Math155 WIM

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Abstract

Fourier analysis is quite effective for obtaining results in analytic number theory. In particular, it plays a key role in the study of the complex-analytic properties of the Riemann zeta function $\zeta(s) = \sum_{n\geq 1} n^{-s}$. This function, defined as this infinite series, converges absolutely and is holomorphic on the open half-plane $\{\text{Re}(\sigma) > 1\}$. However, it is possible to show that this function can be meromorphically extended to one defined on the whole complex plane $\mathbb{C}$ save for a single pole at $s = 1$. The proof of this fact crucially relies on relating $\zeta(s)$ to an auxiliary function $\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s}$ whose analytic behavior for large s is controlled using techniques from Fourier analysis. Motivated by applications such as this, in this paper, we aim to explain the basics of Fourier analysis of periodic functions, culminating in a proof that a Fourier series (whose coefficients do not grow too quickly) converges back to its original function.

1 Introduction

At its core, the main goal of Fourier analysis is to provide a canonical representation for periodic functions which is amenable both to calculations and to theoretic study. This goal is achieved by decomposing any periodic function $f : \mathbb{R} \to \mathbb{C}$ as a (possibly infinite) weighted sum of the classic examples of periodic functions: sin's and cos's. For a fixed $r \in \mathbb{R}$ we say a function $f : \mathbb{R} \to \mathbb{C}$ is **r-periodic** if $f(x+r) = f(x)$ for all $x \in \mathbb{R}$. Hence, given an r-periodic function f, we wish to write something like

$$f(x) = \sum_{n\in\mathbb{Z}} \left[a_n \sin\left(\frac{2\pi nx}{r}\right) + b_n \cos\left(\frac{2\pi nx}{r}\right)\right] \tag{1}$$

with $a_n, b_n \in \mathbb{C}$. There are two common modifications one can make simplify the analysis of such functions. The first is to assume, without loss of generality, that all functions are 1-periodic. Indeed, if $f(x)$ is r-periodic, then $f(x/r)$ is 1-periodic and so we lose nothing by assuming that this is the case. With this assumption in mind, we will call a function simply **periodic** if it is **1-periodic**. The second assumption is to use Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ to have exponentials replace $\sin/\cos$ as our prototypical example of a periodic function. Indeed, using Euler's formula, one easily verifies that

$$a_n \sin(2\pi nx) + b_n \cos(2\pi nx) = \left(\frac{b_n - ia_n}{2}\right)e^{2\pi inx} + \left(\frac{b_n + ia_n}{2}\right)e^{-2\pi inx},$$

so equation (1) can be written entirely in terms of exponentials. Thus, by assuming all functions under consideration are 1-periodic and making use of Euler's formula, our desired equality can be restated as

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx} \tag{2}$$

where $f : \mathbb{R} \to \mathbb{C}$ is periodic and $c_n \in \mathbb{C}$. The numbers $c_n \in \mathbb{C}$ above, when the equality holds, are called the **Fourier coefficients** of f. Our main theorem in this paper will be to show that this is often possible, at least when f is assumed continuous. Specifically, we will prove the following.

Theorem 1.1 (The Main Theorem). *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and continuous. For each $n \in \mathbb{Z}$, let*

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} \mathrm{d}x.$$

Then, the following hold:

(a) *We have $\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_0^1 |f(x)|^2 \mathrm{d}x$, so $c_n \to 0$. Furthermore, if $\sum_{n\in\mathbb{Z}} |c_n|$ converges, then*

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$$

for all $x \in \mathbb{R}$ with partial sums converging uniformly to f.

(b) *Fix any integer $k \geq 2$. If $c_n = O(1/n^k)$ as $n \to \infty$, then f is continuously differentiable $k-2$ times and the termwise derivatives of $\sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$ converge uniformly to the derivatives of f. That is, for $m \leq k-2$, we have*

$$f^{(m)}(x) = \sum_{n\in\mathbb{Z}} c_n (2\pi i n)^m e^{2\pi i n x}.$$

2 Intuition for The Main Theorem

We will prove Theorem 1.1 as a series of lemmas in the next section. Before getting started on that though, a word on intuition: why might one expect equation (2) to hold for any choice of coefficients c_n?

The main point is that it serves as an analogue to the situation of expanding elements of a finite-dimensional vector space in terms of an orthonormal basis. That is, let V be an n-dimensional $\mathbb{C}$-vector space equipped with a Hermitian inner product $(-,-) : V \times V \to \mathbb{C}$, so

- $(v, v) \geq 0$ for all $v \in V$.
- $(v, w) = \overline{(w, v)}$ for all $v, w \in V$ where $^-$ denotes complex conjugation.
- $(av + bu, w) = a(v, w) + b(u, w)$ for all $v, u, w \in V$ and $a, b \in \mathbb{C}$.

