MODULARITY SEMINAR: BASE CHANGE

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Abstract. We will cover the statement of the Local Langlands Correspondence and solvable base change. These will be used in the proof of the Langlands-Tunnell theorem, which produces a automorphic cuspidal representation from a solvable Galois representation.

1. Introduction

For a number field F, let Γ_F denote the absolute Galois group. The goal for the next two talks will be to prove the following:

Theorem 1.0.1 ([CSS97, pg 192])**.** *Suppose F is a number field and suppose $\sigma\colon \Gamma_F \to \mathrm{GL}_2(\mathbb{C})$ is an irreducible representation with solvable image. Then there exists a unique irreducible automorphic cuspidal representation $\pi(\sigma) = \bigotimes \pi_v$ of $\mathrm{GL}_2(\mathbf{A}_F)$ such that*

$$\mathrm{tr}(\sigma(\mathrm{Fr}_v)) = \mathrm{tr}(t_{\pi_v})$$

for almost every place v of F.

Here, t_{π_v} is the *Satake parameter* of the unramified representation π_v, to be defined in §2. Our goal is to review Local Langlands, and state solvable base change, which is a key ingredient in the proof.

2. Local Langlands conjectures

2.1. **For GL_n.** Let F_v be a non-archimedean local field of characteristic zero, i.e., a finite extension of $\mathbb{Q}_p$. Let W_{F_v} be the Weil group of F_v, i.e., the pre-image of $\mathbb{Z}$ under the surjection $\Gamma_{F_v} \to \mathrm{Gal}(F^{\mathrm{unr}}/F) \simeq \widehat{\mathbb{Z}}$. Let the *Weil-Deligne group* of F_v be $W_{F_v} \ltimes \mathbb{C}$, where a Frobenius lift in W_{F_v} acts on $\mathbb{C}$ by multiplication by q_v, the size of the residue field of F_v. It has a homomorphism $\mathrm{val}_{F_v}\colon W_{F_v} \to \mathbb{Z}$ sending a Frobenius lift to 1.

Remark 2.1.1. Some people define the Weil-Deligne group to be $W_{F_v} \times \mathbb{C}$ or $W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$. These definitions are all equivalent by the Jacobson-Morosov theorem.

Then the *Local Langlands Correspondence for* GL_n, proved by [HT01], [Hen00], and [Sch13] is a bijection:[1]

(2.1.2)

$$\mathrm{Irr}\,\mathrm{GL}_n(F_v) := \left\{\begin{matrix}\text{irreducible smooth represen-}\\ \text{tations of } \mathrm{GL}_n(F_v) \text{ over } \mathbb{C}\end{matrix}\right\} \simeq \Phi_n(W_{F_v}) := \left\{\begin{matrix}n\text{-dimensional representations } \rho \text{ of}\\ WD_{F_v} \text{ such that } \rho|_{W_{F_v}} \text{ is semisim-}\\ \text{ple and } \rho|_{\mathbb{C}} \text{ is algebraic}\end{matrix}\right\}$$

satisfying compatibility with parabolic induction, central characters, etc. Denote the correspondence as $\pi \mapsto \rho_\pi$.

Example 2.1.3. Consider the vector space V of smooth functions on $\mathbf{P}^1_{F_v}$. Then since $\mathrm{GL}_2(F_v)$ acts on $\mathbf{P}^1_{F_v}$, it becomes a smooth representation of $\mathrm{GL}_2(F_v)$. It has a subspace $\mathbf{1}$ consisting of

[1] For a proof when $n = 2$, see [BH06].

constant functions, and the quotient $V/\mathbf{1}$ is irreducible, called the *Steinberg representation*. The L-parameter is given by:

$$\begin{aligned} W_F \rtimes \mathbb{C} &\to \mathrm{GL}_2(\mathbb{C}) \\ (w,0) &\mapsto \begin{pmatrix} \|w\|^{1/2} & \\ & \|w\|^{-1/2} \end{pmatrix} \\ (1,x) &\mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}. \end{aligned}$$

The Local Langlands Correspondence translates properties of representations of $\mathrm{GL}_n(F_v)$ to properties of representations of WD_{F_v} in the following way [Bor79]:

Proposition 2.1.4. *Let π be a smooth irreducible representation of $\mathrm{GL}_n(F_v)$, and let $\rho_\pi \colon WD_{F_v} \to \mathrm{GL}_n(\mathbb{C})$ be its Langlands parameter. Then the following hold:*

(1) π is tempered if and only if the image of $\rho_\pi|_{W_F}$ is bounded
(2) π is square-integrable modulo center if and only if ρ_π is irreducible
(3) π is supercuspidal if and only if $\rho_\pi|_{W_F}$ is irreducible.

