let X, V, Z be finite sets, $p_z \in \mathcal{D}(Z)$. Let $f : X \times V \times Z \to X$, $\sigma_0 : X \times V \to \mathbf{R}$ be two functions. Let $\tilde{\sigma} = \{\bar{\sigma}\}$ be a convex compact set of functions $\bar{\sigma} : X \times V \to \mathbf{R}$ such that*

(*) for each $\bar\sigma \in \tilde\sigma$ the IQC $\mathcal{I}[-\bar\sigma(x(t),v(t))] \geq 0$ does not hold for at least one random sequence $(x(t),v(t))$ satisfying (5).

Then the following two conditions are equivalent.

(a) The IC $\mathcal{I}[\sigma_0(x,v)] \geq 0$ holds for all sequences of random variables $x(t)$, $v(t)$, $z(t)$ satisfying (5) and *every IQC $\mathcal{I}[\bar\sigma(x(t),v(t))] \geq 0$ with $\bar\sigma \in \tilde\sigma$.*
(b) There exists a function $H:\ X \to \mathbf{R}$, $\mu \geq 0$ and $\bar\sigma \in \tilde\sigma$ such that

$$\sigma_0(\bar x,\bar v) - \mu\bar\sigma(\bar x,\bar v) + H(\bar x) \geq \sum_{\bar z \in Z} H(f(\bar x,\bar v,\bar z))p_z(\bar z) \quad \forall \ \ \bar x \in X,\ \ \bar v \in V. \tag{16}$$

Proof. The proof of Theorem 3 follows the lines of the proof of Theorem 1, with some minor modifications. The implication (b)⇒(a) is obvious. To prove that (a) implies (b), assume that (b) is false. Then, by assumption (*), (9) does not have a solution H,σ, where $\sigma = \tau\sigma_0 - (1-\tau)\sigma$ with $\tau \in [0,1]$ ranges over the convex hull $\hat\sigma$ of σ_0 and $-\tilde\sigma$. In other words, the maximum $f = f(H,\sigma)$ in (11), which is now a convex function on $\mathcal{H}\times\hat s$, has a strictly positive minimum. Applying standard duality yields existence of $r \in \mathcal{D}(X \times V)$ such that (13) holds, and (12) holds for *all* $\sigma \in \hat\sigma$. Hence the sequence $(x(t),v(t))$ constructed as in the proof of Theorem 1, will satisfy the conditions

$$\mathbf{E}\sigma(x(t),v(t)) = c < 0, \quad \mathbf{E}\sigma(x(t),v(t)) = -c > 0,$$

which contradicts (a).

2.2 Feedback Design for FSA Models

This subsection is devoted to the problem of designing randomized full state feedback for uncertain FSA.

Randomized Feedback in Uncertain FSA. Finite sets X,U,W,Ξ, a probability distribution $p_\xi \in \mathcal{D}(\Xi)$, and a function $f:\ X\times U\times W\times\Xi \to X$ define a FSA $\mathcal{A} = \mathcal{A}(f,p_\xi)$ in which the *input* variable $v(t) = (u(t),w(t)) \in \mathcal{R}(U\times W)$ is partitioned into *control* $u(t) \in \mathcal{R}(U)$ and *disturbance* $w(t) \in \mathcal{R}(W)$. A *randomized feedback* for the FSA is defined by a memoryless automata $u(t) = g(x(t),\theta(t))$ with input $x(t)$ and random number generator $\theta(t)$ with a fixed probability distribution $p_\theta \in \mathcal{D}(\Theta)$. Here $\theta(t) \perp (x(t))$. An alternative way to define a randomized memoryless feedback is by specifying the conditional distribution $p_{u|x}:\ X \to \mathcal{D}(U)$, where

$$p_{u|x}(\bar x,\bar u) = \sum_{\bar\theta:\ g(\bar x,\bar\theta)=\bar u} p_\theta(\bar\theta).$$

By the meaning of the FSA model as an approximation of a complex dynamical system with input $u(t)$, it is not reasonable to assume independence of $w(t)$ and $\theta(t)$. Therefore, to represent the resulting feedback system

