Galois representations associated to modular forms

Vijay Srinivasan

We should remember that modularity of elliptic curves over $\mathbb{Q}$ was a major advance in the global Langlands program for $\mathrm{GL}_2/\mathbb{Q}$. These brief notes are meant to tell us why the Langlands program predicts e.g. that elliptic curves should be associated to holomorphic weight 2 cuspforms and that odd Artin representations should be associated to holomorphic weight 1 cuspforms. We'll focus on GL_1 and GL_2 over $\mathbb{R}$ and $\mathbb{Q}$.

1. Local Langlands for $\mathrm{GL}_n/\mathbb{R}$

There is a compatible family of bijections

$$\left\{\begin{array}{c}\text{irreducible admissible complex}\\ \text{representations of } \mathrm{GL}_n(\mathbb{R})\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}n\text{-dimensional complex}\\ \text{representations of } W_{\mathbb{R}}\end{array}\right\}$$

Basic features of this theory:

- As usual, the LHS is to be understood up to infinitesimal equivalence.
- The *Weil group* $W_{\mathbb{R}}$ of $\mathbb{R}$ fits into an exact sequence
$$0 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \Gamma_{\mathbb{R}} \longrightarrow 0.$$
Concretely, $W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ where $j^2 = -1$ and $j\alpha = \overline{\alpha}j$ for all $\alpha \in \mathbb{C}^\times$.
- The case $n = 1$ is trivial since both sides are continuous characters of $\mathbb{R}^\times$.
- The central character on the LHS corresponds to the determinant of the RHS, and the correspondence is compatible with twisting by characters.
- Conjugating if necessary, we may assume that $\rho(\mathbb{C}^\times)$ lands inside $T(\mathbb{C})$, where T is the diagonal torus of GL_n. Then $\rho|_{\mathbb{C}^\times}$ can be written as $z \mapsto z^a\overline{z}^b$ where $a, b \in X_*(T)_{\mathbb{C}}$ and $a - b \in X_*(T)$. Then a should be the infinitesimal character of π.
- We call ρ *algebraic* if in fact $a, b \in X_*(T)$. We will also call π on the LHS *algebraic* if its associated ρ is algebraic (i.e. if its infinitesimal character is integral).

2. Special case: Local Langlands for $\mathrm{GL}_2/\mathbb{R}$

For $\mathrm{GL}_2(\mathbb{R})$ let's go into more detail. Given characters χ_1, χ_2 of $\mathbb{R}^\times$ we use them to construct a character χ of the split torus in the natural way. We can define the (normalized) parabolic induction

$$\pi(\chi_1, \chi_2) \coloneqq \mathrm{Ind}_{B(\mathbb{R})}^{\mathrm{GL}_2(\mathbb{R})}(\chi \otimes \delta_B^{1/2})$$

By abuse I'll also write $\pi(\chi_1, \chi_2)$ for the associated $(\mathfrak{g}, K)$-module of K-finite vectors (here $K = O(2)$). We write $\chi_i(r) = \mathrm{sgn}(r)^{\varepsilon_i}|r|^{s_i}$, where $\varepsilon_i \in \mathbb{Z}/2\mathbb{Z}$ and $s_i \in \mathbb{C}$. We set $s = \frac{s_1 - s_2 + 1}{2}$, $\lambda = s(1-s)$, $\mu = s_1 + s_2$, and $\varepsilon = \varepsilon_1 + \varepsilon_2 \in \mathbb{Z}/2\mathbb{Z}$. There are two types of infinite dimensional irreducible $(\mathfrak{g}, K)$-modules that can appear:

- If $s \neq k/2$ for some $k \equiv \varepsilon \mod 2$, then $\pi(\chi_1, \chi_2)$ is irreducible and corresponds to the representation $\chi_1 \oplus \chi_2$ of $W_{\mathbb{R}}$.

- If $s = k/2$ for some $k \equiv \varepsilon \mod 2$, then there is a unique infinite dimensional subquotient $D_\mu(k)$ of $\pi(\chi_1, \chi_2)$. In this case we have $\chi_1\chi_2^{-1} = \mathrm{sgn}(r)^\varepsilon |r|^{k-1} = \mathrm{sgn}(r) r^{k-1}$. In this case the representation of $W_\mathbb{R}$ is
$$\mathrm{Ind}_{\mathbb{C}^\times}^{W_\mathbb{R}}(z \mapsto z^{k-1}(z\bar{z})^{s_2}).$$

Algebraic corresponds to $s_1, s_2 \in \mathbb{Z}$. For example the discrete series representations $D_0(k)$ are all algebraic.

