**On the Brauer Groups of Stacky Curves**

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Brauer groups of fields were classically studied objects whose definition was generalized to rings in work of Azumaya, Auslander, and Goldman [6, 3], and then later to schemes in work of Grothendieck [7, 8, 9]. These have been cemented as important cohomological invariants for their applications to class field theory, to understanding $l$-adic cohomology (especially of curves), and to obstructions to rational points on varieties. In recent times, there has been growing interest in extending our understanding of Brauer groups from schemes to stacks. As a starting point, one can study the Brauer groups of `stacky curves`, i.e. a separated, finite-type algebraic stack $\mathcal{X}$ over a field $k$, which is pure of dimension 1 and has finite inertia. To fix ideas, given an algebraic stack $\mathcal{X}$, we define its `Brauer group` to be $\operatorname{Br}\mathcal{X} := \mathrm{H}^2_{\acute{e}t}(\mathcal{X}, \mathbb{G}_m)_{\mathrm{tors}}$.

Previous work along this direction has generally focused on a single stacky curve at a time. For example, there is the breakthrough work of Antieau and Meier [4] who computed the Brauer group of the moduli stack $\mathcal{Y}(1)$ of elliptic curves over a variety of bases of arithmetic interest (e.g. over $\mathbb{Q}, \mathbb{Z}$, or any finite or algebraically closed field of characteristic not 2). For all base schemes $S$ appearing in their main theorem [4, Theorem 1.1], they show

$$\operatorname{Br}\mathcal{Y}(1)_S \cong \operatorname{Br}\mathbb{A}^1_S \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/12\mathbb{Z}). \tag{1}$$

This work was later complemented by work of Shin and of di Lorenzo–Pirisi [12, 10] who were able to compute $\operatorname{Br}\mathcal{Y}(1)_k$ for *any* field $k$. Of note, they found that if $\operatorname{char} k = 2$, then there is an exact sequence

$$0 \to \operatorname{Br}\mathbb{A}^1_k \oplus \mathrm{H}^1_{\acute{e}t}(k, \mathbb{Z}/12\mathbb{Z}) \to \operatorname{Br}\mathcal{Y}(1)_k \to \mathbb{Z}/2\mathbb{Z} \to 0, \tag{2}$$

so $\operatorname{Br}\mathcal{Y}(1)_k$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by the right hand side of (1).

Separately, Bhamidipati, Jha, Ji, Lopez, and I [1] were able to extend the techniques pioneered by Antieau, Meier, and Shin in order to apply them to computing the Brauer group of the moduli stack $\mathcal{Y}_0(2)$ of elliptic curves equipped with an étale subgroup of order 2. We found that, at least when $S$ is $\mathbb{Z}[1/2]$ or a perfect field of characteristic not 2, one has

$$\operatorname{Br}\mathcal{Y}_0(2)_S \cong \operatorname{Br}(\mathbb{A}^1_S - \{0\}) \oplus \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/4\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}. \tag{3}$$

At this point, some explanation of the terms in Eqs. (1) to (3) is in order.

- The schemes $\mathbb{A}^1_S, \mathbb{A}^1_S - \{0\}$ are the respective coarse moduli spaces of the stacks $\mathcal{Y}(1)_S$ and $\mathcal{Y}_0(2)_S$. Their Brauer groups show up as pullbacks along the maps $\mathcal{Y}(1) \to \mathbb{A}^1$ and $\mathcal{Y}_0(2) \to \mathbb{A}^1 - \{0\}$.
- The cohomology groups $\mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/12\mathbb{Z}), \mathrm{H}^1_{\acute{e}t}(S, \mathbb{Z}/4\mathbb{Z})$ ultimately are related to the facts that, for any field $k$ (say, of characteristic not 2), one has $\operatorname{Pic}\mathcal{Y}(1)_k \cong \mathbb{Z}/12\mathbb{Z}$ and $\operatorname{Pic}\mathcal{Y}_0(2)_k \cong \mathbb{Z}/4\mathbb{Z}$.
- The fact that (1) and (2) differ is related to the fact that the stack $\mathcal{Y}(1)$ is generically tame away from characteristic 2, but is nowhere tame in characteristic 2. That is, the extra $\mathbb{Z}/2\mathbb{Z}$ in (2) should be viewed as a "wild" phenomenon.

