Harvard Ax-Schanuel Seminar Notes

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These are notes on talks given in "O-Minimality and Ax-Schanuel Seminar" which took place at Harvard. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here. If you take a look at these notes, and see any mistakes or have any other sorts of improvements you want to suggest, feel free to send me an email. I think after the first few talks, I quickly stopped understanding much of what was happening during the talks, so there's doubtless much which can be improved.

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1 Alexander Betts (1/31/2022): Overview

Plan

- Transcendence Theory
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Last semester we talked about this uniform Mordell stuff, and there was one part of the proof that we had black boxed, showing certain subvarieties of abelian varieties are non-degenerate for the Betti map. This semester, we'll unpack the ingredients going into that proof, and then fill in this black box at the end.

Some of this stuff goes back to a differential geometric argument of Ax. We won't follow this approach. A more modern approach is to use o-minimality and model theory. We'll follow this approach instead. The first few lectures will be on setting up on relevant model theory. Then we'll prove the big technical results we'll need, and then finally prove a few flavors of Ax-Schanuel.

1.1 Classical transcendence theory

Question 1.1. *Which complex numbers are transcendental?*

Example. Both e and π are transcendental numbers. Unknown if they are algebraically independent (over $\mathbb{Q}$). △

Example. $\zeta(3)$ is known to be irrational, but unknown whether or not it's irrational. △

There aren't too too many general results in transcendence theory, but there's a nonzero amount.

Theorem 1.2 (Lindemann-Weierstrass, 1885)**.** *Given $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ which are linearly independent over $\mathbb{Q}$, their exponentials $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.*

Example. $n = 1, \alpha_1 = 1 \implies e$ transcendental. △

Example. $\log 2$ transcendental. Else, take $n = 1, \alpha_1 = \log 2$ to deduce that 2 is transcendental. Similarly, π is transcendental (else take $n = 1$ and $\alpha_1 = 2\pi i$). △

Conjecture 1.3 (**Schanuel's Conjecture**)**.** *Given $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ linearly independent over $\mathbb{Q}$, one has*

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n.$$

This would imply Lindemann-Weierstrass.

Example. $\alpha_1 = 1, \alpha_2 = 2\pi i$ would imply that e, π are algebraically independent. △

In this seminar, we'll talk about the function field analogue of this conjecture.

Theorem 1.4 (**Ax's theorem**, '71). *Let $f_1, \ldots, f_n \in \mathbb{C}[\![t_1, \ldots, t_m]\!]$ be $\mathbb{Q}$-linearly independent modulo constants. Let J be the Jacobian matrix $J_{ij} = \frac{\partial f_i}{\partial t_j}$. Then,*

$$\operatorname{trdeg}_{\mathbb{C}} \mathbb{C}\left(f_1, \ldots, f_n, \exp(f_1), \ldots, \exp(f_n)\right) \geq n + \operatorname{rank} J.$$

By the 10th talk, we'll prove this.

There are many statements one might be referring to when they say "Ax-Schanuel Theorem." Let's see of them.

1.2 Ax-Schanuel for $\mathbb{G}_m^n$

Let $q : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ be the map $(z_1, \ldots, z_n) \mapsto (\exp(z_1), \ldots, \exp(z_n))$ (this is the universal cover). Let $\Gamma \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be the graph of q, i.e. $\Gamma = \{(x, y) : y = q(z)\}$.

Theorem 1.5 (**Ax-Schanuel for $\mathbb{G}_m^n$**). *Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an irreducible, closed algebraic subset (of $\mathbb{A}_{\mathbb{C}}^n \times \mathbb{G}_{m,\mathbb{C}}^n$). Let U be an irreducible analytic component of the intersection $V \cap \Gamma$.*[1] *Suppose that the second projection $\pi(U) \subset (\mathbb{C}^\times)^n$ is not contained in a translate of a proper subtorus. Then,*

$$\dim_{\mathbb{C}} V \geq \dim_{\mathbb{C}} U + n$$

Apparently, this is equivalent to Ax's theorem stated earlier.

Remark 1.6.

(1) One consequence of this is

Theorem 1.7 (**Ax-Lindemann-Weierstrass Theorem**). *Suppose that $V_1 \subset \mathbb{C}^n$ is an irreducible, closed algebraic subset. Then, $\overline{q(V_1)}$ (the Zariski closure of the image under the universal covering map) is always a translate of a subtorus of $(\mathbb{C}^\times)^n$.*

Proof. Let $V_2 = \overline{q(V_1)}$. We may assume wlog that V_2 is not contained in a translate of a proper subtorus of $(\mathbb{C}^\times)^n$. Take $V = V_1 \times V_2$, an algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. Note that $U := V \cap \Gamma$ is the graph of $q|_{V_1}$. Now, Ax-Schanuel gives

$$\dim V = \dim V_1 + \dim V_2 \geq \dim V_1 + n \implies \dim V_2 \geq n \implies V_2 = (\mathbb{C}^\times)^n.$$

■

Example. Take $n = 2$ and $V_1 = \{(z_1, z_2) : 3z_1 = 4z_2\}$. Then, $q(V_1) = \left\{(z_1, z_2) \in (\mathbb{C}^\times)^2 : z_1^3 = z_2^4\right\}$ is already a subtorus. △

(2) Why does Ax-Schanuel imply Ax's theorem?

Proof Sketch. Using some embedding theorem, it suffices to prove Ax's theorem in the case that $f_1, \ldots, f_n \in \mathbb{C}[\![t_1, \ldots, t_m]\!]$ actually converse on some ball $B \subset \mathbb{C}^m$. Assuming this, let $U_0 \subset \mathbb{C}^n$ be the image of $(f_1, \ldots, f_n) : B \to \mathbb{C}^m$, let $U_1 \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be the graph of $q|_{U_0}$, let $V = \overline{U_1}$ (Zariski

[1] Γ not algebraic since the exponential function isn't

closure), and let U be the component of $V \cap \Gamma$ containing U_1. In order to apply Ax-Schanuel, we need to know that the second projection $\pi(U)$ is not contained in a translate of a proper subtorus.

Suppose it were, and note that $q(U_0) \subset \pi(U)$. We'd then get integers $a_1, \ldots, a_n \in \mathbb{Z}$ s.t. $\exp\left(\sum a_i f_i\right)$ is constant (any torus cut out by something like $z_1^{a_1} z_2^{a_2} \ldots z_n^{a_n} = 1$ and then cosets change that constant). This implies that $\sum a_i f_i$ is constant, which violates the assumption that the f_i are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$.

So, Ax-Schanuel implies that $\dim V \geq \dim U + n$. At the same time,

Question: Why?

$$\dim V = \operatorname{trdeg}_{\mathbb{C}} \mathbb{C}(f_1, \ldots, f_n, \exp(f_1), \ldots, \exp(f_n))$$

and

$$\dim U \geq \dim U_0 = \operatorname{rank} J.$$

Last equality holds since U_0 is the image of $(f_1, \ldots, f_n) : B \to \mathbb{C}^n$. ■

Question: Why?

Remark 1.8. rank J is the rank of $\left(\frac{\partial f_i}{\partial t_j} \in \mathbb{C}[[t_1, \ldots, t_m]]\right)$ as a matrix with coefficients in $\mathbb{C}((t_1, \ldots, t_m))$. ○

(3) The inequality $\dim V \geq \dim U + n$ from Ax-Schanuel is equivalent to

$$\operatorname{codim}(U) \geq \operatorname{codim}(V) + \operatorname{codim}(\Gamma)$$

(note RHS is expected codimension of intersection of $V \cap \Gamma$).

Slogan. U has at most the expected dimension, unless $\pi(U)$ is contained in a translate of a proper subtorus.

Theorems of this kind fall under the heading of "unlikely intersections"

○

1.3 More Ax-Schanuel Theorems

What if we replace $\mathbb{G}_m^n$ with an abelian variety?

Let $A/\mathbb{C}$ be an abelian variety. Let $q : \mathbb{C}^g \to A(\mathbb{C})$ be the universal covering, and let $\Gamma \subset \mathbb{C}^g \times A(\mathbb{C})$ be the graph of q.

Theorem 1.9 (**Ax-Schanuel for Abelian Varieties**, Ax '72)**.** *Let $V \subset \mathbb{C}^g \times A(\mathbb{C})$ be an irreducible, closed algebraic subset, and let U be an irreducible analytic component of the intersection $V \cap \Gamma$. Suppose that the second projection $\pi(U) \subset A(\mathbb{C})$ is not contained in a translate of a proper abelian subvariety. Then,*

$$\operatorname{codim}(V) \geq \operatorname{codim}(U) + \underbrace{\operatorname{codim}(\Gamma)}_{=g}.$$

Part of what makes abelian varieties and torii special is that their universal covers have algebraic structures. There are other sorts of varieties (non-group varieties) with this property, e.g. Shimura varieties.

Let $\mathbb{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. On the level of complex points, one has

$$\mathbb{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$$

where

$$\mathbb{H}_g := \{g \times g \text{ complex symmetric matrices } \tau : \mathrm{Im}(\tau) > 0\}$$

is the **Siegal (spelling?) upper half space**. Note that $\mathbb{H}_g \overset{\text{open}}{\subset} \widehat{\mathbb{H}}_g$ where $\widehat{\mathbb{H}}_g$ is the set of $g \times g$ symmetric complex matrices. So here we have a space $\mathbb{A}_g$ with universal cover $\mathbb{H}_g$ with some algebraic structure ($\mathbb{H}_g \overset{\text{open}}{\subset} \widehat{\mathbb{H}}_g$). We need a few more ingredients to state an analogue of Ax-Schanuel.

Definition 1.10. We say $V \subset \mathbb{H}_g$ is **algebraic** if it is of the form $\widehat{V} \cap \mathbb{H}_g$ for some algebraic $\widehat{V} \subset \widehat{\mathbb{H}}_g$.

There is a class of subvarieties of $\mathbb{A}_g$, called the *weakly special subvarieties*.

Example. If L is a CM field of degree $2g$,

$$\{\text{PPAVs A of dimension } g \text{ w/ CM by } \mathscr{O}_L\} \subset \mathbb{A}_g$$

is a (weakly) special subvariety.

Apparently, this is 0-dimensional and all 0-dimensional subvarieties will be weakly special. △

Example. If $g_1 + g_2 = g$, then

$$\{g\text{-diml PPAVs } A \text{ s.t. } A \cong A_1 \times A_2 \text{ for some } g_i\text{-diml PPAVs } A_1, A_2\}$$

is a (weakly) special subvariety in $\mathbb{A}_g$. △

Example. If moreover A_2 is a fixed g_2-dimensional PPAV, then

$$\{[A] \in \mathbb{A}_g : A \cong A_1 \times A_2 \text{ for some } [A_1] \in \mathbb{A}_{g_1}\}$$

is a weakly special subvariety (and not actually special). △

Theorem 1.11 (**Ax-Schanuel for** $\mathbb{A}_g$, Mok-Pila-Tsimerman '17)**.** *Let $q : \mathbb{H}_g \to \mathbb{A}_g(\mathbb{C})$ be the universal covering map, and let $\Gamma \subset \mathbb{H}_g \times \mathbb{A}_g(\mathbb{C})$ be the graph of q. Let $\widehat{V} \subset \widehat{\mathbb{H}}_g \times \mathbb{A}_g(\mathbb{C})$ be an irreducible algebraic subvariety, and let U be an irreducible analytic component of $\widehat{V} \cap \Gamma$. Suppose that the second projection $\pi(U) \subset \mathbb{A}_g(\mathbb{C})$ is not contained in a proper weakly special subvariety. Then,*

$$\mathrm{codim}(U) \geq \mathrm{codim}(\widehat{V}) + \mathrm{codim}(\Gamma).$$

Above, $\mathrm{codim}(\Gamma) = \binom{g}{2}$.

For the uniform Mordell stuff, the real theorem we need is a version of Ax-Schanuel for the universal abelian variety over $\mathbb{A}_g$. That will be stated towards the end of this seminar.

1.4 O-minimality

Definition 1.12. A **structure** on $\mathbb{R}$ consists of, for each n, a family D_n of subsets of $\mathbb{R}^n$ – called the **definable subsets** – which satisfy the conditions

(1) $\emptyset \in D_n$, D_n is closed under binary $\cup, \cap$, complements and coordinate permutations.

(2) If $V_1 \in D_{n_1}$ and $V_2 \in D_{n_2}$, then $V_1 \times V_2 \in D_{n_1+n_2}$.

(3) If $V \in D_n$ and $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is one of the projections, then $\pi(V) \in D_{n-1}$.

(4) The diagonal in $\mathbb{R}^2$ should be definable.

Definition 1.13. A structure is **compatible with field operations** if the graphs

$$\{(x,y,z) \in \mathbb{R}^3 : x + y = z\} \text{ and } \{(x,y,z) \in \mathbb{R}^3 : xy = z\}$$

of addition and multiplication are definable (i.e. $\in D_3$). Moreover, a structure compatible with field operations is **o-minimal** just when D_1 consists exactly of the finite unions of open intervals and single points.

Unclear to me if the *o* is captial or not

Example. A **semialgebraic subset** of $\mathbb{R}^n$ is a subset cut out by finitely many polynomial inequalities and equations, or it's a finite union of such subsets. e.g.

$$\{(x,y,z) \in \mathbb{R}^3 : x = z^2 + 5 \text{ and } x < y < z - 123x\}$$

is a semialgebraic set. If we take $D_n = \{\text{semialgebraic subsets of } \mathbb{R}^n\}$, then this defines a structure called $\mathbb{R}_{\text{alg}}$. In fact, this is *o*-minimal. The most subtle thing to check is condition **(3)** of being a structure. That this holds is a theorem of Tarski-Seidenberg. △

Example. Given some functions $f_i : U_i \to \mathbb{R}$ (with $U_i \subset \mathbb{R}^{n_i}$), there is a smallest structure on $\mathbb{R}$ making the graphs Γ_i of these definable. e.g. take $\mathbb{R}_{\text{very an}}$ to be the structure generated in this many by all real analytic functions $\mathbb{R}^n \to \mathbb{R}$. This structure is not o-minimal. Note that $\Gamma_{\sin(x)} \subset \mathbb{R}^2$ is definable, so $\pi\mathbb{Z} \times 0 = \Gamma_{\sin(x)} \cap \{y = 0\}$ is definable in $\mathbb{R}^2$. Projecting, we see that $\pi\mathbb{Z} \subset \mathbb{R}$ is also definable, but this is not a finite union of points and intervals. △

Principal Example. A **restricted analytic function** is a function $f : [0,1]^n \to \mathbb{R}$ which extends to an analytic function on an open neighborhood of the cube. Let $\mathbb{R}_{\text{an,exp}}$ be the structure generated (the graphs of) by

- $+, \times$
- all restricted analytic functions
- $\exp : \mathbb{R} \to \mathbb{R}$

Fact. $\mathbb{R}_{\text{an,exp}}$ is o-minimal

"Whenever, in practice, anyone is working with an o-minimal structure, they're working with this one."

