MODULARITY SEMINAR: TANGENT SPACES OF DEFORMATION RINGS

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Abstract. I'd like to thank Halloween for sponsoring today's lecture.

These notes are based on [Gee22, Section 3] and [Zho14] and [CHT08, Section 2].

Notation. Let ℓ be a prime greater than 2, and $L/\mathbb{Q}_\ell$ a finite extension with ring of integers $\mathcal{O}$, uniformizer λ and residue field $\mathbb{F}$. We define the category $\mathcal{C}_\mathcal{O}$ whose objects are complete Noetherian local $\mathcal{O}$-algebras A such that $A/\mathfrak{m}_A = \mathbb{F}$, and whose morphisms are local homomorphisms $f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ of local algebras (i.e. $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$). Let G be a profinite group satisfying the following axiom:

Axiom 0.1. *For all finite index subgroups $\Delta \subset G$, $\Delta/\overline{\langle[\Delta, \Delta], \Delta^\ell\rangle}$ is finitely generated.*

Finally, let n be a positive integer such that $\ell \nmid n$, and let $\overline{\rho} : G \to \mathrm{GL}_n(\mathbb{F})$ be a continuous representation.

1. The lifting and deformation functors

Definition 1.1. Let $A \in \mathcal{C}_\mathcal{O}$. A *lift* of $\overline{\rho}$ to A is a continuous representation $\rho : G \to \mathrm{GL}_n(A)$ such that $\rho \bmod \mathfrak{m}_A = \overline{\rho}$. (By $\rho \bmod \mathfrak{m}_A$ we mean the composition of ρ with the projection $\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F})$.)

Recall the *lifting functor* of $\overline{\rho}$, which maps to sets of lifts,

$$\begin{aligned} \mathscr{R}^\square_{\overline{\rho}} : \mathcal{C}_\mathcal{O} &\to \mathrm{Set}, \\ A &\mapsto \{\text{lifts of } \overline{\rho} \text{ to } A\}. \end{aligned}$$

Fact 1.2. *The functor $\mathscr{R}^\square_{\overline{\rho}}$ is represented by some $R^\square_{\overline{\rho}} \in \mathcal{C}_\mathcal{O}$, called the* universal lifting ring.

Equivalently, there exists a continuous representation $\rho^\square : G \to \mathrm{GL}_n(R^\square_{\overline{\rho}})$, called the *universal lifting*, with the following property: for all $A \in \mathcal{C}_\mathcal{O}$ and $\rho : G \to \mathrm{GL}_n(A)$ lifting $\overline{\rho}$, there exists a unique $f_\rho : R^\square_{\overline{\rho}} \to A$ making the following diagram commute. (By abuse of

$$\begin{tikzcd} G \arrow[r, "\rho^\square"] \arrow[dr, "\rho"'] & \mathrm{GL}_n(R^\square_{\overline{\rho}}) \arrow[d, dashed, "f_\rho"] \\ & \mathrm{GL}_n(A) \end{tikzcd}$$

notation, f_ρ also denotes the induced map $\mathrm{GL}_n(R^\square_{\overline{\rho}}) \to \mathrm{GL}_n(A)$.)

These are the main definitions for the lifting functor. For the deformation functor, we first recall that if $\mathrm{End}_{\mathbb{F}[G]}\,\overline{\rho} = \mathbb{F}$ then we say that $\overline{\rho}$ is *Schur*. And if $\overline{\rho} \otimes_\mathbb{F} \overline{\mathbb{F}}$ is irreducible, then we say that $\overline{\rho}$ is *absolutely irreducible*. Moreover, it is absolutely irreducible if and only if it is irreducible and Schur.

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Definition 1.3. Suppose $\overline{\rho}$ is Schur. A *deformation* of $\overline{\rho}$ to A is an equivalence class of liftings, where $\rho \sim \rho'$ if and only if $\rho' = a\rho a^{-1}$ for some $a \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$ (equivalently, some $a \in \mathrm{GL}_n(A)$).

We can define the *deformation functor* of $\overline{\rho}$, which maps to sets of deformations,

$$\begin{aligned} \mathscr{R}_{\overline{\rho}} &: \mathcal{C}_{\mathcal{O}} \to \mathrm{Set}, \\ A &\mapsto \{\text{lifts of } \overline{\rho} \text{ to } A\}/\sim. \end{aligned}$$

Fact 1.4. *If $\overline{\rho}$ is Schur, the functor $\mathscr{R}_{\overline{\rho}}$ is represented by some $R^{\mathrm{univ}}_{\overline{\rho}} \in \mathcal{C}_{\mathcal{O}}$, called the* universal deformation ring*.*

Equivalently, there exists a continuous representation $\rho^{\mathrm{univ}} : G \to \mathrm{GL}_n(R^{\mathrm{univ}}_{\overline{\rho}})$ up to $\sim$-equivalence, called the *universal deformation*, with the following property: for all $A \in \mathcal{C}_{\mathcal{O}}$ and $\rho : G \to \mathrm{GL}_n(A)$ lifting $\overline{\rho}$, there exists a unique $f_\rho : R^{\mathrm{univ}}_{\overline{\rho}} \to A$ such that $f_\rho \circ \rho^{\mathrm{univ}} \sim \rho$.

From the definitions, there is a natural map $R^{\square}_{\overline{\rho}} \to R^{\mathrm{univ}}_{\overline{\rho}}$. In fact, if $\overline{\rho}$ is absolutely irreducible, then $R^{\square}_{\overline{\rho}}$ is isomorphic to a power series ring in (n^2-1) variables over $R^{\mathrm{univ}}_{\overline{\rho}}$.

2. Tangent spaces

Knowing that the rings $R^{\square}_{\overline{\rho}}$ and $R^{\mathrm{univ}}_{\overline{\rho}}$, it is worthy to study their properties such as their Krull dimension, number of generators and relations, etc. In pursuit of this the study of tangent spaces is very useful.

To begin, the adjoint representation $\mathrm{ad}\,\overline{\rho}$ is given by the composite

$$G \to \mathrm{GL}_n(\mathbb{F}) \xrightarrow{\mathrm{ad}} \mathrm{Aut}(M_n(\mathbb{F})),$$

where the map ad is given by

$$g \mapsto (\phi \mapsto g\phi g^{-1}).$$

We also use $\mathrm{ad}\,\overline{\rho}$ to denote $M_n(\mathbb{F})$ as an $\mathbb{F}[G]$-module.