The above gives some geometric explanation for all the terms in Eqs. (1) to (3) *except* for the $\mathbb{Z}/2\mathbb{Z}$ in (3) (note that $\mathcal{Y}_0(2)$ is everywhere tame in characteristic not 2). This brings us to the focus of our talk.

Letting $\overline{k}$ be an algebraically closed field of characteristic not 2, Eqs. (1) and (3) tell us that

$$\operatorname{Br}\mathcal{Y}(1)_{\overline{k}} = 0 \text{ while } \operatorname{Br}\mathcal{Y}_0(2)_{\overline{k}} = \mathbb{Z}/2\mathbb{Z}.$$

Recall that if $X$ is a scheme-y curve over an algebraically closed field, then $\operatorname{Br} X = 0$ always, as a consequence of Tsen's theorem. However, we see above that this result can fail for tame stacky curves.

**Question 1.** *Can one compute the Brauer group of a tame stacky curve over an algebraically closed field?*

In this talk, we address this question, at least for stacks which satisfy the following "locally Brauerless' condition (this includes, for example, all tame modular curves).

**Definition 2.** Let $\mathcal{X}$ be an algebraic stack which is tame in the sense of [5]. We say that $\mathcal{X}$ is `locally Brauerless` if, for any geometric point $x \in \mathcal{X}(\Omega)$ defined over a separably closed field $\Omega$, writing

$$0 \longrightarrow \Delta \longrightarrow \underline{\operatorname{Aut}}_{\mathcal{X}}(x) \longrightarrow G \longrightarrow 0$$

for the connected-étale sequence of its automorphism group, one has $\mathrm{H}^3(G, \mathbb{Z}) = 0$. Here, $G$ is necessarily a constant group and $\mathrm{H}^3(G, \mathbb{Z})$ denotes the group cohomology of this group acting trivially on $\mathbb{Z}$.

**Example 3.** If every geometric automorphism group of $\mathcal{X}$ is of the form $\mu_n$ for some $n$, then $\mathcal{X}$ is tame and locally Brauerless.

**Theorem 4** (A., in preparation [2])**.** *Let $\mathcal{X}$ be a locally Brauerless tame algebraic stack with coarse space $c\colon \mathcal{X} \to X$. Then, $\mathrm{R}^2\, c_*\mathbb{G}_m = 0$.*

The utility of this theorem is that it simplifies the process of computing $\mathrm{H}^2_{ét}(\mathcal{X}, \mathbb{G}_m)$ via the Leray spectral sequence $E_2^{ij} = \mathrm{H}^i_{ét}(X, \mathrm{R}^j\, c_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}_{ét}(\mathcal{X}, \mathbb{G}_m)$. Its proof is rather involved, but its simplest case is exemplified by the following example.

**Example 5.** Say $\mathcal{X} = BG_k$ for a separably closed field $k$ and a finite, constant group $G$ such that $p := \operatorname{char} k \nmid \#G$. Then, $\mathbb{G}_m(k)[1/p]$ is a divisible group and so embeds inside (and hence is a direct summand of) $\mathbb{Q}^{\oplus I} \oplus (\mathbb{Q}/\mathbb{Z})^{\oplus J}$ for some sets $I, J$. Thus,

$$\mathrm{H}^2(BG, \mathbb{G}_m) \simeq \mathrm{H}^2(G, \mathbb{G}_m(k)) \simeq \mathrm{H}^2(G, \mathbb{G}_m(k)[1/p]) \hookrightarrow \mathrm{H}^2(G, \mathbb{Q})^{\oplus I} \oplus \mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})^{\oplus J}.$$

Finally, one can check that $\mathrm{H}^2(G, \mathbb{Q}) = 0$ because $G$ is finite and $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z}) = 0$ if $\mathrm{H}^3(G, \mathbb{Z}) = 0$.

