Brauer groups of stacky curves, via the example of $\mathcal{Y}(1)$

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Brauer groups

Definition

For a scheme or algebraic stack $\mathcal{X}$, we define its `Brauer group` to be

$$\mathrm{Br}\,\mathcal{X} := \mathrm{H}^2_{\text{ét}}(\mathcal{X}, \mathbb{G}_m).$$

If R is a ring, then we set $\mathrm{Br}\,R := \mathrm{Br}(\mathrm{Spec}\,R)$.

Example

Fix a prime/place $p \leq \infty$. Then,

$$\mathrm{inv}_p \colon \mathrm{Br}\,\mathbb{Q}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

which is an isomorphism if $p < \infty$ and has image $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ if $p = \infty$ ($\mathbb{Q}_\infty = \mathbb{R}$).

Arithmetic motivation for studying Brauer groups

There is an exact sequence

$$0 \longrightarrow \operatorname{Br}\mathbb{Q} \longrightarrow \bigoplus_{p\le\infty} \operatorname{Br}\mathbb{Q}_p \xrightarrow{\sum \operatorname{inv}_p} \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

Consequently, given X and $\alpha \in \operatorname{Br} X$, one can define

$$X(\mathbb{A}_\mathbb{Q})^\alpha := \left\{ x \in X(\mathbb{A}_\mathbb{Q}) : \sum_p \operatorname{inv}_p \alpha(x_p) = 0 \right\} \supset X(\mathbb{Q})$$

$$X(\mathbb{A}_\mathbb{Q})^{\operatorname{Br}} := \bigcap_{\alpha\in\operatorname{Br} X} X(\mathbb{A}_\mathbb{Q})^\alpha \supset X(\mathbb{Q}).$$

Note: $X(\mathbb{A}_\mathbb{Q})^{\operatorname{Br}} = \emptyset \implies X(\mathbb{Q}) = \emptyset.$

Remark

$X(\mathbb{A}_\mathbb{Q})^{\operatorname{Br}}$ gives an (often computable) obstruction to the existence of rational points. Variants exist for integral points.

Our star example

Let $\mathcal{Y}(1)$ denote the moduli stack of elliptic curves.

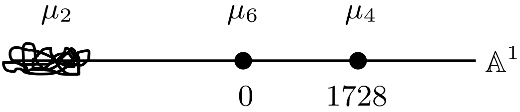

Figure: Artist's rendition of $\mathcal{Y}(1)_k$ (accurate only if $6 \in k^\times$).

Theorem (A., '24)

Let $S/\mathbb{Z}[1/2]$ be a regular, noetherian scheme. Then,

$$\mathrm{Br}\,\mathcal{Y}(1)_S \simeq \mathrm{Br}\,\mathbb{A}^1_S \oplus \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}).$$

Our star example

Let $\mathscr{Y}(1)$ denote the moduli stack of elliptic curves.

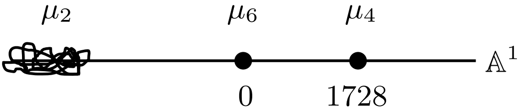

Figure: Artist's rendition of $\mathscr{Y}(1)_k$ (accurate only if $6 \in k^\times$).

Theorem (A., '24)

Let $S/\mathbb{Z}[1/2]$ be a regular, noetherian scheme. Then,

$$\operatorname{Br} \mathscr{Y}(1)_S \simeq \operatorname{Br} \mathbb{A}^1_S \oplus \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}).$$

<u>Note</u>: $\mathscr{Y}(1)$ is everywhere wild in characteristic 2.

<u>Note</u>: Want this computation over $S = \mathbb{Z}, \mathbb{Z}[1/2], \mathbb{Z}[1/6]$, etc. for computing integral Brauer obstructions.

<u>Note</u>: Previously computed for S a field (and a couple other specific cases) by Antieau–Meier, Shin, Di Lorenzo–Pirisi.

Strategy for computing $\operatorname{Br} \mathcal{Y}(1)_S$

(1) Compute $\operatorname{Br} \mathcal{Y}(1)_R$ over strictly henselian local rings R (e.g. $R = \overline{\mathbb{Q}}$) via the coarse moduli space (cms) $c \colon \mathcal{Y}(1)_R \to \mathbb{A}^1_R$.

Remark

"via the cms" here really means "via its Leray spectral sequence".

(2) Compute $\operatorname{Br} \mathcal{Y}(1)_S$ via the structure map $f \colon \mathcal{Y}(1)_S \to S$.