Proposition 2.1. *The following are in natural bijection:*

1. $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\square}_{\overline{\rho}}}/\langle \mathfrak{m}^2_{R^{\square}_{\overline{\rho}}}, \lambda\rangle, \mathbb{F})$
2. $\mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(R^{\square}_{\overline{\rho}}, \mathbb{F}[\epsilon]/\langle \epsilon^2\rangle)$
3. $\mathscr{R}^{\square}_{\overline{\rho}}(\mathbb{F}[\epsilon]/\langle\epsilon^2\rangle) = \{$*lifts of* $\overline{\rho}$ *to* $\mathbb{F}[\epsilon]/\langle\epsilon^2\rangle\}$
4. $Z^1(G, \mathrm{ad}\,\overline{\rho})$*, the continuous* 1*-cocycles.*

Proof. For (1) $\Rightarrow$ (2), define a map $R^{\square}_{\overline{\rho}} \to \mathbb{F}[\epsilon]/\langle\epsilon^2\rangle$ by $a + x \mapsto \overline{a} + f(x)\epsilon$ for all $a \in \mathcal{O}$, $x \in \mathfrak{m}_{R^{\square}_{\overline{\rho}}}$. (Note that $R^{\square}_{\overline{\rho}}/\mathfrak{m}_{R^{\square}_{\overline{\rho}}} \simeq \mathcal{O}/\lambda = \mathbb{F}$. Thus $\mathcal{O} \cap \mathfrak{m}_{R^{\square}_{\overline{\rho}}} = \lambda$, so this is well-defined.)

For (2) $\Rightarrow$ (1), note that under a map $R^{\square}_{\overline{\rho}} \to \mathbb{F}[\epsilon]/\langle\epsilon^2\rangle$, the maximal ideal $\mathfrak{m}_{R^{\square}_{\overline{\rho}}}$ must map to the maximal ideal $\epsilon\mathbb{F}[\epsilon]/\langle\epsilon^2\rangle$, which we identify with $\mathbb{F}$. Moreover, the kernel contains $\mathfrak{m}^2_{R^{\square}_{\overline{\rho}}}$ and also contains λ.

The bijection (2) $\Leftrightarrow$ (3) is by definition.

For (3) $\Rightarrow$ (4), we send a cocycle $\phi \in Z^1(G, \mathrm{ad}\,\overline{\rho})$ to the lifting $\rho : G \to \mathrm{GL}_n(\mathbb{F}[\epsilon]/\langle\epsilon^2\rangle)$, $\rho(g) = (1 + \phi(g)\epsilon)\overline{\rho}(g)$. □

Proposition 2.2. *Suppose $\overline{\rho}$ is absolutely irreducible. Then we have a natural bijection between* $\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\mathrm{univ}}_{\overline{\rho}}}/\langle \mathfrak{m}^2_{R^{\mathrm{univ}}_{\overline{\rho}}}, \lambda\rangle, \mathbb{F})$ *and* $H^1(G, \mathrm{ad}\,\overline{\rho})$*.*

Proof. Replicate the arguments above. □

Corollary 2.3. *We have*

$$\dim_{\mathbb{F}} Z^1(G, \operatorname{ad}\overline{\rho}) = \dim_{\mathbb{F}} \mathfrak{m}_{R^{\square}_{\overline{\rho}}}/\langle \mathfrak{m}^2_{R^{\square}_{\overline{\rho}}}, \lambda\rangle = \dim_{\mathbb{F}} H^1(G, \operatorname{ad}\overline{\rho}) + n^2 - \dim_{\mathbb{F}} H^0(G, \operatorname{ad}\overline{\rho}).$$

Proof. Consider the exact sequence

$$0 \to H^0(G, \operatorname{ad}\overline{\rho}) \to \operatorname{ad}\overline{\rho} \to Z^1(G, \operatorname{ad}\overline{\rho}) \to H^1(G, \operatorname{ad}\overline{\rho}) \to 0,$$

where the middle arrow is $\phi \mapsto (\gamma \mapsto \gamma\phi - \phi)$. □

In particular, if $d = \dim_{\mathbb{F}} Z^1(G, \operatorname{ad}\overline{\rho})$, then we can choose a surjection $\phi : \mathcal{O}[\![x_1, \dots, x_d]\!] \to R^{\square}_{\overline{\rho}}$. Similarly, if $\overline{\rho}$ is absolutely irreducible, we can choose a surjection $\phi' : \mathcal{O}[\![x_1, \dots, x_{d'}]\!] \to R^{\mathrm{univ}}_{\overline{\rho}}$, where $d' = \dim_{\mathbb{F}} H^1(G, \operatorname{ad}\overline{\rho})$.

Lemma 2.4. *If $J = \ker\phi$ or $J = \ker\phi'$, then there is an injection $\operatorname{Hom}_{\mathbb{F}}(J/\mathfrak{m}J, \mathbb{F}) \hookrightarrow H^2(G, \operatorname{ad}\overline{\rho})$, where $\mathfrak{m} = \langle \lambda, x_1, \dots, x_d\rangle$ or $\langle \lambda, x_1, \dots, x_{d'}\rangle$ denotes the maximal ideal of $\mathcal{O}[\![x_1, \dots, x_d]\!]$ or $\mathcal{O}[\![x_1, \dots, x_{d'}]\!]$, respectively.*

Proof. Let $f \in \operatorname{Hom}_{\mathbb{F}}(J/\mathfrak{m}J, \mathbb{F})$. Consider

$$\begin{tikzcd} \rho^{\square} : G \arrow[r] & \mathrm{GL}_n(R^{\square}_{\overline{\rho}}) = \mathrm{GL}_n(\mathcal{O}[\![x_1, \dots, x_d]\!]/J) \\ & \mathrm{GL}_n(\mathcal{O}[\![x_1, \dots, x_d]\!]/\mathfrak{m}J) \arrow[u, two heads] \end{tikzcd}$$

For $g \in G$, let $\widetilde{\rho}(g)$ be a lift of $\rho^{\square}(g)$ to $\mathrm{GL}_n(\mathcal{O}[\![x_1, \dots, x_d]\!]/\mathfrak{m}J)$. Define

$$c_f(g, h) = f(\widetilde{\rho}(gh)\widetilde{\rho}(h)^{-1}\widetilde{\rho}(g)^{-1} - 1_n) \in M_n(\mathbb{F}).$$

One shows that $c_f \in Z^2(G, \operatorname{ad}\overline{\rho})$, i.e. it is a 2-cocycle. We claim that $f \mapsto [c_f]$ is the desired injection.