Let $\{e_i\}_{i=1}^n \subset V$ be an orthonormal set with respect to this inner product, so $(e_i, e_i) = 1$, but $(e_i, e_j) = 0$ when $i \neq j$. Then, orthonormality guarantees that the e_i's are linearly independent, and that fact that $\dim V = n$ then tells us that they form a basis for V. Hence, any $v \in V$ can be uniquely written in the form $v = \sum_{i=1}^n c_i e_i$, and furthermore, one see that

$$(v, e_j) = \left(\sum_{i=1}^n c_i e_i, e_j \right) = \sum_{i=1}^n c_i (e_i, e_j) = c_j,$$

for all j, so v's coefficients are completely determined by the inner product.

The situation surround equation (2) is completely analogous. The set $C(\mathbb{R}/\mathbb{Z})$ of continuous periodic functions $f : \mathbb{C} \to \mathbb{R}$ forms a $\mathbb{C}$-vector space under the operations of pointwise addition and scalar multiplication. One can endow this space with pairing $(-,-) : C(\mathbb{R}/\mathbb{Z}) \times C(\mathbb{R}/\mathbb{Z}) \to \mathbb{C}$ given by

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x,$$

which is easily verified to be a Hermitian inner product. Under this inner product, the functions $e_n(x) = e^{2\pi inx}$ form an orthonormal basis where n ranges over all integers. Indeed, this is a consequence of the facts that $e_0(x) = 1$, $e_n(x)\overline{e_m(x)} = e_{n-m}(x)$, and

$$\int_0^1 e^{2\pi ikx}\mathrm{d}x = \left.\frac{e^{2\pi ikx}}{2\pi ik}\right|_0^1 = 0$$

when $k \neq 0$. Thus, one hopes that $\{e_n\}_{n\in\mathbb{Z}}$ forms a sort of "basis" for $C(\mathbb{R}/\mathbb{Z})$ so that, for any $f \in C(\mathbb{R}/\mathbb{Z})$, we may write

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e_n(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi inx} \text{ where } c_n = (f, e_n) = \int_0^1 f(x)\overline{e_n(x)}\mathrm{d}x = \int_0^1 f(x)e^{-2\pi inx}\mathrm{d}x.$$

Of course, $C(\mathbb{R}/\mathbb{Z})$ is infinite-dimensional and the above sums are infinite whereas the vector space spanned the the e_n's only includes their finite linear combinations, so this is not immediate. There is something prove here, and this, up to a certain growth rate condition, is precisely the content of part **(a)** of Theorem 1.1. Part **(b)** of this theorem takes the above equality one step further by giving sufficient conditions for it to respect derivatives.

3 Proof of The Main Theorem

Now that we have described our main theorem and a little bit of why one might believe it, let's actually prove it. Recall from the previous section that $C(\mathbb{R}/\mathbb{Z})$ denotes the vector space of periodic, continuous functions $f : \mathbb{R} \to \mathbb{C}$ endowed with the Hermitian inner product $(f, g) = \int_0^1 f\overline{g}$. Also recall that the functions $e_n(x) = e^{2\pi inx}$ form an orthonormal set in this space.

We begin with the inequality $\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_0^1 |f(x)|^2 \mathrm{d}x$ in part **(a)** of the main theorem. Using that the e_n's are orthonormal, this will follow almost immediately from the analogous statement for finite-dimensional vector spaces.

Lemma 3.1 (Bessel's inequality)**.** *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and continuous, and let $c_n = \int_0^1 f(x)e^{2\pi inx}\mathrm{d}x$. Then, $\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \int_0^1 |f(x)|^2 \mathrm{d}x$.*

Proof. Fix any $n \geq 0$ and let $V_n = \operatorname{span}\{e_{-n}, \dots, e_n, f\} \subset C(\mathbb{R}/\mathbb{Z})$. Then, $\dim V_n \leq 2n + 1$ is finite and so, since $e_{-n}, \dots, e_n$ are still orthonormal in V_n, we see that

$$\sum_{k=-n}^{n} |c_k|^2 = \sum_{k=-n}^{n} |(f, e_k)|^2 \leq (f, f) = \int_0^1 |f(x)|^2 \mathrm{d}x.$$

Because the above holds for all $n \geq 0$, we can take the limit as $n \to \infty$ to get the desired result. ∎

This brings us to the next part of **(a)** of Theorem 1.1: showing that

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e_n(x) =: g(x)$$

when $\sum_{n\in\mathbb{Z}} |c_n|$ converges. As is usual when working in an algebraic structure (such as a vector space), we would like to reduce to the case when $c_n = 0$ for all n. Let $g_n(x) = \sum_{k=-n}^{n} c_k e_k(x)$. We first claim that $g_n \to g$ uniformly. Indeed, for $n > m \geq 0$, we have

$$|g_n(x) - g_m(x)| = \left| \sum_{k=-n}^{-m-1} c_k e_k(x) + \sum_{k=m+1}^{n} c_k e_k(x) \right| \leq \sum_{k=-n}^{-m-1} |c_k| + \sum_{k=m+1}^{n} |c_k| .$$

