On Brauer groups of stacky curves

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Brauer groups

Definition

For a scheme of algebraic stack $\mathcal{X}$, we define its Brauer group to be

$$\operatorname{Br}\mathcal{X} := \mathrm{H}^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m).$$

If R is a ring, then we set $\operatorname{Br} R := \operatorname{Br}(\operatorname{Spec} R)$.

Example

Fix a prime/place $p \leq \infty$. Then,

$$\mathrm{inv}_p \colon \operatorname{Br}\mathbb{Q}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

which is an isomorphism if $p < \infty$ and has image $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ if $p = \infty$ ($\mathbb{Q}_\infty = \mathbb{R}$).

Arithmetic motivation for studying Brauer groups

There is an exact sequence

$$0 \longrightarrow \operatorname{Br}\mathbb{Q} \longrightarrow \bigoplus_{p\leq\infty} \operatorname{Br}\mathbb{Q}_p \xrightarrow{\sum \operatorname{inv}_p} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Consequently, given X and $\alpha \in \operatorname{Br} X$, one can define

$$X(\mathbb{A}_\mathbb{Q})^\alpha := \left\{x \in X(\mathbb{A}_\mathbb{Q}) : \sum \operatorname{inv}_p \alpha(x_p) = 0\right\} \quad \supset X(\mathbb{Q})$$

$$X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} := \bigcap_{\alpha\in\operatorname{Br} X} X(\mathbb{A}_\mathbb{Q})^\alpha \quad \supset X(\mathbb{Q}).$$

Remark

$X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}}$ gives an (often computable) obstruction to the existence of rational points. Variants exist for integral points.

Stacky curves in nature

Definition (informal)

A stacky curve is a "reasonable" algebraic stack $\mathcal{X}$ whose coarse space X is a curve and whose stabilizer groups are finite.

Example (modular curves)

For $N \geq 1$, there is the stacky modular curve

$$\mathcal{Y}_0(N) = \left\{ (E, C) : \begin{array}{c} E \text{ an elliptic curve} \\ C \subset E \text{ a cyclic subgroup of order } N \end{array} \right\}$$

Example (generalized Fermat)

Consider $S = V(x^a + y^b = z^c) \setminus \{(0,0,0)\} \subset \mathbb{A}^3_{\mathbb{Z}}$. Set

$$\mathcal{X} = [S/\mathbb{G}_m] \text{ where } \lambda \cdot (x, y, z) = (\lambda^{bc}x, \lambda^{ac}y, \lambda^{ab}z).$$

$\mathcal{X}$ is a stacky curve and $S(\mathbb{Z}) \cong \mathcal{X}(\mathbb{Z})/\{\pm 1\}$.

Br $\mathcal{Y}(1)$

Set $\mathcal{Y}(1) := \mathcal{Y}_0(1)$.

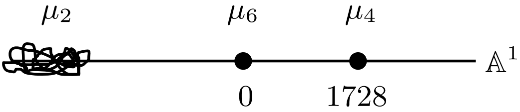


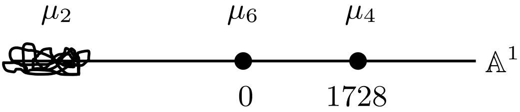
Figure: Artist's rendition of $\mathcal{Y}(1)_k$ (accurate only if $6 \in k^\times$).

Antieau–Meier ('20) and Di Lorenzo–Pirisi ('22) computed that, if k is a perfect field and char $k \neq 2$,

$$\operatorname{Br} \mathcal{Y}(1)_k \cong \operatorname{Br} \mathbb{A}^1_k \oplus \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z}) \cong \operatorname{Br} k \oplus \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z})$$

$\operatorname{Br} \mathcal{Y}(1)$

Set $\mathcal{Y}(1) \coloneqq \mathcal{Y}_0(1)$.



