Math 286Y/Z (Algebraic Curves I/II) Notes

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Fall 2021/Spring 2022

These are my course notes for “Algebraic Curves” at Harvard. Each lecture will get its own ‘chapter’. These notes are live-texed and so likely contain many mistakes. Furthermore, they reflect my understanding (or lack thereof) of the material as the lecture was happening, so they are far from mathematically perfect.[1] Despite this, I hope they are not flawed enough to distract from the underlying mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Joe Harris. Starting 9/22, MIT’s STAGE seminar was running at the same time as this class on Wednesdays, so I don’t have notes for those days (during the Fall semester). Finally, a few of the early lectures were recorded, possibly including some of the ones I missed.

Contents

[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

I don't know what all was in this last lecture, but you can check out this paper for the 'admissible covers' compactifying $\mathcal{H}_{d,g}$ and the details of finishing the computation of $\kappa(\overline{M}_g)$

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1 Lecture 1 (9/1/2021)

1.1 Administrative/Class stuff

Note 1. It seems I have trouble connecting to the internet in this classroom...

Lectures MW 10:30 – 11:45

On Fridays, there will be problem solving sessions Friday at the same time and place (probably, still need to request it).

Remark 1.1. This is a subject which can be dealt with very abstractly, but also very concretely. So we'll try to spend a lot of time on algorithmic/concrete objects, see e.g. first homework. ○

This Friday we'll go over how to do the types of things on the first homework.

There's a course assistant Kai Xu who will be grading homeworks and holding section/office hours.

The homework will be weekly and determines your whole grade.

Prereqs hard to say because two perspectives. More on this in a bit...

1.2 Content

The basic objects can be defined in 2 ways

- compact Riemann surfaces
- smooth, projective algebraic curves over $\mathbb{C}$

(always assume connected)

Warning 1.2. 'projective curve' means can be embedding in $\mathbb{P}^n$, it does *not* mean a curve with a given embedding into $\mathbb{P}^n$ •

These two things are the same, e.g.

- Any compact Riemann surface X can be embedded in projective space.
- Given such an embedding $X \hookrightarrow \mathbb{P}^r$, its image is algebraic.
- If $f : X \to Y$ is a holomorphic map of Riemann surfaces, then it is in fact also a regular map of the associated algebraic curves.

The first of these is the most difficult (and is also false in higher dimensions).

Remark 1.3. Say X is a compact Riemann surface. If you want to embed this into projective space, it best at least support some global meromorphic functions. This is not trivial to prove. Already in the case of surfaces, there are compact complex 2-manifolds with no non-constant meromorphic functions. ○

The second and third statements are what's often called **Chow's Theorem**, projective manifolds are algebraic.

Despite this equivalence, the terminology/notation used in these two perspectives can differ.

Example. On a complex manifold $(X, \mathscr{O}_X)$, its structure sheaf is a sheaf of *holomorphic* functions. When thinking of it as an algebraic curve, we instead endow X with its sheaf $\mathscr{O}_{X,alg}$ of *algebraic* functions. These are two different sheaves, but (e.g. by Serre's GAGA) they have the same cohomology: $\mathrm{H}^i(X, \mathscr{O}_X) = \mathrm{H}^i(X, \mathscr{O}_{X,alg})$. △

Remark 1.4. We will see some constructions in the analytic setting that we can't perform in the algebraic setting, so this duality gives us more options. ○

We'll probably end up using holomorphic/regular and meromorphic/rational roughly interchangeably.

Example. Here's the same theorem from both perspectives.

Theorem 1.5. *Let X be a compact Riemann surface/smooth projective algebraic curve. Then, $f : X \to \mathbb{C}$ a global holomorphic/regular function must be constant.*

Proof. Analytically, this follows from the maximum principle, the modulus of a non-constant holomorphic function has no maximum. However, X is compact, so $|f|$ better have a maximum.

Algebraically, show that the image of a projective variety under a regular map is again projective. Now observe that the only projective subvariety of an affine line is a point. ■

Question: Is this true? Does he secretly mean proper?

Same theorem, but quite different proofs. △

Remark 1.6 (back to prereqs). There are two points of view, but should only be necessary to be comfortable with only one of them (+ being willing to understand some of the other perspective).

Keep in mind Joe will sometimes give proofs using only one of these perspectives, and it won't always be your preferred one. ○

Question 1.7. *How much of the theory of smooth varieties can be extended to singular ones?*

Say $C_0 \subset \mathbb{P}^r$ is a possibly singular algebraic curve. Then, $\exists!$ smooth projective curve C along with a map $f : C \to C_0$ which is generically one-to-one (i.e. birational). Inevitably singular curves will crop up in our work, but we'll be able to recover smooth curves from them. Where does this resolution $C \to C_0$ of singularities come from?

In the algebraic setting, just take the normalization. In the analytic setting, for all $p \in C_0$, there exists a neighborhood U of p s.t. $U \setminus \{p\} \simeq \bigsqcup$(punctured disk) (can find punctured neighborhood of p which is a disjoint union of punctured disks[2]); construct C by completing the punctured disks to discs.

1.3 Linear Systems

Question 1.8. *What and why are linear systems?*

The starting point is the transition from curves in projective space to abstract curves (ca. early 1900s). After making this transition, it is natural to ask: given an abstract curve C, how can we describe all maps $f : C \to \mathbb{P}^r$ to projective space? These maps will be described by linear systems.

There's a problem here. To give a map $C \to \mathbb{P}^r$, we'd really like functions on C. There are no (non-constant) holomorphic functions on C, so those aren't helpful. Instead, we'll introduction meromorphic/rational functions. If we considered all meromorphic functions at once, we'd get a space of functions which is too big to be super useful; we'd get $\mathcal{M}(C) = K(C)$ the field of all meromorphic/rational functions, so we'll look only at functions with bounded singularities (and get f.dim vector spaces).

[2]something something Weierstrass preparation something something. See example on homework/Friday problem session.

Start with a Riemann surface C, and fix your favorite points $\{p_i\}_{i=1}^k$. We'll look at functions with poles of bounded order at these points, but which must be holomorphic elsewhere. So fix integers $m_1, \dots, m_k$, and consider the space

$$\{f \in \mathcal{M}(C) : f \text{ holomorphic on } C \setminus \{p_1, \dots, p_k\} \text{ and } \operatorname{ord}_{p_i}(f) \geq -m_i\}.$$

These are the sorts of spaces of functions we'll be working with.

Remark 1.9. If $m_i < 0$, then we're not actually allowing a pole at p_i. We're requiring a zero (of order at least $-m_i$) there. ○

Definition 1.10. A **divisor** D on C is a formal finite linear combination of points, $D = \sum m_\alpha p_\alpha$. We will call D **effective** if $m_\alpha \geq 0$ for all α. ◇

Remark 1.11. An effective divisor on C is the same thing as a 0-dimensional subscheme of C. ○

We won't actually need much scheme theory in this class it seems.

Given a divisor D, we associate the space

$$\mathscr{L}(D) := \{f \in \mathcal{M}(C) : \operatorname{ord}_{p_\alpha}(f) \geq -m_\alpha\} = \{f \in \mathcal{M}(C) : (f) + D \text{ is effective}\},$$

where for $f \in \mathcal{M}(C)$ we define its divisor to be (zeros/poles of f are isolated so only f.many)

$$(f) = \sum_{p \in C} \operatorname{ord}_p(f) \cdot p.$$

We call such divisors **principal divisors**.

Definition 1.12. Two divisors D, D' are **linearly equivalent**, denoted $D \sim D'$, if $D - D' = (f)$ for some $f \in \mathcal{M}(C)$. ◇

Remark 1.13. If D, D' are linearly equivalent, then we get an iso

$$\cdot f : \mathscr{L}(D) \xrightarrow{\sim} \mathscr{L}(D')$$

via multiplication by f. ○

Definition 1.14. Given a divisor $D = \sum m_\alpha p_\alpha$, its **degree** is

$$\deg D := \sum m_\alpha.$$ ◇

Claim 1.15. *If $f \in \mathcal{M}(C)$, then $\deg(f) = 0$, i.e. global meromorphic functions have the same number of zeros and poles.*

Proof. Analytically, use the residue theorem applied to the meromorphic differential $\frac{\mathrm{d}f}{f}$. The residue of this differential at a point is the order of f at the point, and residue theorem says the sum of residues is 0.

Algebraically, think of f as giving a map $f : C \to \mathbb{P}^1$, and then $(f) = f^*(\infty - 0)$ so $\deg(f) = d \deg(\infty - 0) = 0$ where d is the degree of f the function, not (f) the divisor.

Alternatively algebraically, choose a map $\pi : C \to \mathbb{P}^1$ and consider the induced norm map $K(C) \xrightarrow{\mathrm{Nm}} K(\mathbb{P}^1)$. Use this to reduce to case of $\mathbb{P}^1$ (e.g. $\deg(f) = \deg(\mathrm{Nm}\, f)$, up to some constant).[3] ■

Question 1.16. *What does this have to do with maps $f : C \to \mathbb{P}^r$?*

$\mathbb{P}^r$ is kinda tricky because it doesn't actually have coordinates. To get around this, choose a hyperplane $H \subset \mathbb{P}^r$, so $H \cap f(C)$ will be finitely many points.

Assumption. Always assume maps to $\mathbb{P}^r$ are **non-degenerate**, i.e. their image is not contained in a hyperplane.

After making this choice, can take the divisor $D = f^{-1}(H)$. Now, choose homogeneous coordinates $X_0, \ldots, X_r$ on $\mathbb{P}^r$ so that $H = V(x_0)$. Then, we can think of f as coming from the map

Question: What does X_1/X_0 mean as a function on C? Do we secretly mean $(X_1/X_0) \circ f$ or something?

$$C \setminus f^{-1}(H) \longrightarrow \mathbb{A}^r \text{ given by } \left(\frac{X_1}{X_0}, \ldots, \frac{X_r}{X_0}\right).$$

Note that this is an r-tuple of functions with bounded singularities (supported on $f^{-1}(H)$). We get in this way an r-dim vector space $V = \left\langle \frac{X_1}{X_0}, \ldots, \frac{X_r}{X_0} \right\rangle \subset \mathscr{L}(D)$.

If we chose a different hyperplane $H' = V(L)$ (L some linear function), then $D' := f^{-1}(H') \sim D$ via the rational function L/X_0. This multiplication by this function gives an isomorphism $\mathscr{L}(D) \xrightarrow{\sim} \mathscr{L}(D')$ carrying V onto V'.

