# Homological Stability Notes

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These are my course notes for "a minicourse on homological stability". Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Alexander Kupers, and the course website can be found by clicking this link. This was offered through the electronic computation homotopy theory 'online research community.'

I feel like there's gotta be a shorter term for what this is

## Contents



[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

# List of Figures



# List of Tables

# 1 Day 1 (5/4)

## 1.1 Admin stuff

4 part miniseries. Lectures will be posted, and recordings will maybe be posted. The lecture notes will also maybe be posted. For questions, use chat or speak up.

They have been posted, and are probably more useful than my notes

*Remark* 1.1. There are exercises in the notes on the website.

## 1.2 Intro, Homology of the symmetric group

We want to explain one example of homological stability in detail. At the end, we'll mention other places where homological stability arises.

**Notation 1.2.** We let $\Sigma_n$ denote the permutation group of a set of size $n$.

We want to understand its homology. How is its homology defined? Whenever you have a discrete group $G$, there is a space $BG$ determined uniquely up to homotopy by

$$\pi_*(BG) = \begin{cases} 0 & \text{if } * \neq 1 \\ G & \text{if } * = 1. \end{cases}$$

How does one construct this space? There are many ways, but here's one...

Given a group $G$, let $*//G$ denote the category with a single object $*$ and morphisms $\mathrm{Hom}_{*//G}(*, *) = G$, i.e. there's a morphism $* \xrightarrow{g} *$ for every $g \in G$ and composition of morphisms is given by the group law in $G$. Associated to this is its nerve $N(*//G)$ which is a simplicial set whose $p$-simplies are chains of $p$ composable morphisms

$$* \xrightarrow{g_1} * \to \cdots \to * \xrightarrow{g_p} *.$$

The geometric realization of this nerve is one model of $BG$.

**Question 1.3.** *What is* $\mathrm{H}_*(B\Sigma_n; \mathbb{Z})$*?*

(this is the same as group homology of $\Sigma_n$)

**Example.** When $n = 0, 1$, $\Sigma_n = \{e\}$ and $B\Sigma_n \simeq *$ is contractible.

When $n = 2$, $\Sigma_n \cong C_2$ and $BC_2 \simeq \mathbb{RP}^\infty$. One has

$$\widetilde{\mathrm{H}}_*(\mathbb{RP}^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } * \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

When $n = 3$, $\Sigma_n \cong D_3$ is the dihedral group of order 6. Can compute its homology e.g. via writing it as an extension of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ and then using Serre spectral sequence. One gets

$$\widetilde{\mathrm{H}}_*(BD_3; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } * \equiv 1 \mod 4 \\ \mathbb{Z}/6\mathbb{Z} & \text{if } * \equiv 3 \mod 4 \\ 0 & \text{otherwise.} \end{cases}$$

We see from above that doing these computations by hand quickly gets more involved.

| $d$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 6 | | | | | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^5$ | $\mathbb{Z}_2^5$ |
| 5 | | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^5$ | $\mathbb{Z}_2^5$ |
| 4 | | | | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ |
| 3 | | | $\mathbb{Z}_2$ | $\mathbb{Z}_6$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_{12}$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_{12}$ | $\mathbb{Z}_2^2 \oplus \mathbb{Z}_{12}$ |
| 2 | | | | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 1 | | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | |
| 0 | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Table 1: The homology groups $\mathrm{H}_d(B\Sigma_n;\mathbb{Z})$. Recall the stable range is $d \leq \frac{1}{2}(n-1)$

Alexander put up a table of these groups (A nicer looking version of Table 1 here). There are some noticeable patterns.

**(1)** $\widetilde{\mathrm{H}}_d(B\Sigma_n;\mathbb{Z})$ is finite.

**(2)** In fixed degree $d$, the homology becomes independent of $n$ as $n$ increases.

**(3)** the homology only increases in size.

**(4)** the $p$-power torsion only changes when $p \mid n$

Note that in the above remarks, we're only comparing these as abstract abelian groups. It would be better if we had actual maps between them.

**Notation 1.4.** Let $\underline{n}$ denote the $n$-element set, $\underline{n} = \{1, \ldots, n\}$ if you like.

The inclusion $\underline{n} \hookrightarrow \underline{n+1}$ gives rise to $\sigma : \Sigma_n \to \Sigma_{n+1}$ (extension by id on $n$+1). The nerve construction of $BG$ we gave earlier is functorial, so we get an induces map $\sigma : B\Sigma_n \to B\Sigma_{n+1}$ on classifying spaces, and so a map

$$\sigma_* : \mathrm{H}_*(B\Sigma_n;\mathbb{Z}) \to \mathrm{H}_*(B\Sigma_{n+1};\mathbb{Z})$$

on homology. We can now reformulate some of our earlier observations in terms of these maps.

**(2')** $\sigma_*$ are isomorphisms in a range of degrees tending to $\infty$ w/ $n$

**(3')** $\sigma_*$ is injective

**(4')** $\sigma_*$ is an isomorphism of $p$-power torsion unless $p \mid (n+1)$.

All of these observed properties are in fact true.

Both **(1)** and **(4')** have the same source: transfer maps.

**Proposition 1.5.** *Let $G$ be finite. Then,* ***(i)*** $\widetilde{\mathrm{H}}_*\left(BG;\mathbb{Z}\left[\frac{1}{|G|}\right]\right) = 0$. *Furthermore,* ***(ii)*** *if $H \subset G$ is a subgroup, then* $\mathrm{H}_*\left(BG;\mathbb{Z}\left[\frac{1}{[G:H]}\right]\right)$ *is a retract*[2] *of* $\mathrm{H}_*\left(BH;\mathbb{Z}\left[\frac{1}{[G:H]}\right]\right)$.

[2]There are maps $\mathrm{H}_*\left(BG;\mathbb{Z}\left[\frac{1}{[G:H]}\right]\right) \rightleftarrows \mathrm{H}_*\left(BH;\mathbb{Z}\left[\frac{1}{[G:H]}\right]\right)$ such that the composition starting and stopping at homology of $BG$ is the identity, so homology of $BG$ is a direct summand of that of $BH$

Above, **(ii)** implies **(i)** by taking $H = e$. Can get **(4')** from **(ii)** using $[\Sigma_{n+1}; \Sigma_n] = n+1$ (details left as exercise).

The properties that are really interesting for this minicourse are **(2')** and **(3')**, especially property **(2')**.

**Definition 1.6.** Let $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} X_2 \to \dots$ be a sequence of maps between spaces. This **exhibits homological stability** if the induced maps

$$\sigma_* : \mathrm{H}_*(X_n; \mathbb{Z}) \to \mathrm{H}_*(X_{n+1}; \mathbb{Z})$$

are isomorphisms in a range tending to $\infty$ w/ $n$.

In practice, want an explicit formula for this range (e.g. $* \leq \frac{2}{3}n + 1$ or something), and you may want to vary the coefficients some times.

The goal for the next lecture is the following theorem.

**Theorem 1.7** (Nakaoka)**.** *$B\Sigma_0 \to b\Sigma_1 \to B\Sigma_2 \to \dots$ exhibit homological stability. The induces maps*

$$\sigma_* : \mathrm{H}_*(B\Sigma; \mathbb{Z}) \to \mathrm{H}_*(B\Sigma_{n+1}; \mathbb{Z})$$

*are surjections if $* \leq \frac{n}{2}$ and isomorphisms if $* \leq \frac{n-1}{2}$.*

*Remark* 1.8. **(3')** says these maps are always injective, so the range for iso's can be improved to $* \leq \frac{n}{2}$.