For unramified representations of $\mathrm{GL}_n(F_v)$, the correspondence is particularly simple:

Definition 2.1.5. An irreducible smooth representation π of $\mathrm{GL}_n(F_v)$ is *unramified* if π has a $\mathrm{GL}_n(\mathfrak{o}_v)$-invariant vector.

The *Satake correspondence* is a bijection between subsets of $\mathrm{Irr}\,\mathrm{GL}_n(F_v)$ and $\Phi_n(W_{F_v})$, compatible with the Local Langlands Correspondence (and much easier!):

$$\left\{ \begin{matrix} \text{unramified representations of} \\ \mathrm{GL}_n(F_v) \text{ over } \mathbb{C} \end{matrix} \right\} \simeq S_n \backslash (\mathbb{C}^\times)^n. \tag{2.1.6}$$

Here, $(z_1, \ldots, z_n) \in S_n \backslash (\mathbb{C}^\times)^n$ is viewed as a n-dimensional representations of WD_{F_v} by

$$\begin{aligned} WD_{F_v} &\to \mathrm{GL}_n(\mathbb{C}) \\ (w,x) &\mapsto \mathrm{diag}(z_1^{\mathrm{val}_{F_v}(w)}, \ldots, z_n^{\mathrm{val}_{F_v}(w)}). \end{aligned}$$

Thus, (2.1.6) can be re-written to resemble the general Local Langlands Correspondence:

$$\left\{ \begin{matrix} \text{unramified representations of} \\ \mathrm{GL}_n(F_v) \text{ over } \mathbb{C} \end{matrix} \right\} \simeq \left\{ \begin{matrix} \text{representations of } WD_{F_v} \text{ triv-} \\ \text{ial on } \mathbb{C} \text{ and the inertia } I_{F_v} \end{matrix} \right\}. \tag{2.1.7}$$

Given a n-tuple $(z_1, \ldots, z_n) \in S_n \backslash (\mathbb{C}^\times)^n$, let

$$B_n = \{(x_{ij})_{i,j=1}^n \in \mathrm{GL}_n(F_v) : x_{ij} = 0 \text{ for } i > j\}$$

be the *Borel subgroup* of $\mathrm{GL}_n(F_v)$, the subgroup of upper triangular $n\times n$-matrices. Then $(z_1, \ldots, z_n)$ defines a character $\chi \colon B_n \to \mathbb{C}^\times$ by:

$$\chi(x_{ij})_{i,j=1}^n := z_1^{\mathrm{val}_{F_v}(x_{11})} \cdots z_n^{\mathrm{val}_{F_v}(x_{nn})},$$

where $\mathrm{val}_{F_v} \colon F_v^\times \to \mathbb{Z}$ is the *valuation*. Then the *normalized parabolic induction* of χ, denoted $\mathrm{i}_B^G \chi$, is such that $(\mathrm{i}_B^G \chi)^{\mathrm{GL}_n(\mathfrak{o}_F)}$ is 1-dimensional, hence has a unique unramified subquotient. The subquotient is moreover independent of the order of the z_i's, hence realizes the map from the right to left in (2.1.6). To go from the left to the right, we take eigenvalues of certain Hecke operators on the representation, and we denote the correspondence by $\pi_v \mapsto t_{\pi_v}$.

2.2. **Compatibility with Global Langlands.** We saw last week that the Global Langlands Conjecture predicts a bijection between cuspidal automorphic representations of $\mathrm{GL}_n(\mathbf{A})$ which are algebraic at ∞ and n-dimensional representations of the motivic Galois group $\mathcal{M}_{\mathbb{Q}}$. There should be a map $WD_{\mathbb{Q}_p} \to \mathcal{M}_{\mathbb{Q}}$, making the following diagram commute:

$$\begin{array}{ccc} \left\{\begin{array}{l}\text{cuspidal automorphic representations } \pi = \bigotimes_p \pi_p \otimes \pi_\infty \text{ of}\\ \mathrm{GL}_n(\mathbf{A}_{\mathbb{Q}}) \text{ such that } \pi_\infty \text{ is algebraic}\end{array}\right\} & \xrightarrow{\text{Global Langlands}} & \left\{\begin{array}{l} n\text{-dimensional representations of}\\ \mathcal{M}_{\mathbb{Q}}\end{array}\right\} \\ \Big\downarrow{\scriptstyle \pi \mapsto \pi_p} & & \Big\downarrow{\scriptstyle \text{restriction}} \\ \left\{\begin{array}{l}\text{irreducible smooth represen-}\\ \text{tations of } \mathrm{GL}_n(\mathbb{Q}_p)\end{array}\right\} & \xrightarrow{\text{Local Langlands}} & \left\{\begin{array}{l} n\text{-dimensional representations } \rho \text{ of}\\ WD_{F_v} \text{ such that } \rho|_{W_{F_v}} \text{ is semisim-}\\ \text{ple and } \rho|_{\mathbb{C}} \text{ is algebraic}\end{array}\right\}. \end{array}$$