3. Global Langlands for $\mathrm{GL}_n/\mathbb{Q}$

The global situation is significantly more mysterious. We have the following family of conjectural bijections as $n \geq 1$ varies:

$$\left\{ \begin{array}{c} \text{cuspidal automorphic} \\ \text{representations of } \mathrm{GL}_n(\mathbb{A}), \\ \text{algebraic at } \infty \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{irreducible motives} \\ \text{over } \mathbb{Q} \text{ of rank } n \text{ with} \\ \text{coefficients in } \overline{\mathbb{Q}} \end{array} \right\}$$

It is probably worth mentioning that Clozel, in addition to Langlands, is heavily responsible for this formulation of reciprocity.

- I don't think the "correct" formulation of motive is finalized yet, so we're being deliberately vague. The correct category of motives, whatever it is, should be Tannakian and every Weil cohomology theory should factor through it. The standard conjectures imply the existence of this category.
- Let $\mathcal{M}_\mathbb{Q}$ denote the (conjectural) pro-algebraic group whose irreducible representations are the RHS; this is the *motivic Galois group* of $\mathbb{Q}$.
- Pure motives of weight 0 (Artin motives) form a Tannakian subcategory of the RHS, and there is a resulting short exact sequence
$$0 \longrightarrow \mathcal{M}_\mathbb{Q}^0 \longrightarrow \mathcal{M}_\mathbb{Q} \longrightarrow \Gamma_\mathbb{Q} \longrightarrow 0$$
where $\mathcal{M}_\mathbb{Q}^0$ should be a pro-reductive linear algebraic group.
- Fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. We obtain a compatible diagram of exact sequences
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & W_\mathbb{R} & \longrightarrow & \Gamma_\mathbb{R} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{M}_\mathbb{Q}^0 & \longrightarrow & \mathcal{M}_\mathbb{Q} & \longrightarrow & \Gamma_\mathbb{Q} & \longrightarrow & 0 \end{array}$$
and the global correspondence is required to be compatible with the local correspondence at ∞.
- After fixing an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, we also require compatibility with local Langlands for $\mathbb{Q}_p$, but since we haven't discussed that, let's omit the details.

We are probably used to thinking of Langlands as having Galois representations on one side, so let's mention that. To a pure motive over $\mathbb{Q}$ is associated a continuous ℓ-adic Galois representation on its étale cohomology. What are the restrictions on this representation? It must be unramified almost everywhere and de Rham at ℓ. The Fontaine–Mazur conjecture predicts conversely that any such ℓ-adic Galois representation comes from a motive, so combining the reciprocity conjecture above with that of Fontaine and Mazur leads to the following Galois-theoretic version of Langlands reciprocity:

$$\left\{\begin{array}{c}\text{cuspidal automorphic}\\ \text{representations of } \mathrm{GL}_n(\mathbb{A}),\\ \text{algebraic at } \infty\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{irreducible } \ell\text{-adic } \Gamma_{\mathbb{Q}}\text{-representations}\\ \text{of dimension } n, \text{ unramified a.e.}\\ \text{and de Rham at } \ell\end{array}\right\}$$

4. Special case: Global Langlands for $\mathrm{GL}_1/\mathbb{Q}$

I think we should discuss further the relationship between the two objects that appeared on the RHS of global Langlands above, namely:

- Representations of the complex pro-algebraic group $\mathcal{M}_{\mathbb{Q}}$
- Continuous ℓ-adic representations of $\Gamma_{\mathbb{Q}}$

I think that going from one to the other purely group-theoretically, that is, without actually thinking about motives, is a pretty subtle matter, and I don't really understand it.

To illustrate the subtlety, let's just do class field theory. Let's let $C_K = K^\times \backslash \mathbb{A}_K^\times$ and as usual let Γ_K denote the Galois group. The original formulation of class field theory amounts to something like:

$$\{\text{finite order characters of } C_K\} \longleftrightarrow \{\text{finite order characters of } \Gamma_K\}$$

This proceeds via the Artin map

$$\mathrm{rec} : C_K \longrightarrow \Gamma_K^{\mathrm{ab}}$$

Importantly, even without knowing anything about the map, we notice that the target is profinite and thus the Artin map must be trivial on the connected component of the identity; if C_K^0 denotes this connected component and we set $\pi_0(C_K) \coloneqq C_K/C_K^0$, then we obtain a map

$$\mathrm{rec} : \pi_0(C_K) \xrightarrow{\sim} \Gamma_K^{\mathrm{ab}}$$

and the fact that this map is an isomorphism is the content of class field theory.