**(3')** is slightly special. It occurs in this case as a consequence of transfer maps.

## 1.3 Using homological stability for symmetric groups

**Slogan.** Homological stability can transfer information from finite $n$ to infinite $n$, and vice versa.

### 1.3.1 Alternating groups

Let $X$ be a (based) path-connected space. The Hurewicz map gives a homomorphism $\pi_1(X) \to \mathrm{H}_1(X; \mathbb{Z})$ inducing an isomorphism $\pi_1(X)^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(X; \mathbb{Z})$. We can apply this to $B\Sigma_n$ to get get an iso $\Sigma_n^{\mathrm{ab}} \xrightarrow{\sim} \mathrm{H}_1(B\Sigma_n; \mathbb{Z})$.

We can use this to understand the commutator subgroup of $\Sigma_n$. We have the usual sign homomorphism $\Sigma_n \to C_2$. This induces $B\Sigma_n \to BC_2$ which induces $\mathrm{H}_1(B\Sigma_n; \mathbb{Z}) \to \mathrm{H}_1(BC_2; \mathbb{Z}) \simeq C_2^{\mathrm{ab}} = C_2$. This map is compatible with stabilization (since the sign homomorphism is), so we get a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_1(B\Sigma_n; \mathbb{Z}) & \xrightarrow{\quad\sigma_*\quad} & \mathrm{H}_1(B\Sigma_{n+1}; \mathbb{Z}) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}/2\mathbb{Z} & \end{array}$$

with both vertical maps the sign homomorphism.

Note that the sign homomorphism is an iso when $n = 2$. Unfolding the above diagram gives

$$\begin{array}{ccccc} \mathrm{H}_1(B\Sigma_2;\mathbb{Z}) & \longrightarrow & \mathrm{H}_1(B\Sigma_3;\mathbb{Z}) & \longrightarrow & \cdots \\ {\scriptstyle \mathrm{sign}}\downarrow\wr & & \downarrow & & \downarrow \\ \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

The homological stability result tells us that the first map in the top row is surjective while every map in the top row afterwards is an isomorphism. In fact, the first map is injective by an easy diagram chase, so it is an iso as well. Hence, every map is an isomorphism.

**Corollary 1.9.** *The sign homomorphism* $\mathrm{sign} : \Sigma_n \to \mathbb{Z}/2\mathbb{Z}$ *is the abelianization. Hence,* $[\Sigma_n, \Sigma_n] = \ker(\mathrm{sign}) = A_n$.

### 1.3.2 Group completion

There is always a map

$$\mathrm{H}_*(B\Sigma_n;\mathbb{Z}) \longrightarrow \varinjlim_{n\to\infty} \mathrm{H}_*(B\Sigma_n;\mathbb{Z}).$$

Call the RHS **stable homology**. Homological stability says that this map is an iso in a range tending to $\infty$ w/ $n$. Hence,

- stable homology gives info about $\mathrm{H}_*(B\Sigma_n;\mathbb{Z})$
- $\mathrm{H}_*(B\Sigma_n;\mathbb{Z})$ gives info about stable homology.

This probably would not be very useful unless we had another description of the stable homology. Luckily, stable homology has a stable homotopy-theoretic description as well.

The stabilization maps $\sigma$ from before were constructed using the inclusions $\underline{n} \hookrightarrow \underline{n+1}$. More generally, disjoint union gives homomorphisms

$$\Sigma_n \times \Sigma_m \to \Sigma_{n+m}.$$

These induce maps $B\Sigma_n \times B\Sigma_m \to B\Sigma_{n+m}$ of classifying spaces. You should think of these as 'multiplication maps' on $\bigsqcup_{n\geq 0} B\Sigma_n$. In a suitable model, this is part of the structure of a unital topological monoid which is homotopy-commutative.

**Example.** $\pi_0 := \pi_0\left(\bigsqcup_{n\geq 0} B\Sigma_n\right)$ is a unital commutative monoid. This is $(\mathbb{N}, +)$.

The stabilization maps combine to form $\sigma : \bigsqcup_{n\geq 0} B\Sigma_n \to \bigsqcup_{n\geq 0} B\Sigma_n$ giving $\mathrm{H}_*(\bigsqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})$ the structure of a $\mathbb{Z}[\pi_0]$-module. In the colimit $\varinjlim_{\sigma} \mathrm{H}_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})$ the action of $\pi_0$ becomes invertible. Hence you get an induced map

$$\mathrm{H}_*(\bigsqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})[\pi_0^{-1}] \xrightarrow{\sim} \varinjlim \mathrm{H}_*(\bigsqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})$$

which turns out to be an iso.

The McDuff-Segal group completion theorem tells you how to compute the LHS.

**Theorem 1.10** (McDuff-Segal)**.** *Let $M$ be a homotopy-commutative unital topological monoid. Then,*

$$\mathrm{H}_*(M;\mathbb{Z})[\pi_0^{-1}] \cong \mathrm{H}_*(\Omega BM;\mathbb{Z}).$$

Think of $\Omega BM$ as a way of inverting $\pi_0$ topologically instead of algebraically.

Here, $\Omega$ is the usual based loop space. $BM$ is constructed similarly to $BG$.

*Construction* 1.11. Start with a category $*//M$ enriched in spaces. There will be a single object $*$. The morphism space is $M$. The composition law is multiplication in $M$, and the identity is the unit. Then, $BM = |N(*//M)|$ is the classifying space of the nerve.

**Warning 1.12.** Above construction only correct if $* \hookrightarrow M$ is a cofibration.

There are many techniques for understanding spaces like $BM$. Many of them apply to $M = \bigsqcup_{n\geq 0} B\Sigma_n$. They give the following result.

**Theorem 1.13** (Barratt-Priddy-Quillen-Segal)**.**

$$\Omega B\left(\bigsqcup_{n\geq 0} B\Sigma_n\right) \simeq \Omega^\infty \mathbb{S},$$

*where $\mathbb{S}$ is the sphere spectrum.*

The main thing to know about the sphere spectrum is

$$\pi_i(\Omega^\infty\mathbb{S}) = \varinjlim_{k\to\infty} \pi_{i+k}(S^k),$$

its homotopy groups ares the stable homotopy groups of spheres.

The upshot of all this is that the stable homology of symmetric groups is the same as $\mathrm{H}_*(\Omega_0^\infty\mathbb{S};\mathbb{Z})$ and so related to stable homotopy of spheres (the 0 subscript in $\Omega_0$ denotes taking a single connected component).

### 1.3.3 Serre's theorem and variations

By homological stability for symmetric groups, we have an isomorphism $\mathrm{H}_*(B\Sigma_n;\mathbb{Z}) \cong \mathrm{H}_*(\Omega_0^\infty\mathbb{S})$ for $* \leq \frac{n-1}{2}$. Recall by **(1)** earlier that the LHS is finite for $* > 0$. As a consequence, by taking $n \gg 0$ (RHS independent of $n$), we see that

$$\mathrm{H}_*(\Omega_0^\infty\mathbb{S}) \text{ is finite for } * > 0.$$

This has the following consequence.

**Theorem 1.14** (Serre)**.** *$\pi_*(\mathbb{S})$ are all finite for $* > 0$.*

*Proof Sketch.* In positive degree, $\pi_*(\mathbb{S}) = \pi_*(\Omega_0^\infty\mathbb{S})$. Furthermore, $\Omega_0^\infty\mathbb{S}$ is a **simple space**, i.e. $\pi_1$ abelian and acts trivially on higher homotopy (every infinite loop space is simple). Hence, a Serre class argument gives $\widetilde{\mathrm{H}}_*(\Omega_0^\infty\mathbb{S};\mathbb{Z})$ is finite for all $* \geq 0 \iff \pi_*(\Omega_0^\infty\mathbb{S})$ is finite for all $* \geq 0$. ■

We can do something similar with torsion. Recall **(4')** which, among other things, tells us that $\mathrm{H}_*(B\Sigma_n;\mathbb{Z})$ has no $p$-torsion for $n < p$. Hence the same will be true for stable homology; it has no $p$-torsion for $* \leq \frac{p-1}{2}$. Another Serre class argument gives

**Lemma 1.15.** *$\pi_*(\mathbb{S})$ has no $p$-torsion for $* \le \frac{p-1}{2}$.*

In fact, a much better range is known.