3. Base change

3.1. **The local picture.** Let E_v/F_v be a finite extension with ramification degree e and unramified degree f. Then there is an embedding $\iota\colon W_{E_v} \hookrightarrow W_{F_v}$ with a commutative diagram

$$\begin{array}{ccc} W_{E_v} & \xhookrightarrow{\ \iota\ } & W_{F_v} \\ {\scriptstyle \mathrm{val}_{E_v}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{val}_{F_v}} \\ \mathbb{Z} & \xrightarrow{\ f\ } & \mathbb{Z}, \end{array}$$

where f is the *unramified degree* of E_v/F_v, i.e., the degree of the extension of the residue field k_{E_v}/k_{F_v}. The commuattive diagram induces an inclusion

$$\begin{aligned} WD_{E_v} &\hookrightarrow WD_{F_v} \\ (w, x) &\mapsto (\iota(w), x), \end{aligned}$$

since $(\mathrm{ad}\,\iota(w))x = q_{F_v}^{\mathrm{val}_{F_v}(\iota(w))}x = q_{F_v}^{f\mathrm{val}_{E_v}(w)}x = (\mathrm{ad}\,w)x$ for all $x \in \mathbb{C}$. This gives a map $\Phi_n(W_{F_v}) \to \Phi_n(W_{E_v})$. Under the local Langlands correspondence, we have a map

$$\begin{array}{ccc} \mathrm{Irr}\,\mathrm{GL}_n(F_v) & \xrightarrow{\ \sim\ } & \Phi_n(W_{F_v}) \\ {\scriptstyle \mathrm{BC}_{E_v/F_v}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{res}^{W_{F_v}}_{W_{E_v}}} \\ \mathrm{Irr}\,\mathrm{GL}_n(E_v) & \xrightarrow{\ \sim\ } & \Phi_n(W_{E_v}). \end{array}$$

Remark 3.1.1. In fact, we can be more precise about the image of BC_{E_v/F_v}. The Galois group $\mathrm{Gal}(E_v/F_v)$ acts on $\mathrm{Irr}\,\mathrm{GL}_n(E_v)$ in the obvious way, by pre-composition, denoted by $\Pi \mapsto \Pi^\sigma$,[2] and on $\Phi_n(W_{E_v})$ by conjugation on W_{E_v}. Since the Local Langlands correspondence is equivariant under this action, we expect the representation $\mathrm{BC}_{E_v/F_v}(\pi) \in \mathrm{Irr}\,\mathrm{GL}_n(E_v)$ to satisfy $\mathrm{BC}_{E_v/F_v}(\pi)^\sigma \simeq \mathrm{BC}_{E_v/F_v}(\pi)$ for any $\sigma \in \mathrm{Gal}(E_v/F_v)$.

Example 3.1.2. When π is an unramified representation of $\mathrm{GL}_n(F_v)$, with Satake parameter $t_\pi \in S_n\backslash(\mathbb{C}^\times)^n$, the base-change lift Π is an unramified representation of $\mathrm{GL}_n(E_v)$, with Satake parameter t_π^f.

Thus, we may ask whether there exists a *purely representation-theoretic* characterization/construction of the map $\mathrm{Irr}\,\mathrm{GL}_n(F_v) \to \mathrm{Irr}\,\mathrm{GL}_n(E_v)$. When E_v/F_v, such a characterization was given by [AC89] for cyclic extensions, and was boot-strapped in [CR21] to deal with all *solvable* extensions E_v/F_v.

[2] Given $\Pi\colon \mathrm{GL}_n(E_v) \to \mathrm{GL}(V)$, let $\Pi^\sigma(g) := \Pi(g^\sigma)$

We will describe the characterization explicitly, when E_v/F_v is a cyclic extension of local fields. Throughout the rest of the section, let E_v/F_v be cyclic of degree d and let σ be a generator of the Galois group $\mathrm{Gal}(E_v/F_v)$.

To describe the characterization, we make some preliminary definitions. First, let Π be a smooth irreducible representation of $\mathrm{GL}_n(E_v)$ such that $\Pi^\sigma \simeq \Pi$ (i.e., Π satisfies the conditions of Remark 3.1.1.) We hope to use the intertwining operator $I_\sigma\colon \Pi^\sigma \simeq \Pi$ to characterize base change, but there is no obvious choice for such an isomorphism. One "obvious" requirement we can ask for is that

$$I_\sigma \circ I_\sigma^\sigma \cdots \circ I_\sigma^{\sigma^{d-1}} = 1_\Pi,$$

where $I_\sigma^{\sigma^i}$ is the isomorphism $\Pi^{\sigma^{i+1}} \simeq \Pi^{\sigma^i}$ induced by I_σ. However, this still leaves an ambiguity of a d-th root of unity. [AC89] cleverly solves this problem when Π is generic[3] by requiring compatibility with the unique Whittaker model (which can be chosen to be $\mathrm{Gal}(E_v/F_v)$-invariant).