Note that $[c_f] = 0$ if and only if there exists a set-theoretic map $\widetilde{\rho} : G \to \mathrm{GL}_n(\mathcal{O}[\![x_1, \dots, x_d]\!]/\mathfrak{m}J)$ such that $\widetilde{\rho} \mod J_f$ is a homomorphism where $J_f = \ker(J \to J/\mathfrak{m}J \to \mathbb{F})$. Suppose this is true. By the universal property of $R^{\square}_{\overline{\rho}}$, we can complete the diagram

$$\begin{tikzcd} G \arrow[r, "\rho^{\square}"] \arrow[dr, "\widetilde{\rho}"'] & \mathrm{GL}_n(\mathcal{O}[\![x_1, \dots, x_d]\!]/J) \arrow[d, dashed] \\ & \mathrm{GL}_n(\mathcal{O}[\![x_1, \dots, x_d]\!]/J_f) \end{tikzcd}$$

Suppose $f \neq 0$. Then we have an exact sequence

$$0 \to \mathbb{F} \to \mathcal{O}[\![x_1, \dots, x_d]\!]/J_f \to \mathcal{O}[\![x_1, \dots, x_d]\!]/J \to 0.$$

□

Corollary 2.5. *If $H^2(G, \operatorname{ad}\overline{\rho}) = 0$, then $R^{\square}_{\overline{\rho}} \simeq \mathcal{O}[\![x_1, \dots, x_d]\!]$, where $d = \dim_{\mathbb{F}} Z^1(G, \operatorname{ad}\overline{\rho})$.*

In any case, the Krull dimension of $R^{\square}_{\overline{\rho}}$ is at least

$$1 + n^2 - \dim_{\mathbb{F}} H^0(G, \operatorname{ad}\overline{\rho}) + \dim_{\mathbb{F}} H^1(G, \operatorname{ad}\overline{\rho}) - \dim_{\mathbb{F}} H^2(G, \operatorname{ad}\overline{\rho}).$$

If $\overline{\rho}$ is absolutely irreducible, then the Krull dimension of $R^{\mathrm{univ}}_{\overline{\rho}}$ is at least

$$1 + \dim_{\mathbb{F}} H^1(G, \operatorname{ad}\overline{\rho}) - \dim_{\mathbb{F}} H^2(G, \operatorname{ad}\overline{\rho}).$$

Proof. Combine Corollary 2.3 and Lemma 2.4. □

3. Deformation problems

Definition 3.1. By a *deformation problem* $\mathcal{D}$ we mean a collection of liftings (R,ρ) of $(\mathbb{F},\overline{\rho})$ (with $R\in\mathcal{C}_{\mathcal{O}}$) satisfying the following properties.

- $(\mathbb{F},\overline{\rho})\in\mathcal{D}$.
- If $f:R\to S$ is a morphism in $\mathcal{C}_{\mathcal{O}}$ and $(R,\rho)\in\mathcal{D}$, then $(S,f\circ\rho)\in\mathcal{D}$.
- If $f:R\hookrightarrow S$ is an injective morphism in $\mathcal{C}_{\mathcal{O}}$ then $(R,\rho)\in\mathcal{D}$ iff $(S,f\circ\rho)\in\mathcal{D}$.
- Suppose that $R_1,R_2\in\mathcal{C}_{\mathcal{O}}$ and I_1,I_2 are closed ideals of R_1,R_2, respectively such that there is an isomorphism $f:R_1/I_1\xrightarrow{\sim}R_2/I_2$. Suppose also that $(R_1,\rho_1),(R_2,\rho_2)\in\mathcal{D}$, and that $f(\rho_1 \bmod I_1)=\rho_2 \bmod I_2$.
  Then $(\{(a,b)\in R_1\oplus R_2 \mid f(a \bmod I_1)=b \bmod I_2\},\rho_1\oplus\rho_2)\in\mathcal{D}$.
- If (R,ρ) is a lifting of $(\mathbb{F},\overline{\rho})$ and $I_1\supset I_2\supset\cdots$ is a sequence of ideals of R with $\cap_j I_j=0$, and $(R/I_j,\rho \mod I_j)\in\mathcal{D}$ for all j, then $(R,\rho)\in\mathcal{D}$.
- If $(R,\rho)\in\mathcal{D}$ and $a\in\ker(\mathrm{GL}_n(R)\to\mathrm{GL}_n(\mathbb{F}))$, then $(R,a\rho a^{-1})\in\mathcal{D}$.

Note that each element $a\in\ker(\mathrm{GL}_n(R^{\square}_{\overline{\rho}})\to\mathrm{GL}_n(\mathbb{F}))$ acts on $R^{\square}_{\overline{\rho}}$ via the universal property and by sending $\rho^{\square}$ to $a^{-1}\rho^{\square}a$. (In general, this is not a group action.)

Proposition 3.2. *There is a bijection*

$$\begin{aligned}\{\textit{deformation problems}\}\leftrightarrow\{\ker(\mathrm{GL}_n(R^{\square}_{\overline{\rho}}\to\mathrm{GL}_n(\mathbb{F}))\textit{-invariant radical ideals of } R^{\square}_{\overline{\rho}}\}.\\ \mathcal{D}\mapsto I(\mathcal{D})\\ \mathcal{D}(I)\leftarrow I\end{aligned}$$

which is defined as follows.