"Let me go one step further, and this is where the trouble starts. I'm gonna dump a whole pile of trouble on you guys, and then I'm going to leave." (paraphrase)

Here's the basic correspondence. Start with an abstract curve C.

$$\left\{ \begin{array}{c} \text{nondeg maps} \\ C \to \mathbb{P}^r \text{ of degree } d \end{array} \right\} \Big/ \mathrm{PGL}_{r+1} \longleftrightarrow \left\{ (D, V) \,\middle|\, \begin{array}{c} V \subset \mathscr{L}(D) \\ \text{of dim } r \end{array} \right\} \Big/ \begin{array}{c} \text{linear} \\ \text{equivalence} \end{array}.$$

To make things less hairy, we'll introduce a new object which will incorporate this notion of linear equivalence.

Say D is a divisor on X. We'll introduce a sheaf-y version of $\mathscr{L}(D)$. This is the sheaf $\mathscr{O}_X(D)$ defined on an open set $U \subset X$ via

$$\mathscr{O}_X(D)(U) = \{ f \in \mathcal{M}(U) : \mathrm{ord}_p(f) \geq -\mathrm{ord}_p(D) \ \text{ for all } \ p \in U \}.$$

This bounded-order condition is a local one, so might as well consider the whole sheaf instead of just its global sections $\mathscr{O}_X(D)(X) = \mathscr{L}(D)$.

Fact. As sheaves, $\mathscr{O}_X(D) \simeq \mathscr{O}_X(D') \iff D \sim D'$.

2 Problem Session (9/3)

Note 2. Fixed my internet woes.

Note 3. Not sure if I'll always take notes during these problem sessions.

[3] Something like this, I didn't quite hear

Harris said some stuff about the connection between divisors, invertible sheaves, line bundles, and maps to projective space, but I didn't bother writing it down.

2.1 Problem 1

Dealing with the affine curve $C_0 : y^2 = x^3 + 1$. Note that there is a unique way to extend this to a smooth, projective curve/compact Riemann surface. For example, take the closure in $\mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$), and then normalize the resulting curve. In this case, the closure of $C_0 \hookrightarrow \mathbb{P}^2$ is already smooth.

Let's see a more naive way of compactifying this curve. Consider C_0 as a two-sheeted cover of the x-plane, i.e. consider the map $\pi_0 : C_0 \to \mathbb{A}^1, (x, y) \mapsto x$. This is branched at the three points $-1, -\omega, -\omega^2$ where ω is a cube root of unity (these points only have 1 preimage). Consider some big disc $\Delta_r \subset \mathbb{C}$ of radius r (r big enough so that $-1, -\omega, -\omega^2 \in \Delta_r$). Then, above $\mathbb{C} \setminus \Delta_r$, π_0 becomes a 2-sheeted (unbranched) cover of $\mathbb{C} \setminus \Delta_r \simeq \Delta^*$ (Δ^* = punctured disk). There are only two different 2-sheeted covers of a punctured disk[4]

- $\Delta^* \sqcup \Delta^* \to \Delta^*$
- $\Delta^* \to \Delta^*, z \mapsto z^2$

In the first case, we compactify by adding 2 points. In the second case, we compactify by adding one point.

To figure out which case, look at the monodromy. Take a loop in the base, lift it upstairs, and see if you end up in the same sheet or the other sheet. Note that, for r large, we basically have $y^2 \approx x^3$. As you loop x around once, its argument increases by 2π. So the argument of $x^3 = y^2$ increases by 6π, so the argument of y increases by 3π, i.e. y gets negated. Thus, we end up in the other sheet, so we're in the second case. We only need to add one point to compactify.

After compactifying C_0 to $C = C_0 \cup \{r\}$, we extend π_0 to $\pi : C \to \mathbb{P}^1$ branched at 4 points, the 3 from before + the point at infinity.

Question 2.1. *What does C look like topologically?*

Split the 4 branch points in $\mathbb{CP}^1$ into 2 pairs. Draw arcs A_1, A_2 connecting each point in a pair. The preimage of $\mathbb{P}^1 \setminus (A_1 \cup A_2)$ looks like two disjoint copies (to get from one sheet to another, need a loop with odd total winding number, but that's impossible after removing these arcs).

Note 4. Really hard to take notes on this without being able to draw the pictures he's drawing...

If you take an B transverse to A_1, and look at what it does upstairs, then it starts in one sheet and gets transported to the other sheet. So to go from $\pi^{-1}(\mathbb{P}^1 \setminus (A_1 \cup A_2))$ to all of C, we essentially need to "identify the points on the boundary of the removed arcs." To do this, extend arcs to small opens. Then each $\mathbb{P}^1 \setminus (A_1 \cup A_2)$ upstairs really looks like a sphere with two small balls missing. We identify the boundaries of these missing pieces of each copy, and end up with a torus.

This manual sort of argument will quickly get hairy as our curves become more complicated, so let's see something else. An oriented surface is determined by its genus (equivalently, its Euler characteristic), so we can just compute this. Triangulate $\mathbb{P}^1$ so that each branch point is a vertex in the triangular. Pull

[4]punctured disk homotopy equivalent to a circle

this back along π to a triangulation of C. If The original triangulation has V vertices, E, edges, and F faces, then the one on π will have $2V-4$ vertices (note: 4 branch points), $2E$ edges, and $2F$ facts. Thus,

$$\chi(C) = 2\chi(\mathbb{P}^1) - 4 = 0$$

which is enough to conclude that C is a torus. In general, Riemann-Hurwitz.

Divisor computations Label points $p_1, p_2, p_3 = (-1, 0), (-\omega, 0), (-\omega^2, 0)$ as well as $q_1, q_2 = (0, \pm 1)$ and $s_1, s_2 = (2, \pm 3)$. We want to establish linear equivalences

$$2p_1 \sim 2p_2 \sim 2p_3 \sim 2r$$

and other such things.

Example. What's the divisor of $(x+1)$? It's regular above the x-plane. It has a zero at $p_1 = (-1, 0)$. In fact, it has a double zero there (I missed why). There are no poles above the x-plane, so the only pole is at $r = \pi^{-1}(\infty)$. Since it must have degree zero, we conclude $(x+1) = 2p_1 - 2r$. △

Example. $p_1 + p_2 + p_3 \sim 3r$. Want a rational function with zeros at those three points. Why not take the rational function y? It vanishes at p_1, p_2, p_3 and nowhere else above $\mathbb{C} \subsetneq \mathbb{P}^1$. Hence it must have a triple pole at r, so $(y) = p_1 + p_2 + p_3 - 3r$. △

Describing complete linear system We want to describe the complete linear system $|p_1 + q_1|$. Recall $p_1 = (-1, 0)$ and $q_1 = (0, 1)$. Let's find all meromorphic functions with (at worst simple) poles at just these two points, and which are otherwise holomorphic.

We can describe a function with poles at p_1, q_1, by drawing a line through them, and then taking the reciprocal of its defining equation, e.g. by considering $(y - x - 1)^{-1}$.

Note 5. The guy sitting in front of me keeps sneezing. If I stop taking notes after today, I got covid and died.

Note that $s_1 = (2, 3)$ is also on this line, so

$$\left(\frac{1}{y - x - 1}\right) = 3r - p_1 - q_1 - s_1.$$

Now we want to get ride of that s_1, so let's multiply by a function vanishing at s_1. We don't want the function we choose to introduced new poles (i.e. great if it had poles only at r), so consider the collection

$$\left\{\frac{\alpha x + \beta y + \gamma}{y - x - 1} : 2\alpha + 3\beta + \gamma = 0\right\} \subset \mathscr{L}(p_1 + q_1).$$

This will give a vector space of functions with poles only at p_1, q_1.

We can describe this a little more geometrically. The numerator is a general equation of a line L through s_1. Any such line will pass through C in two more points t_1, t_2. By the above argument, we will have $t_1 + t_2 \sim p_1 + q_1$. Thus, any divisor colinear to s_1 will give one linearly equivalent to $p_1 + q_1$; this gives a 1-parameter (projective) family of divisors linearly equivalent to $p_1 + q_1$.

Are there any other divisor equivalent to $p_1 + q_1$? A prior, who know? But Riemann-Roch will tells us that there are no others (i.e. that $\dim \mathscr{L}(p_1 + q_q) = 2$ so $\dim \mathbb{P}\mathscr{L}(p_1 + q_1) = 1$).

2.2 Problem 2

Start with $C_0 : y^2 = x^6 - 1$ in the affine plane. Can check that this is smooth by looking at partial derivatives. We again want to compactify, so think of this as a two sheeted cover of $\mathbb{A}^1_x$, the x-plane. Then $\pi : C_0 \to \mathbb{A}^1_x$ is degree 2 with branch points $r_\alpha := (\omega^\alpha, 0)$ for $\omega = e^{\pi i/3}$ a 6th root of unity ($\alpha = 0, 1, \dots, 5$).

Play the same disk game as before. Take some large Δ_r ($r > 1$). What does $\pi^{-1}(\mathbb{A}^1 \setminus \Delta_r)$ look like? It will again be a two-sheeted cover, so we only care to know if it's connected. It's not. Why, as x loops around a circle, its argument increases by 2π, so the argument of $x^6 \approx y^2$ goes up by 12π, so the argument of y goes up by 6π, i.e. y ends up where it started. Thus, this is unramified at ∞, so is compactify by adding 2 points $\{p, q\} = \pi^{-1}(\infty)$. Now we have our compact Riemann surface C. What's its genus?

Could do the same sort of concrete analysis as before. You would realize C as the union of 2 spheres, each with three discs removed, such that the boundaries of those disks have been identified. Or you could use Riemann-Hurwitz. In either case, the upshot is $g(C) = 2$ ($\chi(C) = -2$).

Linear equivalence calculations Label the points $s_1, s_2 = (0, \pm i)$.

The first linear equivalence is $p + q \sim 2r_\alpha$ (for all α). The function x on $\mathbb{P}^1$ has a simple pole at $\infty \in \mathbb{P}^1$. Thus it has polar divisor $p + q$ on C ($p + q = x^*(\infty)$). The zeros of x are precisely s_1, s_2, so $(x) = s_1 + s_2 - p - q$. One similarly computes

$$(x - \omega^\alpha) = 2r_\alpha - p - q$$

(factor of 2 since r_α a branched point).