**Theorem 1.16** (Serre)**.** *$\pi_*(\mathbb{S})$ has no $p$-torsion for $* < 2p-3$.*

But we can now argue in the other direction. A Serre class argument shows stable homology has no $p$-torsion in this range. Then homological stability gives information about $\mathrm{H}_*(B\Sigma_n;\mathbb{Z})$. One gets (recall stabilization maps always injective)

**Proposition 1.17.** *$\mathrm{H}_*(B\Sigma_n;\mathbb{Z})$ has no $p$-torsion for $* < 2p-3$.*

Next time we will go over the proof of homological stability for symmetric groups. There will be one connectivity argument that we won't be able to fit in, but we'll include it in the third lecture.

**Question 1.18** (Audience)**.** *Do symmetric groups exhibit homological stability for other coefficients?*

**Answer.** Yes. If you use constant coefficients, this follows e.g. from universal coefficients. We will also see this from the proof which will be 'coefficient agnostic.' This is a good question though because the stabilization range may depend on which coefficients you choose. You could also take coefficients in a local system, so let symmetric groups acts on your coefficients. We'll talk about this a bit in lecture 3 or 4.

**Question 1.19** (Audience)**.** *How does homological stability behave under taking quotients or subgroups?*

**Answer.** This is a difficult question. Say you have extensions

$$1 \longrightarrow H_n \longrightarrow G_n \longrightarrow Q_n \longrightarrow 1.$$

If you have homological stability for $Q_n$ for enough local systems, you may have a chance of proving it for $G_n$. If you have it for $G_n$, and the quotients $Q_n$ are simple (and, say, independent of $n$), then you may have a chance of proving it for $H_n$. There are no general non-technical statements, but there are some special cases where things work out.

**Question 1.20** (Audience)**.** *In practice, do people work one prime at a time?*

**Answer** (paraphrase of what I understood)**.** Not actually. Generally, you do not use too much about your group to get these stability result. Things boil down to showing a certain simplicial complex is highly connected. To need to work one prime at a time, you'd need something like the complex is not highly connected but is homologically connected with certain coefficients; this tends to not be the case in practice.

There were other questions I didn't record.

# 2 Day 2 (5/6)

## 2.1 Admin stuff

Recording from first talk on website along with notes from the first talk. These notes have exercises in them, so give them a go.

Today we want to proof homological stability for symmetric groups.

## 2.2 The Strategy

(Reference: Randal-Williams & Wahl)

We will be following the notation/strategy of a framework for proving homological stability results due to Randal-Williams & Wahl. The general strategy goes back to Quillen.

**History.** This strategy was never published but you can read it in Quillen's unpublished notebooks which have been digitized. Apparently the pages where the strategy starts are blank because they were left out in the sun too long (or something).

Let's remind ourselves of what we want to prove.

**Theorem 2.1** (Nakaoka)**.** $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \to \dots$ *exhibits homological stability. More precisely,*

$$\sigma_* : \mathrm{H}_*(B\Sigma_n;\mathbb{Z}) \to \mathrm{H}_*(B\Sigma_{n+1};\mathbb{Z})$$

*is a surjection if* $* \leq \frac{n}{2}$ *and an isomorphism if* $* \leq \frac{n-1}{2}$.

**Recall 2.2.** We constructed classifying spaces of (discrete) groups as $BG = |N(*//G)|$.

This group category notation $*//G$ suggest that this is some sort of quotient. Indeed, say $G \curvearrowright X$, i.e. $X$ is a "$G$-space." One can take the normal quotient, but it's often better to take the homotopy quotient $X//G$. When $X = *$ is a point, this homotopy quotient is $BG$, up to homotopy equivalence. We won't need to general construction of homotopy quotients, but will need a few properties.

**(1)** homotopy quotients are natural, i.e. if $f : X \to Y$ is a $G$-equivariant map (i.e. $g \cdot f(x) = f(g \cdot x)$), then you get an induced map

$$X//G \longrightarrow Y//G$$

on homotopy quotients.

**(2)** homotopy quotients preserve homological connectivity.

**Definition 2.3.** We say $f : X \to Y$ is **homologically $d$-connected** if the induced map on $\mathrm{H}_i$ is a surjection for $i \leq d$ and an iso for $i < d$.

If $f : X \to Y$ is $G$-equivariant and homologically $d$-connected, then $f//G : X//G \to Y//G$ is homologically $d$-connected as well.

**(3)** homotopy quotients commute with geometric realisations. If $X_\bullet$ is a semi-simplicial space w/ $G$-action (acts on simplicies and face maps are $G$-equivariant), then

$$\|X_\bullet\|//G \simeq \|X_\bullet//G\|$$

are homotopy equivalent.

**(4)** We need a description of homotopy quotients of transitive $G$-sets. Let $S$ be a transitive $G$-set. The induced map

$$B\,\mathrm{Stab}_G(s) \simeq \{s\}//\,\mathrm{Stab}_G(s) \to S//G$$

is a weak equivalence ($s \in S$ any point).

When we try to prove homological stability, we will only care about the homology of symmetric groups in some range. By **(2)** this means that we can replace the point $*$ (in $*//\Sigma_n$) with a space which is $d$-connected. We'll use a semi-simplicial space whose $p$-simplices are transitive $G$-sets. Such a space naturally comes with a filtration and so it's homology is computable via spectral sequences. This will allow us to make an inductive argument.

## 2.3 Injective words/The steps for symmetric groups

**Notation 2.4.** Let $\Delta$ denote the cateogry of non-empty finite ordered sets w/ order-preserving maps. For example, $[p] = (0 < \cdots < p) \in \Delta$.

**Definition 2.5.** A **simplicial set** is a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ and a **simplicial space** is a functor $\Delta^{\mathrm{op}} \to \mathrm{Top}$.

These consist of sets/spaces $X_p = X([p])$. In $\Delta$, one has maps $\delta_i : [p-1] \to [p]$ skipping entry $i$, and $\sigma_i : [p] \to [p-1]$ doubling the $i$th entry. These induce **face maps** $d_i : X_p \to X_{p-1}$ and **degeneracy maps** $s_i : X_{p-1} \to X_p$. This let us form the **geometric realization**

$$|X_\bullet| = \left( \bigsqcup_{p \geq 0} \Delta^p \times X_p \right) / \sim,$$

where the **standard $p$-simplex** is $\Delta^p := \left\{ (t_0, \ldots, t_p) \in [0,1]^{p+1} : \sum t_i = 1 \right\}$ and the equivalence relation $\sim$ is generated by

$$(\delta_i t, x) \sim (t, d_i x) \text{ and } (\sigma_i t, x) \sim (t, s_i x).$$

For the purpose of proving homological stability results, the degeneracy maps play no role, so we can make like simpler by forgetting about them. To do this, we replace $\Delta$ by $\Delta_{inj}$.

**Notation 2.6.** $\Delta_{\mathrm{inj}}$ is the category of non-empty finite ordered sets whose morphisms are *injective* order-preserving maps.

**Definition 2.7.** A **semi-simplicial space/set** is a functor $\Delta_{inj}^{\mathrm{op}} \to \mathrm{Top}$ or $\Delta_{inj}^{\mathrm{op}} \to \mathrm{Set}$

We can still form the **geometric realization**

$$\|X_\bullet\| := \left( \bigsqcup_{p \geq 0} \Delta^p \times X_p \right) / \sim$$

now with $\sim$ generated by $(\delta_i t, x) \sim (t, d_i x)$. Semi-simplicial spaces tend to be smaller than simplicial ones, so it gives us less to keep track of.