Example. Consider $\exp : \mathbb{C} \to \mathbb{C}^\times$. Viewing $\mathbb{C} \cong \mathbb{R}^2$ and $\mathbb{C}^\times \subset \mathbb{R}^2$, is the graph of this function definable in $\mathbb{R}_{an,exp}$? The graph of the whole function will never be definable in a o-minimal structure (think sin issue). However, there is a fundamental domain $F \subset \mathbb{C}$ for exp s.t. F and the graph of $\exp|_F$ are both definable in $\mathbb{R}_{\text{an,exp}}$. You can take $F = \{z \in \mathbb{C} : \text{im}(z) \in [0, 2\pi]\}$. △

What does o-minimality give you?

Theorem 1.14 (**O-Minimal Chow theorem**)**.** *"Closed $\mathbb{C}$-analytic subsets (of $\mathbb{C}^n$?) which are definable in an o-minimal structure are algebraic."*

(Talk 7)

Theorem 1.15 (**Pila-Wilkie Counting Theorem**)**.** *"If $X \subset \mathbb{R}^n$ is definable in an o-minimal structure, then all but very few of the rational points in X lie on semi-algebraic curves."*

(Talk 8/9)

2 Alice Lin (2/7): O-Minimal Structures

Notation 2.1. $\mathbb{N} = \{0, 1, \dots\}$. Also, for a set S, $\mathcal{P}(S)$ denotes its power set.

Definition 2.2. A **structure** $(M, (D_n)_{n\in\mathbb{N}})$ consists of a set M and for each $n \in \mathbb{N}$, a collection $D_n \subset \mathcal{P}(M^n)$ of subsets of M^n s.t.

(1) $\emptyset \in D_n$ and D_n is closed under finite intersections, unions, complements, and permutation of coordinates in M^n

(2) For any $n \geq 1$, the diagonal Δ is an element of D_n

(3) closure under products, i.e. $X_i \in D_{n_i}$ implies $X_1 \times X_2 \in D_{n_1+n_2}$

(4) Closed under projection $M^{n+1} \to M^n$ via any of the $(n+1)$ projections.

The sets in D_n are called the **basic definable sets** of $(M, (D_n))$.

Example. Let $k = \overline{k}$ be a field. For $n \in \mathbb{N}$, let $D_n \subset P(k^n)$ be the constructible subsets of k^n (i.e. finite unions of open subsets of closed subvarieites of $\mathbb{A}^n_k$). Then, $(k, (D_n)_n)$ is a structure. Property **(4)** here is a special case of Chevalley's theorem. △

Example. Take $M = \mathbb{R}$ and let $D_n \subset \mathcal{P}(\mathbb{R}^n)$ be the collection of semialgebraic sets. Then, $(\mathbb{R}, (D_n))$ forms a structure. Property **(4)** is the Tarski-Sedenberg theorem. △

Example. Given arbitrary $D^0_n \subset \mathcal{P}(M^n)$, there will be a minimal structure $(M, (D_n))$ generated by this choice. △

2.1 Model Theory

We won't go too deep into model theory in this seminar, but let's at least get a taster.

Definition 2.3. A **signature** σ consists of two sets $(\mathcal{R}_n)_{n\in\mathbb{N}}$ and $(\mathcal{F}_n)_{n\in\mathbb{N}}$, where $R \in \mathcal{R}_n$ is called an n**-ary relation symbol**, and $f \in \mathcal{F}_n$ is called an n**-ary function symbol**.

Definition 2.4. Given a signature σ, a σ**-structure** is the choice of an underlying set M, and for each $R \in \mathcal{R}_n$, a choice of subset $[R]_M \subset M^n$ of tuples satisfying the relation, and for each function symbol $f \in \mathcal{F}_n$, a function $f^M : M^n \to M$.

Being an element of $[R]_M$ is the definition of satisfying the relation, if I'm under-

Given a σ-structure M, one can define a *structure* $(M, (D_n))$ generated by the graphs of all f^M and $[R]_M$ as basic definable.

Example. Let k be a field, and consider the signature $\sigma_k = \{+, \times, c_\lambda\}$ with $+, \times$ both 2-ary function symbols and c_λ (for $\lambda \in k$) a 0-ary function symbol. One can naturally make k into a σ_k-structure where $+$ is addition, $\times$ is multiplication, and c_λ is the constant $\lambda \in k$.

If $k = \overline{k}$, the structure generated by this σ_k-structure consists precisely of the constructible sets. $\triangle$

Example. Consider $\sigma_{\mathbb{R}}^{+} \{+, \times, c_\lambda, <\}$ with $+, \times, c_\lambda$ as before and $<$ a 2-ary relation symbol. Then, $\mathbb{R}$ naturally has a $\sigma_{\mathbb{R}}^{+}$-structure where e.g. $[<]_{\mathbb{R}} = \left\{(x, y) \in \mathbb{R}^2 : x < y\right\}$. Now, the structure $(\mathbb{R}, (D_n)_n)$ this gives is the structure of semi-algebraic sets. $\triangle$

2.2 Definable vs. Basic definable

Example. Consider the structure on $\mathbb{R}$ generated by the graphs of $+, \times$, i.e. by

$$\{(x, y, z) : x + y = z\} \text{ and } \{(x, y, z) : xy = z\}.$$

A priori, we don't know if singletons like $\{0\}, \{e\}$ are basic definable in this structure. Earlier, they were forced to be basic definable by the inclusion of the c_λ's.

Question 2.5. *What can we find to be basic definable?*

We know $\{0\} \subset \mathbb{R}$ is basic definable since it's the additive identity, i.e. $\{0\} = \mathrm{pr}_2\left(\Gamma_+ \cap \{(x, y, x)\}\right) \subset \mathbb{R}$. A similar process shows that $\{1\} \subset \mathbb{R}$ is basic definable, using that 0 is basic definable (in order to get to $\mathbb{R}^\times$). One can take things further and show that $\{n\}$ is basic definable for $n \in \mathbb{Z}$, $\{r\}$ is for $r \in \mathbb{Q}$, and in fact all varieties over $\mathbb{Q}$ in $\mathbb{R}^n$ are basic definable.

Exercise. All intervals with $\overline{\mathbb{Q}} \cap \mathbb{R}$ end points are basic definable as are $\{\alpha\}$ for $\alpha \in \overline{\mathbb{Q}} \cap \mathbb{R}$

However, $\{\pi\}$ is not basic definable in this structure. This is a little unsatisfying, so we extend from 'basic definable' to 'definable'. $\triangle$

Notation 2.6. Pick some $W \subset M^m \times M^n$. For a point $y \in M^n$, let $W_y \in M^n$ be the fiber above y.

Fix some ambient structure.

Definition 2.7. A subset $V \subset M^n$ is definable when there exists some $m \in \mathbb{N}$, a basic definable set $W \subset M^{m+n}$, and a point $y \in M^m$ such that $V = W_y$.

Slogan. Definable subsets are fibers of basic definable sets

Remark 2.8. Taking a fiber of y in W is the same thing as intersecting $W \cap \{y\} \times M^n$ and then projecting. Hence, if you had the singletons $\{y\}$ to begin with, then definable = basic definable. We will see in a second that definable sets also form a structure, so in fact definable sets are the structure obtained by adding in the singletons. $\circ$

Here are some properties of definable sets

- every basic definable set is definable

- definable sets are closed under intersection, union, complement, etc. In other words, if D'_n are the definable sets in M^n, then $(M, (D'_n)_n)$ defines a structure.

  By the previous remark, this structure is the same as the minimal structure generated by D_n + all the singletons.

2.3 First order sentences and (basic) definable sets

In practice, a set will be basic definable if it is specified by a first order sentence, i.e. conditions on elements of basic definable sets along w/ connectives (e.g. and, or, not, $\in, \exists, \forall, =$), relations (e.g. $<$), and functions (e.g. $+, \times$).

What we're calling basic definable sets in M^n will corresponds to what (in Model theory) are called "sets of M^n which are σ-definable without parameters in a 1st order interpretation of a σ-structures."

Example. Say $W \subset M^2, U \subset M$ are basic definable. Then,

$$V := \{v \in M \mid v \notin U \text{ and } \exists w \in M : (v,w) \in W\}$$

is basic definable. △

Example (How to use $\forall$)**.** Say U, V are basic definable and $W \subset U \times V$ is basic definable. Then,

$$\{u \in U \mid \forall v \in V : (u,v) \in W\}$$

is basic definable. Take negations to see above condition is $\nexists v \in V : (u,v) \notin W$, and then define this set using complements and projections. △

2.4 Definable functions

Definition 2.9. Let U, V be definable sets. Then, $f : U \to V$ is called **definable** if its graph $\Gamma_f \subset U \times V$ is definable.

Lemma 2.10. *Definable sets and functions form a category.*

Lemma 2.11.

- *(pre)images of definable sets under definable functions are definable.*
- *Definable is determined componentwise in the sense that* $f = (f_1, f_2) : U \to V_1 \times V_2$ *is definable iff* f_1, f_2 *are definable.*
- *You can glue definable functions*
- *If a definable function is a bijection, then it has a definable inverse*

2.5 O-minimal structures

O stands for 'Order'.

Definition 2.12. Let R be an ordered field. A structure on R is called **compatible with the ordered field operations** if

(1) addition and multiplication functions $R^2 \to R$ are basic definable.

(2) the set $[<]_R := \{(x, y) : x < y\} \subset R^2$ is basic definable.

Lemma 2.13. *Say we have a structure compatible with the ordered field operations. Then,*

(i) $\{0\}, \{1\}$ *are basic definable*

(ii) $[-1] : R \to R$ *is basic definable*

(iii) $(\cdot)^{-1} : R^\times \to R$ *is basic definable*

(iv) *All polynomial maps* $R^n \to R^m$ *with* R *coefficients are definable (but not necessarily basic definable)*

(v) *For all* $a < b$ $(a, b \in R \cup \{\pm\infty\})$*, the open interval*

$$]a, b[:= \{x \in R : a < x < b\}$$

is definable.

We use this strange (French) notation since (a, b) *looks like a point in* R^2*.*

Definition 2.14. A structure on R compatible w/ the ordered field operations is called **O-minimal** if every definable set of $R = R^1$ is a finite union of points and intervals.

Sounds like it can be quite hard just to prove that a particular structure is O-minimal. That being said, someone has done this work for us.

Example. $\mathbb{R}_{alg}$ generated by $\times, +, <$ (really, their graphs) and singletons in $\mathbb{R}$ is O-minimal. △

Example (Wilkie '96)**.** $\mathbb{R}_{\exp}$ generated by $\mathbb{R}_{alg}$ and the real exponential $\exp : \mathbb{R} \to \mathbb{R}$ is O-minimal. △

Example (Denef, van der Dries '88)**.** $\mathbb{R}_{an}$ generated by $\mathbb{R}_{alg}$ and graphs of overconvergent real analytic functions $[0, 1]^n \to \mathbb{R}$ is O-minimal. △

Example (van der Dries, Miller '94)**.** $\mathbb{R}_{an,\exp}$ generated by $\mathbb{R}_{\exp}$ and $\mathbb{R}_{an}$ is O-minimal. This is non-trivial. △

Non-example. $\sin(x)$ is not definable in any O-minimal structure on $\mathbb{R}$. The main issue is the zeros of $\sin(x)$ form an infinite union of isolated points.

Example. $(x, y) \mapsto \max(x, y)$ is definable in $\mathbb{R}_{alg}$. Think of this as piecewise projection maps △

Theorem 2.15 (**Definable choice**)**.** *Let* $f : U \twoheadrightarrow V$ *be a surjective definable map (in an O-minimal structure). Then, there exists a definable splitting* $g : V \to U$ *so that* $f \circ g = \mathrm{id}_V$*.*

Example (Audience)**.** Consider $\{(x, x^2)\} \xrightarrow{\mathrm{pr}_2} \mathbb{R}$. This has a section since we can talk about positivity. △

View $\overline{R} = R \cup \{\pm\infty\}$ as a basic definable subset of R^2. Specifically, take $\overline{R} = (R \times \{0\}) \cup \{(0, \pm 1)\}$. Note that $[<]_{\overline{R}} \subset R^4$ is basic definable.

Lemma 2.16. *Let V be definable, and consider some definable $U \subset V \times \overline{R}$.*

(1) *Let $V_1 \subset V$ be the set of points $y \in V$ s.t. the fiber U_y has a minimal element $c(y)$. Then,*

- *V_1 is definable*
- *$c : V_1 \to R$ is definable*

(2) *Let $V_2 \subset V$ be the points $y \in V$ s.t. U_y contains an open interval. For $y \in V_2$, let $]a(y), b(y)[$ denote the first maximal (under inclusion) open interval in U_y. Then,*

- *V_2 is definable*
- *$a, b : V_2 \to \overline{R}$ are definable*

Proof. **(1)** Define

$$\begin{aligned} W_1 &:= \{(y, c) \in V_1 \times R \mid c \text{ is minimal element of } U_y\} \\ &= \{(y, c) \in V \times R \mid (y, c) \in U \text{ and } \nexists c' \in R : (c' < c \text{ and } (y, c') \in U)\} \end{aligned}$$

This is defined by a first order sentence, so W_1 is definable. Projecting tells us that V_1 is definable. Also, W_1 is the graph of c, so c is definable.

(2) Consider

$$W_2 := \left\{(y, a, b) \in V \times \overline{R}^2 \mid a < b \text{ and }]a, b[\subset U_y\right\}$$

This is definable since the condition is the first order sentence

$$a < b \text{ and } \nexists c \in R : (a < c < b \text{ and } (y, c) \notin U).$$

Projecting onto the first coordinate shows that V_2 is definable. To show that a, b are definable functions, we consider the tower

$$\begin{array}{ccc} W_2 & = & \left\{(y, a, b) \in V \times \overline{R}^2 \mid a < b \text{ and }]a, b[\subset U_y\right\} \\ \downarrow{\scriptstyle \pi_{12}} & & \\ W_2' & = & \{(y, a) \mid \dots\} \\ \downarrow{\scriptstyle \pi} & & \\ V_2 & = & \{y \mid \dots\} \end{array}$$

Note that the fibers of π are the sets $W'_{2,y}$ of possible left end points of intervals in U_y. This fiber $W'_{2,y}$ has a least element $a(y)$ (by O-minimality), so the function $a : V_2 \to \overline{R}$ is definable (by **(1)** with $\overline{R}$ in place of R). To see that b is definable, note that the fibers $W_{2,(y,a)}$ of π_{12} consist of possible right endpoints of intervals starting at a. Since U_y looks like a finite number of intervals and some points, it has a greatest element $b'(y, a)$. Thus, b' is definable (by **(1)** with greatest in place of least). Finally, define $b(y) := b'(y, a(y))$ which is visibly definable. ■

Proof of Definable Choice. Replace U with $\Gamma_f \subset U \times V \subset R^n \times V$, using that $\Gamma_f \xrightarrow{\sim} U$ is an isomorphism. Second, we induct on n via

$$R^n \times V \to R^{n-1} \times V \to \cdots \to V$$

to see that it suffices to let $n = 1$. Now, define V_2 as the points in V s.t. U_y contains an interval, and we define $V_1 := V \setminus V_2$. Since all fibers are nonempty, fibers above V_1 must consist of finitely many points. Thus, we can defined the splitting $s : V \to R$ as the piecewise function

$$s(y) = \begin{cases} c(y) & \text{if } y \in V_1 \\ \frac{1}{2}(a(y) + b(y)) & \text{if } y \in V_2 \text{ and } a, b \text{ finite} \\ b(y) - 1 & \text{if } y \in V_2, a = -\infty, b \neq \infty \\ a(y) + 1 & \text{if } y \in V_2, a \neq -\infty, b = \infty \\ 0 & \text{if } y \in V_2, a = -\infty, b = \infty \end{cases}$$

■

3 Katia Bogdanova (2/14): Calculus on O-Minimal Structures

Outline

(1) Topology on O-Minimal structures

(2) Calculus on O-Minimal structures

(3) Monotonicity theorem (if there's time)

Recall 3.1. A **structure** is a set M and, for all $n \in \mathbb{N}$, a collection $D_n \subset \mathbb{P}(M^n)$ of subsets of M^n so that

(1) $\emptyset, M^n \in D_n$ and D_n is closed under finite intersection, union, complements, and permutation of coordinates

(2) For all $n \geq 1$, the diagonal $\Delta \in D_n$

(3) For any $V_1 \in D_{n_1}$ and $V_2 \in D_{n_2}$, we have $V_1 \times V_2 \in D_{n_1+n_2}$

(4) For $V \in D_{n+1}$, any projection $\pi(V) \in D_n$

Elements of D_n (for any n) are called **basic definable sets**.