We also want $\sum r_\alpha \sim 3p + 3q$. To start, can we find a function with zeros at the points $r_\alpha = (\omega^\alpha, 0)$? Yes, y. It's zeros are exactly the r_α's. It will also have poles above ∞, so it will have poles at p, q whose order adds up to 6. By looking at local coordinates or arguing via symmetry, the coefficients of p, q better be the same, so

$$(y) \sim \sum r_\alpha - 3p - 3q.$$

Complete linear system Find $|r_0 + r_2 + r_4|$. Can we find a function with poles at these points. Note if you take a function like $x - 1$, it will have a double zero at r_0 (since it's a branch point), so functions like that are no good. We could try $1/y$. This will have simple poles at r_0, r_2, r_4, but will also have poles at r_1, r_3, r_5. We'd like to kill this odd poles, so multiply by function with zeros there, e.g. by $x - \omega^\alpha$ whose divisor is $(x - \omega^\alpha) = 2r_\alpha - p - q$. Observe

$$\left(\frac{(x-\omega)(x-\omega^3)(x-\omega^5)}{y}\right) = [2(r_1 + r_3 + r_5) - 3p - 3q] - \left[\sum r_\alpha - 3p - 3q\right] = r_1 + r_3 + r_5 - r_0 - r_2 - r_4.$$

We now have a two-dimensional vector space of functions instead $\mathscr{L}(r_0+r_2+r_4)$, spanned by the constant function 1 and this function above. Are these all? Yes by Riemann-Roch. Riemann-Roch will tells us

$$\dim \mathscr{L}(r_0 + r_2 + r_4) \leq 3 + 1 - 2 = 2,$$

so we must have found everything.

2.3 Problem 3

The curve is now $C_0 : y^3 = x^5 - 1$.

Remark 2.2. The closure of this curve in $\mathbb{P}^2$ is singular. Could take the normalization, but that's annoying. ○

To compactify, let's consider this as a triple cover of $\mathbb{A}^1_x$. To figure out how many points we have over infinity, take a large disk including all 5 branch points[5], and then ask about the preimage of its complement. It will have 3, 2, or 1 components. Which is it? Same argument argument as before. As $\arg(x)$ increases by 2π, $\arg(y)$ increases by $10\pi/3$. So takes 3 trips around to get back to y, so we have the connected 3-sheeted cover of $\mathbb{C} \setminus \Delta_r$, i.e. we have 1 point above $\infty \in \mathbb{P}^1$ with ramification index 3. Riemann-Hurwitz then tells you that C has genus 4.

3 Lecture 2 (9/8)

Note 6. ~ 12 minutes late

3.1 Divisors, invertible sheaves, and line bundles

Missed some stuff about linear systems and $\mathscr{O}_C(D)$ being a line bundle.

Definition 3.1. A **linear system** on C is a pair (D, V) with $V \subset \mathscr{L}(D)$ up to linear equivalence. ◇

Recall 3.2. The sheaf associated to a divisor D is the sheaf $\mathscr{O}_C(D)$ given on an open $U \subset C$ by

$$\mathscr{O}_C(D)(U) = \{f \in \mathcal{M}(U) : \operatorname{ord}_p f \geq -m_p \text{ for all } p \in U\},$$

where $D = \sum m_p \cdot p$. ⊙

This is locally free of rank 1 (equivalently, invertible) , i.e. for all $p \in C$, there exists neighborhood $U \ni p$ and a meromorphic $\sigma \in \mathcal{M}(U)$ so that $\operatorname{ord}_p \sigma = m_p$ for all $p \in U$. Hence, any $g \in \mathscr{O}_C(D)(U)$ can be written as $f\sigma$ for some $f \in \mathscr{O}_C(U)$. To get U, just choose a neighborhood around p small enough that it contains no other points in $\operatorname{supp} D$.

That brings us up to date on what I missed...

Let $\mathscr{F}$ be any invertible sheaf on C. For all p, there exists some open $U \ni p$ and a section $\sigma \in \mathscr{F}(U)$ so that for all $\tau \in \mathscr{F}(U)$, one can write $\tau = f\sigma$ for some $\sigma \in \mathscr{O}_C(U)$. We define the **order of vanishing of** $\tau \in \mathscr{F}(U)$ **at** p to be $\operatorname{ord}_p(\tau) := \operatorname{ord}_p(f)$.

[5]Each branch point will have ramification index 3

Warning 3.3. If $\mathscr{F} = \mathscr{O}_C(D)$ and $\sigma \in \mathscr{F}(U)$, then this is *not* the same as the order of σ when viewed as a meromorphic function $\sigma \in \mathcal{M}(U)$. The two values differ by m_p, where as usual, $D = \sum m_p \cdot p$. •

Let $\mathscr{F}$ again be an invertible sheaf on C. Can choose a cover $\{U_\alpha\}$ on C so that each $\mathscr{O}_{U_\alpha}$ is free of rank one, generated by $\sigma_\alpha \in \mathscr{F}(U_\alpha)$ (i.e. σ_α a nonvanishing section). From this we get **transition functions**

$$g_{\alpha\beta} := \sigma_\alpha/\sigma_\beta \in \mathscr{O}_C(U_\alpha \cap U_\beta).$$

These functions determine the invertible sheaf $\mathscr{F}$.

What we've said above basically amounts to an equivalence of two sorts of objects (the top two below).

$$\begin{tikzcd}
& \left\{\begin{array}{c}\text{invertible sheaf}\\ \text{on } C\end{array}\right\} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] \\
\left\{\begin{array}{c}\text{linear eq class}\\ \text{of divisors on } C\end{array}\right\} \arrow[dr, leftrightarrow] & \\
& \left\{\begin{array}{c}\text{line bundle}\\ \text{on } C\end{array}\right\}
\end{tikzcd}$$

These are all three equivalent. What is a line bundle?

Definition 3.4. A **line bundle** is a complex manifold L along with a projection map $\pi : L \to C$ whose fibers are complex vector spaces of dimension 1. Furthermore, we require that for all p, there exists a neighborhood U along with a commutative diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr] & & U \times \mathbb{C} \arrow[dl] \\
& U &
\end{tikzcd}$$

◇

Say $L \to C$ is a line bundle along with two open sets U, V above which it trivializes. Then, on the overlap $U \cap V$ we have two different identifications of fibers with $\mathbb{C}$; these will differ by multiplication by a scalar, so we get (holomorphic) transition functions $g_{UV} : U \cap V \to \mathbb{C}^\times$ on overlaps. This is the same data needed to define an invertible sheaf.

Remark 3.5. Can think in terms of line bundles or invertible sheaves as you like. Usually shouldn't make much of a difference. ○

Note both invertible sheaves and line bundles generalize easily to higher dimensions. Can talk about invertible sheaves or line bundles on varieties of higher dimensions. In fact, we can also extend divisors to higher dimensional varieties or complex manifolds.

Example. On $\mathbb{P}^r$, there is the invertible sheaf $\mathscr{O}_{\mathbb{P}^r}(1)$ which is associated to the linear equivalence class of hyperplanes (i.e. zero loci of homogeneous degree 1 polynomials). △

3.2 Linear series on curves

Definition 3.6. A **linear series** on a curve C is a pair $(\mathscr{L}, V)$ with $\mathscr{L}$ an invertible sheaf and $V \subset \mathrm{H}^0(\mathscr{L})$ a vector subspace. We say $(\mathscr{L}, V)$ is **base-point free** if for all $p \in C$, there exists $\sigma \in V$ so that $\sigma(p) \neq 0$ (i.e. in some neighborhood U of p, σ generates $\mathscr{L}|_U$). ◇

Given a basepoint-free linear series $(\mathscr{L}, V)$, we can then define a (non-degenerate) map

$$\varphi : C \longrightarrow \mathbb{P}^r$$

(at least, can defined such a map up to automorphism of $\mathbb{P}^r$) which can be described in the following ways

- (concretely) For all $p \in C$, choose an open $U \ni p$ and a generator τ of $\mathscr{L}|_U$. Choose a basis $\sigma_0, \dots, \sigma_r$ for V, and write $\sigma_i = f_i \tau$ with $f_i : U \to \mathbb{C}$ a regular function. Then we set

  $$\varphi(p) := [f_0(p) : \cdots : f_r(p)] .$$

  This is concrete, but involves choices, so one has to check that it's actually well-defined. In particular, note that changing τ scales the coordinates uniformly (so gives the same element of $\mathbb{P}^r$), and then being base-point free guarantees $f_j(p) \neq 0$ for some j.

- (more intrinsically) Given $(\mathscr{L}, V)$ b.p.f, for all $p \in C$, we define

  $$H_p := \{\sigma \in V : \sigma(p) = 0\} \in \mathbb{P}V^*.$$

  Then can describe φ simply as $\varphi : p \mapsto H_p$.

Exercise.

(a) Verify the above, that the two descriptions of φ are the same.

(b) Suppose $(\mathscr{L}, V)$ is b.p.f. Let $W \subset V$ be a subspace (with $(\mathscr{L}, W)$ also b.p.f). Show there exists a (rational) map $\pi : \mathbb{P}V^* \dashrightarrow \mathbb{P}W^*$ making

$$\begin{tikzcd} C \arrow[r, "\varphi_V"'] \arrow[rr, bend left, "\varphi_W"] & \mathbb{P}V^* \arrow[r, "\pi"'] & \mathbb{P}W^* \end{tikzcd}$$

commute. This π will be projection away from $\mathrm{Ann}(W) \subset V^*$. Note in particular that W being base-point free will give $\mathbb{P}\,\mathrm{Ann}(W) \cap \varphi_{\mathscr{L},V}(C) = \emptyset$.

Question 3.7. *When is $\varphi_{(\mathscr{L},V)} : C \to \mathbb{P}^r$ an embedding?*

Fix an invertible sheaf $\mathscr{L}$. Given a divisor $D = \sum m_p \cdot p$, we can define a new (invertible) sheaf $\mathscr{L}(-D)$ via

$$\mathscr{L}(-D)(U) := \{\sigma \in \mathscr{L}(U) : \forall p \in U, \mathrm{ord}_p\, \sigma \geq m_p\} .$$

Example.

$$\mathrm{H}^0(\mathscr{L}(-p)) = \text{sections of } \mathscr{L} \text{ vanishing of } p.$$ △

Notation 3.8. If $V \subset \mathrm{H}^0(\mathscr{L})$, then we define

$$V(-D) := V \cap \mathrm{H}^0(\mathscr{L}(-D)).$$

Exercise. If $(\mathscr{L}, V)$ is b.p.f, so gives $\varphi : C \to \mathbb{P}^r$, then φ is an embedding if and only if

$$\dim V(-p-q) = \dim V - 2 \text{ for all } p, q \in C$$

(including $p = q$).