We want to replace the point $*$ (in $*//\Sigma_n$) by $\|$some semi-simplicial set$\|$.

In our case, the semi-simplicial set will be something denoted $W_n(\underline{1})_\bullet$. The notation is clunky, but chosen to fit into the Randal-Williams and Wahl framework.

**Notation 2.8.** Let FI denote the category whose objects are finite sets and whose morphisms are injections.

(In general, one uses a category built for their particular application)

**Definition 2.9.** We let $W_n(\underline{1})_\bullet$ be the semi-simplicial set w/ $p$-simplices given by

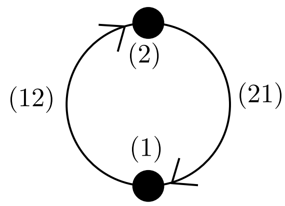
$$W_n(\underline{1})_p := \mathrm{Hom}_{\mathrm{FI}}([p], \underline{n}).$$

Recall $[p] = (0 < \cdots < p)$ and $\underline{n} = \{1, \ldots, n\}$. The face maps are induced by precomposing with $\delta_i$, i.e. delete one element. This is sometimes called the **semi-simplicial set of injective words**.

*Remark* 2.10. $W_n(\underline{1})_p$ consists of sequences $(m_0, m_1, \ldots, m_p)$ of distinct elements of $1, \ldots, n$. The $i$th face map deletes $m_i$.

We would like this to be homologically highly connected.

**Example** $(W_2(\underline{1})_\bullet)$**.** This has 0-simplicies (1) and (2), and it has 1-simplicies (12) and (21). The geometric realization is a circle, see below




**Example** $(W_3(\underline{1})_\bullet)$**.** This has three 0-simplices, $(1), (2), (3)$. It has six 1-simplices and six 2-simplices. It will turn out that this is has a 1-connected geometric realization. Instead of proving this now, we'll show a particular loop is nullhomotopic just to make this sound plausible.

Consider the loop (12)(21) (essentially a copy of $W_1(\underline{1})_\bullet$). Note that we have the 2-simplex (12)(23)(13) as well as one given by (21)(13)(23). These two glue together to give a cone on our loop, so it is nulhomotopic.

Not being able to draw is a real issue

**Proposition 2.11.**

***(1)*** $\|W_n(\underline{1})\|$ *is homologically* $\frac{n-1}{2}$*-connected.*

***(2)*** $W_n(\underline{1})_p$ *is a transitive* $\Sigma_n$*-set, and the stabilizer of any* $x \in W_n(\underline{1})_p$ *is isomorphic to* $\Sigma_{n-p-1}$*.*

Something stronger than this is true, but harder to prove

Remember: A $p$-simplex of $W_n(\underline{1})$ has words of length $p+1$

The upshot is that $\|W_n(\underline{1})_\bullet\|$ can serve to replace $*$ in $*//\Sigma_n$, at least in the range we are interested in.

## 2.4 Geometric realization spectral sequence

The point is that geometric realizations $\|X_\bullet\|$ come with a natural filtration, the **skeletal filtration**. This is

$$F_r\|X_\bullet\| := \left( \coprod_{0 \le p \le r} \Delta^p \times X_p \right) / \sim$$

with equivalence relation the same as before. We naturally get

$$\begin{array}{ccccccc} F_0\|X_\bullet\| & \longrightarrow & F_1\|X_\bullet\| & \longrightarrow & F_2\|X_\bullet\| & \longrightarrow & \cdots \\ \downarrow \wr & & \downarrow & & \downarrow & & \\ X_0 & & \Sigma(X_1)_+ & & \cdots & & \end{array}$$

and the colimit (of the top line) gives the whole geometric realization. In general, the graded pieces are

$$\frac{F_r\|X_\bullet\|}{F_{r-1}\|X_\bullet\|} \simeq \frac{\Delta^r}{\partial\Delta^r} \wedge (X_r)_+.$$

As usual the +-subscript means add a disjoint basepoint.

**Theorem 2.12** (Segal)**.** *There is a strongly-convergent first quadrant spectral sequence*

$$E^1_{p,q} = \mathrm{H}_q(X_p;\mathbb{Z}) \implies \mathrm{H}_{p+q}(\|X_\bullet\|;\mathbb{Z})$$

*with differentials of bidegree $(-r, r+1)$, i.e.*

$$\mathrm{d}^r : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

*In particular, $\mathrm{d}^1$ has an explicit formula,*

$$d^1 = \sum_{i=0}^{p}(-1)^i(d_i)_*.$$

*Furthermore, the edge homomorphism*[3] *is induced by the inclusion $X_0 \to \|X_\bullet\|$.*

## 2.5 Proof of Nakaoka's theorem

Proof is a strong induction over $n$. Assume we know it for $m \leq n$, and we'll prove it for $n+1$. The statments are easy for small $n$, so let's assume that $n+1 \geq 3$.

**Step 1** Replace $*$ by $\|W_n(\underline{1})_\bullet\|$. We saw earlier that $\|W_{n+1}(\underline{1})_\bullet\| \to *$ is homologically $\left(\frac{n}{2}+1\right)$-connected. Thus, the same is true for

$$\|W_{n+1}(\underline{1})_\bullet\|//\Sigma_{n+1} \to *//\Sigma_{n+1}.$$

This is good enough that we may replace RHS by LHS for the purpose of proving Theorem 2.1.

**Step 2** $E^1$-page of geometric realization spectral sequence. Note that

$$\|W_{n+1}(\underline{1})_\bullet\|//\Sigma_{n+1} \simeq \|W_{n+1}(\underline{1})_\bullet//\Sigma_{n+1}\|,$$

so we can take spectral sequence for RHS. This looks like

$$E^1_{pq} = \mathrm{H}_q(W_{n+1}(\underline{1})_p//\Sigma_{n+1}) \implies \mathrm{H}_{p+q}(W_{n+1}(\underline{1})_\bullet//\Sigma_{n+1}).$$

We would like to make the $E^1$-page more explicit, and to compute $d^1$. We say before that $W_{n+1}(\underline{1})_p$ is a transitive $\Sigma_{n+1}$-set, so fix some injection $f : [p] \hookrightarrow \underline{n+1}$. This gives a homotopy equivalence

$$*//\,\mathrm{Stab}_{\Sigma_{n+1}}(f) \xrightarrow{\sim} W_{n+1}(\underline{1})_p/\Sigma_{n+1}.$$

[3] $\mathrm{H}_q(X_0;\mathbb{Z}) = E^1_{0,q} \to E^\infty_{0,q} \to \mathrm{H}_q(\|X_\bullet\|;\mathbb{Z})$

It is important to keep track of which $f$ we use. We will take $f = \iota_p$, the inclusion of the *last* $p+1$ elements, i.e. $(n+1-p \dots n+1)$. Why make this choice? Because now

$$\Sigma_{n-p} = \mathrm{Stab}_{\Sigma_{n+1}}(\iota_p) \to \Sigma_{n+1}$$

is given exactly by iterating our stabilization map (i.e. adding elements to the end of our list).