Remark

- We computed (the stalks of) $\mathrm{R}^2 f_* \mathbb{G}_m$ in part **(1)**.
- This amounts to the usual Hochschild–Serre/Galois descent spectral sequence when $S = \operatorname{Spec} k$ is a field.

Useful cohomological vanishing result

Definition

A tame algebraic stack $\mathcal{X}$ is `locally Brauerless` if, for any of its geometric stabilizer groups G, one has

$$\mathrm{H}^3(G, \mathbb{Z}) = 0.$$

Theorem (A. '24, generalizing Meier '18)

Let $\mathcal{X}$ be a locally Brauerless tame algebraic stack with coarse space map $c \colon \mathcal{X} \to X$. Then, $\mathrm{R}^2 c_* \mathbb{G}_m = 0$.

Example

$\mathrm{H}^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$ for all n, so stacks whose stabilizers are all of the form μ_n are tame and locally Brauerless, e.g. $\mathcal{Y}(1)_S$ is tame and locally Brauerless (if $S/\mathbb{Z}[1/6]$).

Computing $\operatorname{Br} \mathcal{Y}(1)_R$, part I

Let R be a regular, noetherian strictly henselian local $\mathbb{Z}[1/6]$-algebra. Let $c\colon \mathcal{Y}(1)_R \to \mathbb{A}^1_R$ be the coarse space map.

- $\mathcal{Y}(1)_R$ is tame and locally Brauerless $\implies \mathrm{R}^2 c_* \mathbb{G}_m = 0$.
- Leray spectral sequence produces an exact sequence

$$0 \dashrightarrow \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_R, \mathbb{G}_m) \to \mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m).$$

Remark

First map above is injective because there exists a dense open $U \subset \mathbb{A}^1_R$ admitting a section $U \to \mathcal{Y}(1)_R$ of c.

<u>Goal</u>: Compute $\mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m)$.

Computing $\mathrm{Br}\,\mathcal{Y}(1)_R$, part II

Have

$$0 \to \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_R, \mathbb{G}_m) \to \mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m) \quad .$$

<u>Goal</u>: Compute $\mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m)$.

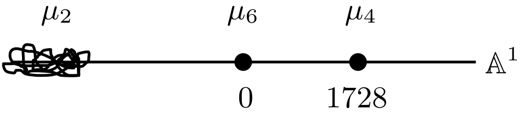

Figure: Artist's rendition of $\mathcal{Y}(1)_R$.

- There is an exact sequence (of sheaves on $\mathbb{A}^1$)

$$\mathrm{R}^1 c_* \mathbb{G}_m \longrightarrow \overbrace{\underline{\mathbb{Z}/2\mathbb{Z}}}^{\mathrm{Pic}(B\mu_2)} \longrightarrow 0.$$

Computing Br $\mathcal{Y}(1)_R$, part II

Have

$$0 \to \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_R, \mathbb{G}_m) \to \mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m) \qquad .$$

<u>Goal</u>: Compute $\mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m)$.

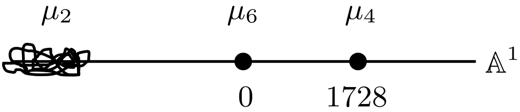

Figure: Artist's rendition of $\mathcal{Y}(1)_R$.

- There is an exact sequence (of sheaves on $\mathbb{A}^1$)

$$0 \longrightarrow \underline{\mathbb{Z}/3\mathbb{Z}}_0 \oplus \underline{\mathbb{Z}/2\mathbb{Z}}_{1728} \longrightarrow \mathrm{R}^1 c_* \mathbb{G}_m \longrightarrow \overbrace{\underline{\mathbb{Z}/2\mathbb{Z}}}^{\mathrm{Pic}(B\mu_2)} \longrightarrow 0.$$

Computing $\operatorname{Br} \mathcal{Y}(1)_R$, part II

Have

$$0 \to \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_R, \mathbb{G}_m) \xrightarrow{0} \mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m) = 0.$$

<u>Goal</u>: Compute $\mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m)$.

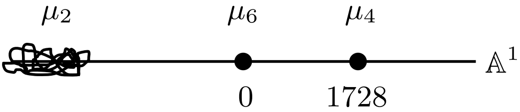

Figure: Artist's rendition of $\mathcal{Y}(1)_R$.