If $\mathcal{D}$ is a deformation problem, then there is a $\ker(\mathrm{GL}_n(R^{\square}_{\overline{\rho}}\to\mathrm{GL}_n(\mathbb{F}))$*-invariant radical ideal $I(\mathcal{D})$ of $R^{\square}_{\overline{\rho}}$ such that $(R,\rho)\in\mathcal{D}$ if and only if the map $R^{\square}_{\overline{\rho}}\to R$ induced by ρ factors through the quotient $R^{\square}_{\overline{\rho}}/I(\mathcal{D})$.*

If I is a $\ker(\mathrm{GL}_n(R^{\square}_{\overline{\rho}}\to\mathrm{GL}_n(\mathbb{F}))$*-invariant radical ideal of $R^{\square}_{\overline{\rho}}$, then*

$$\mathcal{D}(I)=\{(R,\rho)\mid R^{\square}_{\overline{\rho}}\to R\textit{ factors through }R^{\square}_{\overline{\rho}}/I\}$$

is a deformation problem.

Definition 3.3. Let $\widetilde{L}(\mathcal{D})\subseteq Z^1(G,\mathrm{ad}\,\overline{\rho})\simeq\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\square}_{\overline{\rho}}}/\langle\mathfrak{m}^2_{R^{\square}_{\overline{\rho}}},\lambda\rangle,\mathbb{F})$ denote the annihilator of the image of $I(\mathcal{D})$ in $\mathfrak{m}_{R^{\square}_{\overline{\rho}}}/\langle\mathfrak{m}^2_{R^{\square}_{\overline{\rho}}},\lambda\rangle$. Then $\widetilde{L}(\mathcal{D})$ is actually the pre-image of its image $L(\mathcal{D})$ in $H^1(G,\mathrm{ad}\,\overline{\rho})$.

Note that

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R^{\square}_{\overline{\rho}}}/\langle\mathfrak{m}^2_{R^{\square}_{\overline{\rho}}},I(\mathcal{D}),\lambda\rangle,\mathbb{F})\simeq\widetilde{L}(\mathcal{D})$$

and the exact sequence from Corollary 2.3 gives us

$$\dim\widetilde{L}(\mathcal{D})=n^2+\dim L(\mathcal{D})-\dim H^0(G,\mathrm{ad}\,\overline{\rho}).$$

In applications, one often wishes to fix the determinants of the lifts. Given a continuous character $\chi:G\to\mathcal{O}^{\times}$ such that $\chi \bmod \lambda=\det\overline{\rho}$, we let $\mathscr{R}_{\overline{\rho},\chi}(A)$ be the set of liftings to A that have $\det\rho=\chi\otimes_{\mathcal{O}}A$. Similarly we define a functor $\mathscr{R}_{\overline{\rho}}$ with

Fact 3.4. *The functor $\mathscr{R}^{\square}_{\overline{\rho},\chi}$ is represented by a universal object $\rho^{\square}_{\chi}:G\to\mathrm{GL}_n(R^{\square}_{\overline{\rho},\chi})$. If $\overline{\rho}$ is Schur, then $\mathscr{R}_{\overline{\rho},\chi}$ is represented by a universal object $\rho^{\square}_{\chi}:G\to\mathrm{GL}_n(R^{\mathrm{univ}}_{\overline{\rho},\chi})$ up to equivalence.*

Moreover, all of the statements in Section 2 carry over if we replace $\operatorname{ad}\overline{\rho}$ by the subspace $\operatorname{ad}^0\overline{\rho} = \{x \in ad\overline{\rho} \mid \operatorname{trace}(x) = 0\}$. Note that since $\ell \nmid n$, the exact sequence

$$0 \to \operatorname{ad}^0\overline{\rho} \to \operatorname{ad}\overline{\rho} \to \mathbb{F} \to 0$$

is split.

4. Global deformation problems

Fix a finite set S, and for each $v \in S$, a profinite group G_v satisfying Axiom 0.1, together with a continuous homomorphism $G_v \to G$, and a deformation problem $\mathcal{D}_v$ for $\overline{\rho}|_{G_v}$.

Also fix a continuous character $\chi : G \to \mathcal{O}^\times$ such that $\chi \bmod \lambda = \det\overline{\rho}$. Assume that $\overline{\rho}$ is absolutely irreducible, and fix some subset $T \subseteq S$.

Definition 4.1. Fix $A \in \mathcal{C}_\mathcal{O}$. A *$T$-framed deformation* of $\overline{\rho}$ of type $\mathcal{S} = (S, \{\mathcal{D}_v\}_{v\in S}, \chi)$ to A is an equivalence class of tuples $(\rho, \{\alpha_v\}_{v\in T})$, where $\rho : G \to \mathrm{GL}_n(A)$ is a lift of $\overline{\rho}$ such that $\det\rho = \chi \otimes_\mathcal{O} A$ and $\rho|_{G_v} \in \mathcal{D}_v$ for all $v \in S$, and $\alpha_v \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$.

The equivalence relation is defined by decreeing that for each $\beta \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbb{F}))$, we have $(\rho, \{\alpha_v\}_{v\in T}) \sim (\beta\rho\beta^{-1}, \{\beta\alpha_v\}_{v\in T})$.

Fact 4.2. *The functor*

$$\begin{aligned} \mathscr{R}_\mathcal{S}^{\square,T} &: \mathcal{C}_\mathcal{O} \to \mathrm{Set}, \\ & A \mapsto \{T\text{-framed deformations of type } \mathcal{S} = (S, \{\mathcal{D}_v\}_{v\in S}, \chi)\} \end{aligned}$$

is represented by a universal object $(\rho^{\square,T}, \{\alpha_v\}_{v\in T})$ up to equivalence. This means

- $R_\mathcal{S}^{\square,T} \in \mathcal{C}_\mathcal{O}$,
- $\rho_\mathcal{S}^{\square,T} : G \to \mathrm{GL}_n(R_\mathcal{S}^{\square,T})$ *is a lift of $\overline{\rho}$ with determinant equal to χ, and*
- $\alpha_v \in \ker(\mathrm{GL}_n(R_\mathcal{S}^{\square,T}) \to \mathrm{GL}_n(\mathbb{F}))$ *for all $v \in T$.*

Definition 4.3. If $T = \emptyset$ then we will write $R_\mathcal{S}^{\mathrm{univ}}$ for $R_S^{\square,T}$.

To get a better understanding of global Galois deformations, it will be useful to study the rings $R_\mathcal{S}^{\mathrm{univ}}$. Namely, we want to describe how $R_\mathcal{S}^{\mathrm{univ}}$ can be presented in terms of the rings $R^\square_{\overline{\rho}|_{G_v},\chi}$ at the local places.