At this point, we have $E^1_{pq} = \mathrm{H}_q(B\Sigma_{n-p})$. We need to figure out the differential. Recall it is the alternating sum

$$\mathrm{d}^1 = \sum (-1)^i (d_i)_*$$

of the induced maps of the face maps. To trace identifications, first note we have a commuting square

$$\begin{array}{ccc} *//\,\mathrm{Stab}_{\Sigma_{n+1}}(f) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_p//\Sigma_{n+1} \\ {\scriptstyle \mathrm{inc}}\downarrow & & \downarrow{\scriptstyle d_i//\Sigma_{n+1}} \\ *//\,\mathrm{Stab}_{\Sigma_{n+1}}(d_i f) & \xrightarrow{\simeq} & W_{n+1}(\underline{1})_{p-1}//\Sigma_{n+1} \end{array}$$

We still have a problem, $d_p \iota_p \neq \iota_{p-1}$ in general. There will be some $k_i$ s.t. $h_i d_i \iota_p = \iota_{p-1}$ though. Using this will let us making a commuting diagram (inc = inclusion)

$$*//$$

involving some conjugation maps. The one of the right is homotopic to the identity, so no problem. The ones on the left are more annoying, but luckily we get to choose which element $h_i$ we use. So far we have $(d_i)_* = (c_{h_i} \circ \mathrm{inc})_*$. Pick $h_i$ to be transposition swapping $n-p+1$ and $n-p+i$. This choice has the advantage that it acts by the identity on $\Sigma_{n-p}$. Thus,

$$\mathrm{d}^1 = \sum_{i=0}^{p} (-1)^i \sigma_* = \begin{cases} \sigma_* & \text{if } p > 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

where $\sigma_*$ is our stabilization map.

What does our spectral sequence look like? In each column you see the homology of symmetric groups. The differential is alternating between 0 and the stabilization map $\sigma_*$.

$$\begin{array}{c|ccccccc} q & & & & & & & \\ & \vdots & & \vdots & & \vdots & & \\ & \mathrm{H}_q(B\Sigma_n) & \xleftarrow{0} & \mathrm{H}_q(B\Sigma_{n-1}) & \xleftarrow{\sigma} & \mathrm{H}_q(B\Sigma_{n-2}) & \xleftarrow{0} & \dots \\ & \vdots & & \vdots & & \vdots & & \\ \hline & 0 & & 1 & & 2 & & p \end{array}$$

Figure 1: The $E^1$-page of our spectral sequence

We're out of time, so we'll pick up here next time. We'll finish the spectral sequence argument, and prove that $\|W_n(\underline{1})_\bullet\|$ is homologically $\frac{n-1}{2}$-connected. After that, we'll want to formalise this argument.

Didn't type up the questions this time, but some takeaways

- This was not the original argument. The original sounds more involved, something about computing homology with various coefficients.
- The semi-simplicial sets $W_n(\underline{1})_\bullet$ are well understood. In particular, it sounds like all their homology groups are known, but we don't need this much information for the argument we give.

# 3 Day 3 (5/11): homological stability for automorphism groups

## 3.1 Finishing the proof for symmetric groups

Recall we were in the middle of proving

**Theorem 3.1.** *$\sigma_* : \mathrm{H}_*(B\Sigma_n; \mathbb{Z}) \to \mathrm{H}_*(B\Sigma_{n+1}; \mathbb{Z})$ is a surjection for $* \leq \frac{n}{2}$ and an isomorphism for $* \leq \frac{n-1}{2}$.*

The proof was via strong induction over $n$, and we had done two steps.

- replace $*$ by $\|W_n(\underline{1})_\bullet\|$
- understand $E^1$ page of geometric realization spectral sequence (see Figure 1 or look below)

$$
\begin{array}{c|ccccccc}
q & & & & & & & \\
 & \vdots & & \vdots & & \vdots & & \\
 & \mathrm{H}_q(B\Sigma_n) & \xleftarrow{0} & \mathrm{H}_q(B\Sigma_{n-1}) & \xleftarrow{\sigma} & \mathrm{H}_q(B\Sigma_{n-2}) & \xleftarrow{0} & \dots \\
 & \vdots & & \vdots & & \vdots & & \\
\hline
 & 0 & & 1 & & 2 & & p
\end{array}
$$

There will be two cases, depending on $n$ being odd or even. We'll do two particular cases which are illustrative of the general case.

($n = 8$) What does the $E^2$-page look like? We can a lot of vanishing from the inductive hypothesis.

- The first column remains intact since differential into it is 0
- There will be a range (below a line of slope 1/2) where terms cancel out since $\sigma_*$ is an iso (or a surjection).

  Looking at the statement of Theorem 3.1, we see that in column 1, for example, the incoming $\sigma_* : \mathrm{H}_q(B\Sigma_6) \to \mathrm{H}_q(B\Sigma_7)$ is surjective whenever $q \leq 6/2 = 3$. Similarly, in column 2 the outgoing $\sigma_*$ will be an isomorphism (so injective) whenever $q \leq (6-1)/2 = 2.5$.

The resulting picture of the $E^2$-page is thus

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $q$ | | | | | | | |
| 4 | $\vdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | |
| 3 | $\vdots$ | 0 | $*$ | $*$ | $*$ | $*$ | |
| 2 | $\mathrm{H}_q(B\Sigma_8)$ | 0 | 0 | 0 | $*$ | $*$ | |
| 1 | $\vdots$ | 0 | 0 | 0 | 0 | 0 | |
| 0 | $\vdots$ | 0 | 0 | 0 | 0 | 0 | |
| | 0 | 1 | 2 | 3 | 4 | 5 | $p$ |

This is converging to $\mathrm{H}_{p+q}(B\Sigma_9)$. We see that the stabilization map (first left edge to $E_\infty$) is an isomorphism for degrees $n \leq 3$ but only a surjection in degree $n = 4$ (have a nontrivial differential $* \to \mathrm{H}_4(B\Sigma_8)$

$(n = 9)$ Start with a similar picture for $E^2$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $q$ | | | | | | | |
| 4 | $\vdots$ | $*$ | $*$ | $*$ | $*$ | $*$ | |
| 3 | $\vdots$ | 0 | 0 | $*$ | $*$ | $*$ | |
| 2 | $\mathrm{H}_q(B\Sigma_9)$ | 0 | 0 | 0 | 0 | $*$ | |
| 1 | $\vdots$ | 0 | 0 | 0 | 0 | 0 | |
| 0 | $\vdots$ | 0 | 0 | 0 | 0 | 0 | |
| | 0 | 1 | 2 | 3 | 4 | 5 | $p$ |

In this case, you get an iso for all degrees up to 4

This finishes the proof save for one step we skipped last time.

**Proposition 3.2.** $\|W_n(\underline{1})_\bullet\|$ *is homologically* $\frac{n-1}{2}$*-connected.*

*Remark* 3.3. In fact, it is $(n-2)$-connected, but we don't need this. References for this fact are in the lecture notes.

**Recall 3.4.** This is the semi-simplicial set with $W_n(\underline{1})_p = \mathrm{Hom}_{\mathrm{FI}}([p], \underline{n})$ and face maps given by precomposition (i.e. "deleting a point").

**Notation 3.5.** We write $W(\underline{n})_\bullet$ for this semi-simplicial set, and observe that $W(S)_\bullet$ makes sense for any set $S$ in place of $\underline{n}$.

*Proof of Proposition 3.2.* We'll use strong induction over $n$, assuming the cases $n \leq 3$ (we did $n \leq 2$ last time. $n = 3$ is an exercise). By construction $\|W(\underline{n})_\bullet\|$ has a CW-structure, and we can explicitly write down an $(n-2)$-skeleton for it. Let $S_i = \underline{n} \setminus \{i\}$ $(i = 1, \ldots, n)$. Then we have

$$\bigcup_{i=1}^{n} \|W(S_i)_\bullet\| \hookrightarrow \|W(\underline{n})_\bullet\|.$$

This is an $(n-2)$-skeleton. Hence, its induced map on homology is a surjection for $* \leq n-2$ (and an iso for $* \leq n-3$). Since we assumed $n \geq 4$, we have $n-2 \geq \frac{n-1}{2}$, so it suffices to prove the map on homology is 0 for $* \leq \frac{n-1}{2}$.