Remark 3.9. Being base point free already gives $\dim V(-p) = \dim V - 1$. For intuition in the above exercise, the condition with $p \neq q$ says that $\varphi(p) \neq \varphi(q)$ (i.e. φ separates points). The condition with $p = q$ says that φ is an immersion at p (has nonvanishing derivative, i.e. separates tangent vectors). ∘

3.3 Canonical bundle

There is always one special invertible sheaf/line bundles/linear equivalence class.

Say C is a compact Riemann surface. A **holomorphic 1-form** (resp. **meromorphic 1-form**) is locally given by $f(z)\mathrm{d}z$ where f is holomorphic (resp. meromorphic). In a coordinate neighborhood, have the 1-form $\mathrm{d}z$, and all others are multiples of it by some {holo, mero}morphic function. Given p in this neighborhood, we define

$$\mathrm{ord}_p(\omega) := \mathrm{ord}_p(f).$$

Note that if ω, η are two one-forms, then their ratio ω/η is a global meromorphic function. Thus, defining

$$(\omega) = \sum_{p \in C} \mathrm{ord}_p(\omega) \cdot p,$$

then $(\omega) \sim (\eta)$. In this way, we get a well-defined linear equivalence class of divisors, the **canonical class**. As a line bundle, this is simply to cotangent bundle; as an invertible sheaf, it's the sheaf of holomorphic 1-forms. When thinking of it as a divisor (class), we denote it by K or K_C.

Here are a couple questions.

Question 3.10.

(1) *Given a curve C along with a divisor D of degree d, what can we say about* $\ell(D) := \dim \mathscr{L}(D) = \dim \mathrm{H}^0(\mathscr{O}_C(D)) =: h^0(\mathscr{O}_C(D)) = h^0(D)$.

"We now write $h^0(\mathscr{O}_C(D))$. Originally, it was $\ell(D)$, but then someone came along and said, 'Why denote this with 4 strokes [of the pen] when you could use 9?'" (paraphrase)

(2) *Given C, does there exists a divisor D of degree d for which $h^0(D) \geq r + 1$ (for given r).*

4 Problem Session (9/10)

Note 7. roughly 3 minutes late

4.1 Divisor/line bundle/invertible sheaf correspondence

Recall 4.1. We have equivalences

$$\begin{tikzcd}
 & \left\{\begin{matrix}\text{invertible sheaf}\\ \text{on } C\end{matrix}\right\} \arrow[dd, leftrightarrow] \\
\left\{\begin{matrix}\text{linear eq class}\\ \text{of divisors on } C\end{matrix}\right\} \arrow[ur, leftrightarrow] \arrow[dr, leftrightarrow] & \\
 & \left\{\begin{matrix}\text{line bundle}\\ \text{on } C\end{matrix}\right\}
\end{tikzcd}$$

In particular, recall a line bundle is a family of lines parameterised by C. ⊙

I think he's in the middle of talking about constructing line bundles from transition functions. So we start with an open cover $\{U_\alpha\}$ of our base curve C, and in each member of the cover, we take a trivial line bundle $U_\alpha \times \mathbb{C}$. Over the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$, we have two ways of identifying fibers with $\mathbb{C}$, either view $U_{\alpha\beta}$ as a subset of U_α or as a subset of U_β. We need to identify these two perspectives using a linear automorphism of $\mathbb{C}$, i.e. an element of $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, i.e. for every $p \in U_{\alpha\beta}$, we get some $g_{\alpha\beta}(p) \in \mathbb{C}^\times$ giving the desired isomorphism. These identifications are consistent in the sense that

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$$

(on the triple overlap $U_{\alpha\beta\gamma}$). Such data is called a **1-cocycle**. This is exactly the same data needed to define an invertible sheaf.

Remark 4.2. Every invertible sheaf is the sheaf of sections of some line bundle, and the sheaf of sections of a line bundle is always an invertible sheaf. ∘

Recall 4.3. Given a divisor D, we can naturally write down the corresponding invertible sheaf

$$\mathscr{O}_C(D)(U) = \{f \in \mathcal{M}(U) : (f) + D \geq 0\}. \qquad ⊙$$

You can also go from a divisor straight to a line bundle (via writing down the appropriate transition functions), but this is maybe a less immediately clear correspondence. Since $\mathscr{O}_C(D)$ is locally free of rank 1, there is a cover $\{U_\alpha\}$ with sections $f_\alpha \in \mathcal{M}(U_\alpha)$ so that $(f_\alpha) = D$ in U_α.[6] In other words, $\mathscr{O}_C(D)|_{U_\alpha} = f_\alpha^{-1} \cdot \mathscr{O}_C|_{U_\alpha}$. With this in mind, the **line bundle associated to a divisor** D is then given by the transition functions $g_{\alpha\beta} = f_\alpha/f_\beta$.[7]

An example Say we have a curve C and a point $p \in C$. Consider the inclusion $\mathscr{O}_C(-p) \hookrightarrow \mathscr{O}_C$ of invertible sheaves.

Notation 4.4. One may also write $\mathscr{I}_{p/C} = \mathscr{O}_C(-p)$, thinking of it as the ideal sheaf of p (sheaf of functions vanishing at p).

[6] f_α is like a local equation for D

[7] Potentially this is backwards, but that's not super important.

Now consider the corresponding line bundles. The line bundle associated to $\mathscr{O}_C(-p)$ should map to the trivial bundle $C \times \mathbb{C}$ corresponding to $\mathscr{O}_C$.

Warning 4.5. This map (of line bundles) is not an injection. In the map of line bundles, the fibers of $\mathscr{O}_C(-p)$ map isomorphically onto the fibers of $\mathscr{O}_C$ *except* the fiber above $p \in C$. This fiber of $\mathscr{O}_C(-p)$ gets collapsed to 0. •

So 'injective' maps of line bundles are the not the same as 'injective' maps of invertible sheaves.

4.2 Example of computing complete linear systems: Problem 3 on the HW

There are two objectives in these computations: **(1)** finding linear equivalences (i.e. finding sections of the appropriate sheaf) and **(2)** knowing when you've found them all. The second part comes from Riemann-Roch. The first part can be made algorithmic, but in practice, you can usually just play around with things (he said something like this. I was distracted).

Let $C_0 = V(y^3 - x^5 + 1) \subset \mathbb{A}^5$ with compactification C. Note that $C \setminus C_0 = \{r\}$ and that we have a degree 3 cover $\pi : C \to \mathbb{P}^1$ branched over the 5th roots of unity and over ∞ (each point completely ramified). Triangulating $\mathbb{P}^1$ (with branch points as vertices) and pulling this back to a triangulation of C, we see that $\chi(C) = 3\chi(\mathbb{P}^1) - 2(6) = -6$ so $g(C) = 4$.

The next part of the problem asks to show $3r_\alpha \sim 3p$ and $\sum r_\alpha \sim 5p$. These come from the divisors $(x - \omega^\alpha) = 3r_\alpha - 3p$ and $(y) = \sum r_\alpha - 5p$.

Now, the question that was actually asked (by the audience): find $\mathrm{H}^0(K)$, the space of holomorphic 1-forms on C. The space of meromorphic 1-forms is 1-dimensional over $\mathcal{M}(C)$, so let's start with a meromorphic 1-form and see which multiple of it are holomorphic.

We'll start with $\mathrm{d}x$. This differential is non-vanishing on $\mathbb{A}^1$. Note that, near r_α $(= (\omega^\alpha, 0))$, π looks like $z \mapsto z^3$, so $\mathrm{d}x$ will have a double zero at each of these points (i.e. $\mathrm{d}(z^3) = 3z^2\mathrm{d}z$). Hence, $(\mathrm{d}x) = 2\sum r_\alpha -???$. Now, we could do a local calculation at ∞ to figure out the poles. However, we don't have to since it's a fact that $\deg(\mathrm{d}x) = 2g - 2 = 6$. Thus, we must have $??? = 4p$, i.e. $(\mathrm{d}x) = 2\sum r_\alpha - 4p$. As a consequence,

$$\mathrm{H}^0(K) = \mathscr{L}\left(2\sum r_\alpha - 4p\right) \cdot \mathrm{d}x$$

(need to cancel out the poles of $\mathrm{d}x$ without introducing too many new poles in order to get something holomorphic).

Note that we can kill the poles of $\mathrm{d}x$ by dividing by y (recall $(y) = \sum r_\alpha - 5p$) in order to get $w_0 = \frac{\mathrm{d}x}{y}$ with divisor $(w_0) = \sum r_\alpha + p$. What do we do next? Well, w_0 still vanishes at all the r_α's so can do the same thing: define $w_1 = \frac{\mathrm{d}x}{y^2}$. This has divisor $(w_1) = 6p$. Then take $w_2 = x\mathrm{d}x/y^2$ and $w_3 = x^2\mathrm{d}x/y^2$. Since x has a triple pole at p (and no other poles), these are also holomorphic. We know $\dim \mathrm{H}^0(K) = g = 4$, so we've found everything:

$$\mathrm{H}^0(K) = \mathrm{span}\left\{\frac{\mathrm{d}x}{y}, \frac{\mathrm{d}x}{y^2}, x\frac{\mathrm{d}x}{y^2}, x^2\frac{\mathrm{d}x}{y^2}\right\}.$$

Let's now look at **(g)** of this homework problem. Let $D = r_0 + \cdots + r_4$. We're first asked to show $h^0(K_C - D) = 1$. We saw above that $K_C \sim (\mathrm{d}x) = D + p$, so $K_C - D \sim p$. We know $h^0(\mathscr{O}(p)) = 1$. It's at least one-dimension since it contains the constant function 1; if there were more sections, we could

Question: Did I write down the wrong thing when taking notes? I'm pretty sure this just is get a degree 1 map to $\mathbb{P}^1$, forcing C to be rational. Riemann-Roch then gives $h^0(\mathscr{O}_C(D)) = 3$. We could even find a basis if we wanted.

