Lower and upper bounds for optimal L2 gain nonlinear robust control of first order linear system

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***Abstract*— The following feedback design problem is considered: given a family of first order LTI discrete time plants, desig, when possible, a single nonlinear feedback law which, when applied to each of the plants, satisfies a given (plant-dependent) closed loop L2 gain bound. Sufficient conditions of feasibility and infeasibility of the adaptive control problem are given.**

Key Words: nonlinear feedback, robust control, adaptive control, L2 gain bounds.

I. INTRODUCTION

Considering the wealth of results available for adaptive control design for uncertain linear systems, it appears that little is known about the fundamental limitations a problem setup imposes on the quality of transient response in a closed loop adaptive system. In particular the problem of minimizing L2 gains of adaptive systems appears to be far from being solved.

In this paper, a special case of the following general setup is considered. Let $\{P_S\}_{S\in\mathcal{S}}$ be a family of discrete time systems with fixed dimensions of control input $u = u(t)$, disturbance input $w = w(t)$, cost output $z = z(t)$, sensor output $y = y(t)$, and state vector $x = x(t)$, governed by equations

$$\begin{bmatrix} x(t+1) \\ z(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ w(t) \\ u(t) \end{bmatrix}, \quad t = 0, 1, 2, \ldots, \qquad (1)$$

and parameterized by matrices S ranging over a given set $\mathcal{S}$. The objective is to construct a single nonlinear feedback law

$$u(t) = K_t(y(t), y(t-1), \ldots, y(0)), \qquad (2)$$

such that for every $S \in \mathcal{S}$ there exists a constant c such that the inequality

$$\sum_{t=0}^{T} |z(t)|^2 \leq c|x(0)|^2 + \gamma(S)^2 \sum_{t=0}^{T} |w(t)|^2 \qquad (3)$$

is satisfied for all $T \geq 0$, $S \in \mathcal{S}$, and for all solutions of (1),(2), where $\gamma : \mathcal{S} \mapsto \mathbf{R}_+$ is a given "desired L2 gain profile" of the closed loop system.

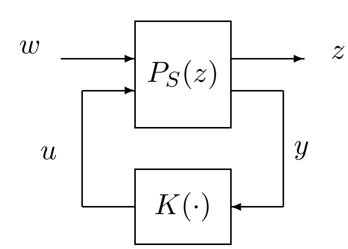

Fig. 1. Feedback design setup

When S is a compact set of matrices each of which defines a *stabilizable* plant one would expect that, for a sufficiently small positive constant γ, the design objective can be satisfied. However, even for a basic statement like this, the authors were not able to find a proof in the literature. The conference paper [1] contains the claim without a proof, referencing instead an unpublished manuscript. The more recent paper [2] proves a number of results about induced gains for a particular adaptive control algorithm.

In this paper, first order plants are considered, and sufficient conditions for feasibility and infeasibility of the adaptive control setup are given. More precisely, for the family of plants described by equations

$$x(t+1) = ax(t) + u(t) + w(t), \tag{4}$$

$$x(0) = 0, \quad z(t) = y(t) = x(t),$$

where $a \in \mathcal{A} \subset \mathbf{R}$ is an unknown parameter it is shown that a simple controller produces closed loop L2 gain $\gamma = \gamma(a)$ which is not larger than $(2 + 32M^2 + \epsilon)^{1/2}$ for every $a \in \mathcal{A} = [-M, M]$, and it is shown that a controller which makes $\gamma(a) < \infty$ for all a from an interval of length L, where $L \geq 100$, must have a gain larger than $L/20$ for $a = 0$. In particular, this shows that it is impossible to find a controller which makes $\gamma(a) < \infty$ for all $a \in (0, \infty)$. On the other hand, it also shows that the ratio of the upper and lower performance bounds for a ranging over $[-M, M]$ remains bounded while the bounds grow linearly with M as $M \to \infty$.