$\mathcal{F} = \mathcal{F}(f, p_\xi, g, p_\theta)$ in the form (5), let us define $x_c(t) = (x(t), u(t)) \in X \times U$ as the *state* of $\mathcal{F}$, $z(t) = (\xi(t), \theta(t+1))$ as its random number generator with distribution $p_z(\bar{\xi}, \bar{\theta}) = p_\xi(\bar{\xi})p_\theta(\bar{t})$, $w(t)$ as the disturbance input, and

$$f_{cl}\left(\begin{bmatrix}\bar{x}\\ \bar{u}\end{bmatrix}, \bar{w}, \begin{bmatrix}\bar{\xi}\\ \bar{\theta}\end{bmatrix}\right) = \begin{bmatrix} f(\bar{x}, \bar{u}, \bar{w}, \bar{\xi}) \\ g(f(\bar{x}, \bar{u}, \bar{w}, \bar{\xi}), \bar{\theta})\end{bmatrix}.$$

A typical objective of feedback design is to satisfy an integral constraint $\mathcal{I}[\sigma(x(t), u(t), w(t))] \geq 0$, where σ is either a given function (when w plays the role of *external disturbance*, and optimization of *nominal performance* is the goal) or can be selected from a given convex set $\hat{\sigma} = \{\sigma\}$ of functions (in the case when some components of w model dynamical uncertainty, and hence *robust performance* is to be optimized).

Design Feasibility of Integral Constraints. The following result gives necessary and sufficient conditions of feasibility in a feedback optimization problem for FSA with a single integral constraint defining the design objective.

Theorem 4. *Let finite sets X, U, W, Ξ, a distribution $p_\xi \in \mathcal{D}(\Xi)$, and functions $f: X \times U \times W \times \Xi \to X$ and $\sigma: X \times U \times W \to \mathbf{R}$ be given. The following conditions are equivalent.*

(a) There exists a randomized feedback $u(t) = g(x(t), \theta(t))$ such that the integral constraint $\mathcal{I}[\sigma(x(t), u(t), w(t))] \geq 0$ holds for all solutions of the closed loop system $\mathcal{F} = \mathcal{F}(f, p_\xi, g, p_\theta)$.
(b) There exists a function $H: X \to \mathbf{R}$ such that

$$H(\bar{x}) \geq \min_{p \in \mathcal{D}(U)} \max_{\bar{w} \in W} \sum_{\bar{u} \in U} \left\{ -\sigma(\bar{x}, \bar{u}, \bar{w}) + \sum_{\bar{\xi} \in \Xi} H(f(\bar{x}, \bar{u}, \bar{w}, \bar{\xi})) p_\xi(\bar{\xi}) \right\} p(\bar{u}) \tag{17}$$

for all $\bar{x} \in X$.

Note that the optimal distributions $p \in \mathcal{D}(U)$ in (17), one for each $\bar{x} \in X$, define the conditional distribution $p_{(u|x)}: X \to \mathcal{D}(U)$ of the desired randomized feedback.

Practically, the search for the *control storage function* H in (17) is frequently reduced to the *value iteration* procedure $H_{n+1} = \beta(H_n)$, $H_0 = 0$, where the function $\beta: \mathcal{H} \to \mathcal{H}$ is defined by $\beta(H)(\bar{x}) =$

$$\max\left\{0, \min_{p \in \mathcal{D}(U)} \max_{\bar{w} \in W} \sum_{\bar{u} \in U} \left\{ -\sigma(\bar{x}, \bar{u}, \bar{w}) + \sum_{\bar{\xi} \in \Xi} H(f(\bar{x}, \bar{u}, \bar{w}, \bar{\xi})) p_\xi(\bar{\xi}) \right\} p(\bar{u}) \right\}.$$

The iterative techniques suggested earlier for FSA storage function analysis extend naturally to the design case.

Proof of Theorem 4. The implication (b)$\Rightarrow$(a) in Theorem 4 is straightforward, since the optimal distributions $p \in \mathcal{D}(U)$ in (17) define the conditional distribution $p_(u|x): X \to \mathcal{D}(U)$ of a desired randomized feedback such that

$$\mathbf{E}\sigma(x(t), u(t), w(t)) \geq \mathbf{E}H(x(t+1)) - \mathbf{E}H(x(t))$$

for all random variables $w(t)$.