Definition 3.1.3 ([AC89, §1])**.** For $x \in \mathrm{GL}_n(E_v)$, let $\mathcal{N}x$ be the conjugacy class of $\mathrm{GL}_n(F_v)$ which is conjugate to $xx^\sigma \cdots x^{\sigma^{d-1}}$ in $\mathrm{GL}_n(E_v)$. We will say that $x \in \mathrm{GL}_n(E_v)$ is *σ-semisimple* if $\mathcal{N}x$ is semisimple.

The following definition is due to Shintani:

Definition 3.1.4 ([AC89, Definition 6.1])**.** Let E_v/F_v be a finite cyclic extension of local fields, whose Galois group is generated by $\sigma \in \mathrm{Gal}(E_v/F_v)$. Let π be an irreducible smooth representation of $\mathrm{GL}_n(F_v)$ and let Π be a representation of $\mathrm{GL}_n(E_v)$, such that $\Pi^\sigma \simeq \Pi$. Then Π is the *base change lift* of π if, for every σ-semisimple $g \in \mathrm{GL}_n(E_v)$,

$$\mathrm{tr}(\Pi(g)I_\sigma) = \mathrm{tr}\,\pi(\mathcal{N}g).$$

Now, [AC89, Theorem 6.2] states:

Theorem 3.1.5. *Let π, Π denote irreducible tempered representations of $\mathrm{GL}_n(F_v)$ and $\mathrm{GL}_n(E_v)$, respectively.*

(a) Any tempered irreducible representation π of $\mathrm{GL}_n(F_v)$ has a unique base-change lift Π of $\mathrm{GL}_n(E_v)$. The representation Π is tempered.
(b) Conversely, assume $\Pi^\sigma \simeq \Pi$ is an irreducible tempered representation of $\mathrm{GL}_n(E_v)$. Then there is at least one representation π of $\mathrm{GL}_n(F_v)$ such that Π is the base-change of π. Here π is tempered.

3.2. **The global picture.** Given a finite extension E/F of number fields, we can define base-change in the following way:

Definition 3.2.1. Let $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and let $\Pi = \bigotimes_w \Pi_w$ be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$. Then Π is a *base-change lift* of π, denoted $\mathrm{BC}_{E/F}(\pi)$, if for any place v of F and any place w of E lying above it, $\rho_{\Pi_w} = \rho_{\pi_v}|_{W_{E_w}}$, i.e., Π_w is a base-change lift of π_v.

Let E/F is a cyclic Galois extension with Galois group generated by $\sigma \in \mathrm{Gal}(E/F)$. Again using the uniqueness of Whittaker models, given a cuspidal automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_E)$ such that $\Pi^\sigma \simeq \Pi$, we have a *canonical* choice of an intertwiner $I_\sigma\colon \Pi^\sigma \to \Pi$. Now, we again have a character-theoretic characterization of base-change, again due to Shintani:

Definition 3.2.2. Let π be an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ and let Π be an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_E)$ such that $\Pi^\sigma \simeq \Pi$. Then Π is the *base-change lift* of π if, for every σ-semisimple $g \in \mathrm{GL}_n(\mathbf{A}_E)$,

$$\mathrm{tr}(\Pi(g)I_\sigma) = \mathrm{tr}\,\pi(\mathcal{N}g).$$

[3] i.e., has a non-degenerate Whittaker model.

Now for global cyclic extensions, we have ([CSS97, §6.1]):

Theorem 3.2.3. *(1) Every cuspidal automorphic representation* π *of* $\mathrm{GL}_2(\mathbf{A}_F)$ *has a unique base change lift to* $\mathrm{GL}_2(\mathbf{A}_E)$*. The lift is also cuspidal unless* E/F *is quadratic*

(2) If two cuspidal representations π *and* π' *of* $\mathrm{GL}_2(\mathbf{A}_F)$ *have the same base-change lift to* E*, then* $\pi' \simeq \pi\otimes$ *for some character* ω *of* $F^\times N_{E/F}(\mathbf{A}_E^\times)\backslash\mathbf{A}_F^\times$*.*

(3) A cuspidal representation Π *of* $\mathrm{GL}_2(\mathbf{A}_E)$ *is the base-change lift of some representation* π *of* $\mathrm{GL}_2(\mathbf{A}_F)$ *if and only if* $\Pi^\sigma \simeq \Pi$*.*

Moreover, similar to the local case, base-change lifting can be boot-strapped to solvable extensions.

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