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Antieau–Meier ('20) and Di Lorenzo–Pirisi ('22) computed that, if k is a perfect field and $\operatorname{char} k \neq 2$,

$$\begin{aligned} \operatorname{Br} \mathcal{Y}(1)_k &\cong \operatorname{Br} \mathbb{A}^1_k \oplus \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z}) \cong \operatorname{Br} k \oplus \mathrm{H}^1(k, \mathbb{Z}/12\mathbb{Z}) \\ \operatorname{Br} \mathcal{Y}(1)_{\overline{k}} &\cong 0 \end{aligned}$$

Note: Tsen's theorem implies that $\operatorname{Br} X_{\overline{k}} = 0$ if X is a curve.

$\operatorname{Br} \mathcal{Y}_0(2)$

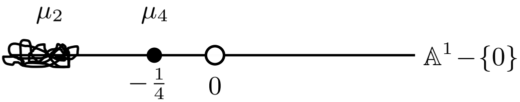

Figure: Artist's rendition of $\mathcal{Y}_0(2)_k$.

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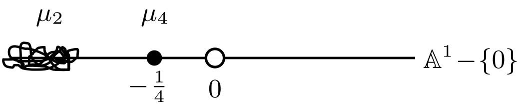

Figure: Artist's rendition of $\mathcal{Y}_0(2)_k$.

A.–Bhamidipati–Jha–Ji–Lopez ('24) computed that, if k is a perfect field and char $k \neq 2$,

$$\operatorname{Br} \mathcal{Y}_0(2)_k \cong \operatorname{Br}(\mathbb{A}^1_k \setminus \{0\}) \oplus \mathrm{H}^1(k, \mathbb{Z}/4\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}$$

$$\operatorname{Br} \mathcal{Y}_0(2)_{\overline{k}} \cong \mathbb{Z}/2\mathbb{Z}.$$

Question

Why is $\operatorname{Br} \mathcal{Y}(1)_{\overline{k}}$ *trivial, but* $\operatorname{Br} \mathcal{Y}_0(2)_{\overline{k}}$ *not?*

Prelude to the main result I: tameness

Goal: We want to generalize the previous computations.

Example (linearly reductive groups)

- μ_n is linearly reductive over any field F.
- If G is a finite étale F-group, then G is linearly reductive if and only if $\operatorname{char} F \nmid \# G$.

Definition (slightly informal)

An algebraic stack $\mathcal{X}$ is `tame` if every point $x \in \mathcal{X}(F)$ over any field F has a finite linearly reductive stabilizer group $\underline{\mathrm{Aut}}_{\mathcal{X}}(x)/F$.

Example

- $\mathcal{Y}(1)_k$ is tame iff $\operatorname{char} k \neq 2, 3$.
- $\mathcal{Y}_0(2)_k$ is tame iff $\operatorname{char} k \neq 2$.

Prelude to the main result II: Brauerlessness

Definition

A tame algebraic stack $\mathcal{X}$ is `locally Brauerless` if,
for any of its geometric stabilizer groups G/F^s,
the group $\pi_0(G)$ of G's connected components satisfies

$$\mathrm{H}^3(\pi_0(G), \mathbb{Z}) = 0.$$

Theorem (A., in preparation)

Let $\mathcal{X}$ be a locally Brauerless tame algebraic stack with coarse space map $c\colon \mathcal{X} \to X$. Then, $\mathrm{R}^2 c_* \mathbb{G}_m = 0$.

Example

$\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ for all n, so stacks whose stabilizers are all of the form μ_n are tame and locally Brauerless.

Statement of the main result

Notation

If G is a commutative, finite group scheme over a field k, we write $G^\vee := \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ for its Cartier dual.

Theorem (A., in preparation)

Suppose we're given

$$\mathcal{X} \xrightarrow{G\text{-gerbe}} \mathcal{Y} \xrightarrow{\text{root stack}} \underbrace{X}_{\text{smooth curve}/k=\overline{k}}$$

for some finite, commutative linearly reductive group G/k. If $\mathcal{X}$ is locally Brauerless, then $\mathrm{Br}\,\mathcal{X} \simeq \mathrm{H}^1(X, G^\vee)$.

Remark

The above isomorphism $\mathrm{H}^1(X, G^\vee) \xrightarrow{\sim} \mathrm{Br}\,\mathcal{X}$ is totally non-explicit, but it would be interesting to remedy this.