Because $\sum_{n\in\mathbb{Z}} |c_n|$ converges, its tails must vanish, i.e. the upper bound above approaches 0 for large n, m independent of x. Hence, $g_n(x) \to g(x)$ uniformly, so $g(x)$ is well-defined and continuous, and $f(x) - g(x)$ has Fourier coefficients

$$(f - g, e_k) = (f, e_k) - (g, e_k) = c_k - \lim_{n\to\infty} (g_n, e_k) = c_k - c_k = 0.$$

Thus, we are reduced to the case when $c_n = 0$.

Remark 3.1. Because of Euler's formula $e_n(x) = \cos(2\pi nx) + i \sin(2\pi nx)$, we can write $\cos(2\pi nx) = (e_n(x) + e_{-n}(x))/2$ and $\sin(2\pi nx) = (e_n(x) - e_{-n}(x))/(2i)$. Hence, if $c_n = (f, e_n) = 0$ for all n, then also $(f, \cos(2\pi nx)) = 0 = (f, \sin(2\pi nx))$ for all n. Furthermore, any polynomial in $\cos(2\pi nx), \sin(2\pi nx)$ can be rewritten as a (finite) sums of e_n's, so also $(f, p) = 0$ where p is any trigonometric polynomial with coefficients in $\mathbb{Z}$.

Lemma 3.2. *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and continuous, and suppose that $c_n = \int_0^1 f(x)e^{2\pi inx}\mathrm{d}x = 0$ for all $n \in \mathbb{Z}$. Then, $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. We will first handle the case where f is real-valued, and then show how to get the general case from this case. Fix any $x_0 \in \mathbb{R}$. In order to show that $f(x_0) = 0$, we can equivalently show that $f(x + x_0) = 0$ when $x = 0$. Hence, we lose no generality in only showing that $f(0) = 0$. We will proceed by contradiction, so assume $f(0) > 0$ (if $f(0) < 0$ then replace f with $-f$).

By the remark above this lemma, $(f, p) = 0$ for all trigonometric polynomials $p(x)$ with integral coefficients. At the same time $f(0) > 0$, so if we construct a real-valued such p where $p(x)$ is concentrated around $x = 0$, then the fact that $f(x) > 0$ for x near 0 (by continuity) will force $(f, p) = \int f(x)\overline{p(x)}\mathrm{d}x = \int f(x)p(x)\mathrm{d}x$ to be nonzero. This is our plan.

First, we need to decide what we mean by "x near 0." Since f is continuous, we may choose some $\delta \in (0, 1/2)$ so that $f(x) > f(0)/2$ whenever $|x| < \delta$. Let

$$p_1(x) = \varepsilon + \cos(2\pi x)$$

where $\varepsilon > 0$ is small enough that $|p(x)| < 1 - \varepsilon/2$ whenever $\delta \leq |x| \leq \frac{1}{2}$.[1] On the other hand, since $p_1(0) = \varepsilon + 1 > 1 + \varepsilon/2$, we can use continuity of $p_1(x)$ to obtain a positive $\eta < \delta$ such that $p(x) \geq 1 + \varepsilon/2$ for all x with $|x| < \eta$. Now, for any $k > 1$, set $p_k(x) = p_1(x)^k$. Finally, since f is continuous, it is bounded on the compact set $[-1/2, 1/2]$ so we can fix $B > 0$ large enough that $|f(x)| \leq B$ for all $x \in [-1/2, 1/2]$. By

[1] Any $\varepsilon < \frac{2}{3}(1 - \cos(2\pi\delta))$ should work since $\cos(2\pi x) \leq \cos(2\pi\delta)$ for x in this range.

construction, each p_k is a real-valued trigonometric polynomial, so

$$(f,p) = \int_0^1 f(x)p_k(x)\mathrm{d}x = 0$$

for all $k \geq 1$. At the same time, our various chosen parameters give us the following integral estimates