**Remark 6.** As was more-or-less shown already in [11], the DM case of Theorem 4 can essentially be reduced to Example 5.

In relation to Question 1, Theorem 4 allows us to prove the following.

**Setup 7.** Fix $k = \bar{k}$ be a field. Suppose we're given

$$\mathcal{X} \xrightarrow{\pi} \mathcal{Y} \xrightarrow{\rho} X, \quad c = \rho \circ \pi : \mathcal{X} \to X$$

where

- $X$ is a smooth $k$-curve.
- There exists distinct, closed $x_1, \ldots, x_r \in X$ and $e_1, \ldots, e_r > 1$ so that
$$\mathcal{Y} \simeq \sqrt[e_1]{x_1/X} \times_X \ldots \times_X \sqrt[e_r]{x_r/X} \xrightarrow{\rho} X$$
- There is a commutative, finite linearly reductive group $G/k$ so that $\mathcal{X} \to \mathcal{Y}$ is a $G$-gerbe.

**Theorem 8** (A., in preparation [2])**.** *If $\mathcal{X}$ is locally Brauerless, then* $\operatorname{Br} \mathcal{X} \simeq \mathrm{H}^1_{ét}(X, G^\vee)$*. Here, $G^\vee$ is the Cartier dual of $G$.*

*Proof Idea.* One first uses the Leray spectral sequence $E_2^{ij} = \mathrm{H}^i_{ét}(X, \mathrm{R}^j\, c_*\mathbb{G}_m) \implies \mathrm{H}^{i+j}_{ét}(\mathcal{X}, \mathbb{G}_m)$ to compute that $\mathrm{H}^2_{ét}(\mathcal{X}, \mathbb{G}_m) \simeq \mathrm{H}^1_{ét}(X, \mathrm{R}^1\, c_*\mathbb{G}_m)$. One then uses the Grothendieck spectral sequence $F_2^{ij} = \mathrm{R}^i\, \rho_*\, \mathrm{R}^j\, \pi_*\mathbb{G}_m \implies \mathrm{R}^{i+j}\, c_*\mathbb{G}_m$ to produce an exact sequence $0 \to \mathrm{R}^1\, \rho_*\mathbb{G}_m \to \mathrm{R}^1\, c_*\mathbb{G}_m \to G^\vee \to 0$. Finally, $\mathrm{R}^1\, \rho_*\mathbb{G}_m$ is supported on a finite $k$-scheme and so is acyclic; therefore, $\mathrm{H}^1_{ét}(X, \mathrm{R}^1\, c_*\mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^1_{ét}(X, G^\vee)$. ☐

**Example 9.** If $\mathcal{X} = \mathcal{Y}(1)$ (and $\operatorname{char} k \nmid 6$), then one can take $G = \mu_2$ and Theorem 8 shows that $\operatorname{Br} \mathcal{Y}(1)_{\bar{k}} \simeq \mathrm{H}^1_{ét}(\mathbb{A}^1_{\bar{k}}, \mathbb{Z}/2\mathbb{Z}) = 0$.

**Example 10.** If $\mathcal{X} = \mathcal{Y}_0(2)$ (and $\operatorname{char} k \nmid 2$), then one can take $G = \mu_2$ and Theorem 8 shows that $\operatorname{Br} \mathcal{Y}_0(2)_{\bar{k}} \simeq \mathrm{H}^1_{ét}(\mathbb{A}^1_{\bar{k}} - \{0\}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

This gives a geometric explanation of the $\mathbb{Z}/2\mathbb{Z}$ in (3).

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