Recall 3.2. "Fibers of basic definable sets are definable"

We call $V \subset M^n$ a **definable set** if there is some $m \in \mathbb{N}$ and a basic definable set $W \subset M^{m+n}$ along with some point $y \in M^m$ s.t. $V = W_y$ is the fiber over y. The definable sets form another structure on M.

Recall 3.3. If U, V are (basic) definable sets, a function $f : U \to V$ is called (basic) definable when its graph $\Gamma_f \subset U \times V$ is (basic) definable.

Recall 3.4. Let R be an ordered field. A structure on R is said to be **compatible with the ordered field operations** if

(1) addition and multiplication $+, \times : R \times R \to R$ are basic definable; and

(2) $[<]_R := \{(x, y) \in R^2 : x < y\}$ is basic definable.

Moreover, a structure compatible with the field operations is called **o-minimal** if every definable set in R (i.e. every element of D_1) is a finite union of points and open intervals.

3.1 Topology

Setup. Fix an ordered field R along with an o-minimal structure on it. We give R the order topology, i.e. the topology with basis consisting of open intervals $]a, b[= \{x \in R : a < x < b\}$. We then give R^n the product topology, so it has a basis consisting of sets of the form $]a_1, b_1[\times \ldots \times]a_n, b_n[$. Finally, any subset $V \subset R^n$ is given the product topology.

Lemma 3.5.

(1) *Let $f : A \to R^m$ be a definable function (i.e. its graph Γ_f is definable), with $A \subset R^n$. Then, A and $f(A)$ are definable.*

(2) *Let $V \subset R^n$ be definable, and choose some definable $U \subset V$. Then, the interior* $\operatorname{int}(U)$*, the closure* $\operatorname{clos}(U)$*, and the boundary $\partial U = \operatorname{clos}(U) \setminus \operatorname{int}(U)$ (all three in V) are definable.*

(3) *Say $f : U \to V$ is definable. Then,*

$$\{x \in U : f \textit{ is continuous at } x\}$$

is definable.

Proof. **(1)** Write $A = \pi_1(\Gamma_f)$ and $f(A) = \pi_2(\Gamma_f)$ as projections of the graph.

(2) We will describe both the closure as a first order formula, and this will suffice to show that it is definable:

TODO: Fix formatting

$$\operatorname{clos}(U) = \{(x_1, \ldots, x_n) \in V \mid \forall y_1, \ldots, y_n, z_1, \ldots, z_n : (y_1 < x_1 < z_1, \ldots, y_n < x_n < z_n) \implies (\exists (u_1, \ldots, u_n) \in U : (y_1 < u_1$$

Now, the interior of U is the complement of the closure of the complement of U, so it is definable. Finally, the boundary is the difference between the interior and the closure, so it is definable.

(3) Let $\varphi(x, y)$ be a formula defining f. Then,

$$\{x \in U \mid f \text{ is cont. at } x\} = \{a = (a_1, \ldots, a_m) \in U \mid \forall z_1, \ldots, z_m, z'_1, \ldots, z'_m, b_1, \ldots, b_m : (\varphi(a, b) \wedge z_i < b_i < z'_i) \implies (\exists x_1, \ldots$$

In words, f is continuous at a if for any open W around $f(a)$, there's an open contained in $f^{-1}(W)$. ■

Definition 3.6. $X \subset R^n$ is called **definably connected** if X is definable and X is not the disjoint union of two nonempty open definable subsets of X.

Lemma 3.7.

(1) *The definably connected subsets of R are the $\emptyset$ and those of the form $]a,b[$, $[a,b]$, $]a,b]$, $[a,b[$ for $a,b \in R$ with $a < b$.*

(2) *The image of a definably connected set by a definable continuous map is definably connected.*

Question 3.8 (Audience)**.** *Can you give an example of a definable set which is definable connected but not connected?*

Answer (Assuming I heard correctly)**.** This is hard to find. For the reals $\mathbb{R}$, definable connected = connected, but these can differ for bigger fields. Sounds like if you have a field R with infinitesimal elements (those between 0 and $1/n$ for all $n \in \mathbb{N}_{\geq 1}$), then the set of infinitesimals will be open, and it's possible that something like $[-1, 1]$ is definably connected but not connected.

Corollary 3.9 (**Definable Intermediate Value Theorem**)**.** *If $f : [a,b] \to R$ is definably continuous (i.e. definable and continuous) with $f(a) < 0 < f(b)$, then there exists $c \in [a,b]$ such that $f(c) = 0$.*

Proposition 3.10. *Let $f : [a,b] \to R$ be a definably continuous function. Then, f attains a maximum and a minimum.*

Proof. Without loss of generality, we may assume $\operatorname{clos}(f([a,b])) = [0,c]$ for some $c \in \overline{R}$. We want to show that $c \in f([a,b])$, so suppose the contrary. Then, for any $s \in]0,c[$, we know $f^{-1}([s,c[)$ is a nonempty definable closed set. Thus, this preimage has a minimum, which we call $g(s)$. Then, $g :]0,c[\to [a,b]$ is definable (can write the condition that a value is a minimum as a first order formula) and weakly increasing. Now, let $m \in [a,b]$ be the supremum of the image of g, so $g(s) \to m$ as $s \to c$. Thus, $fg(s) \to f(m)$ as $s \to c$. By definition of g, we have $fg(s) \geq s$ which implies $f(m) \geq c$ and so in fact $f(m) = c$ (in particular, $c \in R \subsetneq \overline{R}$). ■

3.2 Differentiable definable functions

Definition 3.11. We define a norm $|\cdot| : R^n \to R$ as

$$|x| := \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{when } n = 1.$$

When $n > 1$, we use $|x| = \max_i \{|x_i|\}$ for $x = (x_1, \ldots, x_n) \in R^n$.

Definition 3.12. For $U \subset R^n$ an open, definable and $f : U \to R^m$, we say that f is **differentiable** at $x = (x_1, \ldots, x_n) \in U$ with partial derivatives $y_1, \ldots, y_n \in R^m$ when for any $\varepsilon > 0$ ($\varepsilon \in R$), there exists $\delta > 0$ ($\delta \in R$) so that

$$\left| f(x') - f(x) - \sum y_n(x'_n - x_n) \right| < \varepsilon \, |x' - x| \text{ for any } x' \in U \text{ with } |x' - x| < \delta.$$

Similarly, we say that f is continuously differentiable at x if it is differentiable on an open neighborhood V of x and its partial derivatives are continuous functions on V.

Lemma 3.13. *Let $U \subset R^n$ be a definable open, and let $f : U \to R^m$ be a definable function. Fix $k \in \mathbb{N}$. Then*

$$U^{(k)} := \{x \in U \mid f \text{ is } k \text{ times continuously differentiable at } x\}$$

is a definable subset of U. Furthermore, the k-fold partial derivatives of f are all definable functions $U^{(k)} \to R^m$.

Proof for $k = 1$. Let $W = \{(x, y) \in U \times R^{n+m} : f$ is differentiable at x w/ partial derivatives $y\}$. This set is definable since it can be described using first order formulas. Now, $U' := \pi_U(W)$ is the set of points where f is differentiable, so the set of such points is definable. At the same time, W is the graph of the partial derivatives of f, so these partial derivatives are also all definable. Finally, $U^{(1)} = \text{int}(U') \subset U$, so $U^{(1)}$ is also definable. ■

Theorem 3.14 (**Rolle's theorem**)**.** *Let $f : [a, b] \to R$ be definable, and continuously differentiable on $]a, b[$. Then, there exists $c \in]a, b[$ so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. If $f(a) = f(b) = 0$, we take c to be a maximum of the function. Then, $f'(c) = 0$. Else, set

$$g(x) := f(x) + \frac{f(a) - f(b)}{b - a}(x - a)$$

and note that $g(a) = 0 = g(b)$. Hence, get c s.t. $g'(c) = 0$ which is so say $f'(c) = [f(b) - f(a)] / (b-a)$. ■

Question 3.15 (Audience)**.** *Is there a good example of a fact from classical analysis that doesn't carry over directly to this context?*

Answer. The fact that the reals are archimedean (e.g. no real less than $1/n$ for all n) doesn't hold in general. We've also seen that the sin function can't exist in an o-minimal structure; this will mean that lot's of differential equations aren't solvable and lots of infinite series don't converge.

3.3 Monotonicity theorem

Definition 3.16. We'll say $f :]a, b[\to R$ is **strictly monotone** if it is either constant, strictly increasing, or strictly decreasing. We will say it is **locally monotone** at a point $x \in]a, b[$ if there's an open around that point on which is is strictly monotone. It will be locally monotone on the whole interval if it is locally monotone on each point.

Theorem 3.17 (**Monotonicity Theorem**)**.** *Let $f : \underbrace{]a, b[}_{I} \to R$ be definable. Then, there exists a partition*

$$a = a_0 < a_1 < \cdots < a_n = b$$

of I s.t. $f|_{]a_i, a_{i+1}[}$ is continuous and strictly monotone for all $i < n$.

Corollary 3.18. *For all $c \in [a, b]$, both limits $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist in $\overline{R}$.*

Note 1. There was some discussion about whether a, b above can be in $\overline{R}$ or if they need to lie in R.

We will prove this theorem using two lemmas.

Lemma 3.19. *There exists a subinterval of I on which f is strictly monotone.*

Lemma 3.20. *If f is strictly monotone, then f is continuous on a subinterval of I.*

Proof of Theorem 3.17 assuming both lemmas above. Let $A := \{x \in]a,b[\mid f$ is continuous and locally strictly monotone at $x\}$. This is definable by usual first order argument. Now, $]a,b[\setminus A$ is definable and finite; otherwise, there will be a subinterval $J \subset]a,b[\setminus A$ (by O-minimality). By the above lemmas, we'd then get a subinterval of J s.t. f is continuous and strictly monotone, contradicting the definition of A.

We may assume that $A =]a,b[$ as well as that f is locally constant or locally increasing or locally decreasing. In the first case, for any point $x_0 \in]a,b[$, consider

$$s(x_0) := \sup\{x \mid x_0 < x < b \text{ and } f \text{ is constant on } [x,b]\}.$$

Then, $s(x_0) = b$ (otherwise contradict locally constant assumption), so f is constant on $[x_0, b[$. One can similarly show that f is constant on $]a, x_0]$ so it's constant on all of $]a,b[$.

A similar argument works in the other two cases. ■

This just leaves proving Lemma 3.19 and 3.20.

Proof of Lemma 3.20. We need to show that f is continuous on a subinterval of I. Assume that f is strictly increasing. Then, $f(I)$ is infinite (since I is), so it must contain some interval $K \subset f(I)$. Now, choose $c, d \in K$ with $c < d$ and let c', d' be their preimages. Then, f is continuous on $]c', d'[$. This follows quickly since we're working in the order topology. ■

Proof of Lemma 3.19. Let $\Delta(I) := \{(x,y) \in I^2 : x < y\}$. For $* \in \{=,<,>\}$, similarly define $\Delta_*(f) := \{(x,y) \in I^2 : f(x) * f(y)\}$. We want to find a subinterval $I' \subset I$ such that $\Delta(I') \subset \Delta_*(f)$ for some $* \in \{=,<,>\}$. We will in fact prove a more general statement. Note that $I^2 = \Delta_=(f) \cup \Delta_>(f) \cup \Delta_<(f)$, so it suffices to prove Lemma 3.21 below. ■

Lemma 3.21 (**O-minimal pigeonhole principle**). *Let $S_1, \ldots, S_n \subset R^2$ be definable and assume $I^2 \subset S_1 \cup \cdots \cup S_n$. Then, there exists k and a subinterval $I' \subset I$ s.t. $\Delta(I') \subset S_k$.*

Proof. First define

$$A_k := \{x \in I \mid \exists x' > x \text{ such that }]x, x'[\subset (S_k)_x\}.$$

Since the S_k's cover I^2, we conclude that there exists some fixed k s.t. A_k contains an interval $J \subset I$. Now, consider the function

$$\begin{array}{rccc} g: & J & \longrightarrow & \overline{R} \\ & x & \longmapsto & sup\{x' \in I \mid x' > x \text{ and }]x,x'[\subset (S_k)_x\} \end{array}.$$

We claim that there exists a bounded interval $I' \subset J$ and an element $d > \sup(I')$ such that $g(x) > d$ for any $x \in I'$. Once we have this claim, for any $x, x' \in I'$ with $x < x'$, we'll have $x' < d$ and so $]x, x'[\subset (S_k)_x$, i.e. $(x, x') \in S_k$, so $\Delta(I') \subset S_k$.

To prove the claim, define $A := \{y \in J \mid \forall x \in J : (x < y) \implies g(x) \leq g(y)\}$, and then break into two cases

(1) First suppose there's an interval $J' \subset A$. Then, for any $c \in J'$, we have $g(c) > c$ so there exists $d, d' \in]c, g(c)[$. Take them so that $d < d'$ and define $I' := J' \cap]c, d'[$. For any $x \in I'$, we get that $g(x) \geq g(c)$ (by definition of A) and $g(c) > d$ (so $g(x) > d$). This proves the claim in this case.

(2) Alternatively, suppose there's an interval $J' \subset J \backslash A$. Then, for any $c \in J'$, there exists $x_1 \in]\inf J', c[$ such that $g(x_1) > g(c)$ (since $c \notin A$). Repeating let's us find an infinite sequence

$$x_1 > x_2 > \cdots > \inf J' \text{ with } g(x_{i+1}) > g(x_i) > \cdots > g(c) > c.$$

This tells us we have an infinite, definable set, so O-minimality ensures that there's an interval $I' \subset]\inf J', c[$ such that for all $x \in I'$, $g(x) > g(c) > c$. Take $g(c) = d$ and we win. ■

4 Sasha Petrov (2/28): Cell Decomposition

Let $(R, <)$ be a totally ordered field.

Setup. Fix some O-minimal structure $(D_n \subset R^n)_{n \geq 1}$ on R. Recall that this means, among other things, that D_1 consists of finite unions of points and intervals.