- There is an exact sequence (of sheaves on $\mathbb{A}^1$)

$$0 \longrightarrow \underline{\mathbb{Z}/3\mathbb{Z}}_0 \oplus \underline{\mathbb{Z}/2\mathbb{Z}}_{1728} \longrightarrow \mathrm{R}^1 c_* \mathbb{G}_m \longrightarrow \overbrace{\underline{\mathbb{Z}/2\mathbb{Z}}}^{\operatorname{Pic}(B\mu_2)} \longrightarrow 0.$$

- Taking cohomology $\implies \mathrm{H}^1(\mathbb{A}^1_R, \mathrm{R}^1 c_* \mathbb{G}_m) = 0.$
- $c^* \colon \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^2(\mathcal{Y}(1)_R, \mathbb{G}_m).$

Computing Br $\mathcal{Y}(1)_S$, part I

Let S be a regular, noetherian $\mathbb{Z}[1/6]$-scheme. Consider

$$\overbrace{\mathcal{Y}(1)_S \xrightarrow[c]{} \mathbb{A}^1_S \xrightarrow[g]{} S}^{f}.$$

Lemma (previous slide)

$\mathrm{R}^2 f_* \mathbb{G}_m \xleftarrow{\sim} \mathrm{R}^2 g_* \mathbb{G}_m$
(on stalks: $\mathrm{H}^2(\mathcal{Y}(1)_R, \mathbb{G}_m) \simeq \mathrm{H}^2(\mathbb{A}^1_R, \mathbb{G}_m)$*).*

Lemma

$\mathrm{R}^1 f_* \mathbb{G}_m = \underline{\mathbb{Z}/12\mathbb{Z}}$ *(Fulton–Olsson, '10)*
$\mathrm{R}^1 g_* \mathbb{G}_m = 0$ *(on stalks:* Pic $R[x] = 0$*)*

Lemma

$f_* \mathbb{G}_m = g_* \mathbb{G}_m = \mathbb{G}_m.$ *(on stalks:* $R[x]^\times = R^\times$*)*

Computing $\operatorname{Br} \mathcal{Y}(1)_S$, part II

$$\begin{array}{ccc} \mathcal{Y}(1)_S & \xrightarrow{\quad c \quad} & \mathbb{A}^1_S \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S. & \end{array}$$

- Consider spectral sequences

$$E_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j f_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathcal{Y}(1)_S, \mathbb{G}_m)$$
$$F_2^{ij} = \mathrm{H}^i(S, \mathrm{R}^j g_* \mathbb{G}_m) \implies \mathrm{H}^{i+j}(\mathbb{A}^1_S, \mathbb{G}_m)$$

- Compute that

$$E_\infty^{02} \simeq F_\infty^{02} \text{ and } E_\infty^{20} \simeq F_\infty^{20}$$
$$\text{but } E_\infty^{11} \simeq \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \not\simeq 0 \simeq F_\infty^{11}$$

- Conclude existence of short exact sequence

$$0 \to \mathrm{H}^2(\mathbb{A}^1_S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_S, \mathbb{G}_m) \to \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \to 0.$$

Additional remarks

- One can construct an explicit splitting of
$$0 \to \mathrm{H}^2(\mathbb{A}^1_S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathcal{Y}(1)_S, \mathbb{G}_m) \to \mathrm{H}^1(S, \mathbb{Z}/12\mathbb{Z}) \to 0.$$
- We needed $S/\mathbb{Z}[1/6]$ for $\mathcal{Y}(1)_S$ to be tame. However, in characteristic 3, it's tame away from j-invariant 0 so passing from $6 \rightsquigarrow 2$ only requires carrying out an additional computation "at a single point" (to ensure vanishing of $\mathrm{R}^2 c_* \mathbb{G}_m$).
- This strategy applies to other stacky curves and simplifies if you work over a perfect field k. In the simplest case,

Theorem (A., '24, informally stated here)

If $\mathcal{X}$ is a tame, locally Brauerless stacky curve over $\overline{k}$ with coarse space X and generic stabilizer G/k, then $\mathrm{Br}\,\mathcal{X} \simeq \mathrm{H}^1(X, G^\vee)$, $G^\vee :=$ Cartier dual of $G = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$.

Summary

- We defined Brauer groups $\operatorname{Br} \mathcal{X} = \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$. These give rise to obstructions to points on varieties.
- For future applications to computing such obstructions for integral points, one would like to compute these for stacks over $\mathbb{Z}$ and its localizations.
- Under some geometric assumptions (being tame and locally Brauerless), we showed that one can carry out such computations for stacky curves.
- We worked out the example of computing $\operatorname{Br} \mathcal{Y}(1)_S$, where $\mathcal{Y}(1)$ is the moduli space of elliptic curves, over fairly general S.

Thank you!