We first prove the following:

$$\bigoplus_{i=1}^{n} \mathrm{H}_*(\|W(S_i)_\bullet\|) \to \mathrm{H}_*\left(\bigcup_{i=1}^{n} \|W(S_i)_\bullet\|\right)$$

is surjective for $* \leq \frac{n-1}{2}$. To do this, we recognize it as an edge homomorphism in a **Mayer-Vietoris spectral sequence**:

$$E^1_{pq} = \bigoplus_{1 \leq i_0 < \cdots < i_p \leq n} \mathrm{H}_q\left(\bigcap_{j=0}^{p} \|W(S_{i_j})_\bullet\|\right) \implies \mathrm{H}_{p+q}\left(\bigcup_{i=1}^{n} \|W(S_i)\|_\bullet\|\right).$$

Note that

$$\bigcap_{j=0}^{p} \|W(S_{i_j})_\bullet\| = \left\| W\left(\bigcap_{j=0}^{p} S_{i_j}\right)_\bullet \right\| \cong \|W(\underline{n-p-1})_\bullet\|$$

which is homologically $\left(\frac{n-p-2}{2}\right)$-connected. Thus this spectral sequence will again have a triangle of 0 (with nonzero entries on the edges). On the bottom edge, one sees the cellular chain complex of $\partial\Delta^{n-1}$ so its homology vanishes for $0 < p < n-2$ (so get some 0's on the $E^2$-page on the bottom row). At this point, one just needs to do some numerology to get the desired map being a surjection.

Now, in our range of interest (for $* \leq \frac{n-1}{2}$), we have

$$\bigoplus_{i=1}^{n} \mathrm{H}_*(\|W(S_i)_\bullet\|) \twoheadrightarrow \mathrm{H}_*\left(\bigcup_{i=1}^{n} \|W(S_i)_\bullet\|\right) \twoheadrightarrow \mathrm{H}_*(\|W(\underline{n})_\bullet\|).$$

Hence, we only need to show this composition is the zero map in positive degrees. This is because the inclusions $\|W(S_i)_\bullet\| \hookrightarrow \|W(\underline{n})_\bullet\|$ are null-homotopic (exercise[4]). ■

Connectivity arguments like this are the crux of homological stability results, and are usually pretty difficult.

This finishes our proof of homological stability for symmetric groups, so let's now see things in a more general light.

## 3.2 Framework of Randal-Williams and Wahl

*Setup.* Start with a symmetric monoidal groupoid $G$.

- **groupoid** = category where all morphisms are isomorphisms
- **symmetric monoidal** = have $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\mathbf{1} \in \mathcal{C}$ and a swap map $\mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$

**Example.** $G = FB$, finite sets with bijections. Symmetric monoidal structure given by disjoint union $\sqcup : FB \times FB \to FB$ and empty set $\emptyset \in FB$.

[4] Use the removed element $\{i\} \in \underline{n}$ to cone things off

Given objects $A, X \in G$, we can form the group $G_n := \mathrm{Aut}_G(A \oplus X^{\oplus n})$. This naturally come with stabilization maps

$$G_n = \mathrm{Aut}_G(A \oplus X^{\oplus n}) \xrightarrow[-\oplus \mathrm{id}_X]{\sigma} \mathrm{Aut}_G(A \oplus X^{\oplus (n+1)}) = G_{n+1}.$$

**Question 3.6.** *When does the sequence*

$$BG_0 \xrightarrow{\sigma} BG_1 \xrightarrow{\sigma} BG_2 \to \dots$$

*exhibit homological stability?*

**Theorem 3.7** (**Randal-Williams-Wahl** (two people))**.** *Let $G, A, X$ be as above. Suppose $G$ satisfies*

***(i)*** *the monoid of iso classes of objects in $G$ under $\oplus$ satisfies cancellation*

***(ii)*** $\mathrm{Aut}(A) \xrightarrow{-\oplus \mathrm{id}_B} \mathrm{Aut}(A \oplus B)$ *is always injective.*

Really only need this for direct sums with $X$, not all $B$

*Then there are semi-simplicial sets $W_n(A, X)_\bullet$ such that if there is a $k \geq 2$ with $\|W_n(A, X)_\bullet\|$ homologically $\left(\frac{n-2}{k}\right)$-connected for all $n \geq 2$, then the maps*

$$\sigma_* : \mathrm{H}_*(BG_n) \longrightarrow \mathrm{H}_*(BG_{n+1})$$

*are surjective for $* \leq \frac{n}{k}$, and isos for $* \leq \frac{n-1}{k}$.*

*Remark* 3.8. The conditions on $G$ are "mild" and usually satisfied. The crux is the connectivity result for the $W$'s.

*Remark* 3.9. It's not necessary for $G$ to by symmetric. It suffices for it to be braided (its symmetry doesn't have to square to the identity).

I didn't get the entire explanation of why, but sounds like if you willing to work with $E_2$-algebras and to replace the $W_n$'s with semi-simplicial spaces (as opposed to sets), then you can remove both of the "mild" conditions.

What are these $W_n(A, X)_\bullet$'s? The first step is to build a different category from $G$ (think $FB \rightsquigarrow \mathrm{FI}$).

**Notation 3.10.** Let $\mathcal{U}G$ be the category whose objects are those of $G$, but whose morphisms $X \to Y$ are pairs $(f, Z)$ where $f : Z \oplus X \to Y$, defined only up to the equivalence relation $(f, Z) \sim (f', Z')$ if there is a diagram

$$\begin{array}{ccc} Z \oplus X & \xrightarrow{g \oplus \mathrm{id}_X} & Z' \oplus X \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f'} & \\ & Y & \end{array}$$

(think of a morphism as an inclusion of $X$ into $Y$ with a choice of complement)

**Notation 3.11.** Now, $W_n(A, X)_\bullet$ is the semi-simplicial set with $p$-simplicies

$$W_n(A, X)_p := \mathrm{Hom}_{\mathcal{U}G}\left(X^{\oplus [p]}, A \oplus X^{\oplus n}\right).$$

**Example.** Say $G = FB$, finite sets and bijections. Then $\mathcal{U}G = \mathrm{FI}$, finite sets and injections (with $Z$ playing the role of the complement). One then sees $W_n(\emptyset, \underline{1})_\bullet = W_n(\underline{1})_\bullet$.

**Example.** Take $G$ to be f.g. abelian groups with isomorphisms (this satisfies both mild conditions). What is $W_n(0,\mathbb{Z})$? It's $p$-simplicies are given by inclusions

$$W_n(0,\mathbb{Z})_p = \left\{\mathbb{Z}^{\oplus[p]} \hookrightarrow \mathbb{Z}^{\oplus n} \text{ with choice of complement}\right\}.$$

Ruth Charney proved that these semi-simplicial sets are highly connected (with $k=1$). Feeding her result into this theorem of Randal-Williams and Wahl gives homological stability for $BG_n = B\operatorname{Aut}(0 \oplus \mathbb{Z}^{\oplus n}) = B\mathrm{GL}_n(\mathbb{Z})$.

Let's see an application of this. Recall that in the first lecture we showed homological stability for $\Sigma_n$ gives finiteness for the stable homotopy groups of spheres. We'll see something of a similar flavor now, except this will be a result that, as far as I know, doesn't have a proof not using stability is some form.