This is not quite the same thing as global Langlands for GL_1, because not every automorphic character of GL_1/K is finite order. For example we have the norm character $\|\cdot\| : C_K \to \mathbb{R}_{>0}$. So, for example, every $s \in \mathbb{C}$ yields an automorphic character $\|\cdot\|^s$.

Let's now specialize to $K = \mathbb{Q}$. The idele class group is just

$$\{\pm 1\}\backslash(\mathbb{R}^\times \times \widehat{\mathbb{Z}}^\times) \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$$

and from an idele class character χ we obtain a character χ_∞ of $\mathbb{R}^\times$ via the embedding $\mathbb{R}^\times \hookrightarrow C_K$, and likewise a character χ_p for each $p < \infty$. We will say χ is *algebraic* if $\chi_\infty(r) = r^n$ for some $n \in \mathbb{Z}$ and all $r > 0$.

To obtain an ℓ-adic Galois character from an algebraic Hecke character, fix an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$. We consider the new character $\chi' : \mathbb{A}_{\mathbb{Q}}^\times \to \overline{\mathbb{Q}}_\ell^\times$ given by $x = (x_\infty, (x_p)_{p<\infty})$ by

$$\chi'(x) = \iota^{-1}(\chi(x)x_\infty^{-n}) \cdot x_\ell^n$$

This is continuous, still makes sense as an idele class character, and is *trivial* at ∞. Thus the Artin map yields a corresponding Galois character. This highlights the importance of *algebraicity*; we could not have performed this maneuver if χ_∞ involved raising to an arbitrary complex power.

The motivic picture is clearer. We are looking for rank 1 motives over $\mathbb{Q}$, with coefficients in $\overline{\mathbb{Q}}$. Motives of weight 0 correspond to Artin representations, so rank 1 motives of weight

0 correspond to characters $\Gamma_{\mathbb{Q}} \to \overline{\mathbb{Q}}^{\times}$. But there is also the Lefschetz motive $\mathbb{L}$, and its ℓ-adic realization is precisely the inverse of the ℓ-adic cyclotomic character. The automorphic character of $C_{\mathbb{Q}}$ corresponding to $\mathbb{L}$ is precisely the norm character.

5. Special case: Global Langlands for $\mathrm{GL}_2/\mathbb{Q}$

Let's look at $n = 2$. What if we have a complex Galois representation? This corresponds to a representation of $\mathcal{M}_{\mathbb{Q}}$ which is trivial on $\mathcal{M}_{\mathbb{Q}}^0$. In particular, ρ_∞ must be trivial on $\mathbb{C}^\times$. There are two choices; either π_∞ is $D_0(1)$ or it is $\pi(1,1)$. In the former case, we have $\rho_\infty(j) \sim \left[\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}\right]$ and in the latter case we have $\rho_\infty(j) = \left[\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right]$. Thus $D_0(1)$ corresponds to *odd* Galois representations while $\pi(1,1)$ corresponds to *even* Galois representations. In classical terms, a holomorphic cuspform of weight 1 corresponds to an odd Galois representation while a Maass cuspform of weight 0 with Laplace eigenvalue $\frac{1}{4}$ corresponds to an even Galois representation.

What happens if we want to know about holomorphic cuspforms of weight ≥ 2? If f is holomorphic of weight 2, then π_∞ is $D_0(2)$. The resulting representation of $W_{\mathbb{R}}$ is given by

$$\mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}} \mathrm{id} \simeq \left(\rho_\infty : W_{\mathbb{R}} \to \mathrm{GL}_2(\mathbb{C}), \qquad \rho_\infty|_{\mathbb{C}^\times} = \begin{bmatrix} z & \\ & \overline{z} \end{bmatrix}, \rho_\infty(j) = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\right)$$

In particular, the resulting Hodge structure is of type $(0,1)+(1,0)$, and so should correspond to a summand in the H^1 of an abelian variety. If this motive in fact had coefficients in $\mathbb{Q}$, then for reasons of rank it would correspond to an elliptic curve. Conversely, we see that an elliptic curve should correspond to a modular form of weight 2. A holomorphic form of weight $k \geq 2$ will in general just correspond to a Hodge structure of type $(0, k-1)+(k-1, 0)$. These motives are instead found as summands in Kuga–Sato varieties.

When we pass to ℓ-adic Galois representations, the conditions on the Hodge numbers at infinity are thus transferred to constraints on the *Hodge–Tate weights* of the $\Gamma_{\mathbb{Q}_\ell}$-representation. We see that holomorphic forms of weight $k \geq 2$ have distinct HT weights 0 and $k-1$ while a holomorphic form of weight 1 has HT weight 0 with multiplicity 2.