Remark. When we write $R_\mathcal{S}^{\square,T}$ we should interpret this as something like $R^{\square,T}_{\overline{\rho},S,\{\mathcal{D}_v\},\chi}$.

5. Presenting global deformation rings over local lifting rings

To reiterate, we assume that $\overline{\rho}$ is absolutely irreducible.

Since $\alpha_v^{-1}\rho^{\square,T}|_{G_v}\alpha_v : G \to \mathrm{GL}_n(A)$ is a well-defined element of $\mathcal{D}_v$, we have a tautological homomorphism $R^\square_{\overline{\rho}|_{G_v},\chi}/I(\mathcal{D}_v) \to R_\mathcal{S}^{\square,T}$. Define

$$R^{\mathrm{loc}}_{\mathcal{S},T} = \widehat{\bigotimes}_{v\in T} R^\square_{\overline{\rho}|_{G_v},\chi}/I(\mathcal{D}_v).$$

Here $\widehat{\bigotimes}$ denotes the completed tensor product, which is the pushout in the category of complete local Noetherian $\mathcal{O}$-algebras. We have a natural map $R^{\mathrm{loc}}_{\mathcal{S},T} \to R_\mathcal{S}^{\square,T}$. It turns out that $R_\mathcal{S}^{\square,T}$ is finitely presented as the quotient of a power series ring over $R^{\mathrm{loc}}_{\mathcal{S},T}$ in some number of variables. To compute this number, we must compute

$$\dim_\mathbb{F} \mathfrak{m}_{R_\mathcal{S}^{\square,T}}/\langle \mathfrak{m}^2_{R_\mathcal{S}^{\square,T}}, \mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}, \lambda\rangle.$$

This quantity will give the number of generators for $R_{\mathcal{S}}^{\square,T}$ as an algebra over $R_{\mathcal{S},T}^{\mathrm{loc}}$, as was done in the case for $R_{\overline{\rho}}^{\square}$ over $\mathcal{O}$.

Given a group G and an $\mathbb{F}[G]$-module M, let $C^i(G,M)$ be the space of functions $G^i \to M$, and let $\partial : C^i(G,M) \to C^{i+1}(G,M)$ be the usual coboundary map. Also, write $H^i(G,M)$ for the cohomology groups of the complex $C^\bullet(G,M)$.

We define a complex $C^\bullet_{\mathcal{S},T,\mathrm{loc}}(G,\mathrm{ad}^0\overline{\rho})$ by

$$\begin{aligned}
C^0_{\mathcal{S},T,\mathrm{loc}}(G,\mathrm{ad}^0\overline{\rho}) &= \bigoplus_{v\in T} C^0(G_v,\mathrm{ad}\,\overline{\rho}) \oplus \bigoplus_{v\in S\setminus T} 0,\\
C^1_{\mathcal{S},T,\mathrm{loc}}(G,\mathrm{ad}^0\overline{\rho}) &= \bigoplus_{v\in T} C^1(G_v,\mathrm{ad}^0\overline{\rho}) \oplus \bigoplus_{v\in S\setminus T} C^1(G_v,\mathrm{ad}^0\overline{\rho})/\widetilde{L}(\mathcal{D}_v),\\
C^i_{\mathcal{S},T,\mathrm{loc}}(G,\mathrm{ad}^0\overline{\rho}) &= \bigoplus_{v\in S} C^i(G_v,\mathrm{ad}^0\overline{\rho}) \text{ for all } i\geq 2.
\end{aligned}$$

We define another complex $C^\bullet_0(G,\mathrm{ad}^0\overline{\rho})$ by

$$\begin{aligned}
C^0_0(G,\mathrm{ad}^0\overline{\rho}) &= C^0(G,\mathrm{ad}\,\overline{\rho}),\\
C^i_0(G,\mathrm{ad}^0\overline{\rho}) &= C^i(G,\mathrm{ad}^0\overline{\rho}) \text{ for all } i\geq 1.
\end{aligned}$$

Finally, we let

$$C^\bullet_{\mathcal{S},T}(G,\mathrm{ad}^0\overline{\rho}) = C^\bullet_0(G,\mathrm{ad}^0\overline{\rho}) \oplus C^{\bullet-1}_{\mathcal{S},T,\mathrm{loc}}(G,\mathrm{ad}^0\overline{\rho})$$

where the coboundary map is given by

$$(\phi,(\psi_v)_{v\in S}) \mapsto (\partial\phi,(\phi|_{G_v}-\psi_v)_{v\in S}).$$

Write $H^i_{\mathcal{S},T,\mathrm{loc}}$, H^i_0, $H^i_{\mathcal{S},T}$ for the cohomology of the complexes $C^\bullet_{S,T,\mathrm{loc}}$, $C^\bullet_0$, $C^\bullet_{\mathcal{S},T}$, respectively.

Then we have an exact sequence of complexes

$$0 \to C^{\bullet-1}_{\mathcal{S},T,\mathrm{loc}}(G,\mathrm{ad}^0\overline{\rho}) \to C^\bullet_{\mathcal{S},T}(G,\mathrm{ad}^0\overline{\rho}) \to C^\bullet_0(G,\mathrm{ad}^0\overline{\rho}) \to 0,$$

and the corresponding long exact sequence in cohomology is

$$\begin{aligned}
&0 \longrightarrow H^0_{\mathcal{S},T} \longrightarrow H^0(G,\mathrm{ad}\,\overline{\rho}) \longrightarrow \textstyle\bigoplus_{v\in T} H^0(G_v,\mathrm{ad}\,\overline{\rho})\\
&\longrightarrow H^1_{\mathcal{S},T} \longrightarrow H^1(G,\mathrm{ad}^0\overline{\rho}) \longrightarrow \textstyle\bigoplus_{v\in T} H^1(G_v,\mathrm{ad}^0\overline{\rho}) \oplus \bigoplus_{v\in S\setminus T} H^1(G_v,\mathrm{ad}^0\overline{\rho})/L(\mathcal{D}_v)\\
&\longrightarrow H^2_{\mathcal{S},T} \longrightarrow H^2(G,\mathrm{ad}^0\overline{\rho}) \longrightarrow \textstyle\bigoplus_{v\in S} H^2(G_v,\mathrm{ad}^0\overline{\rho})\\
&\longrightarrow H^3_{\mathcal{S},T} \longrightarrow \cdots
\end{aligned}$$

Let's double-check why the groups $H^i_{\mathcal{S},T}$ are worth studying.