II. A LOWER BOUND FOR ADAPTIVE L2 GAIN

Theorem 1: Let $\gamma_0 \geq 5$ be a real number. Let $\mathcal{I}$ be a closed interval in $\mathbf{R}$ such that the length of its intersection with $\mathbf{R}_+$ or $\mathbf{R}_-$ is at least $10\gamma_0$. For a given family $K = \{K_t\}_{t=0}^{\infty}$ of functions

$$K_t : \ \mathbf{R}^{t+1} \mapsto \mathbf{R}, \quad t = 0, 1, 2, \dots,$$

let $\Gamma = \Gamma(a, K)$ be the infimum of those γ for which

$$\gamma^2 \sum_{t=1}^{T} |x(t) - ax(t-1) - u(t-1)|^2 \tag{5}$$

$$\geq \sum_{t=1}^{T} |x(t)|^2 \quad \forall\ T, x(\cdot) : \ x(0) = 0,$$

whenever

$$u(t) = K_t(x(t), x(t-1), \dots, x(1), x(0)) \quad \forall\ t.$$

If $\Gamma(0) \leq \gamma_0$ then there exists at least one $a \in \mathcal{I}$ such that $\Gamma(a) = \infty$.

Proof Let

$$X(T) = \sum_{t=1}^{T} |x_{t-1}|^2,$$

$$Y(T) = \sum_{t=1}^{T} x(t-1)[u(t-1) - x(t)],$$

$$Z(T) = \sum_{t=1}^{T} |u_{t-1} - x(t)|^2.$$

Assuming, without loos of generality, that $x(1) \neq 0$, define

$$\theta(T) = -\frac{Y(T)}{X(T)},$$

$$\kappa(T) = \frac{Z(T)X(T) - Y(T)^2}{X(T)^2},$$

$$h(T) = \frac{x(T)}{X(T)^{1/2}},$$

$$q(T) = \frac{u(T) + \theta(T)x(T)}{X(T)^{1/2}(1 + h(T)^2)^{1/2}},$$

$$p(T) = \frac{x(T+1) + \theta(T)x(T)}{X(T)^{1/2}(1 + h(T)^2)^{1/2}}.$$

Then

$$\kappa(t+1) = \frac{\kappa(t) + p(t)^2}{1 + h(t)^2},$$

$$\theta(t+1) = \theta(t) - \frac{h(t)p(t)}{(1 + h(t)^2)^{1/2}},$$

$$h(t+1) = q(t) - p(t),$$

and the inequality in (5) can be re-written as

$$\gamma^2[\kappa(T) + (a - \theta(T))^2] \geq (1 + h(T)^2).$$

For a given a, the setup describes a game in which $q(t)$ is the control action decided at time t with the knowledge of $h(t)$, $\kappa(t)$, $\theta(t)$ with the aim of satisfying (5) perpetually, and $p(t)$ is the anthagonist disturbance action decided at time t with the knowledge of $h(t)$, $\kappa(t)$, $\theta(t)$, and $q(t)$, with the aim of making (5) fail eventually.

Let us show that if the control action guarantees that (5) is satisfied with $\gamma = \gamma_0$ for $a = 0$ then (5) cannot be guaranteed (no matter how large γ is) for some $a \in \mathcal{I}$. Let a_c be the closest to zero point of $\mathcal{I}$. Without loss of generality, assume that $a_c \geq 0$. Then by assumption $[a_c, a_c + 10\gamma_0] \in \mathcal{I}$. Define $p = p(t)$ according to the following law:

(a) $p(t) = 0$ if

$$\begin{aligned} &|q(t)| \geq 2\gamma_0, && \text{and} \\ &\theta(t) - a_c \in [5\gamma_0, 9.02\gamma_0], && \text{and} \\ &\kappa(t) \leq 4.03, \end{aligned}$$

(b) $p(t) = 0.01\gamma_0 \text{sgn}(h(t)(\theta(t) - 7.01\gamma_0))$ if

$$\begin{array}{ll} |q(t)| \geq 2.01\gamma_0, & \text{but} \\ \theta(t) - a_c \notin [5\gamma_0, 9.02\gamma_0], & \text{or} \\ \kappa(t) \geq 4.03, & \end{array}$$

(c) $q(t) = 4.01\gamma_0 \text{sgn}(h(t)(\theta(t) - 7.01\gamma_0))$ otherwise.