Note that the 'building blocks' of D_1 are points and intervals. We want to make sense of something similar for higher dimensional definable sets.

Definition 4.1. For all integers $m \geq 1$, and all $(\iota_1, \ldots, \iota_m) \in \{0,1\}^m$, there is a class of $(\iota_1, \ldots, \iota_m)$**-cells** in D_m defined inductively via

(1) (0)-cells are singletons $\{a\} \subset R$ and (1)-cells are (open, nonempty) intervals $]a, b[\subset R$ (note $a, b \in \overline{R} = R \cup \{-\infty, +\infty\}$)

(2) For every $m \geq 1$, an $(\iota_1, \ldots, \iota_m, 0)$-cell is anything of the form $\Gamma_f \subset R^{m+1}$, where $f : X \to R$ is a definable, continuous function on a $(\iota_1, \ldots, \iota_m)$-cell $X \subset R^m$.

Similarly, an $(\iota_1, \ldots, \iota_m, 1)$-cell is anything of the form

$$\{(x, y) \in R^m \times R \mid x \in X \text{ and } f_1(x) < y < f_2(x)\}$$

where $X \subset R^m$ is an $(\iota_1, \ldots, \iota_m)$-cell and $f_1, f_2 : X \to R$ functions satisfying $f_1 < f_2$ on all of X. Here, we require each f_1, f_2 to either be a definable, continuous function or to be a constant functions with value $\pm\infty$.

Think: the dimension of an $(\iota_1, \ldots, \iota_m)$-cell is $\sum_j \iota_j$.

Example. A $(0,0)$-cell is the graph of a function from a point, so a $(0,0)$-cell is a point $(a, b) \in R^2$.
A $(1,0)$-cell is the graph of a (continuous, definable) function on an interval.
A $(0,1)$-cell looks like a point cross an interval: $\{a\} \times]b, c[$
A $(1,1)$-cell is the interior of a region bounded by the graphs of two functions on an interval △

Warning 4.2. The notion of being a cell is sensitive to the choice of coordinates. For example, if you take a $(1,1)$-cell and then swap the coordinates on the ambient R^2 (do like a diagonal reflection of the picture) the result will likely no longer be a $(1,1)$-cell.

Remark 4.3.

- A cell C is open (in the order topology) iff it is of type $(1, 1, \ldots, 1)$.

- Every cell is locally closed, i.e. $C \overset{\text{open}}{\subset} \overline{C}$ (= closure of C).
- For any non-open cell C, there exists a coordinate projection $\pi : R^m \to R^{m-1}$ so that $\pi|_C : C \xrightarrow{\sim} \pi(C)$ is a bijection onto its image, and $\pi(C) \subset R^{m-1}$ is a cell.

  (I think) get this by projecting along some coordinate where a 0 appears int he cell type.

○

Definition 4.4. A **cell decomposition**

(1) of R^1 is an expression

$$R^1 = C_1 \sqcup \cdots \sqcup C_N$$

where each C_i is a cell.

(2) of R^{m+1} (for $m \geq 1$) is a decomposition $R^{m+1} = C_1 \sqcup \cdots \sqcup C_N$ where each C_i is a cell and s.t. $\{\pi(C_1), \ldots, \pi(C_N)\}$ form a cell decomposition of R^m,[2] where $\pi : R^{m+1} \to R^m$ is forgetting the last coordinate.

We extend this definition

Note 2. Sasha drew an example of a cell decomposition of R^2, but I don't have something to draw with... so exercise: come up with an example

TODO: Replace this note with a picture?

Our main goal is to establish a structure theorem for arbitrary definable sets, so we should probably state what sort of structure we would like.

Definition 4.5. Let $A \subset R^m$ be definable. We say a cell decomposition $R^m = C_1 \sqcup \cdots \sqcup C_N$ **partitions** A if $A = C_{i_1} \sqcup \cdots \sqcup C_{i_k}$ for some $1 \leq i_1 < \cdots < i_k \leq N$.

Our goal is to prove the following result

Theorem 4.6. *Let $A_1, \ldots, A_k \subset R^m$ be definable subsets. Then, there exists a cell decomposition of R^m which partitions each A_i.*

The proof will be inductive, and so we will find it useful to name the various statements we'll inductively make use of

(CD_m) Let $A_1, \ldots, A_k \subset R^m$ be definable subsets. Then, there exists a cell decomposition of R^m which partitions each A_i.

(PC_m) For $A \subset R^m$ and $f : A \to R$ both definable, there exists a partition of A s.t. f is continuous on every cell in A

(UF_m) Fix $A \subset R^m$ definable. Consider projection $\pi : R^m \to R^{m-1}$ and assume that A is **finite over** R^{m-1} in the sense that $\#A_x < \infty$ for all $x \in R^{m-1}$ ($A_x = \pi^{-1}(x) \cap A$). Then, A is **uniformly finite over** R^{m-1} in the sense that there's some $k \in \mathbb{N}$ s.t. $\#A_x < k$ for all x.

Remark 4.7.

[2] In particular, for any i, j, we have $\pi(C_i) = \pi(C_j)$ or $\pi(C_i) \cap \pi(C_j) = \emptyset$

- (CD_1) follows from the definition of o-minimality.
- (PC_1) follows from the monotoncity theorem 3.17. Recall, this said that for definable $f :]a, b[\to R$, one can choose a partition $a = a_0 < \cdots < a_n = b$ of the domain s.t. f is continuous and monotone on every $]a_i, a_{i+1}[$.
- It's not hard to show that $(CD_{m+1}) \implies (PC_m)$ by applying the former to the graph Γ_f of a function $f : A \to R$ (with $A \subset R^m$).

$\circ$

How will the proof go? We already know (CD_1) and (PC_1). We will prove

$$(\mathbf{PC_m}) + (\mathbf{CD_m}) \implies (\mathbf{UF_{m+1}}) \text{ and } (\mathbf{PC_m}) + (\mathbf{CD_{m+1}}) + (\mathbf{UF_{m+1}}) \implies (\mathbf{CD_{m+1}}), (\mathbf{PC_{m+1}})$$

Proof of $(\mathbf{UF_2})$. [3] We will assume that R is uncountable since this simplifies the proof and suffices for our applications. Say we have $A \subset R^2$ s.t. $\pi : A \to R$ is finite.

We will say that

- $(a, b) \in R \times \overline{R}$ is *normal* if there exists intervals I, J in R s.t. $(a, b) \in I \times J$ and $(I \times J) \cap A = \emptyset$ or $(I \times J) \cap A = \Gamma_f$ for some continuous $f : I \to R$.
- $a \in R$ is *good* if for all b, the point $(a, b) \in R^2$ is normal.

  We let $\mathcal{G} \subset R$ denote the set of good points, and we let $\mathcal{B} = R \setminus \mathcal{G}$ be the complement.

We first remark that $\mathcal{G}$ is definable. I missed the argument, but sounded like the main point is that being the graph of a function can be checked e.g. using the vertical line test and that a function being continuous is a definable condition.

Next, we claim that $\mathcal{B}$ is finite. If not, there's an interval $I \subset \mathcal{B}$ by O-minimality. Let $X_n = \{x \in R : \#A_x = n\}$ which is definable (can express this using an n-variable formula). By assumption, $R = \bigsqcup_{n \geq 0} X_n$. We ultimately want to show that $X_m = \emptyset$ for $m \gg 0$. Define functions

$$f_1 < f_2 < \cdots < f_n : A \to R$$

s.t. for any $x \in X_n$, $\{f_1(x), \ldots, x_n(x)\} = A_x$. These are all definable functions. Now we use our assumption that R is uncountable (so I is as well). If each X_n is finite, then their union R would be at most countable; thus, there must be an n s.t. $X_n \cap I$ contains an interval J. Now, $(\mathbf{PC_1})$ for $f_1, \ldots, f_n$ on J tells us that, after shrinking J, all $f_1, \ldots, f_n$ are continuous.

At this point, we have an interval $J \subset \mathcal{B}$ consisting solely of bad points. Over it, we have the continuous functions $f_1 < f_2 < \cdots < f_n : J \to R$. Now, for any $b \in J$, we can draw small disjoint boxes around each $f_i(b)$ in order to see that (a, b) must be good for all $a \in R$. Thus, $J \subset \mathcal{G}$, a contradiction. Thus, $\mathcal{B}$ must in fact be finite.

Now, we claim that $x \mapsto \#A_x$ is locally constant on $\mathcal{G}$. Fix some $a \in \mathcal{G}$. Around each point in the fiber A_a, we can fix a small box whose intersection with A is the graph of f_i. Thus, for b near a (b in the intersection of the projections of these boxes), we have $\#A_b \geq n$. We need to deal with the possibility

[3] I had a hard time following this proof, so these notes may be less than helpful

that $\#A_b > n$. The function $g : J \to R$ given by $g(x) = \min\{A_x \setminus \Gamma_{f_1} \cup \cdots \cup \Gamma_{f_n}\}$ is definable (with domain consisting of nearby points with $\geq n+1$ preimages). The domain must be definable. If $a \notin \overline{J}$, then $x \mapsto \#A_x = n$ in a neighborhood of a, so we're happy. If $a \in \overline{J}$, things are more annoying. Can can pick some $b < a$ s.t. g is continuous monotone on $]b, a[$ by Theorem 3.17. If $y := \lim_{x \to a^-} g(x) \in \overline{R}$, then (a, y) is not normal, a contradiction. ■

Proposition 4.8. $(\mathbf{PC_m}) + (\mathbf{CD_m}) + (\mathbf{UF_{m+1}}) \implies (\mathbf{CD_{m+1}})$

Proof. Fix $A_1, \ldots, A_k \subset R^{m+1}$. Consider

$$\partial_m A = \{(x, y) \in R^m \times R \mid y \in \partial(A_x)\}.$$

Note that each $\partial_m A_1, \ldots, \partial_m A_k$ are finite over R^m. By uniform finiteness, there is some N s.t. for all $\ell = 1, \ldots, k$ and any $x \in R^m$, one has $\#(\partial_m A_\ell)_x \leq N$. For convenience, define $Y := \partial_m A_1 \cup \cdots \cup \partial_m A_k$ and let $X_n = \{x \in R^m : \#Y_x = n\}$. Consider functions

$$f_{n1} < f_{n2} < \cdots < f_{nn} : X_n \to R \text{ such that } \{f_{n1}(x), \ldots, f_{nn}(x)\} = Y_x.$$

Also introduce the constant function $f_{n0} \equiv -\infty$. Let

$$C_{\ell,i,j} = \{x \in R^m \mid f_{ij}(x) \in (A_\ell)_x\} \text{ and } D_{\ell,i,j} = \{x \in R^m \mid]f_{ij}(x), f_{i,j+1}(x)[\subset (A_\ell)_x\}.$$

These are keeping track of the potential boundaries/interiors of our cell decomposition. Now, $(\mathbf{CD_m}) + (\mathbf{PC_m})$ applied to $C_{\ell,i,j}$, $D_{\ell,i,j}$ and f_{ij} gives a cell decomposition $R^m = B_1 \sqcup \cdots \sqcup B_t$. If I've not completely lost myself, the cells inbetween the graphs of the f_{ij}'s should contained completely in A_ℓ or be disjoint from A_ℓ (for any particular ℓ), and this can be used to show that the cells bounded by these graphs (and the cells giving the boundaries) give a cell decomposition partitioning $A_1, \ldots, A_k$. ■

This leaves proving $(\mathbf{PC_{m+1}})$. We don't have time to give all the details, but one of the key ingredients is the following lemma

Lemma 4.9. *Let X be a topological space, and say $f : X \times R \to R$ is a function s.t. for all $(x, r) \in X \times R$, both $f(x, -)$ is continuous + monotone and $f(-, r)$ is continuous. Then, f is continuous.*

Question 4.10 (Audience)**.** *You mentioned at one point some applications of this you wanted to mention. Could you if there's time?*

Answer. For example, I think it follows from this that if you take the $\mathbb{C}$-points of an algebraic variety, then this has a triangulation.

(there was some further discussion of alternate proofs of this fact, but I didn't bother typing any of it down)

5 Sam Marks (3/21): Dimension Theory

Note 3. Roughly 12 minutes late and sitting in a less than ideal spot, so these notes will be even worse than usual...

Fix a totally ordered field R with a given o-minimal structure.

Definition 5.1. If C is a $(i_1, \ldots, i_n)$-cell in R^n, then $\dim C := i_1 + \cdots + i_n$. If $X \subset R^n$ is definable, then

$$\dim X := \max \{\dim C : C \subset X \text{ is a cell}\}.$$

Finally, $\dim \emptyset = -\infty$.

Recall 5.2. A cell decomposition of R^n is a finite partition of R^n into cells $\{A\}$ such that $\{\pi(A)\}$ is a cell decomposition of R^{n-1}.

Recall 5.3. If $A_1, \ldots, A_k \subset R^n$ are definable, there exists a cell decomp of R^n adapted to them.

(there was another recall here that I missed)

Recall 5.4. A $(i_1, \ldots, i_n)$-cell is open iff it has maximal dimension (i.e. $i_1 = \cdots = i_n = 1$). Furthermore, if $C \subset R^n$ is a cell, there is a coordinate projection $\pi_C : R^n \to R^{\dim C}$ s.t. $\pi_C \xrightarrow{\sim} C' \overset{\text{open}}{\subset} R^{\dim C}$.

Observation 5.5. Say $X \subset R^n$ is definable. Then,

(1) $\dim X = n \iff X$ contains an open cell

(2) $\dim X = 0 \iff X$ is nonempty and finite

this comes from the decomposition theorem.

Lemma 5.6. *If $C \subset R^n$ is an open cell and $f : C \hookrightarrow R^n$ is a definable injection, then $f(C)$ contains an open cell.*

(Note that both n's above are the same)

Proof. Induct on n. If $n = 1$, then C infinite $\implies$ $f(C)$ infinite $\implies$ $f(C)$ contains an interval. Now suppose $n > 1$. By cell decomposition and piecewise continuity, we can reduce to the case of $f : C \hookrightarrow D \subset R^n$ with D a cell and f continuous. Futzing with cell decompositions, we may assuming further that $f(C) = D$.

We want to show that D is an open cell. If not, we can find a coordinate projection $\pi_D : D \xrightarrow{\sim} D' \subset R^{n-1}$. Since C is open, it contains some $B \times]a, b[$ with B a box in R^{n-1}. Let $g : B \times]a, b[\to R^{n-1}$ be composition of the relevant functions. Fixing some $c \in]a, b[$, the image of $g(-, c) : R^{n-1} \times R^{n-1}$ contains some box in R^{n-1}. I didn't follow, but fixing the other coordinate and applying the inductive hypothesis once more, you can contradict injectivity. ■

TODO: Come back and think about this

Corollary 5.7. *Let $X, Y \subset R^n$ be definable. Then,*

(1) *If $f : X \hookrightarrow Y$ is a definable injection, then $\dim X \le \dim Y$. In particular, if f is a bijection, then $\dim X = \dim Y$.*

(2) $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$

Proof. (**1**) Suffices to prove the claim about injections. By cell decomposition theorem, we can reduce to $C \hookrightarrow D$ with C, D cells and $C \overset{\text{open}}{\subset} R^n$. Consider a homeomorphic coordinate projection $\pi_D : D \xrightarrow{\sim} D' \subset R^{\dim D}$. Consider composition

$$C \hookrightarrow D \xrightarrow{\sim} D' \overset{\text{open}}{\subset} R^{\dim D}.$$

If $\dim D < \dim X$, we get a closed injection $R^{\dim D} \hookrightarrow R^{\dim X}$. Adding this to the above composition, the obtained map $C \to R^{\dim X}$ contradicts the previous lemma (image can't contain an open cell), so $\dim X \le \dim D \le \dim Y$.