Proposition 5.1. *There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R_{\mathcal{S}}^{\square,T}}/\langle \mathfrak{m}^2_{R_{\mathcal{S}}^{\square,T}}, \mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}, \lambda\rangle, \mathbb{F}) \simeq H^1_{\mathcal{S},T}.$$

Thus, if $d = \dim_{\mathbb{F}} H^1_{\mathcal{S},T}(G,\mathrm{ad}^0\overline{\rho})$, *then there is a surjection* $\phi : R^{\mathrm{loc}}_{\mathcal{S},T}[\![x_1,\ldots,x_d]\!] \to R_{\mathcal{S}}^{\square,T}$.

Proof. As before, we have a natural isomorphism

$$\mathrm{Hom}_{\mathbb{F}}(\mathfrak{m}_{R_{\mathcal{S}}^{\square,T}}/\langle\mathfrak{m}^2_{R_{\mathcal{S}}^{\square,T}}, \mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}, \lambda\rangle, \mathbb{F}) \simeq \mathrm{Hom}_{\mathcal{C}_{\mathcal{O}}}(R_{\mathcal{S}}^{\square,T}/\langle \mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}\rangle, \mathbb{F}[\epsilon]/\langle\epsilon^2\rangle).$$

Taking the quotient by $\mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}$ amounts to requiring that the lifting be trivial at $v \in T$. Thus this space is identified with the set of T-framed deformations to $\mathbb{F}[\epsilon]/\langle\epsilon^2\rangle$ of type $\mathcal{S}$ that give trivial liftings at each of the places in T.

Such a deformation is given by

$$((1_n + \phi\epsilon)\overline{\rho}, \{1_n + \psi_v\epsilon\}_{v\in T})$$

where we use the data of a 1-cocycle $\phi \in Z^1(G, \mathrm{ad}^0\overline{\rho})$ and elements $\{\psi_v\} \in \mathrm{ad}\,\overline{\rho}$. By decreeing that ϕ is a deformation of type $\mathcal{S}$, we are saying that $\phi|_{G_v} \in \widetilde{L}(\mathcal{D}_v)$ for all $v \in S$. By decreeing that $\{\psi_v\}_{v\in T}$ gives trivial liftings at the places in T, we are saying that

$$(1_n - a_v\epsilon)(1_n + \phi\epsilon)\overline{\rho}|_{G_v}(1_n + \psi_v\epsilon) = \overline{\rho}|_{G_v}$$

for all $v \in T$, that is,

$$\phi|_{G_v} = (\mathrm{ad}\,\overline{\rho}|_{G_v} - 1_n)\psi_v$$

for all $v \in T$.

Two such pairs $(\phi, \{\psi_v\})$ and $(\phi', \{\psi'_v\})$ are considered equivalent if and only if there exists $\beta \in \mathrm{ad}\,\overline{\rho}$ with

$$\phi' = \phi + (1_n - \mathrm{ad}\,\overline{\rho})\beta$$

and

$$\psi'_v = \psi_v + \beta$$

for all $v \in T$.

Using this description, one identifies the space of such data with $H^1_{\mathcal{S},T}$. □

We wish to express each of these quantities in terms of standard cohomology groups and local deformation rings.

Before continuing, define the "negative Euler characteristics"

$$\chi(G, \mathrm{ad}^0\overline{\rho}) = \sum_{i\geq 0}(-1)^{i+1}\dim_{\mathbb{F}} H^i(G, \mathrm{ad}^0\overline{\rho}),$$

$$\chi_{\mathcal{S},T,\mathrm{loc}} = \sum_{i\geq 0}(-1)^{i+1}\dim_{\mathbb{F}} H^i_{\mathcal{S},T,\mathrm{loc}},$$

$$\chi_0 = \sum_{i\geq 0}(-1)^{i+1}\dim_{\mathbb{F}} H^i_0,$$

$$\chi_{\mathcal{S},T} = \sum_{i\geq 0}(-1)^{i+1}\dim_{\mathbb{F}} H^i_{\mathcal{S},T}.$$

We know that $\chi_{\mathcal{S},T} = \chi_0 - \chi_{\mathcal{S},T,\mathrm{loc}}$. But we want an expression in terms of standard group cohomology. Now

$$\chi_0 = \chi(G, \mathrm{ad}^0\overline{\rho}) + (\dim H^0(G, \mathrm{ad}\,\overline{\rho}) - \dim H^0(G, \mathrm{ad}^0\overline{\rho}))$$

and the last term is -1 because $\operatorname{ad}\overline{\rho} = \operatorname{ad}^0\overline{\rho} \oplus \mathbb{F}$. And by definition

$$\begin{aligned}
\chi_{\mathcal{S},T,\mathrm{loc}} &= \sum_{v\in S}\chi(G_v, \operatorname{ad}^0\overline{\rho}) - \sum_{v\in T}(\dim H^0(G_v, \operatorname{ad}\overline{\rho}) - \dim H^0(G_v, \operatorname{ad}^0\overline{\rho})) \\
&\quad + \sum_{v\in S\setminus T}(\dim H^0(G_v, \operatorname{ad}^0\overline{\rho}) - \dim L(\mathcal{D}_v))) \\
&= \sum_{v\in S}\chi(G_v, \operatorname{ad}^0\overline{\rho}) - \#T + \sum_{v\in S\setminus T}(\dim H^0(G_v, \operatorname{ad}^0\overline{\rho}) - \dim L(\mathcal{D}_v)))
\end{aligned}$$

So in total,

$$\begin{aligned}
\chi_{\mathcal{S},T} &= \chi_0 - \chi_{\mathcal{S},T,\mathrm{loc}} \\
&= (\#T-1) + \chi(G, \operatorname{ad}^0\overline{\rho}) - \sum_{v\in S}\chi(G_v, \operatorname{ad}^0\overline{\rho}) - \sum_{v\in S\setminus T}(\dim H^0(G_v, \operatorname{ad}^0\overline{\rho}) - \dim L(\mathcal{D}_v)))
\end{aligned} \tag{5.1}$$

6. The number field case

We specialize to the case where F is a number field, and S is a finite set of finite places including the places lying over ℓ, and we set $G = G_{F,S}$, $G_v = G_{F_v}$ for $v \in S$. We cite a few results on Galois cohomology.