To prove that (a) implies (b), note first that, according to Theorem 1, the IC $\mathcal{I}[\sigma(x(t), u(t), w(t))] \geq 0$ is satisfied for the closed loop system if and only if there exists a function $\hat{H}: X \times U \to \mathbf{R}$ such that

$$\hat{H}(\bar{x}, \bar{u}) \geq -\sigma(\bar{x}, \bar{u}, \bar{w}) + \sum_{\hat{u} \in U, \xi \in \Xi} \hat{H}(f(\bar{x}, \bar{u}, \bar{w}, \bar{\xi}), \hat{u}) p_{u|x}(f(\bar{x}, \bar{u}, \bar{w}, \bar{z}), \hat{u}) p_z(\bar{z})$$

for all $\bar{x} \in X$, $\bar{u} \in U$, $\bar{w} \in W$. For a fixed $\bar{w} \in W$, multiplying these inequalities by $p_{u|x}(\bar{x}, \bar{u})$ and summing up over all $\bar{u}$ yields (17) for

$$H(\bar{x}) = \sum_{\bar{u} \in U} \hat{H}(\bar{x}, \bar{u}) p_{u|x}(\bar{x}, \bar{u}).$$

Example: Single Bit Memory Stabilization of Double Integrator. It is known that memoryless output feedback $u_c(\tau) = K(y_c(\tau))$ is not capable of stabilizing the double integrator system

$$\ddot{y}_c(\tau) = u_c(\tau). \tag{18}$$

However, the stabilization can be accomplished with a single bit of memory. The problem of finite memory stabilization can be reduced to robust FSA feedback design in the following way.

Consider the sampled data feedback control law

$$u_c(\tau) = -\omega(u(t))^2 y_c(\tau) \quad \text{for} \quad tT \leq \tau \leq (t+1)T, \tag{19}$$

where $T \in (0, \pi/4)$ is a fixed sampling rate, $t = 0, 1, 2, \ldots$ is the discrete time, $w: \{0,1\} \to \mathbf{R}$ is a given function,

$$\omega(0) = 1, \quad \omega(1) = \omega_1 \in \left(1, \frac{\pi}{4T}\right),$$

and $u(t) \in \{0,1\}$ is the output of a FSA (to be designed) with input

$$s(t) = \text{sign}(y_c(tT)).$$

The feedback design objective is to maximize the stabilization rate, hence $h(t)$ can be defined as the amount by which the logarithm of the state state vector length decreases over a sampling interval $[tT, tT+T]$:

$$h(t) = 0.5 \log\left(\frac{|y_c(tT)|^2 + |\dot{y}_c(tT)|^2}{|y_c(tT+T)|^2 + |\dot{y}_c(tT+T)|^2}\right) - \gamma,$$

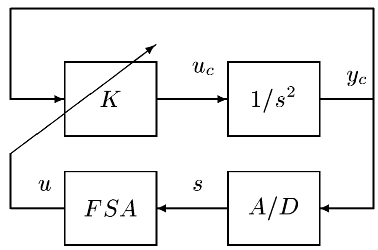

Fig. 5. Finite Memory Feedback for Double Integrator

where $\gamma > 0$ is the parameter to be maximized. The setup is shown on Figure 5, where the A/D block represents the sign sampler.

Let us use the observer with a single bit state $m(t) \in M = \{-1, 1\}$ and output $x(t) \in X = \{0, 1\}$ defined by

$$m(t+1) = s(t), \quad x(t) = 0.25(m(t) - s(t))^2.$$

Thus $x(t)$ is the indicator that the sign of y_c has changed over the last sampling interval.

In terms of this paper, the transformation of the quantized control input $u(t) \in \{0, 1\}$ into the observer output $x(t) \in \{0, 1\}$ defines a "complex" open loop model H. A simplified model of H (its *abstraction*) can be defined by the equations

$$x(t+1) = w(t).$$

To make this simplified model useful for design and analysis, one has to put integral constraints on the uncertain input variable $w(t)$, and to provide a lower bound for the average value of $h(t)$ in terms of $x(t), u(t), w(t)$.

Let

$$M = \begin{bmatrix} \cos(\omega_1 T) & \omega_1^{-1}\sin(\omega_1 T) \\ -\omega_1 \sin(\omega_1 T) & \cos(\omega_1 T) \end{bmatrix}, \quad e_q = \begin{bmatrix} \sin(q) \\ \cos(q) \end{bmatrix},$$

$$\rho_1 = -\log\frac{\|M^2 e_0\|}{\|M e_0\|}, \quad \rho_2 = \log\|M\|, \quad \phi_0 = \arccos \max_{q \in [0,\pi]} \frac{|e_q' M e_q|}{\|M e_q\|},$$

where $\|L\|$ denotes the largest singular value of matrix L. Here M is the matrix of the linear transformation $x_c(tT+T) = M x_c(tT)$ of the analog state

$$x_c(\tau) = \begin{bmatrix} y_c(\tau) \\ \dot{y}_c(\tau) \end{bmatrix}$$

of the system over a single time sampling interval $\tau \in [tT, tT+T]$ when $u(t) = 1$; $-\rho_1$ is the maximal possible increment of $\log\|x_c(\tau)\|$ over a single time sampling interval when $u(t) = x(t) = 1$; ρ_2 is the maximal possible increment of $\log\|x_c(\tau)\|$ over a single time sampling interval when $u(t) = 1$ (note that