Example. Say we have f a meromorphic function with poles at $r_0, \ldots, r_4$ and otherwise holomorphic. Consider the meromorphic differential $fw_0 = f\mathrm{d}x/y$. This is a holomorphic differential vanishing at p. From our computation of $\mathrm{H}^0(K)$, we know the only holomorphic differential vanishing at p are w_0, w_1, w_2, so $fw_0 \in \operatorname{span}\{w_0, w_1, w_2\}$. Equivalently,

$$f \in \operatorname{span}\left\{\frac{w_0}{w_0}, \frac{w_1}{w_0}, \frac{w_2}{w_0}\right\} = \mathrm{H}^0(\mathscr{O}(D)). \qquad \triangle$$

He explained something about making this algorithmic but I wasn't paying attention and missed it...

4.3 Something about embeddings? I couldn't hear the question asked over the sound of the AC

Say we have $(\mathscr{L}, V)$ with $V \subset \mathrm{H}^0(\mathscr{L})$ base point free. This gives a map

$$\varphi : C \longrightarrow \mathbb{P}^r \text{ where } \dim V = r + 1.$$

(defined up to linear automorphism of $\mathbb{P}^r$). Fixing a basis $\sigma_0, \ldots, \sigma_r$ of V, this map is simply $\varphi(c) = [\sigma_0(p), \ldots, \sigma_r(p)]$. More intrinsically, can think of this as a map $\varphi : C \to \mathbb{P}V^*$ sending $p \mapsto \{\sigma \in V : \sigma(p) = 0\} =: H_p$.

Question 4.6. *What's the condition that φ is 1-1?*

This is simply that $H_p \neq H_q$ for any pair $p \neq q$. In other words, $H_p \cap H_q$ is neither H_p nor H_q. Since H_p, H_q are both codimension 1 subspaces of V, this is saying that $H_p \cap H_q = V(-p-q)$ is codimension 2, i.e. that $\dim V(-p-q) = 2$.

Question 4.7. *What's the condition that φ is an immersion?*

Non-example. The map

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & \mathbb{A}^2 \\ t & \longmapsto & (t^2, t^3) \end{array}$$

is 1-1 but not an immersion (image looks like a cuspidal cubic $y^2 = x^3$, i.e. like $\prec$) $\quad \triangledown$

Claim 4.8. *The condition that φ be an immersion at p is precisely that* $\dim V(-2p) = \dim V - 2$.

If φ is not an immersion at p, then every function vanishing at p does so to order ≥ 2.

Proposition 4.9. *φ is an embedding $\iff$ $\dim V(-D) = \dim V - 2$ for all degree 2 effective divisors D.*

(in particular, this implies $V(-p) = \dim V - 1$ for all $p \in C$).

5 Lecture 3 (9/13)

Note 8. Roughly 7 minutes late

5.1 Two small items

(1) Homework 2 posted later today, due 9/20. In general, homeworks due on Mondays

(2) Degree of a line bundle/invertible sheaf

Any line bundle can be written as $\mathscr{L} = \mathscr{O}_C(D)$, so can define $\deg \mathscr{L} = \deg D$.

Alternatively, in topology/dif geo can associate to any complex vector bundle on a space X its **chern classes** $c_k(L) \in \mathrm{H}^{2k}(X, \mathbb{Z})$. In the case of a compact Riemann surface, we have $\deg L = c_1(L) \in H^2(C; \mathbb{Z}) = \mathbb{Z}$.

5.2 Basic properties: genus, Riemann-Roch, etc.

Genus Let's define genera (genii? genuses? genes? genies?).

Over the complex numbers, C is a compact Riemann surface, so in particular is an oriented two-dimensional real surface. Hence it is topologically classed by its topological genus g, essentially its number of handles. This only works of (connected) curves over $\mathbb{C}$ though, so we'd like a more algebraic definition.

Fact. $\deg(K_C) = 2g - 2$

We can define the canonical bundle over any field, so this is the definition we'll use in general, even for disconnected curves.

Example. $g(\mathbb{P}^1 \sqcup \mathbb{P}^1) = -1$. △

There's another definition that works algebraically: $g = h^0(K_C) = 1 - \chi(\mathscr{O}_C)$. There is one more definition we want to mention. Say we have an embedding $C \hookrightarrow \mathbb{P}^r$. This embedded has an associated Hilbert polynomial p_C, and one can define $g = 1 - p_C(0)$.

It's not immediately obvious that all these various characterizations are equivalent, but they are.

Remark 5.1. Eventually, we will have to deal with singular curves, so we'll eventually need a definition of genus for them as well. We won't worry about this just yet. ○

Riemann-Roch Both Riemann and Roch were 19th century mathematicians, so let's first state this theorem in a form closer to how they would have thought about it.

Theorem 5.2 (**Riemann-Roch**). *Let D be a degree d divisor on C. Then,*

$$h^0(D) = d - g + 1 + h^0(K_C - D).$$

(in particular, $h^0(K_C) = \deg(K_C) - g + 2$ so $h^0(K_C) = g \iff \deg(K_C) = 2g - 2$, giving one of the equivalences from before).

Corollary 5.3. *Say $\deg D \geq 2g - 1$. Then, $h^0(D) = d - g + 1$.*

Proof. $\deg(K_C - D) < 0 \implies h^0(K_C - D) = 0$. ■

Corollary 5.4. *Say $\deg D \geq 2g+1$. Then, $|D|$ defines an embedding $\varphi_D : C \hookrightarrow \mathbb{P}^r$ into projective space. In particular, any curve can be embedding into projective space, as a curve of degree $2g + 1$.*

Question 5.5. *Can we do better than $2g+1$? Given a curve of genus g, what is the smallest degree of an embedding of that curve into projective space?*

Suppose we have an embedding $C \hookrightarrow \mathbb{P}^r$. Consider the exact sequence

$$0 \longrightarrow \mathscr{I}_{C/\mathbb{P}^r} \longrightarrow \mathscr{O}_{\mathbb{P}^r} \longrightarrow \mathscr{O}_C \longrightarrow 0.$$

Twisting by m and taking cohomology gives (note $r > 1$)

$$0 \to \mathrm{H}^0(\mathscr{I}_{C/\mathbb{P}^r}(m)) \to \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^r}(m)) \xrightarrow{\rho_m} \mathrm{H}^0(\mathscr{O}_C(m)) \to \mathrm{H}^1(\mathscr{I}_{C/\mathbb{P}^r}(m)) \to 0 = \mathrm{H}^1(\mathscr{O}_{\mathbb{P}^r}(m)).$$

Definition 5.6. The **Hilbert function** is $h_C(m) := \operatorname{rank} \rho_m$. $\diamond$

Think: how many conditions does it take to say a degree m polynomial vanishes on C? This is $\dim \ker \rho_m$.

Theorem 5.7. *For $m \gg 0$, $h_C(m) = p_C(m)$ for some polynomial p_C.*

As a consequence of a theorem of Serre (in FAC) for $m \gg 0$, $h^1(\mathscr{I}_{C/\mathbb{P}^r}(m)) = 0$, i.e. $h_C(m) := h^0(\mathscr{O}_C(m))$. We saw earlier that for $m \gg 0$, we have $h^0(\mathscr{O}_C(m)) = md - g + 1$. Put together, this says that $p_C(m) = md - g + 1$, so in particular $1 - p_C(0) = g$.

In general, this p_C is the **Hilbert polynomial** $p_C(m) = \chi(\mathscr{O}_C(m))$

Serre Duality Here's a nice result.

Theorem 5.8 (**Serre Duality**)**.** *There is a perfect pairing*

$$\mathrm{H}^0(D) \otimes \mathrm{H}^1(K - D) \longrightarrow \mathrm{H}^1(K) = \mathbb{C}$$

given by cup products. Hence, $\mathrm{H}^0(D) = \mathrm{H}^1(K-D)^*$, so $h^1(D) = h^0(K-D)$.

This allows us to give another formulation of Riemann-Roch:

$$\chi(\mathscr{L}) = \deg \mathscr{L} - g + 1 = \deg \mathscr{L} + \chi(\mathscr{O}_C)$$

for an arbitrary line bundle $\mathscr{L}$. In this form, this generalizes to higher dimensional varieties and higher rank vector bundles, can get a formula for the Euler characteristic of a vector bundle on a variety in terms of some function of that bundle and the Euler characteristic of the structure sheaf.[8]

Remark 5.9. This form of Riemann-Roch (for line bundles on curves) is easy to prove. Taking cohomology of the exact sequence

$$0 \longrightarrow \mathscr{L}(-p) \longrightarrow \mathscr{L} \longrightarrow \mathscr{L}_p \longrightarrow 0$$

tells you that RR holds for $\mathscr{L} \iff$ it holds for $\mathscr{L}(-p)$, i.e. that $\chi(\mathscr{L}) = \chi(\mathscr{L}(-p)) + 1$. Thus, to prove RR, only need to prove it for a single line bundle. $\circ$

Riemann-Hurwitz Say we have a (non-constant) map $f : C \to X$ between compact Riemann surfaces. Let d be the degree of this map, so outside a finite set $q_1, \dots, q_\ell$ of points of X, this map is a d-sheeted (unramified) covering space. The points $q_1, \dots, q_\ell$ (where $\# f^{-1}(q_i) < d$) are called **branch points**.

[8]Technically, Joe didn't say all of this, but I'm pretty sure general RR formulas are always of this rough shape

Remark 5.10. If $p \in C$ and $q = f(p) \in X$, then there exists local coordinates z around p and w around q such that f is given by $w = z^m$ for some m. ○

Definition 5.11. The **ramification index** of f at p is $\mathrm{ram}(f,p) = m - 1$ with m as above. ◇

Since there are only finitely many ramification points, we can define the **ramification divisor**

$$R = \sum_{p \in C} \mathrm{ram}(f,p) \cdot p \in \mathrm{Div}(C)$$

as well as the **branch divisor**

$$B = f_* R = \sum_{q \in X} \left(\sum_{p \in f^{-1}(q)} \mathrm{ram}(f,p) \right) q \in \mathrm{Div}(X).$$

Note that at a point $q \in X$, one has $\mathrm{ord}_q B = d - \# f^{-1}(q)$. We also set $b = \deg R = \deg B$.