(**2**) Let $d = \dim(X \cup Y)$. Let $C \subset X \cup Y$ be a d-dimensional cell. Consider suitable coordinate projection $\pi_C : C \xrightarrow{\sim} C' \subset R^d$. Note that $C = (C \cap X) \cup (C \cap Y)$ and apply coordinate projection, so $(C \cap X) \cup (C \cap Y) \xrightarrow{\sim} \pi_C(C \cap X) \cup \pi_C(C \cap Y)$. Taking a cell decomposition of C', at least one of $\pi_C(C \cap X), \pi_C(C \cap Y)$ must contain an open cell. Wlog say $C' \cap \pi_C(X)$ contains an open cell. Then,

$$d = \dim \pi_C(C \cap X) = \dim(C \cap X) \le \dim X \le \dim(X \cup Y) = d,$$

so we win. ■

Proposition 5.8. *Let $X \subset R^m \times R^n$ be definable. For $-\infty \le d \le n$, set*

$$X(d) := \{a \in R^m \mid \dim X_a = d\} \subset R^m.$$

Then, $X(d)$ is definable and

$$\dim \left(X_{X(d)}\right) = \dim \left(\bigcup_{a \in X(d)} \{a\} \cup X_a \right) = \dim X(d) + d.$$

Proof. First suppose that X is an $(i_1, \dots, i_{n+m})$-cell. Let $\pi : R^m \times R^n \to R^m$ be the natural projection. By definition of cells, we automatically get that $\pi(X) \subset R^m$ is a $(i_1, \dots, i_m)$-cell, and that $X_a \subset R^n$ is an $(i_{m+1}, \dots, i_{m+n})$-cell for any $a \in \pi(X)$. In this case, $X(d)$ is either $\pi(X)$ (if $d = i_{m+1} + \cdots + i_{m+n}$) or empty.

From the definition of cell decompositions, one sees that $X(d)$ will be a union of cells in R^m. The general result then follows from the result for cells. ■

Corollary 5.9.

(1) *If $X \subset R^m \times R^n$ is definable, then*

$$\dim X = \max_{0 \le d \le n} (\dim X(d) + d)$$

(2) *Let $f : X \to R^m$ be definable ($X \subset R^n$). Then, for $0 \le d \le n$, the set*

$$X_f(d) := \left\{a \in R^m : \dim f^{-1}(a) = d\right\}$$

is definable and $\dim f^{-1}\left(X_f(d)\right) = \dim X_f(d) + d$.

(3) *If $f : X \to Y$ is definable, then* $\dim f(X) \le \dim X$

(4) *If X, Y are definable, then* $\dim(X \times Y) = \dim X + \dim Y$

Proof. (**1**) Note $X = \bigcup_{0 \le d \le n} X_{X(d)}$, so its dimension is the maximum of $\dim X_{X(d)} = \dim X(d) + d$.

(**2**) Apply the proposition to the (transpose of the) graph $\{(f(x), x) : x \in X\} \subset R^m \times R^n$.

(**3**), (**4**) both follow from (**2**). ■

Proposition 5.10. *Let $X \subset R^n$ be definable. Let $\partial X := \operatorname{clos}(X) \setminus X$ be the* ***frontier*** *of X. Then, $\dim(\partial X) < \dim X$, so $\dim \operatorname{clos}(X) = \dim X$.*

Before the proof, first a lemma.

Lemma 5.11. *Let $\pi : R^n \to R$ be projection onto the first coordinate. Then,*

$$S := \{x \in R \mid \partial(X_x) \neq (\partial X)_x\}$$

is finite.

Proof. We can equivalently write

$$S = \{x \in R \mid \operatorname{clos}(X_x) \subsetneq \operatorname{clos}(X)_x\}.$$

Let $B := \left\{(a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}) \in R^{2(n-1)} \mid a_i < b_i \text{ for all } i\right\}$, so B parameterizes boxes in R^{n-1}. Given $z \in B$, let $B(z) \subset R^{n-1}$ be the corresponding box. Consider also the 'incidence set'

$$T := \{(x, z) \in R \times B \mid \operatorname{clos}(X_x) \cap B(z) = \emptyset \text{ but } \operatorname{clos}(X)_x \cap B(z) \neq \emptyset\}.$$

$$S \longleftarrow T \longrightarrow B$$

Note that the left map is surjective by definitions. If $x \in S$, then T_x is nonempty and in fact contains an open cell (perturb the box witnessing the disagreement), so $\dim T_x = 2(n-1)$. On the other hand, if $z \in B$, then $T_z \subset S \subset R$ is finite. Else, by o-minimality, it must contain an interval $I \subset S$, but

$$\emptyset = \operatorname{clos}(X_x) \cap B(z) \supset X_x \cap B(z) \text{ for all } x \in I.$$

This means that $(I \times B(z)) \cap X = \emptyset$ which in turn means that $(I \times B(z)) \cap \operatorname{clos}(X) = \emptyset$ (since $I \times B(z)$ is open), a contradiction.

Thus, $\dim S + 2(n-1) = \dim T \leq \dim B + \max_z \dim T_z = \dim B = 2(n-1) \implies \dim S = 0$, so S is finite. ∎

This just leaves us with the proposition...

Proof of Proposition 5.10. We want to show that $\dim(\partial X) < \dim X$. We induct on n. If $n = 1$, then ∂X is finite (since X a finite union of intervals and points). To keep life simple, let's only write out the case $n = 2$.

Let S_1, S_2 be the sets of bad points in each copy of R, coming from the two projections $X \subset R^2 \rightrightarrows R$. Convince yourself that

$$(\partial X) \subset (\partial X)_{S_1} \cup (\partial X)_{S_2}.$$

Both of these pieces have dimension ≤ 1. For example,

TODO: Come make sense of/fix this

$$(\partial X)_{S_1} = \bigcup_{x \in R \setminus S_1} \{x\} \times (\partial X)_x = \bigcup_{x \in R \setminus S_1} \{x\} \times \partial(X_x) = \dim \partial(X_x) < \dim X_x$$

(last ineqality by induction) ∎

6 Elliot Glazer (3/28): Elementary Extensions – I'm out of town

7 Yujie (4/4): O-Minimal Chow Lemma

(I'm not 100% sure, but I think we're implicitly working with $\mathbb{R}_{\text{an,exp}}$ throughout this talk?)

7.1 Affine Version

Recall 7.1 (Classical Chow Lemma)**.** If X is a proper complex algebraic variety, and $Y \subset X$ is a closed complex analytic subvariety, then Y is algebraic.

Remember: For us, varieties are always reduced

Theorem 7.2 (**affine O-minimal Chow**, Peterzil-Starchenko)**.** *Let $\mathcal{Y} \subset (\mathbb{A}^n)^{\text{an}}$ be a closed analytic subvariety whose underlying set is definable. Then, $\mathcal{Y}$ is algebraic, i.e. $\mathcal{Y} = Y^{\text{an}}$ for an algebraic subvariety $Y \subset \mathbb{A}^n$.*

Remark 7.3. O-minimal Chow $\implies$ usual Chow (exercise). ○

Remark 7.4. There is a proof of Theorem 7.2 which uses an analyticity criterion of Bishop + usual Chow. We will not go over this. ○

Instead, we will give a different proof making use of the following lemma.

Lemma 7.5. *Any definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is algebraic.*

Proof. Induct of n. First say $n = 1$. One can use the Casorati-Weierstrass theorem from complex analysis to show that an entire definable function $f : \mathbb{C} \to \mathbb{C}$ is algebraic. Now say $n > 1$ and assume the lemma for $n-1$. Decompose $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and consider coordinates $(z, w) \in \mathbb{C} \times \mathbb{C}^{n-1}$. Then, for each fixed w, $f(z, w)$ is a polynomial in z by the $n = 1$ case. Now, recall the following lemma

Lemma 7.6. *Let $f : U \to V$ be a definable map with finite fibers. Then, the sets*

$$V_n := \left\{v \in V : \#f^{-1}(v) = n\right\}$$

are definable. Hence, the size of the fibers are uniformly bounded.

This implies that $\deg_z f(z, w)$ is uniformly bounded in w. Thus, for some N, one may write

$$f(z, w) = \sum_{k=0}^{N} \frac{\partial^k f}{\partial z^k}(0, w)\frac{z^k}{k!}.$$

The partial derivatives above are definable functions $\mathbb{C}^{n-1} \to \mathbb{C}$ and hence algebraic by the inductive hypothesis. ■

Proof of Theorem 7.2. We first fix some notation. Let $\mathcal{P}^n := (\mathbb{P}^n)^{\text{an}}$, choose some $\mathbb{C}^n \subset \mathcal{P}^n$, and denote its complement $\mathcal{P}^{n-1}$, the plane at infinity. We induct on $\dim_{\mathbb{C}} \mathcal{Y} := d$.

(Step 1) The boundary $\partial\mathcal{Y} = \overline{\mathcal{Y}} \setminus \mathcal{Y} \subset \mathcal{P}^{n-1}$ is definable and has real dimension $\leq 2d - 1$ since $\dim_{\mathbb{R}}(\partial\mathcal{Y}) < \dim_{\mathbb{R}} \mathcal{Y} = 2d$.

(Step 2) There is a linear projection $\pi : \mathbb{C}^n \to \mathbb{C}^d$ s.t. the restriction $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C}^d$ is proper.

Proof. First note that every point of $\mathcal{P}^{n-1}$ gives a linear projection $\mathbb{C}^n \twoheadrightarrow \mathbb{C}^{n-1}$. Since $d < n$, this implies that $\dim_{\mathbb{R}}(\partial\mathcal{Y}) \leq 2d-1 \leq 2(n-1)-1 < 2n-2 = \dim_{\mathbb{R}} \mathcal{P}^{n-1}$. Hence, there must exist a projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ s.t. each fiber L has bounded intersection with Y. This gives a projection $\mathcal{Y} \to \mathbb{C}^{n-1}$ which is proper. By Remmert's proper mapping theorem, its image will be a closed definable analytic variety. Iterate to go down to $\mathbb{C}^d$. ■

From audience discussion, sounds like one should only need the usual version of Remmert's theorem, and not an o-minimal version of it

(Step 3) The locus $\mathcal{Y}_0 \subset \mathcal{Y}$, where $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C}^d$ is not étale, is a closed algebraic subvariety of $\mathbb{C}^n$ ($\mathcal{Y}_0 = Y_0^{\text{an}}$)

Proof. $\mathcal{Y}_0$ is analytic and definable (locus where fiber size is non-generic). Since $\dim \mathcal{Y}_0 < \dim \mathcal{Y}$, the induction hypothesis tells us that $\mathcal{Y}_0$ is algebraic. ■

(Step 4) $\mathcal{Y}$ is algebraic.

Proof. Decompose $\mathbb{C}^n = \mathbb{C}^{n-d} \times \mathbb{C}^d$, and recall we have $\pi_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{C}^d$. Let $N := \det \pi_{\mathcal{Y}}$. Let $\pi : \mathbb{C}^n \to \mathbb{C}^d$ be the coordinate projection, and let $Z := \pi(Y_0) \subset \mathbb{C}^d$, a closed algebraic subvariety. Consider function

$$\begin{array}{cccc} F: & \mathbb{C}^d \setminus Z^{\text{an}} & \longrightarrow & \operatorname{Sym}^N \mathbb{C}^{n-d} \\ & z & \longmapsto & \pi_{\mathcal{Y}}^{-1}(z). \end{array}$$

Note that F is a holomorphic, definable map. Since $\pi_{\mathcal{Y}}$ is finite, F is locally bounded around Z^{an}. Hence, the pullbacks of coordinate functions[4] on $\operatorname{Sym}^N \mathbb{C}^{n-d}$ along F extend to holomorphic functions $\mathbb{C}^d \to \mathbb{C}$. By Lemma 7.5 these functions are algebraic, and so $\mathcal{Y} \setminus \mathcal{Y}_0$ is algebraic[5], so $\mathcal{Y}$ is as well. ■

■

7.2 Definable topological spaces

Definition 7.7. An S**-definable topological space** (S some O-minimal structure) M is

- a topological space M
- a finite open covering $\bigcup V_i \supset M$
- homeomorphisms $\varphi_i : V_i \xrightarrow{\sim} U_i \subset \mathbb{R}^n$ s.t.
  - **(1)** The U_i and the pairwise intersections $U_{ij} = \varphi_i(V_i \cap V_j)$ are definable
  - **(2)** The transition functions $\varphi_{ij} := \varphi_j \circ \varphi_i^{-1} : U_{ij} \to U_{ji}$ are definable

The collection $\{(V_i, \varphi_i)\}$ is called a **definable atlas**. A **morphism of definable topological spaces** $f : M \to M'$ is a continuous map f such that for any i, i', the composition

$$\varphi_i\left(V_i \cap f^{-1}(V'_{i'})\right) \xrightarrow{\varphi_i^{-1}} f^{-1}(V'_{i'}) \xrightarrow{f} V'_{i'} \xrightarrow{\varphi'_{i'}} U'_{i'}$$

is definable.

[4]Sounds like this means the elementary symmetric functions

[5]If I'm not too confused by what's been said (which I very possibly am), potentially $\mathcal{Y} \setminus \mathcal{Y}_0$ is the fiber product of $\mathbb{C}^d$ and $\mathbb{C}^{n-d}$ over $\mathbb{C}^{N+1}$ (with the maps to $\mathbb{C}^{N+1}$ given by elementary symmetric functions)

Example. Let X be a real algebraic variety. Consider $X(\mathbb{R})$ with the Euclidean topology. This carries a canonical $\mathbb{R}_{\text{alg}}$-definable topological space structure by covering by finitely many affine varieties. △

Example. Let X be an (affine) complex algebraic variety. Then, $X(\mathbb{C}) = \left(\operatorname{Res}_{\mathbb{C}/\mathbb{R}} X\right)(\mathbb{R})$, and so $X(\mathbb{C})$ has a definable topological space structure by the previous example. We call this canonical $\mathbb{R}_{\text{alg}}$-definable topological space structure the **definabilization of** X, denoted X^{def}. △

Let Top_S denote the category of S-definable topological spaces,[6] and let $\text{Var}_{\mathbb{C}}$ denote the category of algebraic varieties over $\mathbb{C}$.[7] We have a diagram of functors

$$\begin{array}{ccc} \text{Var}_{\mathbb{C}} & \xrightarrow{(-)^{\text{def}}} & \text{Top}_S \\ & {\scriptstyle (-)^{\text{Eucl}}} \searrow \quad \swarrow {\scriptstyle (-)^{\text{top}}} & \\ & \text{Top} & \end{array}$$

(The above is over $\mathbb{C}$. There is a similar picture over $\mathbb{R}$)

7.3 Basic definable analytic spaces

We identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Given a definable open $U \subset \mathbb{C}^n$, we set

$$\mathscr{O}_{\mathbb{C}^n}(U) := \left\{ \begin{matrix} \text{definable holomorphic functions} \\ U \to \mathbb{C} \end{matrix} \right\}.$$

If I am following the discussion, this will be a sheaf for a Grothendieck topology where covers are finite definable covers. Consider $I \subset \mathscr{O}_{\mathbb{C}^n}(U)$ some finitely generated ideal.