$\|x_c(\tau)\|$ does not change when $u(t) = 0$); ϕ_0 is the minimal increment in the phase of $x_c(\tau)$ over a single time sampling interval when $u(t) = 1$ (the increment equals T when $u(t) = 0$).

By construction, $h(t) \geq \rho_1 - \gamma$ when $x(t) = u(t) = 1$. On the other hand, $h(t) = 0$ when $u(t) = 0$, and the inequality $h(t) \geq -\rho_2 - \gamma$ always holds. Therefore a lower bound for $h(t)$ can be derived in terms of $x(t), u(t), w(t)$ according to

$$h(t) \geq \sigma_0(x(t), u(t), v(t)) = [\rho_1 x(t) - \rho_2\{1 - x(t)\}]u(t) - \gamma.$$

On the other hand one switch of the sign of $y_c(\tau)$ occurs (on average) for every π radians increment of the phase of $x_c(\tau)$. Moreover, not more than one sign switch can take place during a single sample interval, and hence the total number of sign switches over the time interval $\tau \in [0, Tt_0]$ equals the sum of $x(t)$ from $t = 0$ to $t = t_0$. Hence the integral constraint $\mathcal{I}[\sigma(x(t), u(t), w(t))] \geq 0$ holds for

$$\sigma(x(t), u(t), w(t)) = \pi x(t) - \phi_0 u(t) - T(1 - u(t)).$$

With these functions σ_0, σ, the FSA/IC design formulation is complete, and Theorems 1-3 can be applied to optimize a randomized feedback law.

Indeed, for the design feasibility of the reduced single bit model, the performance condition $\mathcal{I}[\sigma_0] \geq 0$ must be satisfied subject to $\mathcal{I}[\sigma] \geq 0$ and the FSA equation. According to Theorem 3, this means existence of $\mu \geq 0$ such that the IC $\mathcal{I}[\sigma_0 - \mu\sigma] \geq 0$ holds subject to the FSA equation $x(t+1) = w(t)$. According to Theorem 4, this means existence of a storage function satisfying (17), i.e.

$$H(\bar{x}) \geq \min_p \max_{\bar{w}} \sum_{\bar{u}} \{-\bar{\sigma}(\bar{x}, \bar{u}) + H(\bar{w})\} p(\bar{u}),$$

where

$$\bar{\sigma}(\bar{x}, \bar{u}) = [(\rho_1 + \rho_2)\bar{x} - \rho_2]\bar{u} - \gamma - \mu[\pi\bar{x} - (\phi_0 - T)\bar{u} - T].$$

Here the maximum with respect to $\bar{w}$ is achieved when $H(\bar{w}) = \max H$, independently of what p is, and hence the minimum with respect to the distribution p is achieved at an atomic distribution. If we assume (without loss of generality) that $\max H = 0$, (17) further collapses to

$$0 \geq H(\bar{x}) \geq \min_{\bar{u}} \{-\bar{\sigma}(\bar{x}, \bar{u})\}.$$

Finally, the largest achievable γ equals $-\hat{d}$, where $\hat{d}$ is the minimum (with respect to $\mu \geq 0$) of

$$d(\mu) = \max\{\min\{-\mu T, -\mu\phi_0 + \rho_2\}, \min\{\mu(\pi - T), \mu(\pi - \phi_0) - \rho_1\}\}.$$

For example, for $T = 0.22$, $\omega_1 = \pi/(4T) \approx 3.57$ this yields the maximal $\gamma \approx 0.067$ at $\mu \approx 0.31$, with the optimal control $u(t) = x(t)$.

References

1. A. Megretski and A. Rantzer, System Analysis via Integral Quadratic Constraints, IEEE Transactions on Automatic Control, volume 47, no. 6, pp. 819–830, June 1997.
2. Dimitri P. Bertsekas, Dynamic Programming and Optimal Control: 2nd Edition, Athena Scientific, 2000/2001.