Theorem 5.12 (Riemann-Hurwitz).

$$2g(C) - 2 = d(2g(X) - 2) + b.$$

Proof. **(1)** Let ω be an meromorphic differential on the target X (so $\deg(\omega) = 2g(X) - 2$). Consider the pullback $f^*\omega$, a meromorphic 1-form on C. What is its divisor? It is

$$(f^*\omega) = f^{-1}((\omega)) + R.$$

The point is that the zeros/poles of ω give zeros/poles of $f^*\omega$, but also even if ω has no zero a point q, if q is a branch point, then $f^*\omega$ will acquire zeros at the ramified points over it (essentially because $\mathrm{d}z^m = mz^{m-1}\mathrm{d}z$ which has a zero or order $m - 1$ even though $\mathrm{d}z$ does not).

(2) Alternatively, one can play around with triangulations. First triangulate X so that all points in B are vertices. Say this triangulation has f_X faces, e_X edges, and v_X vertices. Then this pulls back to a triangulation of C with $f_C = df_X$ faces, $e_C = de_X$ edges, and $v_C = dv_X - b$ vertices. Taking topological Euler characteristics, we win. ■

5.3 Canonical Map

Recall the connection between linear series and maps into projective space, essentially

$$\left\{ \begin{matrix} \text{linear series} \\ \text{on } C \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{maps} \\ C \longrightarrow \mathbb{P}^r \end{matrix} \right\}.$$

Every curve comes equipped with a particular (usually) non-trivial line bundle, the canonical bundle, and so comes equipped with a natural associated map $\varphi_K : C \to \mathbb{P}^{g-1}$. Given a basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-forms, this is the map

$$\varphi_k = [\omega_1 : \cdots : \omega_g].$$

Before looking at the geometry of this map, we better hope $|K|$ is base-point free. Note that $p \in C$ is a base-point (vanishes at every section of $\mathscr{O}(K)$) iff $h^0(K-p) = h^0(K)$. Riemann-Roch tells us that

$$h^0(K-p) = 2g - 3 - g + 1 + h^0(\mathscr{O}_C(p)).$$

Remark 5.13. $h^0(\mathscr{O}_C(p)) = 1$ if $g > 0$. This is because two independent sections would give a degree 1 map to $\mathbb{P}^1$. ○

Hence (assuming $g > 0$),

$$h^0(K-p) = g - 1,$$

so $|K|$ is base-point free.[9] Thus, φ_K is at least a morphism (assume $g > 0$ from now on).

Question 5.14. *Is φ_K an embedding?*

Answer. Yes iff $h^0(K_C - D) = h^0(K_C) - 2 = g - 2$ for all effective divisors $D = p + q$ on C. ⋆

Another application of Riemann-Roch gives

$$h^0(K-D) = g - 3 + h^0(\mathscr{O}_C(D))$$

so the question becomes: is $h^0(\mathscr{O}_C(D))$ equal to 1 or 2?

Claim 5.15. *We always have $h^0(\mathscr{O}_C(D)) = 1$ unless C admits a degree 2 map $\varphi_D : C \to \mathbb{P}^1$.*

(A degree two subspace of $L(D)$ is such a degree 2 map)

Definition 5.16. We say C is **hyperelliptic** if there exists a degree 2 map $f : C \to \mathbb{P}^1$. ◇

Thus, given a curve C, φ_K is an embedding or C is hyperelliptic.

Question 5.17. *Given a genus g, are there any hyperelliptic curves of genus g? Similarly, are there any non-hyperelliptic curves of genus g?*

In genus ≤ 2, every curve is hyperelliptic. In genus ≥ 3, most curves are not hyperelliptic. We'll make this more precise later.

Let's end with an application of using the canonical embedding.

Question 5.18. *Say C is a non-hyperelliptic curve of genus g. Say $D = p_1 + \cdots + p_d$ is an effective divisor of degree d (say the p_i are distinct). Does there exist a meromorphic function on C with at most simple poles at the points p_i (and is holomorphic elsewhere)?*

Answer. Think of the curve C as embedding $\varphi_K : C \hookrightarrow \mathbb{P}^{g-1}$ canonically. Then the answer is yes iff the points $p_1, \ldots, p_d \in C \subset \mathbb{P}^{g-1}$ are linearly dependent. ⋆

6 Lecture 4 (9/15)

Outline for today

[9]When $g = 0$, $\mathrm{H}^0(K) = 0$

- canonical curves + geometric Riemann-Roch
- moduli spaces
- rational spaces

Recall 6.1. We said last time that "most curves of genus ≥ 3 are not hyperelliptic." What does this actually mean? The answer will involve moduli spaces. ⊙

"I should warn you that I looked over my notes for the class today, and uh, decided not to do that."

6.1 Canonical Curves

Let C be a smooth curve of genus g which is non-hyperelliptic, so its canonical bundle is very ample (given en embedding $\varphi_K : C \hookrightarrow \mathbb{P}^{g-1}$ into projective space).

Recall 6.2. The canonical series never has base points if $g \neq 0$. ⊙

Remark 6.3. If C is hyperelliptic the canonical map $\varphi_K : C \to \mathbb{P}^{g-1}$ is the hyperelliptic map $C \xrightarrow{2} \mathbb{P}^1$ hollowed by the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$, $[x:y] \mapsto [x^{g-1}, x^{g-2}y, \ldots, xy^{g-2}, y^{g-1}]$. ○

Let's talk geometric Riemann-Roch.

Let $D = p_1 + \cdots + p_d$ be a degree d divisor with the p_i distinct.

Exercise. Think about how to make sense of the following when D has points appearing with multiplicity > 1.

Identify C with its image in $\mathbb{P}^{g-1}$. Note that Riemann-Roch says

$$h^0(D) = d - g + 1 + h^0(K - D).$$

What is $h^0(K - D)$ thinking of C under its canonical embedding?

First note that every canonical divisor K on C is the intersection of $C \hookrightarrow \mathbb{P}^{g-1}$ with some hyperplane in $\mathbb{P}^{g-1}$. We are now asking how many of these hyperplanes contain the points of D. Naively, we'd expect a $(g-d)$-dimensional vector space (d linear conditions given by the d points). However, this only holds if the p_i are linearly independent. In general, we have

$$h^0(K - D) = g - d + \#\text{linear relations among the } p_i\text{'s}.$$

Notation 6.4. $r(D) := h^0(D) - 1 = \dim |D| = \#$ linear relations among the points p_i.

Question: Why is this counting linear relations?

Recapping, we have obtained

Theorem 6.5 (**Geometric Riemann-Roch**)**.** *Let C be a curve of genus $g \geq 2$, and let $\varphi : C \to \mathbb{P}^{g-1}$ be its canonical map. Then,*

$$r(D) = d - 1 - \dim \overline{\varphi(D)}.$$

(this holds even if C is hyperelliptic)

Question 6.6. *Is C expressible as a 3-sheeted cover of $\mathbb{P}^1$?*

Such curves are called **trigonal**.

Answer. C is trivial iff $C \hookrightarrow \mathbb{P}^{g-1}$ contains 3-colinear points (including e.g. something like $2p+q$ where the tangent to C at p contains q).

Need a divisor of degree 3 which moves in a 1-dimensional linear series. ⋆

6.2 Canonical maps

- In genus $g = 0$, they don't exist since $h^0(K) = 0$.
- In genus $g = 1$, the canonical map $\varphi_K : C \to \mathbb{P}^0$ is simply collapsing the curve to a point.
- In genus $g = 2$, $\varphi_K : C \xrightarrow{2} \mathbb{P}^1$ is a hyperelliptic map.

  From here on, assume the curve is non-hyperelliptic
- In genus $g = 3$, we get an embedding $\varphi_K : C \hookrightarrow \mathbb{P}^2$ with image a smooth plane quartic curve.

  Note that any line in $\mathbb{P}^2$ will meet C in 4 points, any 3 of which are collinear. Hence, C is automatically trigonal.

  More geometrically, if these four points are s, p, q, r, then we have $p+q+r \sim K - s$, and we can see that there is a 1-dimensional linear series of linear equivalent divisors: consider the lines through s, i.e. $K - s \sim p' + q' + r'$ where p', q', r' lie on a line passing through s.

Question 6.7. *Is a non-hyperelliptic curve C of genus 4 trigonal?*

Start with the canonical embedding $\varphi_K : C \hookrightarrow \mathbb{P}^3$. Note that C is a curve of degree 6. This is not enough by itself to know much about the equations that define it.

Remark 6.8. The map associated to a linear series is always non-degenerate, the image never lies in a hyperplane. ○

Does C lie on a quadric surface? To answer this, consider the restriction map $\mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_C(2)$. C will lie on a quadric surface iff the induced map on global sections is non-injective (an element of the kernel is a degree 2 polynomial vanishing along C).

Note $6 = 2 \cdot 3$ so reasonable to expect C to a lie on a quadric surface and a cubic surface

Note 9. Joe said some more stuff here, but I couldn't hear over the sound of the AC.

To figure out if $\rho_2 : \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2)) \to \mathrm{H}^0(\mathscr{O}_C(2)) = \mathrm{H}^0(2K_C)$ has a kernel, let's start by computing some dimensions. Stars and bars tells us that $\dim \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2)) = \binom{5}{2} = 10$ while Riemann-Roch tells us that

$$\dim \mathrm{H}^0(\mathscr{O}_C(2)) = 12 - 4 + 1 = 9.$$

Thus, C lies on a (unique) quadric surface Q. Why unique? If C lied on two (irreducible) quadrics, then it's lie in their intersection, a curve of degree 4. Note that C can't lie on a reducible quadric.

Question: Why?

Answer: Because C does not lie on a hyperplane, so can't lie on a quadric that's a product of two linear

Does C lie on any (irreducible) cubic surfaces? Look at

$$\rho_3 : \underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(3))}_{\dim=20} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(3))}_{\dim=15}$$

with dimensions as indicated. Thus, C lies on (at least) 5 linearly independent cubics. We know C lies on Q, so it also lies on the cubics X_0Q, X_1Q, X_2Q, X_3Q with the X_i's coming from $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(1))$ (a

4-dimensional space). Thus, there must be some other cubic surface S, not containing Q, in which C lies. Thus, $C \subset Q \cap S$ with $Q \cap S$ a degree 6 curve by Bezout. That is, $C = Q \cap S$ is defined by two independent relations, one quadric and one cubic.