$$\begin{aligned}
\left| \int_{\delta \leq |x| \leq \frac{1}{2}} f(x)p_k(x)\mathrm{d}x \right| &\leq \int_{\delta \leq |x| \leq \frac{1}{2}} |f(x)|\,|p_k(x)|\,\mathrm{d}x &&\leq B\left(1 - \frac{\varepsilon}{2}\right)^k \\
\int_{\eta \leq |x| < \delta} f(x)p_k(x)\mathrm{d}x &\geq \frac{f(0)}{2} \int_{\eta \leq |x| < \delta} p_k(x)\mathrm{d}x &&\geq 0 \\
\int_{|x| < \eta} f(x)p_k(x)\mathrm{d}x &\geq \int_{|x|<\eta} \left(\frac{f(0)}{2}\right)\left(1 + \frac{\varepsilon}{2}\right)^k \mathrm{d}x &&= \frac{\eta f(0)}{2}\left(1 + \frac{\varepsilon}{2}\right)^k
\end{aligned}$$

for all $k \geq 1$. As $k \to \infty$, $(1-\varepsilon/2)^k \to 0$ and $(1+\varepsilon/2)^k \to \infty$ so, combining the three estimates above, we see that

$$0 = (f, p_k) = \int_0^1 f(x)p_k(x)\mathrm{d}x = \int_{-1/2}^{1/2} f(x)p_k(x)\mathrm{d}x \to \infty,$$

a contradiction. Thus $f = 0$ after all.

When f is not assumed real-valued, we write $f(x) = u(x) + i(v)$ with $u, v : \mathbb{R} \to \mathbb{R}$ continuous, periodic. Then, $u(x) = \frac{1}{2}(f(x) + \overline{f}(x))$ and $v(x) = -\frac{i}{2}(f(x) - \overline{f}(x))$. Hence, the Fourier coefficients of u are

$$(u, e_n) = \frac{1}{2}((f, e_n) + (\overline{f}, e_n)) = \frac{1}{2}((f, e_n) + \overline{(f, e_{-n})}) = 0$$

since f's Fourier coefficients are all 0. Similarly, $(v, e_n) = 0$ for all n, so $u = v = 0$ and hence $f = 0$ as well. ■

Hence, by the discussion preceding this lemma, we have succeeded in proving part **(a)** of Theorem 1.1. Part **(b)** of that theorem, as we shall soon see, follows from **(a)** by an induction argument.

Lemma 3.3. *Let $f : \mathbb{R} \to \mathbb{C}$ be periodic and continuous, For each $n \in \mathbb{Z}$, let $c_n = \int_0^1 f(x)e^{-2\pi nx}\mathrm{d}x$. Suppose that $c_n = O(1/n^k)$ for some fixed integer $k \geq 2$. Then, f is continuous differentiable $k-2$ times and its derivative can be computed termwise.*

Proof. First suppose $k = 2$. Then, the fact that $c_n = O(n^{-2})$ entails that $\sum |c_n|$ converges, so $f = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}$ (with the partial sum converging uniformly) by Theorem 1.1**(a)**. Now suppose $k \geq 3$ and that the claim holds for $k-1$. Replacing f with $f^{(k-1)}$, we are reduced to showing that f is continuously differentiable once when $k = 3$. In this case, let $f_n(x) = \sum_{m=-n}^{n} c_m e_m(x)$, so $f_n \to f$ uniformly. By a general theorem of analysis, to show f' exists and that $f_n' \to f'$ uniformly, it suffices to show that f_n' has a uniform limit. Note that

$$f_n'(x) = \sum_{m=-n}^{n} 2\pi i m c_m e_m(x) = 2\pi i \sum_{m=-n}^{n} m c_m e_m(x),$$

so the Fourier coefficients $c_m' = mc_m = (f_n, e_m)$ of f_n (when $|m| \leq n$) satisfy $c_m' = O(m/m^3) = O(m^{-2})$ and so are absolutely summable. Thus, $\lim f_n' = \sum_{m \in \mathbb{Z}} c_m' e_m(x) = \sum_{m \in \mathbb{Z}} c_m e_m'(x)$ is a continuous function, is the uniform limit of f_n', and hence is equal to f'. This proves the claim. ■

4 Concluding Remarks

It is worth briefly recapping what we have done. Given a continuous function $f : \mathbb{R} \to \mathbb{C}$ which is also periodic, we wanted to be able to express it as an infinite sum

$$f(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}. \tag{3}$$

for some appropriately chosen coefficients $c_n \in \mathbb{C}$. By finding an appropriately large orthonormal set of periodic, continuous functions $e_n(x) = e^{2\pi i n x}$, we were able to guess that one should take

$$c_n = (f, e_n) = \int_0^1 f(x) e^{-2\pi i n x} \mathrm{d}x.$$

We then showed that when the c_n's are absolutely summable, this choice of coefficients does indeed yield (3), and furthermore, that this is the only choice of coefficients which works in this case. Finally, by applying the same sort of reasoning to the derivatives $f^{(k)}$ of f, we were able connect the growth rate of the c_n's directly to the smoothness of f; namely, we should that $c_n = O(n^{-k}) \implies f$ has $(k-2)$ continuous derivatives which can computed from its Fourier series exactly as one would hope.