Notation 7.8. Let $X = |V(I)|$ denote the (definable) topological space of $V(I)$.

There is a sheaf $\mathscr{O}_U / I\mathscr{O}_U$ on U which is supported on X. We define $\mathscr{O}_X := (\mathscr{O}_U / I\mathscr{O}_U)|_X$, and we call the data $(U \subset \mathbb{C}^n, I)$ a **basic definable analytic space**.

Remark 7.9. This data has an associated locally $\mathbb{C}$-ring definable space $(X, \mathscr{O}_X)$. This is a definable topological space X equipped w/ a locally $\mathbb{C}$-ringed sheaf $\mathscr{O}_X$ on the definable site $\underline{X}$. This is the site with

- objects: definable open subsets of X
- morphisms: inclusions
- covers: finite open covers by definable open subsets

○

Definition 7.10. A **morphism between basic definable analytic spaces** $(X \subset C \subset \mathbb{C}^n) \to (Y \subset V \subset \mathbb{C}^n)$ consists of a

- definable holomorphic map $U \to V$ pulling I_Y back to I_X

[6] Yujie used the notation S-TopSp instead

[7] Yujie used the notation AlgVar $/\mathbb{C}$

- a morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ as locally $\mathbb{C}$-ringed definable spaces coming from $U \to V$

The corresponding category is denoted S-BasicDefAnSp/$\mathbb{C}$.

There is an analytification functor

$$(-)^{\mathrm{an}} : S\text{-BasicDefAnSp}/\mathbb{C} \longrightarrow \mathrm{AnSp}/\mathbb{C}.$$

This sends X to the analytic space cut out by I on U.

Notation 7.11. We let $\mathrm{Mod}(\mathscr{O}_X)$ denote the category of $\mathscr{O}_X$-modules, and we let $\mathrm{Coh}(\mathscr{O}_X)$ denote the full subcategory of coherent $\mathscr{O}_X$-modules.

Theorem 7.12. *Let X be a basic definable analytic space. Then,*

$$(-)^{\mathrm{an}} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$$

is faithful and exact.

Corollary 7.13. *The forgetful functor S-BasicDefAnSp/$\mathbb{C}$ $\to$ (locally $\mathbb{C}$-ringed definable spaces) is fully faithful.*

Note 4. My laptop's at 8%, so potentially it will die before the talk is over...

7.4 Definable analytic space

Definition 7.14. A locally $\mathbb{C}$-ringed definable space $(X, \mathscr{O}_X)$ is an (S-)**definable analytic space** if locally it is the locally $\mathbb{C}$-ringed definable space associated to some basic definable analytic space. The category of such objects is denoted S-DefAnSp/$\mathbb{C}$; this is a full subcategory of the category of locally $\mathbb{C}$-ringed definable spaces.

We have functors

- $(-)^{\mathrm{an}} : S\text{-DefAnSp}/\mathbb{C} \to \mathrm{AnSp}/\mathbb{C}$
- $(-)^{\mathrm{an}} : \mathrm{Coh}(X) \to \mathrm{Coh}(X^{\mathrm{an}})$ when X is a definable analytic space. This is faithful and exact.

Theorem 7.15 (Definable GAGA). *Let X be an algebraic space. Then,*

$$(-)^{def} : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}(X^{def})$$

is fully faithful and exact. Moreover, it's (essential) image is closed under subobjects and quotients.

Corollary 7.16 (O-minimal Chow). *Let X be an algebraic space, and let $\mathcal{Y} \subset X^{def}$ be a closed, definable analytic subspace. Then, $\mathcal{Y}$ is algebraic, i.e. $\mathcal{Y} = Y^{def}$.*

8 Si Ying (4/11): Pila-Wilkie Counting

Two part talk. Today the part of the talk not using O-minimality.

Let X be a definable subset of $\mathbb{R}^n$.

Notation 8.1. We let X^{alg} denote the union of all connected semialgebraic subsets of X of positive dimension.

Warning 8.2. X^{alg} may not be semialgebraic, and may not even be definable.

(Sounds like 'semialgebraic' means definable in $\mathbb{R}_{\text{alg}}$)

Example. Consider

$$X = \left\{(x, y, z) \in \mathbb{R}^3 : z = x^y \text{ and } x, y \in [2, 3]\right\}.$$

This is definable in $\mathbb{R}_{\exp}$, but X^{alg} is the infinite union of the (disjoint) curves $z = x^y$ where $y \in \mathbb{Q}$, so it not definable. △

Our next goal is

Theorem 8.3 (**Pila-Wilkie Theorem**).

(Version 1) Let $X \subset \mathbb{R}^n$ be definable. Then, for all $\varepsilon > 0$, there exists a constant $c(X, \varepsilon)$ such that for all $T \geq 1$,

$$\#(X - X^{alg})(\mathbb{Q}, T) =: N(X - X^{alg}, T) \leq c(X, \varepsilon)T^{\varepsilon}$$

where $(X - X^{alg})(\mathbb{Q}, T) := \left\{P \in (X - X^{alg})(\mathbb{Q}) : H(P) \leq T\right\}$. Here, H is the naive height on $\mathbb{Q}^n$, i.e.

I think this is the definition

$$H\left(\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}\right) = \max_i \{|a_i|, |b_i|\}$$

(every fraction in lowest terms)

(Version 2) Let $Z \subset \mathbb{R}^n \times \mathbb{R}^m$ be definable. Then, for every $\varepsilon > 0$, there's a $c(X, \varepsilon)$ and definable $W(Z, \varepsilon) \subset Z$ so that

(1) *$W_y \subset (Z_y)^{alg}$ for all $y \in \mathbb{R}^m$*

(2) *For all $T \geq 1$, $N(Z_y - W_y, T) \leq c(Z, \varepsilon)T^{\varepsilon}$ for all $y \in \mathbb{R}^m$*

In version 2, think of Z as a family over $\mathbb{R}^m$ (the second factor of $\mathbb{R}^n \times \mathbb{R}^m$). If I'm following, Version 2 is giving a uniform bound for the number of non-algebraic points of small height for definable families.

Remark 8.4 (Audience). In the previous example, most points in X^{alg} are... ○

The main workhorse will be

Lemma 8.5 (Main Lemma (I)). *Let $Z \subset (0, 1)^n \times \mathbb{R}^m$ be definable with fiber dimension $k < n$. Let $\varepsilon > 0$. Then, there exists $d(\varepsilon, k, n) \in \mathbb{N}$ and $K(Z, \varepsilon)$ constant such that for all $y \in \mathbb{R}^m$,*

$Z_y(\mathbb{Q}, T)$ is contained in the union of at most $K(Z, \varepsilon)T^{\varepsilon}$ hypersurfaces of degree d.

Note that $d = d(\varepsilon, k, n)$ and $K = K(Z, \varepsilon)$ are both independent of $y \in \mathbb{R}^m$.

(to be covered next time)

Warning 8.6. The role of n, m have been swapped a couple times during the talk, so potentially I will have them backwards at some parts in my notes. Noticing any such typos is left as an exercise for the reader

There's a second version of the main lemma which we will state. Before this, we need a new definition.

Definition 8.7. A **k-cylinder of degree d contained in $\mathbb{R}^m$** is an intersection

$$\bigcap_{\sigma \in S} \pi_\sigma^{-1}(H_\sigma) \text{ where } S = \{\sigma \subset \{1, \ldots, n\} : \#\sigma = k+1\},$$

H_σ is a hypersurface of degree d in $\mathbb{R}^{k+1}$, and π_σ is the projection from $\mathbb{R}^m \to \mathbb{R}^\sigma \cong \mathbb{R}^{k+1}$ (using the chosen coordinates).

Lemma 8.8 (Main Lemma (II)). *Let $Z \subset (0,1)^n \times \mathbb{R}^m$ be definable with fiber dimension $\leq k$ ($< m$?). Let $\varepsilon > 0$. There exists $d = d(n, k, \varepsilon)$ and constant $c_2(Z, \varepsilon)$ such that*

$$Z_y(\mathbb{Q}, T) \text{ is contained in a union of at most } c_2(Z, \varepsilon)T^\varepsilon \text{ } k\text{-cylinders of degree } d.$$

This follows from Main Lemma (I), it sounds by applying (I) to each projection $\pi_\sigma(Z_y)$. It is version (II) that we will use in this talk.

Proof of Theorem 8.3 Version 2, assuming Lemma 8.8. Observe that if $C = A \cup B$ with A, B definable and we know the theorem for A, B, then we can conclude it for C.[8] Furthermore, since $x \mapsto \pm x^{-1}$ doesn't change heights of points, we may assume that $Z \subset (0,1)^m \times \mathbb{R}^n$ as required for Lemma 8.8. Now we proceed by induction on k, the maximal dimension of the fibers of $Z \to \mathbb{R}^n$. First say $k = 0$. From Sasha's talk, there exists a uniform bound C for the # of points in each fiber, so $\#Z_y(\mathbb{Q}, T) \leq C$ for all $y \in \mathbb{R}^n$.

Now say $k > 0$ and the theorem holds for families with fibers of dimension $\leq k - 1$. Choose a fiber $X := Z_y$ and consider cases.

(Case I: $k = m$) Consider the set[9]

$$r_m(X) = \text{subset of } C^1\text{-smooth points of } X \text{ of dimension } m$$

of points on X which are regular of dimension m. In other words, these are points for which there exists an open ball $U \subset \mathbb{R}^m$ containing x s.t. $U \cap X$ is a (definable) C^1-manifold of dimension m. Note that $x \in r_m(X) \implies \exists$ open ball in $\mathbb{R}^m$ contained in X which forces $x \in X^{\text{alg}}$ (since open balls are semi-algebraic). Now,

$$A := \{(x, y) \in (0,1)^m \times \mathbb{R}^n : x \in r_m(Z_y)\}$$

is a definable set. The complement $Z - A$ has fiber dimension $< m$,[10] so we can apply induction to it, and then win (since the theorem obviously holds for A as $A_y \subset Z_y^{\text{alg}}$).

(Case II: $k < m$) In this case we can apply Lemma 8.8. Again consider $X = Z_y$ some fiber. Lemma 8.8 tells us that $X(\mathbb{Q}, T)$ is contained in some union of $c_1(Z, \varepsilon)T^{\varepsilon/2}$ many k-cylinders. We want a

[8] Take $c(C, \varepsilon) = c(A, \varepsilon) + c(B, \varepsilon)$ and $W(C, \varepsilon) = W(A, \varepsilon) \cup W(B, \varepsilon)$

[9] Si Ying used the notation $\text{reg}_m(X)$, but that takes longer to type

[10] If it had dimension m, you'd get a fiber containing an open cell, but open cells contain C^1-smooth points of maximal dimension

uniform bound on the number of non-algebraic points of $X(\mathbb{Q},T)\cap$ (a k-cylinder). This will be of the form $c_3(Z,\varepsilon)T^{\varepsilon/2}$, so we win by taking $c(Z,\varepsilon)=c_1(Z,\varepsilon)c_3(Z,\varepsilon)$.

Note that real hypersurfaces of degree d in $\mathbb{R}^{k+1}$ are parameterized by $\mathbb{P}^\nu(\mathbb{R})$ where $\nu=\binom{k+1+d}{d}-1$. If I heard correctly, we claim we can embed this semi-algebraically inside some $T\subset\mathbb{R}^p$ for some p sufficiently large. Recall $S=\{\sigma:|\sigma|=k+1\}$ and say $\#S=q$. For $t\in T^q$, let (t_σ) correspond to the k-cylinder $L(t_\sigma)$ of degree d. Define

$$\Sigma=\{(x,y,t):\pi_\sigma(x)\in L(t_\sigma)\ \text{ for all }\ \sigma\in S\}\subset\mathbb{R}^m\times(\mathbb{R}^n\times\mathbb{R}^{pq}).$$

Note that $\Sigma_{(y,t)}$ has (fiber) dimension $\leq k$. Set

$$Z'=\{(x,y,t):(x,y)\in Z\}\subset\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{pq},$$

and note that it has the same fibers as Z did. Furthermore, $Z'\cap\Sigma$ has fiber dimension $\leq k$. If it had fiber dimension $<k$, we could apply induction and win. Note if we had a Pila-Wilkie type result for this family $Z'\cap\Sigma$, then we'd be the uniform bound on non-algebraic points we wanted, and so win.

Consider the subsets

$$\begin{aligned}A_1&=\{(x,y,t)\in Z'\cap\Sigma\mid x\notin r_k(Z'\cap\Sigma)_{y,t}\}\\A_2&=\{(x,y,t)\in Z'\cap\Sigma\mid x\notin r_k(Z')_{y,t}\}\\A_3&=\{(x,y,t)\in Z'\cap\Sigma\mid x\notin r_k(\Sigma)_{y,t}\}\end{aligned}$$

Note that each of these will have fiber dimensions $\leq k-1$. Hence, we can apply the inductive hypothesis to each of them. Now, let $B=(Z'\cap\Sigma)\setminus(A_1\cup A_2\cup A_3)$, so $(x,y,t)\in B$ means that x is regular in $Z',\Sigma,Z'\cap\Sigma$. Hence, there is some $\Delta\subset\mathbb{R}^m$ s.t. $\Delta\cap Z'_{y,t}$, $\Delta\cap\Sigma_{y,t}$, $\Delta\cap(Z'\cap\Sigma)_{y,t}$ are all C^1-manifolds of dimension k. If we shrink Δ further, we may assume that these three sets all coincide. Now, a point $x\in\Delta\cap\Sigma_{y,t}$ must be algebraic since Σ was defined algebraically. Therefore, $(x,y,t)\in(Z'\cap\Sigma)^{\mathrm{alg}}$. ∎

This still leaves the problem of proving the main lemma. The key to its proof is the existence of a 'uniform r-parameterization'. Say $X\subset(0,1)^n$ is definable and $\dim X=k$. Then, an **r-parameterization** is a set S of functions $\varphi:(0,1)^k\to X$ s.t.

(1) each φ is $C^{(r)}$;

(2) $\bigcup_\varphi \mathrm{im}(\varphi)=X$; and

(3) for all $\varphi\in S$,

$$\left|\varphi^{(\alpha)}(x)\right|\leq 1$$

for all $\alpha=(\alpha_1,\dots,\alpha_n)$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\leq r$

(Some bound on partial derivatives)

The main idea for making use of such a thing is to say that

$$\det\left(\varphi(X)^{(\alpha)}\right)_{\varphi,\alpha} = 0 \iff \text{ it lies on a hypersurface}$$

(Above, $(x_1,\ldots,x_n)^{(\alpha)} = x_1^{\alpha_1}\ldots x_n^{\alpha_n}$). An r-parameterization will give an upper bound for the (absolute value of the) determinant. Say $\varphi(x) \in Z(\mathbb{Q},T)$. Then also $\det \in \frac{1}{S}\mathbb{Z}$ with $S \leq T^{|\alpha|}$. So a sufficiently small upper bound will actually force the determinant to be 0.