Recall 6.9. We asked if C is trigonal. This is the case iff it contains 3 colinear points. $\odot$

Suppose we had 3 collinear points on C. Then, any quadratic polynomial vanishing along C would vanish at 3 points of that line, and so vanish identically along that line (i.e. the line would lie in the quadric surface cut out by the polynomial). The converse holds as well: any line in Q will intersect C is 3 (colinear) points.

Fact. If Q is smooth, then $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ with lines given by the fibers of its two projection maps.

Corollary 6.10. *There are two linear series of degree* 3 *and dimension* 1 *on* C*.*

Consider the maps $C \hookrightarrow Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \mathbb{P}^1$.

Fact. If Q is singular, then Q is a cone (think $x^2 + y^2 + z^2 = 0$). In this case, C is expressible as a 3-sheeted cover of $\mathbb{P}^1$ in 1 way.

6.3 Moduli problems

Slogan. The objects we work with are often parameterized by some geometric space.

Definition 6.11. A **moduli problem** consists of two things

- A class of objects (e.g. varieties, sheaves, subvarieties of a given variety, etc.)
- A notion of what it means to have a *family* of such objects over a given base B. $\diamond$

Example. Take objects to be {isom classes of smooth, projective curves of genus g}. What is a family of such things? It is simply a smooth, projective morphism $\mathcal{C} \to B$ whose fibers are smooth projective curves of genus g. $\triangle$

Example. The objects {curves $C \hookrightarrow \mathbb{P}^r$ of degree d, genus g}. A family is a subscheme $\mathcal{S} \hookrightarrow B \times \mathbb{P}^r$ ($= \mathbb{P}^r_B$) so that the projection map $\pi : \mathcal{C} \to B$ is flat (+ the fibers are curves of degree d and genus g). $\triangle$

Example. Fix a curve C. Can take objects {effective divisors of degree d on C}. A family here is a subscheme $\mathcal{D} \subset B \times C$ which is flat over B, and whose fibers are as above. $\triangle$

Example. Fix curve C. Can take objects {line bundles $\mathscr{L}$ of degree d on C}. A family will be a line bundle $\mathscr{L}$ on $B \times C$, more or less...

We want think of a line bundle on $B \times C$ as a family of line bundles on C, parameterized by B. The fibers $\mathscr{L}_b$ are line bundles on C, but they don't change if $\mathscr{L}$ is tensored by the pullback of a line bundle on B. Hence, we'll say...

A family here is an element of
$$\frac{\text{line bundle on } B \times C}{\otimes \pi^*(\text{l.b. on } B)},$$
where $\pi : B \times C \to B$ the projection. $\triangle$

Given a moduli problem, we would love to find a variety/scheme M whose points are "naturally" in bijection with the objects of the moduli problem. We don't just want any old bijection. We want one compatible with the notion of families. That is, by "natural", we mean that given a family over B, we get a morphism $B \to M$ sending $b \mapsto [\text{fiber}/b]$.

This definition of moduli space does not quite work. This condition, it turns out, does not uniquely determine M. Furthermore, we are wanting the set map $b \mapsto [\text{fiber}/b]$ to be a morphism, but a morphism of schemes is not determined by the underlying map on topological spaces. Hence, there is more to be required.

Take 2: require that for any B, we have a natural bijection

$$\left\{\begin{matrix}\text{families of objects}\\ \text{over } B\end{matrix}\right\} \longleftrightarrow \operatorname{Hom}(B, M).$$

Taking $B = *$ ($= \operatorname{Spec}\mathbb{C}$), a point, this includes that points of M correspond to objects of the moduli problem. Natural here is in the sense of 'natural transformation,' i.e. given a morphism $B' \to B$ we can pull families over B back to families over B', and we want the following diagram to commute

$$\begin{tikzcd} \left\{\begin{matrix}\text{families of objects}\\ \text{over } B\end{matrix}\right\} \arrow[r] \arrow[d] & \operatorname{Hom}(B, M) \arrow[d] \\ \left\{\begin{matrix}\text{families of objects}\\ \text{over } B'\end{matrix}\right\} \arrow[r] & \operatorname{Hom}(B', M) \end{tikzcd}.$$

What we're saying is that we have two (contravariant) functors Sch $\to$ Set given by

$$B \mapsto \{\text{families}/B\} \quad\text{and}\quad B \mapsto \operatorname{Hom}(B, M),$$

and M is moduli space iff these two functors are naturally isomorphic. That is, a **(fine) moduli space** is a space representing the functor given by your moduli problem.

Example (isom classes of genus g curves)**.** This moduli problem does not have a fine moduli space. However, there's the next best thing: a coarse moduli space M_g. More on this later.

When $g \geq 2$, this is irreducible of dimension $3g - 3$. △

Example (curves $C \subset \mathbb{P}^r$ of degree d and genus g)**.** There is a (fine) moduli space, the Hilbert scheme H.

Known to exist, but really mysterious in general. △

Example (divisors of degree d on C)**.** Again, there is a moduli space $C_d = \operatorname{Sym}^d C := C^d/S_d$. Note it is easy to quotient a variety by a finite group.

Irreducible of dimension d. △

Example (line bundles of degree d on C)**.** There is a moduli space here too, the Picard variety $\operatorname{Pic}^d(C)$.

Irreducible of dimension g. △

7 Problem Session (9/17)

7.1 Line bundles on projective space

Fact. On $\mathbb{P}^1$, any two divisors of the same degree are linearly equivalent, i.e. there's a unique line bundle $\mathscr{O}_{\mathbb{P}^1}(d)$ of degree d.

Fact. Every curve of genus 0 is isomorphic to $\mathbb{P}^1$.

Example (tautological line bundle on $\mathbb{P}^n$)**.** Recall $\mathbb{P}^n = \{\text{one-dimensional subspaces } \ell \subset \mathbb{C}^{n+1}\}$. The **tautological line bundle** $\mathscr{O}(-1)$ has total space

$$\left\{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in \ell\right\}.$$

Similarly, $\mathscr{O}(1)$ has total space

$$\mathscr{O}(1) = \{(\ell, m) : m \in \ell^*\}.$$

Do these have sections? $\mathscr{O}(1)$ does because a section is just a choice of linear functional on each line. To get one, just choose a linear functional on $\mathbb{C}^{n+1}$ and then restrict it to each line. $\mathscr{O}(-1)$ has no sections, i.e. no nice way to choose a single point of every line (unless you just always choose 0). △

Let's get back to $\mathbb{P}^1$. The complete linear system $|\mathscr{O}_{\mathbb{P}^1}(d)|$ given an embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$. Every linear series of degree d is a subseries of $\mathscr{O}(d)$, so the induced map always fits into a diagram

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\mathscr{O}(d)"'] \arrow[rr, bend left] & \mathbb{P}^d \arrow[r, "\text{project}"'] & \mathbb{P}^n \end{tikzcd} .$$

Note that $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is the veronese embedding $[x : y] \mapsto [x^d : x^{d-1}y : \cdots : y^d]$ and the image is a so-called **rational normal curve**. Furthermore, there's only one rational normal curve of a given degree.

7.2 Rational quartics in $\mathbb{P}^3$

TODO: Understand the geometry in this example

Consider the map $[F_0(t) : F_1(t) : F_2(t) : F_3(t)] : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ with the F_i's 4 linearly independent quartic polynomials with no common zeros. What is the equation defining the image of this map? Given these F_i's, what relations do they satisfy?

We can ask, what surfaces in $\mathbb{P}^3$ contain these curves? No hyperplanes do, so what about quadrics to start? Given any quadratic polynomial on $\mathbb{P}^3$, applying it to the F_i's gives a homogeneous polynomial of degree 8 on $\mathbb{P}^1$. On other words, we have a map

$$\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2)) \longrightarrow \mathrm{H}^0(\underbrace{\mathscr{O}_C(2)}_{\mathscr{O}_{\mathbb{P}^1}(8)}).$$

Does it have a kernel? These spaces have dimensions 10 and 9, respectively. Thus, it does have a kernel, i.e. $C \subset Q$ for some quadric surface $Q \subset \mathbb{P}^3$. Since Q contains a non-deg curve, it is an irreducible quadric (not a union of planes), so it's a smooth quadric surface or a cone. Say for now that it's smooth.

Any smooth quadric surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. What is the class of the curve C on $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$? Since C has degree 4 in $\mathbb{P}^3$, it must be of type $(2, 2)$ or type $(1, 3)$ in Q. It turns out that it is of type $(1,3)$. We'll see this in a bit.

Now look at cubics containing C. Note that $\dim \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(3)) = 20$ while $\dim \mathrm{H}^0(\mathscr{O}_C(3)) = 13$, so C lies on ≥ 7 linearly independent cubics. Four of these 7 cubics come from the quadric Q (e.g. Q union a hyperplane). Each cubic surface containing C will intersect Q in C + two lines of type $(1,0)$ (the intersection has type $(3,3)$).

7.3 Rational quintics in $\mathbb{P}^3$

Say $C \hookrightarrow \mathbb{P}^3$ is genus 0 and degree 5. Does C lie on a quadric surface. In this case, we have $h^0(\mathscr{O}_{\mathbb{P}^3}(2)) = 10$ and $h^0(\mathscr{O}_C(2)) = 11$, so it's not immediately clear.

"We don't have all the tools to answer this yet, but I'll go over it anyways. This is part of why this is a terrible homework problem" (paraphrase)

Suppose $C \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then, C has type (a,b) for some $a+b=5$, i.e. type $(1,4)$ or $(2,3)$. When is a curve defined by such a bihomogeneous polynomial rational? If $(a,b)=(1,4)$, then C is indeed rational, e.g. consider $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \rightrightarrows \mathbb{P}^1$ (one of these compositions will be degree 1). This shows that C can lie on a quadric (choose a general bihomgeneous poly of degree $(1,4)$. Will cut out a smooth curve by Bertini-type argument).

Suppose $C \subset Q$ with Q now a singular quadric.

Exercise. Get a contradiction.

Hint: look at the blowup of Q at the vertex. This is the Hirzebruch surface $\mathbb{F}_2$. Look at possible divisor classes of curves on it.

Back to the case of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ smooth and C of type $(1,4)$. One ruling will cut out a pencil of quartic polynomials on C. Choose quadratic polynomials $F(t), G(t)$ on $\mathbb{P}^1$. Look at the map $\mathbb{P}^1 \mapsto \mathbb{P}^3$ given by $[t_0F : t_1F : t_0G : t_1G]$. This comes from a pencil of quartics, and we see that the image lies on the quadric surface $WX - YZ$ with homogeneous coordinates $[X:Y:Z:W]$ on $\mathbb{P}^3$.