Note first that, by construction,

$$|h(t)| \geq 2\gamma_0 \quad \text{for all} \quad t > 1.$$

Indeed, if $|q(t)| \geq 2.01\gamma_0$ then (a) or (b) are used to define $p(t)$, in which case $|p(t)| \leq |q(t)| - 2\gamma_0$ and

$$|h(t+1)| = |q(t) - p(t)| \geq 2\gamma_0.$$

Otherwise $|q(t)| \leq 2.01\gamma_0$ and $|p(t)| \geq 4.01\gamma_0$, which also imlies $|h(t+1)| \geq 2\gamma_0$.

Now let us show that $\kappa(t) \leq 4.03$ for all sufficiently large t. Indeed, $|p(t)| \leq 4.01\gamma_0$ for all $t > 0$. Hence

$$\kappa(t+1) - 4.025 = \frac{\kappa(t) + p(t)^2}{1 + h(t)^2}$$

$$\leq \frac{\kappa(t) + 16.1\gamma_0^2}{1 + 4\gamma_0^2} - 4.025$$

$$\leq \frac{\kappa(t) - 4.025}{1 + 4\gamma_0^2}$$

for all $t > 1$. Hence $\max\{0, \kappa(t) - 4.025\}$ is exponentially decreasing. Hence there exists t_* such that $\kappa(t) \leq 4.03$ for $t \geq t_*$.

Now for $t \geq t_*$ the definition of $p(t)$ is such that

$$|\theta(t+1) - a_c - 7.01\gamma_0| \leq |\theta(t) - a_c - 7.01\gamma_0| - 0.01\gamma_0$$

whenever $\theta(t) - a_c \notin [5\gamma_0, 9.02\gamma_0]$. Hence $\theta(t) - a_c \in [5\gamma_0, 9.2\gamma_0]$ for sufficiently large t.

Note that, in order to maintain (5) for $a = 0$,$\gamma = \gamma_0$, the control action must satisfy

$$\gamma_0^2 \left[\frac{\kappa(t)}{1 + h(t)^2} + \frac{\theta(t)}{h(t)^2} \right] - 1$$

$$\geq \left| q(t) - \frac{\theta(t)(1 + h(t)^2)^{1/2}}{h(t)} \right|$$

whenever $h(t) \neq 0$. Indeed, otherwise using

$$p(t) = \frac{\theta(t)(1 + h(t)^2)^{1/2}}{h(t)}$$

yields $\theta(t+1) = 0$ and

$$\kappa(t+1) = \frac{\kappa(t) + p(t)^2}{1 + h(t)^2}$$

$$= \frac{\kappa(t)}{1 + h(t)^2} + \frac{\theta(t)}{h(t)^2}.$$

In order to maintain the gain bound of γ_0 at $a = 0$, the inequality

$$\gamma_0^2 \kappa(t+1) \geq 1 + h(t+1)^2$$

$$= 1 + \left(q(t) - \frac{\theta(t)(1 + h(t)^2)^{1/2}}{h(t)} \right)$$

must hold, which implies the desired constraint on $q(t)$.

Combining the constraint with the lower bound $\theta(t) \geq 5\gamma_0$ implies that at every sufficiently large time instance

$$|q| \geq \frac{|\theta|(1 + h^2)^{1/2}}{|h|} - \frac{\gamma_0 \kappa^{1/2}}{(1 + h^2)^{1/2}} - \frac{\gamma_0|\theta|}{|h|}$$

$$\geq \frac{|\theta|}{2} - \frac{\gamma_0 4.03^{1/2}}{1 + 4\gamma_0^2} \geq 2\gamma_0.$$

Hence $p(t) = 0$ and $\theta(t) = \theta_0 \in \mathcal{I}$ is constant for sufficiently large t, and therefore $\kappa(t) \to 0$ exponentially as $t \to \infty$. Hence the L2 gain is infinite for $a = \theta_0$. ■