Fact 6.1 (Cohomological vanishing). *We have the following.*

(a) *Suppose M is a finite $\mathbb{F}[G_{F_v}]$-module. Then $H^i(G_{F_v}, M)$ is finite, and $H^i(G_{F_v}, M) = 0$ for all $i \geq 3$.*

(b) *Suppose v is a real place, and $G_{F_v} = \{1, c\}$ acts on a module M whose order is a power of a prime $\ell \neq 2$. Then $H^i(G_{F_v}.M) = 0$ for all $i \geq 0$.*

(c) *Suppose M is a finite $\mathbb{F}[G_{F,S}]$-module. Then $H^i(G_{F,S}, M)$ is finite, and*

$$H^i(G_{F,S}, M) \simeq \bigoplus_{v\ \mathit{real}} H^i(G_v, M)$$

for all $i \geq 3$. Thus, if the prime $\ell \neq 2$, then $H^i(G_{F,S}, M) = 0$ for all $i \geq 3$.

Fact 6.2 (Local/global Euler characteristic). *We have the following.*

(a) *Suppose M is a finite $\mathbb{F}[G_{F_v}]$-module. Then*

$$\chi(G_{F_v}, M) = \dim_{\mathbb{F}}(\mathcal{O}/$$

(b) *Suppose M is a finite $\mathbb{F}[G_{F,S}]$-module. Then*

$$\chi(G_{F,S}, M) = [F:\mathbb{Q}]\dim_{\mathbb{F},M} - \sum_{v|\infty}\dim_{\mathbb{F}} H^0(G_{F_v}, \operatorname{ad}^0\overline{\rho}).$$

Let $G = G_{F,S}$ or G_v, and let M be a finite $\mathbb{F}[G]$-module. Let $M^\vee = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F})$. Let $M(1) = M \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(\epsilon_\ell)$, where $\epsilon_\ell : G \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character. Thus $M^\vee = M^\vee \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(\epsilon_\ell)$

We have the following.

Fact 6.3 (Tate local duality). *Let M be a finite $\mathbb{F}[G_{F_v}]$-module. For $i = 0, 1, 2$, the cup product gives a non-degenerate pairing*

$$H^i(G_{F_v}, M^\vee(1)) \times H^{2-i}(G_{F_v}, M) \to \mathbb{F}.$$

Fact 6.4 (Poitou–Tate theorem). *The following nine-term sequence is exact:*

$$0 \to H^0(G_{F,S}, M) \to \bigoplus_{v \text{ real}} \widehat{H}^0(G_{F_v}, M) \times \prod_{v \in S} H^0(G_{F_v}, M) \to H^2(G_{F,S}, M^\vee(1))^\vee$$
$$\to H^1(G_{F,S}, M) \to \bigoplus_{v \in S} H^1(G_{F_v}, M) \to H^1(G_{F,S}, M^\vee(1))^\vee$$
$$\to H^2(G_{F,S}, M) \to \bigoplus_{v \in S} H^2(G_{F_v}, M) \to H^0(G_{F,S}, M^\vee(1))^\vee \to 0.$$

We apply these facts to the $G_{F,S}$-module $M = \mathrm{ad}^0 \overline{\rho}$ possessing the cohomology long exact sequence above.

- $\overline{\rho}$ is absolutely irreducible, so that $H^0(G_{F,S}, \mathrm{ad}\, \overline{\rho}) = \mathbb{F}$, so $H^0_{\mathcal{S},T} = \mathbb{F}$.
- By cohomological vanishing, only the groups $H^0_{\mathcal{S},T}$, $H^1_{\mathcal{S},T}$, $H^2_{\mathcal{S},T}$, and $H^3_{\mathcal{S},T}$ are nonzero.
- Meanwhile, by the local/global Euler characteristic formula we find that

$$\chi(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) - \sum_{v \in S} \chi(G_v, \mathrm{ad}^0 \overline{\rho}) = [F : \mathbb{Q}](n^2 - 1) - \sum_{v \mid \infty} \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) - \sum_{v \mid \ell} (n^2 - 1)[F_v : \mathbb{Q}_\ell]$$
$$= -\sum_{v \mid \infty} \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho})$$

  because the last term is equal to $(n^2 - 1)[F : \mathbb{Q}]$.
- Note that $M = \mathrm{ad}^0 \overline{\rho}$ is self-dual under the trace pairing $M \times M \to \mathbb{F}$, $(x, y) \mapsto \mathrm{trace}(xy)$ (i.e. it is a perfect pairing), which means $(\mathrm{ad}^0 \overline{\rho}) \simeq (\mathrm{ad}^0 \overline{\rho})^\vee$ and $(\mathrm{ad}^0 \overline{\rho})(1) \simeq (\mathrm{ad}^0 \overline{\rho})^\vee(1)$.
- For all $v \in S$, let $H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \times H^1(G_{F_v}, (\mathrm{ad}^0 \overline{\rho})(1)) \to \mathbb{F}$ be the pairing of local Tate duality (here we use that $\mathrm{ad}^0 \overline{\rho}$ is self-dual). Consider $L(\mathcal{D}_v) \subset H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho})$ and let $L(\mathcal{D}_v)^\perp \subset H^1(G_{F_v}, (\mathrm{ad}^0 \overline{\rho})(1))$ be its annihilator under the pairing.
- The last 6 terms of the Poitou–Tate sequence read

$$H^1(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) \to \bigoplus_{v \in S} H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \to H^1(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))^\vee$$
$$\to H^2(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) \to \bigoplus_{v \in S} H^2(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \to H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))^\vee \to 0.$$