9 Niven Achenjang (4/19): Reparametrisation of Definable Sets – notes here

10 (4/26): Proof of Ax-Schanuel – I'm out of town

11 Daniel ... Fill in later

Note 5. 5ish minutes late

11.1 Shimura Varieties and Definability

Definition 11.1. A **connected Shimura datum** is a semisimple group $G/\mathbb{Q}$ and a $G(\mathbb{R})^+$-conjugacy class Ω of homomorphisms $h : R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G^{\text{ad}}_{\mathbb{R}}$ s.t.

(1) The only characters of $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ in $\text{Lie}(G)_{\mathbb{C}}$ are $z/\overline{z}, 1, \overline{z}/z$

(2) $\text{ad}\, h(i)$ is a Cartan invariant on $G^{\text{ad}}_{\mathbb{R}}$

(3) G^{ad} has no factor H where h is trivial

Fact.

- $K := \text{Stab}_{G(\mathbb{R})^+}(h) \subset G(\mathbb{R})^+$ is always a maximal compact subgroup
- $\Omega \cong G(\mathbb{R})^+/K$ is a hermitian symmetric domain
- Ω depends only on $G^{\text{ad}}_{\mathbb{R}}$

Think of Ω as a generalization of the upper half plane.

Lemma 11.2. *There exists a canonical parabolic $P \subset G_{\mathbb{C}}$ w/*

$$P(\mathbb{C}) \cap G(\mathbb{R})^+ = K$$

such that the induced map

$$\Omega \hookrightarrow (G/P)(\mathbb{C}) =: \widehat{\Omega}$$

(note $\widehat{\Omega} = G(\mathbb{C})/P(\mathbb{C})$ since $\mathbb{C}$ algebraically closed) is a semi-algebraic holomorphic open embedding. We call $\widehat{\Omega}$ a ***compact dual****.*

In particular, Ω is a semi-algebraic subset of $\widehat{\Omega}$.

Let Γ be an arithmetic subgroup of $G(\mathbb{R})^+$ w/ torsion-free image in $G^{\mathrm{ad}}(\mathbb{R})$.

Theorem 11.3 (Baily-Borel)**.** *$S := \Gamma\backslash\Omega$ is naturally a quasi-projective variety over $\mathbb{C}$. We call this the* ***Shimura variety*** *attached to our Shimura datum.*

Notation 11.4. We let $q : \Omega \to S$ denote the natural quotient map, and let $D \subset \Omega \times S$ be its graph.

Note that q is not definable in an o-minimal structure, e.g. since it has infinite fibers.

Proposition 11.5 (Borel)**.** *There is an explicit semialgebraic $\Sigma \subset \Omega$ such that some union of finitely many $G(\mathbb{Q})$-translates of Σ forms a fundamental domain $\mathcal{F}$ for $\Gamma \curvearrowright \Omega$. This Σ is called a* ***Siegel set****.*

(All that matters from above is that there exists a semi-algebraic fundamental domain, if I'm following audience discussion)

Warning 11.6. Above, 'fundamental domain' really means something like $q : \Sigma \to S$ is quasi-finite and its closure isn't too bad.

Proposition 11.7 (Klingler-Ullmo-Yafaev)**.** *The restriction $q : \Sigma \to S$ is definable in $\mathbb{R}_{an,exp}$.*

Example. Say $G = \mathrm{SL}_2$, so $G^{\mathrm{ad}} = \mathrm{PGL}_2$. Take

$$h(a+bi) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \implies K = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\} = \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i),$$

where $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane $\mathcal{H}$ in the usual way. Hence, $\Omega = \mathcal{H}$. One can show that P is the Borel (upper triangular matrices?) so $\widehat{\Omega} = \mathbb{P}^1_{\mathbb{C}}$ parameterizes lines in $\mathbb{C}^2$. Finally, the map $\Omega \hookrightarrow \widehat{\Omega}$ is the natural inclusion. Further note that S here parameterizes elliptic curves with Γ-level structure, and so it a quasi-projective variety. To form a fundamental domain in this case, one can start with

$$\Sigma = \left\{ z \in \mathbb{C} : |\mathrm{Re}\, z| \le \frac{1}{2} \text{ and } \mathrm{Im}\, z > t \right\} \text{ for small enough } t.$$

Finally, $q : \Sigma \to S$ here is taking the j-invariant $j(z) = j(e^{2\pi i z})$. Note that, restricted to Σ, $e^{2\pi i z}$ is definable; the real parts of the parameters are restricted, so you get restrict $\sin, \cos$ functions (the an in $\mathbb{R}_{\mathrm{an,exp}}$) and the imaginary parts just give real exponentials (the exp in $\mathbb{R}_{\mathrm{an,exp}}$). The $j(-)$ part of this is a convergent power series, and that suffices for definability in $\mathbb{R}_{\mathrm{an,exp}}$? △

11.2 Weakly special subvarieties and Ax-Schanuel

Definition 11.8. Say we have $(G', \Omega') \hookrightarrow (G, \Omega)$ and injection of Shimura data ($G' \hookrightarrow G$ inducing map $\Omega' \to \Omega$). Say

$$((G')^{\mathrm{ad}}, \Omega') = (H_1, \Omega_1) \times (H_2, \Omega_2).$$

Let $x_2 \in \Omega_2$. The image of $\Omega_1 \times \{x_2\}$ in $S = \Gamma\backslash\Omega$ is called a **weakly special subvariety**.

Proposition 11.9 (Deligne)**.** *Let Y be a smooth, irreducible algebraic variety over $\mathbb{C}$, and say we have an algebraic map $Y \to S$. Assume $\Gamma \curvearrowright \Omega$ is a free action (so $\pi_1(S) = \Gamma$). If the image of $\pi_1(Y)$ in $\Gamma = \pi_1(S)$ is not Zariski dense in G, then the image of Y in S lies in a proper weakly special subvariety.*

Definition 11.10. An **irreducible algebraic subvariety** of $\Omega \times S$ is an irreducible component of

$$\widehat{X} \cap (\Omega \times S),$$

where $\widehat{X} \subset \widehat{\Omega} \times S$ is an algebraic subvariety.

Theorem 11.11 (Mok-Pila-Tsimerman)**.** *Let $X \subset \Omega \times S$ be an algebraic subvariety. Let U be an irreducible component of $X \cap D$ (recall $D \subset \Omega \times S$ is the graph of $q : \Omega \to S$). If $\dim X < \dim U + \dim S$, then*

$$\mathrm{pr}_2(U) \text{ lies in a proper weakly special subvariety.}$$

Note above that (these all say that the dimension of the intersection is bigger than expected)

$$\dim X < \dim U + \dim S \iff \operatorname{codim} U < \operatorname{codim} X + \operatorname{codim} D \iff \dim U > \dim X - \dim D,$$

and that $\dim D = \dim S$.

Proof. May assume that X is irreducible (take component containing U). By replacing Γ with a finite-index subgroup, we may assume that it is torsion-free and that it acts on Ω freely. Now, we induct on the **type**

$$(\Omega, X, U) := (\dim \Omega, \dim X - \dim U, \dim U)$$

of a triple. We will inductive by increasing the first two (i.e. $\dim \Omega, \dim X - \dim U$), but decreasing $\dim U$.

- First say $\dim \Omega = 0$.

  It's always been points.

- Say $\dim U = 0$.

  Points are always weakly special.

- $\dim U = \dim \Omega$ is as big as possible.

  Both U and D are irreducible, and they have the same dimension, so $D = U \subset X$. Hence, the fibers in X over S each contain a Γ-orbit. From the Shimura datum setup, Γ is Zariski dense in G. Combining this with the fact that X is algebraic, one can show that the fibers over $\mathrm{pr}_2 : X \to S$ are everything $\Omega \times \{s\}$. Since $X \twoheadrightarrow S$ (as $D \twoheadrightarrow S$), we conclude that $X = \Omega \times S$.

- Time to induct.

  Take the definable set

  $$I := \left\{ g \in G(\mathbb{R})^+ \mid \dim(gX \cap D \cap (\mathcal{F} \times S)) = \dim U \right\}.$$

  Note that for $\gamma \in \Gamma$, we have

  $$\gamma X \cap D \cap (\mathcal{F} \times S) = \gamma \left(X \cap D \cap \gamma^{-1}(\mathcal{F} \times S) \right) \supset \gamma(U \cap \gamma^{-1}(\mathcal{F} \times S)),$$

  so the dimension equality holds as soon as $U \cap \gamma^{-1}\mathcal{F} \times S \neq \emptyset$. We would like to apply Pila-Wilie counting, but first we'll need a couple facts.

Lemma 11.12. *For all $g \in I$, the volume of*

$$gX \cap D \cap (\mathcal{F} \times S)$$

is uniformly bounded.

Proof Sketch. Klingler-Ullmo-Yafaev showed $\Sigma \overset{\text{open}}{\subset} \prod_{i=1}^{d} J_i$, where the J_i have volume $(1,1)$-forms ω_i s.t. $\int_{J_i} \omega_i < \infty$ and $\sum_{i=1}^{d} \omega_i$ dominates the Kähler form on Ω (after restricting both to Σ?), so it suffices to show $gX \cap D \cap (\Sigma \times S)$ projects to

$$J_I := \prod_{i \in I} J_i \text{ for any } I \subset \{1, \dots, d\} \text{ with } \#I = \dim U$$

with uniformly bounded finite fibers. This follows from definability. ■

Fact. A result of Hwang-To + the $\dim U > 0$ case $\implies$ $\mathrm{vol}(U \cap B(R))$ ($B(R)$ a hyperbolic ball of radius R around some fixed point in U) grows exponentially in R.

The above shows $U \cap \gamma^{-1}\mathcal{F} \times S$ has uniformly bounded volume, so you need exponentially many to cover $U \cap B(R)$. Hence, $N(I,T)$ will grow polynomially[11] in T, so Pila Wilkie (theorem 8.3) will tell you that I contains an irreducible semi-algebraic real curve C w/ one non-identity $\gamma \in \Gamma$.

Remark 11.13 (Me trying to follow audience discussion on above). Since $\mathcal{F}$ is a fundamental domain, we can cover $U \cap B(R)$ using sets of the form $U \cap \gamma^{-1}\mathcal{F} \times S$ with $\gamma \in \Gamma$. Each γ with $U \cap \gamma^{-1}(\mathcal{F} \times S) \neq \emptyset$ gives an integral point of I. By above, need exponentially (in R) many such γ to cover $U \cap B(R)$. This should translate to into a polynomial lower bound for $N(I,T)$ somehow... ○

Assume $cX \neq X$ for some $c \in C$. Two cases

- First assume $U \subset cX$.

  Then, $U \cap X \cap cX$. Note X, cX are irreducible and distinct, so $\dim(X \cap cX) < \dim X$. Apply inductive hypothesis to $(\Omega, X \cap cX, U)$.

- Now assume $U \not\subset cX$

  Take $X^{\#}$ to be the Zariski closure of CX. Since $U \not\subset cX$, the irreducible component $U^{\#}$ of $X^{\#} \cap D$ containing U is bigger than U (since it contains $CU \supset cU \neq U$), and so has dimension $\dim U^{\#} = \dim U + 1$ (also $\dim X^{\#} = \dim X + 1$). Apply inductive hypothesis to $(\Omega, X^{\#}, U^{\#})$.

Still need to handle the case where $cX = X$ for all $c \in C$. This is where one uses the existence of γ. Note that γ stabilizes X and generates an infinite subgroup $\langle \gamma \rangle$ of Γ (since Γ torsion-free).

Claim 11.14. *This implies the theorem.*

Proof Sketch. Let $M :=$ be the Hilbert scheme of $\widehat{X}$. Define

$$A := \left\{(x,s,m) \in \Omega \times S \times M : (x,s) \in X_m \cap D \text{ and } \dim_{(x,s)}(X_m \cap D) \geq \dim U\right\}.$$

[11]When relating things to heights, you take log's at some point, and this is why we have polynomially here instead of exponentially.

The definability of q and o-minimal Chow will tell you that

$$T := \bigcup_{m\text{'s in } A} \mathrm{pr}_2(X_m \cap D) \subset S$$

is algebraic. If this were contained in a proper weakly special subvariety, we'd be done, so suppose it is not. Hence, its monodromy must be Zariski dense in G (by theorem of Deligne). One can prove that for very general[12] $m \in M_0 := \mathrm{pr}_3(A)$, X_m cannot be preserved by infinite subgroup of Γ. ■

■

12 Salim Tayou (5/9): Mixed Ax-Schanuel & Mordell-Lang

Goal. Explain where in the proof of Mordell-Lang there is a use of the mixed Ax-Schanuel theorem.

Recall 12.1. Fix some $g \geq 2$ and $\ell \geq 3$. Everything below over $\overline{\mathbb{Q}}$

- Let $\mathbb{M}_g$ be the moduli space of genus g curves
- Let $\mathbb{A}_g$ be the moduli of PPAVs of dimension g
- $\mathcal{A}_g \to \mathbb{A}_g$ is the universal family
- $\mathcal{C}_g \to \mathbb{M}_g$ is the universal family
- $\mathrm{Jac}(\mathcal{C}_g/\mathbb{M}_g) \to \mathbb{M}_g$ is the universal Jacobian
- Have quasi-finite Torelli morphism $\tau : \mathbb{M}_g \to \mathbb{A}_g$ defined so as to fit into the diagram

$$\begin{array}{ccc} \mathrm{Jac}(\mathcal{C}_g/\mathbb{M}_g) & \longrightarrow & \mathcal{A}_g \\ \big\downarrow & \ulcorner & \big\downarrow \\ \mathbb{M}_g & \xrightarrow{\ \tau\ } & \mathbb{A}_g. \end{array}$$

- Choose suitable height functions $h : \mathcal{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ and $h_{\mathbb{M}_g} : \mathbb{M}_g(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$.

To prove uniform Mordell one wanted to

- bound points with large height $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$, $P \in \mathcal{C}_{g,s}(\overline{\mathbb{Q}})$ where $h(P) \gg h_{\mathbb{M}_g}(s)$
- bound points with small height, those like above but where $h(P) \ll h_{\mathbb{M}_g}(s)$

To do this, one need some non-degeneracy coming from $\mathcal{C}_g$ in order to obtain a height inequality.

We'll focus on this non-degeneracy thing.