20th century progress for GL_2. Some parts of the global Langlands correspondence are understood for GL_2, but a complete understanding (even in the motivic setting to which we have restricted ourselves) is still elusive. In particular, the Maass forms are very poorly understood.

- holomorphic cusp form of weight 2 $\longrightarrow$ abelian variety: Eichler–Shimura
- holomorphic cusp form of weight > 2 $\longrightarrow$ ℓ-adic Galois representation: Deligne
- holomorphic cusp form of weight 1 $\longrightarrow$ Galois representation: Deligne–Serre
- Galois representation with solvable image $\longrightarrow$ automorphic form: Langlands–Tunnell
- holomorphic cusp form of weight > 2 $\longrightarrow$ motive: Scholl
- elliptic curve/$\mathbb{Q}$ $\longrightarrow$ holomorphic cusp form of weight 2: Wiles, Taylor–Wiles, ...

In the 21st century, things have happened, but I'm not entirely clear on what! Certainly there has been progress on the Fontaine–Mazur conjecture as well as the Artin conjecture, but I think the question of Maass forms in the icosahedral case is still pretty open. Also I don't think much is known about associating a Galois representation to a Maass form.

Discussion: the map $W_{\mathbb{R}} \to \mathcal{M}_{\mathbb{Q}}$

We didn't say before what the map $W_{\mathbb{R}} \to \mathcal{M}_{\mathbb{Q}}$ was. To give such a map is equivalent to functorially assigning to a motive over $\mathbb{Q}$ (with coefficients in $\overline{\mathbb{Q}}$) a representation of $W_{\mathbb{R}}$, compatibly with tensor products etc. Given a motive $M/\mathbb{R}$ with coefficients in $\overline{\mathbb{Q}}$, let $H_B(M)$ denote its Betti realization, a $\overline{\mathbb{Q}}$-vector space. We have

$$H_B(M) \otimes_{\iota_\infty} \mathbb{C} \cong \bigoplus H^{p,q}$$

given by Hodge theory. The $\overline{\mathbb{Q}}$-vector space $H_B(M)$ admits an action by complex conjugation $\sigma \in \Gamma_{\mathbb{R}}$ (via the conjugation action on $\mathbb{C}$-points of varieties over $\mathbb{R}$). The Hodge structure determines a representation $\rho : \mathbb{C}^\times \to \mathrm{End}(H_B(M)_{\mathbb{C}})$ in the usual way, and we extend this to a representation of $W_{\mathbb{R}}$ by setting

$$\rho(j)|_{H^{p,q}} = (-1)^q \cdot \sigma$$

which is indeed a $\mathbb{C}$-linear automorphism of $H_B(M)_{\mathbb{C}}$ commuting with the action of $\mathbb{C}^\times$ in the correct way and which squares to $\rho(-1)$ (to see this, observe that σ interchanges $H^{p,q}$ and $H^{q,p}$).

Discussion: CM elliptic curves and Hecke characters

For this we will work over a general number field L, and we will only use global Langlands for GL_1.

Suppose we had an E/L with CM by the ring of integers $\mathcal{O}$ in an imaginary quadratic field $K \subset \mathbb{C}$. Let $M = h^1(E)$ denote the relevant motive over L.

First suppose that L contains K. Then the motive $M = h^1(E)$ decomposes as a sum of rank 1 motives $M = M^+ \oplus M^-$ according to the two embeddings $K \hookrightarrow \overline{\mathbb{Q}}$ and the two motives are interchanged by complex conjugation on L. Thus we expect an algebraic Hecke character ψ_E defined on $\mathbb{A}_L^\times$ such that

$$L(E/L, s) = L(\psi_E, s)L(\overline{\psi}_E, s).$$

Note that the L-functions on the RHS are *not* Artin L-functions.

Now suppose that L does not contain K. Then M is irreducible. Let $L' = LK$ and let c generate $\mathrm{Gal}(L'/L)$. Now we have $M_{L'} = M_{L'}^+ \oplus M_{L'}^-$ and the two motives are interchanged by c. Let $\tau : \operatorname{Spec} L' \to \operatorname{Spec} L$; then we have a natural map $\tau_* M_{L'}^+ \to M$, which is an isomorphism (as can be checked on Betti or ℓ-adic realizations). If ψ'_E denotes the expected Hecke character over L' corresponding to $M_{L'}^+$, then we have

$$L(E/L, s) = L(\tau_* M_{L'}^+, s) = L(M_{L'}^+, s) = L(\psi'_E, s)$$

The construction of ψ_E and ψ'_E in the two cases, and proof of the L-function identities, are old theorems of Deuring. They can be found in Silverman *Advanced Topics...* II.9.