  If we define

$$H^1_{\mathcal{S},T}(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1)) \coloneqq \ker(H^1(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1)) \to \bigoplus_{v \in S \setminus T} H^1(G_{F_v}, (\mathrm{ad}^0 \overline{\rho})(1))/L(\mathcal{D}_v)^\perp)),$$

  then we can dualize this expression, and edit the above second and third terms and have an exact sequence

$$H^1(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) \to \bigoplus_{v \in T} H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \oplus \bigoplus_{v \in S \setminus T} H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho})/L(\mathcal{D}_v) \to H^1_{\mathcal{S},T}(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))^\vee$$
$$\to H^2(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) \to \bigoplus_{v \in S} H^2(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \to H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))^\vee \to 0.$$

- The last 6 nonzero terms of our long exact cohomology sequence read

$$H^1(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) \to \bigoplus_{v \in T} H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \oplus \bigoplus_{v \in S \setminus T} H^1(G_{F_v}, \mathrm{ad}^0 \overline{\rho})/L(\mathcal{D}_v) \to H^2_{\mathcal{S},T}$$
$$\to H^2(G_{F,S}, \mathrm{ad}^0 \overline{\rho}) \to \bigoplus_{v \in S} H^2(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) \to H^3_{\mathcal{S},T} \to 0.$$

These two exact sequences coincide at all but the third and sixth positions. Thus

$$H^3_{\mathcal{S},T} \simeq H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))^\vee$$

and

$$H^3_{\mathcal{S},T} \simeq H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))^\vee.$$

Combining with (5.1), we see that

$$\dim_{\mathbb{F}} H^1_{\mathcal{S},T} = \#T - \sum_{v \mid \infty} \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) + \sum_{v \in S \setminus T} (\dim L(\mathcal{D}_v) - \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}))$$
$$+ \dim H^1_{\mathcal{S},T}(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1)) - \dim H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1)).$$

The expression $H^1_{\mathcal{S},T}(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1))$ was just manufactured by us, so let's see what conclusion we get in the nice case where $T = \emptyset$.

Proposition 6.5. *The Krull dimension of $R^{\mathrm{univ}}_{\mathcal{S}}$ is at least*

$$1 + \sum_{v \in S} (\dim_{\mathrm{Krull}}(R^{\square}_{\overline{\rho}|_{G_{F_v}},\chi}/I(\mathcal{D}_v)) - n^2) - \sum_{v \mid \infty} \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) - \dim H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1)).$$

Proof. Let $d = \dim H^1_{\mathcal{S},\emptyset}$, and let J be the kernel of the surjection $R = R^{\mathrm{loc}}_{\mathcal{S}}[\![x_1, \ldots, x_d]\!] \to R^{\mathrm{univ}}_{\mathcal{S}}$. We define an injective map

$$\mathrm{Hom}(J/\mathfrak{m}_{R^{\mathrm{univ}}_{\mathcal{S}}} J, \mathbb{F}) \to H^2_{\mathcal{S},\emptyset}(G, \mathrm{ad}^0 \overline{\rho})$$

as follows. Pick a lift of $\rho^{\mathrm{univ}}_{\mathcal{S}}$ to $\mathrm{GL}_n(R)$, denoted $\widetilde{\rho}$, and define for $\gamma, \delta \in G$

$$c_f(\gamma, \delta) = f(\widetilde{\rho}(\gamma\delta)\widetilde{\rho}(\delta)^{-1}\widetilde{(\rho)}(\gamma)^{-1} - 1_n) \in \mathrm{ad}^0 \overline{\rho}.$$

Also for $v \in S$, pick a lift $\widehat{\rho}_v$ of $\rho^{\mathrm{univ}}_{\mathcal{S}}|_{G_v}$ and define for $\gamma \in G_v$

$$d_{f,v}(\gamma) = f(\widetilde{\rho}(\gamma)\widehat{\rho}(\gamma)^{-1} - 1_n)$$

One shows that this gives a well-defined element of $H^2_{\mathcal{S},\emptyset}$ and that the associated $f \mapsto [(c_f, d_{f,v})]$ is injective.

Hence setting $T = \emptyset$, we find that

$$\dim_{\mathrm{Krull}} R^{\mathrm{univ}}_{\mathcal{S}} \geq \dim_{\mathrm{Krull}} R^{\mathrm{loc}}_{\mathcal{S},\emptyset} + \dim H^1_{\mathcal{S},\emptyset} - \dim H^2_{\mathcal{S},\emptyset}$$
$$= 1 - \sum_{v \mid \infty} \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) + \sum_{v \in S} (\dim L(\mathcal{D}_v) - \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}))$$
$$- \dim H^0(G_{F,S}, (\mathrm{ad}^0 \overline{\rho})(1)).$$

where the 1 comes from $\mathcal{O}$. However,

$$\dim L(\mathcal{D}_v) - \dim H^0(G_{F_v}, \mathrm{ad}^0 \overline{\rho}) = \dim_{\mathrm{Krull}} R^{\square}_{\overline{\rho}|_{G_v},\chi}/I(\mathcal{D}_v) - n^2.$$

$\square$

In addition, we can also give a lower bound for the Krull dimension of $R_S^{\square,T}$ if we assume that $\mathcal{D}_v$ is *liftable* for all $v \in S \backslash T$.

Definition 6.6. $\mathcal{D}_v$ is *liftable* if for each $R \in \mathcal{C}_{\mathcal{O}}$, for each ideal $I \subset R$ with $\mathfrak{m}_R I = \langle 0 \rangle$ and for each lifting ρ to R/I in $\mathcal{D}_v$, there is a lifting of ρ to R. This is equivalent to $R_{\overline{\rho}}^{\square}/I(\mathcal{D}_v)$ being a power series ring over $\mathcal{O}$.

7. Maps between global Galois deformation rings

Suppose that F'/F is a finite extension of number fields, and that S' is the set of places of F' lying over S. Assume that $\overline{\rho}|_{G_{F',S'}}$ is absolutely irreducible. Then restricting the universal deformation ρ^{univ} of $\overline{\rho}$ to $G_{F',S'}$ gives a ring homomorphism

$$R^{\text{univ}}_{\overline{\rho}|_{G_{F',S'}}} \to R^{\text{univ}}_{\overline{\rho}}$$

Proposition 7.1. *The ring $R^{\text{univ}}_{\overline{\rho}}$ is a module-finite $R^{\text{univ}}_{\overline{\rho}|_{G_{F',S'}}}$-algebra.*

Proof. □

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