Some consequences

Example (our favorite modular curves)

- Take $\mathcal{X} = \mathcal{Y}(1)_k$ with $6 \in k^\times$. Then, $G = \mu_2$ so $\operatorname{Br} \mathcal{Y}(1)_{\overline{k}} \cong \mathrm{H}^1(\mathbb{A}^1, \mathbb{Z}/2\mathbb{Z}) = 0$.
- Take $\mathcal{X} = \mathcal{Y}_0(2)_k$ with $2 \in k^\times$. Then, $G = \mu_2$ so $\operatorname{Br} \mathcal{Y}_0(2)_{\overline{k}} \cong \mathrm{H}^1(\mathbb{A}^1 \setminus \{0\}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

Remark (over a perfect field)

Say k is a perfect field. Then, there is a spectral sequence relating $\mathrm{H}^i(\mathcal{X}, \mathbb{G}_m)$ to $\mathrm{H}^i(k, \mathrm{H}^j(\mathcal{X}_{\overline{k}}, \mathbb{G}_m))$. If $\mathcal{X}(k) \neq \emptyset$ and X is both proper and geometrically integral, then it produces the short exact sequence

$$0 \longrightarrow \operatorname{Br} k \longrightarrow \ker\left(\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^1(X_{\overline{k}}, G^\vee)\right) \longrightarrow \mathrm{H}^1(k, \operatorname{Pic} \mathcal{X}_{\overline{k}}) \longrightarrow 0$$

Proof sketch

Recall the coarse space $c : \mathcal{X} \to X$ factors as $\mathcal{X} \xrightarrow[G]{\pi} \mathcal{Y} \xrightarrow[\text{root}]{\rho} X$.

(1) Use the Leray spectral sequence $\mathrm{H}^i(X, \mathrm{R}^j c_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{X}, \mathbb{G}_m)$ to produce

$$\mathrm{H}^2(X, \mathbb{G}_m) = 0 \longrightarrow \ker\big(\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \to \mathrm{H}^0(X, \mathrm{R}^2 c_* \mathbb{G}_m)\big) \longrightarrow \mathrm{H}^1(X, \mathrm{R}^1 c_* \mathbb{G}_m) \longrightarrow 0 = \mathrm{H}^3(X, \mathbb{G}_m)$$

(2) Since $\mathcal{X}$ is tame and locally Brauerless, we have $\mathrm{R}^2 c_* \mathbb{G}_m = 0$ from which we deduce $\mathrm{H}^2(\mathcal{X}, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^1(X, \mathrm{R}^1 c_* \mathbb{G}_m)$.

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(3) Use the Grothendieck spectral sequence $\mathrm{R}^i \rho_* \mathrm{R}^j \pi_* \mathbb{G}_m \implies \mathrm{R}^{i+j} c_* \mathbb{G}_m$ to produce

$$0 \longrightarrow \mathrm{R}^1 \rho_* \mathbb{G}_m \longrightarrow \mathrm{R}^1 c_* \mathbb{G}_m \longrightarrow G^\vee \longrightarrow 0 = \mathrm{R}^2 \rho_* \mathbb{G}_m$$

(4) Note that $\mathrm{R}^1 \rho_* \mathbb{G}_m$ is supported on a finite k-scheme and so is acyclic. We deduce $\mathrm{H}^1(X, \mathrm{R}^1 c_* \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^1(X, G^\vee)$.

Summary

- We defined Brauer groups $\operatorname{Br}\mathcal{X} = \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$. These give rise to obstructions to points on varieties.
- Stacky curves are essentially curves w/ finite stabilizer groups attached to each point (e.g. modular curves, generalized Fermat curves).
- A stack is tame if all its stabilizer groups are finite linearly reductive groups (e.g. μ_n).
- We identified a condition ('locally Brauerless') which guarantees that that a tame stacky curve $\mathcal{X}/k = \overline{k}$ with coarse space X and 'generic stabilizer' G/k has Brauer group $\operatorname{Br}\mathcal{X} \simeq \mathrm{H}^1(X, G^\vee)$.

Thank you!