**Application** (finite generation of $K_*(\mathbb{Z})$)**.** The $K$-theory of the integers is $K_*(\mathbb{Z}) = \pi_* K(\mathbb{Z})$ for some connective spectrum $K(\mathbb{Z})$ satisfying $\mathrm{H}_*(\Omega_0^\infty K(\mathbb{Z})) = \varinjlim_{n\to\infty} \mathrm{H}_*(B\mathrm{GL}_n(\mathbb{Z}))$. Hence, it suffices to prove that this homology is finitely generated in each degree (and then make a Serre class spectrum). By homological stability, for any fixed degree $d$,

$$\mathrm{H}_d(B\mathrm{GL}_m(\mathbb{Z})) \xrightarrow{\sim} \varinjlim_{n\to\infty} \mathrm{H}_d(B\mathrm{GL}_n(\mathbb{Z}))$$

for $m \gg 0$. Thus, we only need to prove that the groups $\mathrm{H}_d(B\mathrm{GL}_n(\mathbb{Z}))$ are finitely generated. This is a consequence of a result of Borel and Serre: $B\mathrm{GL}_n(\mathbb{Z})$ is homotopically equivalent to a CW-complex with finitely many cells in each dimension.

In the notes, you can find an argument that the (algebraic) $K$-theory groups of the sphere spectrum are also finitely generated.

*Remark* 3.12. There are also homological stability results w/ local coefficients. Generally for local coefficient systems that are "abelian" or "polynomial."

The next lecture will be the last one. We'll mainly be interested in the following question:

**Question 3.13.** *What can you say if homological stability fails?*

**Answer.** There are other sorts of stability phenomena which may still hold. One could have e.g. 'representation stability' or 'secondary stability'.

Forgot the type up the audience questions again...

# 4 Day 4 (5/13): more subtle stability phenomena

We want to at least introduce

- representation stability
- higher-order stability

In both cases, we'll go over a particular illustrative example. We won't have enough time to prove everything, or go over more general statements.

There are a few other topics which could be interesting, but we won't have time to talk about today.

- stability near "top dimension"
- stable stability

## 4.1 Representation Stability

To illustrate this, we'll look at configuration spaces.

**Definition 4.1.** Let $X$ be a topological space (e.g. $X = \mathbb{R}^d$). The **configuration space of** $n$ **unordered points in** $X$ is the space

$$C_n(X) := \{(x_1, \ldots, x_n) \in X^n : i \neq j \implies x_i \neq x_j\} / \Sigma_n.$$

By construction, this a quotient of the space

$$\mathrm{Conf}_n(X) = \{(x_1, \ldots, x_n) \in X^n : i \neq j \implies x_i \neq x_j\},$$

the **configuration space of** $n$ **ordered points in** $X$.

These are related to configuration spaces of symmetric groups via the following result.

**Lemma 4.2.**

***(i)*** $\mathrm{Conf}_n(\mathbb{R}^d)$ *is* $(d-2)$*-connected.*

***(ii)*** *There is a* $(d-1)$*-connected map* $C_n(\mathbb{R}^d) \to B\Sigma_n$

*Proof.* **(ii)** follows from **(i)**. To prove **(i)**, we'll prove a more general statement:

$$\mathrm{Conf}_n(\mathbb{R}^d \setminus k \text{ points}) \text{ is } (d-2)\text{-connected.}$$

We do this by induction over $n$ and $k$.

$(n = 1)$ In this case

$$\mathrm{Conf}_1(\mathbb{R}^d \setminus k \text{ pts}) = \mathbb{R}^d \setminus k \text{ pts} \simeq \bigvee_k S^{d-1}$$

is $(d-2)$-connected.

$(n - 1 \rightsquigarrow n)$ Only remembering the first point gives a fibration

$$\mathrm{Conf}_{n-1}(\mathbb{R}^d \setminus k+1 \text{ pts}) \hookrightarrow \mathrm{Conf}_n(\mathbb{R}^d \setminus k \text{ pts}) \twoheadrightarrow \mathbb{R}^d \setminus k \text{ pts}.$$

We know both the base and fiber are $(d-2)$-connected, so the same is true for the total space.

∎

These maps $C_n(\mathbb{R}^d) \to B\Sigma_n$ fit into a homotopy-commutative diagram

$$\begin{array}{ccc} C_n(\mathbb{R}^d) & \xrightarrow{\sigma} & C_{n+1}(\mathbb{R}^d) \\ {\scriptstyle (d-1)-ctd}\downarrow & & \downarrow{\scriptstyle (d-1)-ctd} \\ B\Sigma_n & \xrightarrow{\sigma} & B\Sigma_{n+1} \end{array}$$

The subtly above is defining the stabilization map $\sigma$ on the top. Morally, it is obtained by adding an extra point away from the $n$ points already there.

**Corollary 4.3.** *One has homological stability for* $C_0(\mathbb{R}^d) \to C_1(\mathbb{R}^d) \to C_2(\mathbb{R}^d) \to \dots$ *in degrees* $* < d-1$.

One can remove the $* < d-1$ condition above.

Seeing this, here's a natural question.

**Question 4.4.** *Do the configuration spaces* $\mathrm{Conf}_n(\mathbb{R}^d)$ *of ordered points satisfy homological stability?*

**Answer.** No. We can compute $\mathrm{H}_{d-1}(\mathrm{Conf}_n(\mathbb{R}^d); \mathbb{Q}) \cong \mathbb{Q}^{n-1}$ (lowest nonzero (reduced) homology group). Hence, if you fix a degree and let $n$ vary, you never get a stable value.

*Remark* 4.5. Keep in the mind the symmetric group acts on $\mathrm{H}_{d-1}(\mathrm{Conf}_n(\mathbb{R}^d); \mathbb{Q})$, i.e. $\Sigma_n \curvearrowright \mathbb{Q}^{n-1}$. What is this action? Turns out, as a $\mathbb{Q}[\Sigma_n]$-module $\mathrm{H}_{d-1}(\mathrm{Conf}_n(\mathbb{R}^d); \mathbb{Q})$ is the **reduced regular representation**

$$\ker\left(\mathbb{Q}[\underline{n}] \xrightarrow{\varepsilon} \mathbb{Q}\right).$$

Hence, taking into account the representation, this homology group does stabilise (we have this uniform description).

This phenomenon is called 'representation stability'. To give a more precise statement of this, let's replace $\mathbb{R}^d$ by a $d$-dimensional connected manifold $M$ of finite type (where $d \geq 2$).

What do we really mean by a 'uniform description' of these representations? To answer this, we need the classification of f.dim $\mathbb{Q}[\Sigma_n]$-modules. These are all semisimple, and the irreducible ones are in bijection with Young diagrams w/ $n$ boxes (equivalently, partitions of $n$ into positive integers).

**Example.** The trivial representation corresponds to the partition $n = n$.

The reduced regular representation $\leftrightarrow (n-1, 1)$ (i.e. to $n = (n-1) + 1$).

We can "stabilise" these partitions by adding 1 to the first (i.e. largest) entry. This allows us to consistently take iso classes of rational $\Sigma_n$-reps $V_n$ to iso classes of rational $\Sigma_{n+1}$-reps $V_{n+1}$ by stabilising the corresponding partitions.

Now, given a sequence $\{V_n\}_{n\geq 0}$ of rational $\Sigma_n$-reps, we can ask whether the multiplicity of the partition $(i_1 + k, i_2, \dots, i_k)$ of $n + k$ stabilises. We call this '**multiplicity stability**'.

**Theorem 4.6 (Church).** *For* $M$ *as above,* $\{\mathrm{H}^d(\mathrm{Conf}_n(M); \mathbb{Q})\}_{n\geq 0}$ *exhibit multiplicity stability.*

We could have formulated this for homology as well, e.g. using UCT. However, we prefer cohomology. We did not say the manifold was open, so there is no stabilization map which brings in a point from far away (like we did for $\mathbb{R}^d$). Instead, we have forgetful maps

$$\mathrm{Conf}_n(M) \to \mathrm{Conf}_{n-1}(M)$$

which deletes/forgets the $i$th point. This induce maps on cohomology where the index $n$ increases:

$$\mathrm{H}^d(\mathrm{Conf}_n(M); \mathbb{Q}) \leftarrow \mathrm{H}^d(\mathrm{Conf}_{n-1}(M); \mathbb{Q}).$$

*Remark* 4.7. This approach to representation stability relied on the classification of rational representations of symmetric groups. One does not have such a nice classification in general, so a different setup is needed for other groups.