Recall 12.2. Let $\pi : \mathcal{A} \to S$ be an abelian scheme (e.g. $S = \mathbb{A}_g$). Let $\Delta \subset S$ be a contractible analytic open subset. For any $s \in \Delta$, one has

$$\mathrm{H}_1(\mathcal{A}_s, \mathbb{R}) \hookrightarrow \mathrm{H}_1(\mathcal{A}_s, \mathbb{C}) = \mathrm{H}_{-1,0} \oplus \mathrm{H}_{0,-1} \to \mathrm{H}_{-1,0},$$

[12] complement of countably many algebraic subvarieties

with the above composition an $\mathbb{R}$-linear isomorphism onto the holomorphic tangent space of $\mathcal{A}_s$. Quotienting by the lattice $\mathrm{H}_1(\mathcal{A}_s, \mathbb{Z})$, one obtains (real analytic) isomorphisms

$$\mathbb{R}^{2g}/\mathbb{Z}^{2g} = \mathbb{T}^{2g} = \mathrm{H}_1(\mathcal{A}_s, \mathbb{R})/\,\mathrm{H}_1(\mathcal{A}_s, \mathbb{Z}) \longrightarrow \mathrm{H}_{-1,0}\,/\,\mathrm{H}_1(\mathcal{A}_s, \mathbb{Z}) = \mathcal{A}_s.$$

This same construction can be done in families (over a contractible base). One gets

$$\mathcal{A}_\Delta = \mathcal{V}/\,\mathrm{H}_1(\mathcal{A}_\Delta, \mathbb{Z}) \xrightarrow{\sim} \mathbb{T}^{2g} \times \Delta$$

and the induced projection

$$b_\Delta : \mathcal{A}_\Delta \longrightarrow \mathbb{T}^{2g}$$

is the **Betti map**. Some properties:

- For $\Lambda \in \mathbb{T}^{2g}$, $b^{-1}(\Lambda)$ is complex analytic
- For any $X \subset \mathcal{A}$ algebraic irreducible and any $x \in X^{sm}(\mathbb{C})$, the **Betti rank at** x

  $$\mathrm{rank}_{\mathbb{R}}\left(\mathrm{d}b_{X,x}\right) \leq \max(2g, 2\dim X).$$

  The **generic rank** is

  $$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_X) := \max_{x \in X^{sm}(\mathbb{C})} \mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{X,x}).$$

  We say X is **non-degenerate** if its generic rank is $2\dim X$.

Theorem 12.3. *Let*

$$\mathcal{D}_M : \underbrace{\mathcal{A}_g \times_S \ldots \times_S \mathcal{A}_g}_{(m+1)\ factors} \longrightarrow \underbrace{\mathcal{A}_g \times_S \ldots \times_S \mathcal{A}_g}_{m\ factors} =: \mathcal{A}_g{}^{(m)}$$

be the ***Faltings-Zhang map*** *sending* $(P_0, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0)$. *Then,*

(1) *For any* $m \geq \dim(\mathcal{C}_g) = 1 + \dim(\mathcal{M}_g)$, $\mathcal{D}_m\left(\mathcal{C}_g{}^{(m+1)}\right)$ *is non-degenerate inside* $\mathcal{A}_g{}^{(m)}$.

(2) *For any* $m \geq \dim S$, *the generic rank of* $\mathrm{d}b_\Delta^{(m)}|_{\mathcal{C}_g^{(m)}}$ *is* $2\dim(\mathcal{C}_g^{(m)})$ *(assuming there is a section* $S \to \mathcal{C}_S = \mathcal{C}_g \times_{\mathbb{M}_g} S$*)*

I think all the $\mathcal{C}_g$'s in (**2**) should be $\mathcal{C}_S$'s?

Theorem 12.4. *Let* $\pi : \mathcal{A} \to S$ *be an abelian scheme. Choose* $X \subset \mathcal{A}$ *dominating* S *and assume* $\iota : S \to \mathbb{A}_g$ *is quasi-finite. Assume*

(a) $\dim X > \dim S$

(b) *There exists* $s \in S$ *so that* X_s *generates* $\mathcal{A}_s$

(c) *For any non-isotrivial abelian scheme* $\mathcal{A}' \subset \mathcal{A} \to S$, $\mathcal{A}' + X \not\subset X$

Then,

(1) $\mathcal{D}_m^{\mathcal{A}}(X^{(m+1)})$ *is non-degenerate for all* $m \geq \dim X$

(2) $X^{(m)}$ *is non-degenerate for any* $m \geq \dim S$

Remark 12.5. Theorem 12.4 implies Theorem 12.3 by applying it the $S = \mathbb{M}_g$ and $X = \mathcal{C}_g$ sitting in its Jacobian (reason for requiring a section in (**2**) of Theorem 12.3). ○

Let's try doing some further reduction. Let $\mathcal{A}_X$ be the translate of an abelian variety by a trosion section which contains X, and which is minimal for this property.

Theorem 12.6. *For any ℓ,* $\operatorname{rank}(\mathrm{d}b|_X) < 2\ell$ *if and only if there exists an abelian scheme $B \subset \mathcal{A}_X \to S$ of relative dimension g_B s.t. for the quotient*

$$\begin{array}{ccccc} \mathcal{A}_X & \xrightarrow{p_B} & \mathcal{A}_X/B & \xrightarrow{\iota_B} & \mathcal{A}_{g-g_B} \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & S & \longrightarrow & \mathbb{A}_{g-g_B}, \end{array}$$

one has $\dim(\iota_B \circ p_B(X)) < \ell - g_B$.

Remark 12.7. Suppose for a minute that $\mathcal{A}_X = \mathcal{A}$ (e.g. if condition (b) of Theorem 12.4 holds). If X has $\dim X < 2\ell$, then the condition of Theorem 12.6 is satisfies with $B = 0$. ○

We'll late show that Theorem 12.6 implies Theorem 12.4.

Theorem 12.8. *A closed irreducible $Z \subset \mathcal{A}$ is called a* ***generically special subvariety of Sg type*** *(Sg = subgroup?) if there exists some covering $S' \to S$ so that there is an abelian subscheme $B' \subset \mathcal{A}' := \mathcal{A} \times_S S'$ along with a torsion section $\sigma' : S' \to \mathcal{A}'$ and a constant section $\sigma'_0 : S' \to \mathcal{A}'$ such that*

$$Z = \rho\left(B' + \sigma' + \sigma'_0\right) \ \text{ where } \ \rho : \mathcal{A}' \to \mathcal{A}.$$

Above, if we have $C' \times S' \subset \mathcal{A}'$ the largest trivial abelian subscheme and $c' \in C'(\mathbb{C})$, then a section of the form $S' \xrightarrow{\sim} c' \times S' \subset \mathcal{A}'$ is called a ***constant section****.*

Notation 12.9. For any $Y \subset \mathcal{A}$, we let $\langle Y \rangle_{Sg}$ denote the smallest generically special subvariety of Sg type of $\mathcal{A}_{\pi_S(Y)} = \pi_S^{-1}(\pi_S(Y))$.

Definition 12.10. Say $t \in \mathbb{Z}$, and let $X \subset \mathcal{A}_g$ be a closed and irreducible. The t**th degenracy locus** is

$$X^{deg}(t) := \bigcup_Y Y$$

with Y ranging over positive-dimensional irreducible closed subvarieties of X with $\dim \langle Y \rangle_{Sg} - \dim \pi_S(Y) < \dim Y + t$. For $t = 0$, we write $X^{deg} := X^{deg}(0)$.

Fact. X is degenerate if $\overline{X^{deg}} = X$.

Theorem 12.11. *For $\ell \leq \dim X$, generic rank $< 2\ell \iff X^{deg}(\ell - \dim X)$ is Zariski dense in X.*

Proof. Let $\mathbb{H}_g$ be the Siegel upper half space, and let $\mathcal{X}_{2g} = \mathbb{C}^g \times \mathbb{H}_g$. Consider the diagram

$$\begin{array}{ccccc} \mathbb{R}^{2g} & \xleftarrow[\widetilde{b}]{} & \mathcal{X}_{2g} & \longrightarrow & \mathbb{H}_g \\ & & \downarrow u & & \downarrow u_G \\ & & \mathcal{A}_g & \longrightarrow & \mathbb{A}_g \end{array}$$

Let $\widetilde{X} \subset u^{-1}(X)$ be an irreducible component, and let $d = \dim X$. For $\ell \in \{1, \ldots, d\}$, define

$$\widetilde{X}_{<2\ell} = \left\{ \widetilde{x} \in \widetilde{X} : \operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{x}} < 2\ell \text{ and } u(\widetilde{x}) \in X^{sm}(\mathbb{C}) \right\}.$$

Then,

$$X^{deg}(\ell - d) \cap X^{sm}(\mathbb{C}) \subset u(\widetilde{X}_{<2\ell}).$$

Suppose not, so there's some $\widetilde{x} \in \widetilde{X}$ with $u(\widetilde{x}) \in X^{deg}(\ell - d) \cap X^{sm}(\mathbb{C})$ and $\operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{x}}) \geq 2\ell$. Then, there exists $Y \subset X$ containing $x = u(\widetilde{x})$ s.t.

$$\dim \langle Y \rangle_{Sg} - \dim \pi(Y) < \dim Y + \ell - d.$$

Choose $\widetilde{Y} \subset u^{-1}(Y)$ with $\widetilde{x} \in \widetilde{Y}^{sm}(\mathbb{C})$. Then,

I'm not sure I copied down any of the below correctly... but I'm also not sure what's going on, so I don't know how to fix it...

$$\operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{X},\widetilde{x}}) \leq \operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{Y},\widetilde{x}}) + 2(\dim X - \dim Y),$$

so

$$2\ell - 2(d - \dim Y) \leq \operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{X},\widetilde{x}}) - 2(\dim X - \dim Y) < \operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{Y},\widetilde{x}}.$$

Let $\left\langle \widetilde{Y} \right\rangle_{Sg}$ be the irreducible component of $u^{-1}(\langle Y \rangle_{Sg})$ containing x. Then,

$$\operatorname{rank}(\widetilde{b}_{\langle \widetilde{Y} \rangle_{Sg}})_{\widetilde{x}} = 2(\dim \langle Y \rangle_{Sg} - \dim \pi(Y)) < 2\ell - 2(d - \dim Y) \leq \operatorname{rank}(\mathrm{d}\widetilde{b}_{\widetilde{Y}})_{\widetilde{x}},$$

a contradiction. Thus, $X^{deg}(\ell - d) \cap X^{sm}(\mathbb{C}) \subset u(\widetilde{X}_{<2\ell})$. Note that this means that if $X^{deg}(\ell - \dim X)$ is Zariski dense, then so it $u(\widetilde{X}_{<2\ell})$, so the generic rank is $< 2\ell$.

Next, we want to show that if the generic rank is $< 2\ell$, then there exists some nonempty open $U \subset \widetilde{X}^{sm}(\mathbb{C})$ s.t. $u(U) \subset X^{deg}(\ell - d)$ (so $X^{deg}(\ell - d)$ contains an analytic open set and hence is Zariski dense). This will finish the proof, and is where we need mixed Ax-Schanuel.

Note that $\mathcal{X}_{2g} = \mathbb{C}^g \times \mathbb{H}_g$ has an embedding $\mathcal{X}_{2g} \hookrightarrow \widehat{\mathcal{X}}_{2g}$ into some algebraic variety (can embed $\mathbb{H}_g \hookrightarrow$ a certain flag variety, and then $\widehat{\mathcal{X}}_{2g}$ will be the total space of a suitable vector bundle over that flag variety) and this embedding is semi-algebraic. To set up Ax-Schanuel, we make some definitions

Definition 12.12.

(1) $\widehat{Y} \subset \mathcal{X}_{2g}$ is **irreducible algebraic** $\iff$ $\widehat{Y}$ is an irreducible component of $(\mathcal{X}_{2g} \cap W)$ for some algebraic $W \subset \widehat{\mathcal{X}}_{2g}$.

(2) $Y \subset \mathcal{A}_g$ is **bi-algebraic** if it is algebraic and any (one) irreducible component of $u(Y)$ is algebraic.

(3) For $Z \subset \mathcal{A}_g$, we call Z^{biZar} the smallest bi-algebraic subvariety that contains Z

Proposition 12.13 (Gao)**.** *Say $B \subset \mathbb{A}_g$ and $\mathcal{A}_{g/B} = \pi^{-1}(B)$. Then, the generically special subvareities of Sg type in $\mathcal{A}_{g/B}$ are exactly the irreducible components of $\mathcal{A}_{g/B} \cap F$ with F an irreducible bi-algebraic subvariety s.t. $B \subset \pi(F)$.*

Consequently, $\langle Y \rangle_{Sg}$ is an irreducible component of $\mathcal{A}_{g/\pi(Y)} \cap Y^{biZar}$ and hence $\dim \langle Y \rangle_{Sg} - \dim \pi(Y) = \dim Y^{biZar} - \dim \pi(Y)^{biZar}$.

Theorem 12.14 (Mixed Ax-Schanuel). *Let $\widetilde{Z}$ be a complex analytic irreducible subvariety of the graph of $u \subset \mathcal{X}_{2g} \times \mathcal{A}_g$. Let $Z = \mathrm{pr}_{\mathcal{A}_g}(\widetilde{Z})$. Then,*

$$\dim \widetilde{Z}^{Zar} - \dim \widetilde{Z} \geq \dim Z^{biZar}.$$

Remark 12.15. If $\widetilde{Z} \subset \mathcal{X}_{2g}$ is irreducbie analytic, then

$$\dim \widetilde{Z}^{Zar} - \dim \left(u(\widetilde{Z})^{Zar} \right) \geq \dim \widetilde{Z} + \dim u(\widetilde{Z})^{biZar}.$$

This follows mixed Ax-Schanuel by considering $\widetilde{Z} \times u(\widetilde{Z}) \hookrightarrow \mathcal{X}_{2g} \times \mathcal{A}_g$. ○

Back to the problem at hand... say $\widetilde{x} \in \widetilde{X}_{<2\ell}$ and set $r = b(\widetilde{x}) \in \mathbb{R}^{2g}$. Then,

$$\dim_{\mathbb{R}} \left(b^{-1}(r) \cap \widetilde{X} \right)_{\widetilde{x}} > 2(d - \ell)$$

for some analytic open $U \ni \widetilde{x}$. Recall $\mathcal{X}_{2g} = \mathbb{R}^{2g} \times \mathbb{H}_g$ and note $b^{-1}(\{r\}) = \{r\} \times \mathbb{H}_g$, so $b^{-1}(\{r\} \cap \widetilde{X}$ has an analytic irreducible $\{r\} \times \widetilde{W} \subset \widetilde{X}$ (containing $\widetilde{x}$). Let

$$Y = u \left(\{r\} \times \widetilde{W} \right)^{Zar} \subset \mathcal{A}_g.$$

By Ax-Schanuel,

$$\dim \left(\{r\} \times \widetilde{W} \right)^{Zar} + \dim Y \geq \dim(\{r\} \times \widetilde{W}) + \dim Y^{biZar}.$$

Note $\widetilde{W}^{Zar} \subset \widetilde{W}^{biZar} \subset u_G(\widetilde{W})^{biZar} \subset \pi(Y)^{biZar}$.

Note 6. Laptop dying. Will shut down in a matter of minutes.

Claim 12.16. $(\{r\} \times \widetilde{W})^{Zar} = \{r\} \times \widetilde{W}^{Zar}$

This is because...

$$\dim(\{r\} \times \widetilde{W})^{Zar} \leq \dim \widetilde{W}^{Zar} \leq \dim \langle Y \rangle_{Sg}$$

■

Theorem 12.17. *$X^{deg}(t)$ is Zariski closed in X for all t.*

(We won't prove this)

We claim that Theorems 12.11 + 12.17 together imply Theorem 12.6.

13 List of Marginal Comments

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