III. AN UPPER BOUND FOR ADAPTIVE L2 GAIN

Theorem 2: Let $0 \leq \lambda < 1$. For the linear system

$$x(k+1) = ax(k) + u(k) + w(k) \qquad x(0) = 0$$

with $|a| \leq A$ and $w(0) \neq 0$, the non-linear control law

$$u(k) = -\hat{a}(k)x(k),$$

where

$$\hat{a}(k) = \text{sat}_A \left(\frac{\sum_{j=0}^{k-1} \lambda^{k-j} x(j)[x(j+1) - u(j)]}{\sum_{j=0}^{k-1} \lambda^{k-j} x(j)^2} \right),$$

$$\text{sat}_A(y) = \begin{cases} y, & |y| \leq A, \\ Ay/|y|, & |y| > A, \end{cases}$$

gives the stable closed loop performance

$$\sum_{k=0}^{n} x(k)^2 \leq \left(\frac{32A^2}{1 - \lambda} + 2 \right) \sum_{k=0}^{n} w(k)^2$$

Proof For breif notation, we will write x and x_+ instead of $x(k)$ and $x(k+1)$. The system equation then reads

$$x_+ = ax + u + w$$

Define S, T, W and X through the difference equations

$$\begin{aligned} X_+ &= \lambda X + x^2 & X(0) &= 0 \\ T_+ &= \lambda T + xw & T(0) &= 0 \\ W_+ &= \lambda W + w^2 & W(0) &= 0 \\ S_+ &= \lambda S + x(x_+ - u) & & \\ &= \lambda S + x(ax + w) & S(0) &= 0 \end{aligned}$$

and note that $S = aX + T$. Let

$$\widehat{a} = A\,\mathrm{sat}\left(\frac{S}{AX}\right) \qquad \tilde{a} = a - \widehat{a}$$

Then the control law can be written $u = -\widehat{a}x$ and the closed loop system

$$x_+ = \tilde{a}x + w$$

The proof is completed by verifying that the Lyapunov function

$$V = x^2 + dW$$

with $d = 32A^2/(1-\lambda)$ satisfies

$$V_+ - V \le (d+2)w^2 - x^2$$

To see this, some calculations are needed

$$\begin{aligned} |\tilde{a}| = |\,a - \widehat{a}\,| &= \left| a - A\,\mathrm{sat}\left(\frac{S}{AX}\right)\right| \\ &= \left| a - A\,\mathrm{sat}\left(a + \frac{T}{AX}\right)\right| \le \left|\frac{T}{X}\right| \end{aligned}$$

The definition of T, W and X shows that

$$0 \le \begin{bmatrix} X & T \\ T & W \end{bmatrix}$$

Consequently $T^2 \le XW$ and

$$\tilde{a}^2 \le \frac{T^2}{X^2} \le \frac{W}{X}$$

At the same time

$$\tilde{a}^2 \le (|a| + |\widehat{a}|)^2 \le 4A^2$$

Together, the two bounds give

$$\begin{aligned} x_+^2 &= (\tilde{a}x + w)^2 \le 2\tilde{a}^2x^2 + 2w^2 \\ &\le 8A^2X_+ + 2W_+ \end{aligned}$$

$$x^2 \le 8A^2X + 2W$$

$$\begin{aligned} \tilde{a}^2x^2 &\le 8\tilde{a}^2A^2X + 2\tilde{a}^2W \\ &\le 8A^2W + 2(4A^2)W = 16A^2W \end{aligned}$$

$$\begin{aligned} V_+ - V &= (x_+^2 + dW_+) - (x^2 + dW) \\ &\le (\tilde{a}x + w)^2 - x^2 + d(\lambda W + w^2 - W) \\ &\le 2\tilde{a}^2x^2 + 2w^2 - x^2 + d(\lambda W + w^2 - W) \\ &\le 32A^2W + 2w^2 - x^2 + d(\lambda W + w^2 - W) \\ &= (d+2)w^2 - x^2 \end{aligned}$$

Adding over k gives

$$\begin{aligned} V(n+1) &= \sum_{k=0}^{n}[V(k+1) - V(k)] \\ &\le \sum_{k=0}^{n}\left[(d+2)w(k)^2 - x(k)^2\right] \end{aligned}$$

and the desired performance bound is proved. ■

IV. REFERENCES

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