#### 4.1.1 Categorical Representation Theory

Observe that these rational cohomology groups $\mathrm{H}^d(\mathrm{Conf}_n(M);\mathbb{Z})$ along with the forgetful maps we mentioned assemble to give a functor

$$\begin{array}{ccc} \mathrm{FI} & \longrightarrow & \mathrm{Ab} \\ S & \longmapsto & \mathrm{H}^d(\mathrm{Conf}_S(M);\mathbb{Z}) \end{array}$$

out of the category of finite sets with injections. If you have $T \hookrightarrow S$, to get a map $\mathrm{Conf}_T(M) \leftarrow \mathrm{Conf}_S(M)$ by just taking the points in $T$, and then this induces

$$\mathrm{H}^d(\mathrm{Conf}_T(M);\mathbb{Z}) \to \mathrm{H}^d(\mathrm{Conf}_S(M);\mathbb{Z}).$$

**Definition 4.8.** Such a functor $\mathrm{FI} \to \mathrm{Ab}$ is called an **FI-module**.

One can now rephrase representation stability for integral coefficients by saying it means that this is a finitely-generated FI-module.

**Theorem 4.9** (Church-Ellenberg-Farb[5])**.** *The cohomology of configurations spaces of a fixed manifold $M$, with integral coefficients, is a finitely generated* FI*-module.*

*Remark* 4.10. Ordinary homological stability "is" representation stability with FI replaced by $N_{\leq}$, i.e. a functor $N_{\leq} \to \mathrm{Ab}$ is finitely generated iff it is eventually constant.

### 4.2 Higher-order stability

Our example for this case will be mapping class groups.

**Definition 4.11.** Let $\Sigma_{g,1}$ denote a genus $g$ surface with 1 boundary component. Let $\mathrm{Diff}_\partial(\Sigma_{g,1})$ be the space of diffeomorphisms fixing the boundary pointwise, in $C^\infty$-topology. We define the **mapping class group** $\Gamma_{g,1} := \pi_0 \mathrm{Diff}_\partial(\Sigma_{g,1})$.

(The boundary components are contractible).

*Remark* 4.12. $B\Gamma_{g,1}$ is homotopy equivalent to $\mathcal{M}_{g,1}$, the moduli space of genus $g$ curves w/ 1 unordered pt with a nonzero tangent vector.

Note that, given $\Sigma_{g,1}$, one can take the boundary connected sum with a surface of genus 1. Extending diffeomorphisms by the identity on this attached piece gives a homomorphism $\Gamma_{g,1} \xrightarrow{\sigma} \Gamma_{g+1,1}$ of mapping class groups.

**Theorem 4.13.** *The induced maps*

$$\sigma_* : \mathrm{H}_*(B\Gamma_{g,1};\mathbb{Z}) \to \mathrm{H}_*(B\Gamma_{g+1,1};\mathbb{Z})$$

*are a surjection for $* \leq \frac{2g}{3}$ and an iso for $* \leq \frac{2g-2}{3}$.*

*Remark* 4.14. $2/3 \neq 1/k$, so this result (in this form) cannot be deduced from the general result we talked about last time.

[5] Assuming I heard correctly

Note there's an implicit claim when we attempt to apply last lecture's result here: we need to know that the $\Gamma_{g,1}$'s assemble to a braided monoidal groupoid.

Here's one way of thinking about Theorem 4.13. It says that the homology in fixed degree is eventually constant, so can think of this as saying it "eventually has vanishing first derivative." In this analogy then, 'higher-order stability' is "eventually has vanishing higher derivatives." Can we make this precise?

The "first derivative" will be the relative homology groups $\mathrm{H}_*(B\Gamma_{g+1,1}, B\Gamma_{g+1}; \mathbb{Z})$. Then 'second-order stability' should be that the second derivative eventually vanishes, i.e. that the first derivative is eventually constant. So secondary stability should be a stability result for these relative homology groups. However, these groups are eventually zero, so it may seem like this is uninteresting. Keep in mind, though, that we first need maps between these groups before we can really talk about stability.

**Claim 4.15.** *There are (non-unique!) maps*

$$\varphi_* : \mathrm{H}_d(B\Gamma_{g,1}, B\Gamma_{g-1,1}; \mathbb{Z}) \to \mathrm{H}_{d+2}(B\Gamma_{g+3,1}, B\Gamma_{g+2,1}; \mathbb{Z})$$

*(note these change both degree and genus).*

**Theorem 4.16 (Galatius-K.-Randal-Williams).** *$\varphi_*$ is a surjection for $* \leq \frac{3g}{4}$ and an isomorphism for $* \leq \frac{3g-4}{4}$.*

*Remark* 4.17. Note that we have stability in a larger range here ($\frac{2}{3} < \frac{3}{4}$), and that these maps change both degree and genus.

*Remark* 4.18. If you can prove that $\varphi_*$ is an iso and the zero map, then this secondary stability result gives an improved range for primary homological stability.

### 4.2.1 Strategy and ingredients

Here's a key observation: when $G$ is a braided monoidal groupoid, $|NG|$ is an $E_2$-algebra.

*Remark* 4.19. In homotopy theory there's a spectrum of increasingly commutative algebras from $E_1$-algebras to $E_\infty$-algebras, where $E_1$ = coherently associative and $E_\infty$ = coherently commutative. If I heard correctly, $E_k$-algebras have the space of multiplication increasingly highly connected (or something like that).

The official definition makes use of operads.

**Definition 4.20.** $E_2$ is the operad whose space of $n$-ary operations is $E_2(n)$ = rectilinear embeddings of $n$ copies of $D^2$ into $D^2$ with disjoint image.[6] An $E_2$-algebra $A$ involves the data of maps like $E_2(n) \times A^n \to A$.

How can mapping class groups be made into an $E_2$-algebra?

*Remark* 4.21. Here's the idea. If you have an element of the $E_2$-operad along with a bunch of diffeomorphisms $f_i$ (which are the identity on the boundary) on surfaces of genii $g_i$, attach each surface to the big disk at the small disks, and then extend to a diffeomorphism of the whole picture. See Figure 2 of an illustration of this idea.

This gives algebraic structure on $\bigsqcup_{g \geq 0} B\Gamma_{g,1}$. This extra structure is what one uses to prove the higher order stability result quoted earlier. Let's give a basic outline.

[6] So it looks like $n$ (scaled/translated) small disks within a fixed larger disk

Figure 2: Turning mapping class groups into an $E_2$-algebra

**(1)** try to build a custom $E_2$-algebra $A$ which approximates (captures stability phenomena in a range) $\bigsqcup_{g\geq 0} B\Gamma_{g,1}$.

**(2)** this will be a small cellular $E_2$-algebra.

That is, you take iterated pushouts along $\mathrm{Free}^{E_2}(S^{k-1}) \hookrightarrow \mathrm{Free}^{E_2}(D^k)$, where $\mathrm{Free}^{E_2}(blah)$ is the free $E_2$-algebra on blah (think: attaching cells to form a CW-complex).

**(3)** need to know which $E_2$-cells are required.

This ends up becoming the connectivity of some semi-simplicial set.

**(4)** need to know how to deduce ordinary/secondary stability

F.Cohen computed homology of free $E_2$-algebras w/ field coefficients. This knowledge allows you to do a computation in $A$ to deduce stability results.

# 5 List of Marginal Comments

# Index