Question 7.1. *The above shows we can have rational quintics lying on a quadric surface. Do all of them? Can we have rational quintic curves not lying on a quadric surface?*

At this point, it'd be good to make a dimension count. Everything in sight moves in some moduli space, so we can compare the sizes of the relevant spaces.

Question 7.2. *How many rational qunitics are there?*

These are parameterized by (an open subset of) a Hilbert scheme, so it makes sense to talk about the dimension of this family. Note a rational quintic is the image of a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by a 4-tuple of homogeneous quintic polynomials. What's the dimension of the space of degree 5 maps $\mathbb{P}^1 \to \mathbb{P}^3$? There's a 6 dimensional vector space of homogeneous polynomials of degree 5, so there's a 24-dimensional vector space of 4-tuples of homogeneous polynomials of degree 5. However, the underlying map is invariant under scaling, so we end up with a 23-dimensional (projective) space of candidate degree 5 maps $\mathbb{P}^1 \to \mathbb{P}^3$. The actual space of degree 5 maps will be an open in this projective space (so still 23-dimensional).

Question: Why is being a map an open condition

Now, we have a map

$$\left\{\begin{matrix}\text{degree 5 maps}\\ \mathbb{P}^1 \to \mathbb{P}^3\end{matrix}\right\} \longrightarrow \mathcal{H}$$

with $\mathcal{H}$ the relevant Hilbert scheme. The fibers are $\cong \mathrm{PGL}_2$ and so have dimension 3. Hence, $\dim \mathcal{H} = 20$.

Answer: Not a map if there's a common zero, so

Could alternately think about linear series. These correspond to maps up to automorphic of the target projective space. One would get a 5-dimensional space of linear series here, corresponding to $\mathscr{H}/\operatorname{PGL}_4$. Since $\dim \operatorname{PGL}_4 = 15$, one would still see $\dim \mathcal{H} = 20$.

Now let's try to compute the dimension of the family of rational quintics that do lie on a quartic. Call this moduli space $K \subset \mathcal{H}$. Observe: if $C \subset Q$, then Q is unique (e.g. by bezout?). Hence, get a map $K \longrightarrow \mathbb{P}^9 = \{\text{quadrics}\}$. How many rational quintic curves lie on a given quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. We answered this already:

The smooth quadrics will be an open in the base, so enough to look at them

$$\left\{\begin{matrix}\text{rational quintics}\\ \text{on } Q\end{matrix}\right\} = \mathbb{P}(\text{bihom poly of bidegree } (1,4) \text{ or } (4,1)).$$

There's a $(10 = 2 \cdot 5)$-dimensional vector space of bidegree $(1,4)$. Thus, the above space is $\mathbb{P}^9 \sqcup \mathbb{P}^9$, a disjoint union of $\mathbb{P}^9$'s. Thus, we conclude that $\dim K = 18 < 20 = \dim \mathcal{H}$.

Probably use adjunction to show that if C has digree $(1,4)$ then it's won't be genus 0

Corollary 7.3. *There exist rational quintics not lying on a quadric. In fact, the dimension counts show that for a general rational quintic curve, the map* $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2)) \to \mathrm{H}^0(\mathscr{O}_C(2))$ *is injective.*

This is true in general. A general rational curve will have the ranks of the above surjection maps maximal, proved recently (past few years) by Eric Larson.

Joe added some remarks about this giving the Hilbert function of a general rational curve, but that the possible Hilbert functions of any rational curve are still unknown? Something like this?

Question 7.4 (Audience)**.** *How did we rule out bidegree* $(2,3)$*?*

Answer. There's a **genus formula** for smooth curves $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of digree (a,b):

$$g(C) = (a-1)(b-1).$$

⋆

8 Lecture 5 (9/20)

Today

- moduli spaces + dimension counting
- examples in genus $0, 1$

As we move onto curves in $\mathbb{P}^3$, a large part of understanding them depends on understanding surfaces containing those curves. In particular, if we want to understand the canonical bundles of these curves, the main tool for doing so will be via comparison with the canonical bundles of surfaces containing them. We'll see this on Wednesday; in particular, we'll talk about the adjunction formula. After that, we'll get into Jacobians.

8.1 Moduli spaces

4 basic examples

- isom classes of curves of genus g, M_g

- curves $C \subset \mathbb{P}^r$ of degree d and genus g, Hilbert scheme $\mathcal{H}$
- divisors on a given curve C
- line bundles on a given curve C

Note that we have not constructed any of these spaces yet. On Wednesday, we'll start talking about constructing the bottom two moduli spaces. Constructing Hilbert schemes is difficult and won't be done in this class. A fine moduli space of isom classes of genus g curves does not exist.

Recall 8.1. A fine moduli space M has the following property: for any scheme B, there is a natural bijection

$$\{\text{families}/B\} \xrightarrow{\sim} \mathrm{Hom}(B, M). \qquad \odot$$

This M_g space we have been alluding to is *not* a fine moduli space, but it's close.

Warning 8.2. There exist families $\mathcal{C} \to B$ with all fibers isomorphic to a given fixed curve C, but which are non-trivial, i.e. $\mathcal{C} \not\cong C \times B$. This is a problem because both of these would correspond to the constant map $B \to M$. •

Warning 8.3. There exists maps $B \to M_g$ that do *not* come from any family. In particular, there is no universal family over M_g. •

However, if $B \xrightarrow{\varphi} M_g$ is any map, there always exists a finite, flat map $B' \xrightarrow{\pi} B$ so that $\varphi \circ \pi$ does come from a family. Also, for any families $\mathcal{C}, \mathcal{C}' \to B$ so that the corresponding morphisms $B \to M_g$ are the same, then there's a finite, flat cover $B' \to B$ so that $\mathcal{C}_{B'} \cong \mathcal{C}'_{B'}$.

Can see examples of this phenomenon e.g. in the book 'Geometry of Schemes'

Definition 8.4. The above conditions express that M_g is a **coarse moduli space**. ◇

Question 8.5 (Audience)**.** *Can we take $B' \to B$ to be étale?*

Answer (I missed some bits)**.** We can't; it's exactly the ramification that we need. $M_1 = \mathbb{A}^1_j$ is the j-line. Whenever there is a family of genus 1 curves $\mathcal{C} \to B$, if the associated j-map $B \to \mathbb{A}^1_j$ has a zero, then it must be a zero of order divisible by 3. So to get this phenomenon, we'll want a map ramified at 0. ⋆

In general, given a moduli problem, we'd like to show a moduli space M exists, and then describe M (what's its dimension? Is it irreducible? smooth? projective? etc.). Most of the work describing these spaces was down over a century before anyone proved they existed (e.g. M_g was proven to exist only in 1969 by Deligne and Mumford). However, much of what we'll see today was known already to Riemann.

Question 8.6. *If you had a compact Riemann surface on your desk, and you wanted to specify it to a friend – over the phone, or in a letter, or over email – how much bandwidth would you need?*

Giving an atlas and a sheaf of functions would work, but that's a lot of data and not immediately insightful. However, if you embed in projective space, then it's just cut out by finitely with equations with finitely many coefficients, so then you could just specify the relevant coefficients.

Slogan. Abstract curves are hard to get a handle on, but ones given with extra data (e.g. a map to projective space) are easier to handle.

Instead of looking directly at M_g, let's introduce an auxiliary space

$$H_{d,g} = \Big\{(C, f) : C \text{ genus } g \text{ and } f : C \xrightarrow{d} \mathbb{P}^1 \text{ simply branched of degree } d\Big\}.$$

Definition 8.7. By **simply branched** we mean that every branch point was exactly one ramification point over it, with ramification index 2. This is the simplest possible ramification. ◇

This $H_{d,g}$ is called a **Hurwitz space**. Here's the plan

- Describe $H_{d,g}$
- Use the map $H_{d,g} \to M_g$ to describe M_g.

Say we have a point $[C \xrightarrow{\pi} \mathbb{P}^1] \in H_{d,g}$. By Riemann-Hurwitz, the branch divisor B of π has degree $b = 2d + 2g - 2$. Since π is simply branched, this divisor is reduced, i.e. it consists of b distinct points. In other words, the branch divisor is a unordered b-tuple of distinct points in $\mathbb{P}^1$, so we get a map

$$H_{d,g} \longrightarrow \left\{ \begin{array}{c} \text{unordered } b\text{-tuples} \\ \text{of distinct pts in } \mathbb{P}^1 \end{array} \right\} =: U \overset{\text{open}}{\subset} \mathbb{P}^b = \text{poly of degree } b \text{ in } \mathbb{P}^1.$$

The b-tuples of *distinct* points correspond to polynomials with no multiple roots, i.e. the complement of the discriminant locus.

Claim 8.8. *$H_{d,g} \to U$ is a finite covering space.*

Proof Sketch. Picture $\mathbb{P}^1$ as the Riemann sphere with branch points $p_1, \dots, p_b$. Fix some auxilliary point p. Let $\gamma_1, \dots, \gamma_b$ be arcs from p to p_1. Let $S = \mathbb{P}^1 \setminus \bigcup \gamma_i$. Then, S is simply connected, so $\pi^{-1}(S) \to S$ is a d-sheeted cover of a simply connected space, i.e. $\pi^{-1}(S)$ is a disjoint union of d copies of S. Label these copies $1, 2, \dots, d$. Take a simple loop σ around a single branch point p_i. As you move around this loop, you swap two of the sheets, so you get a transposition τ_i (transposition because ramification degree 2 upstairs at a single point in the fiber, i.e. we're really using simply branched here). Thus, the cover $C \to \mathbb{P}^1$ is described by a b-tuple of transpositions $\tau_1, \dots, \tau_b$ satisfying $\tau_1 \dots \tau_b = \text{id}$ (fundamental group of sphere minus b points). Since C is connected, we need $\langle \tau_1, \dots, \tau_b \rangle \subset S_d$ to be a transitive subgroup.

How does this tuple depend on choices? If we chose a different labeling of the sheets, we'd simultaneously conjugate each of the τ_i, so to each cover we associate the data

$$\left\{ \begin{array}{c} \tau_1, \dots, \tau_b = \text{id} \in S_d \\ \langle \tau_1, \dots, \tau_b \rangle \ \text{transitive} \end{array} \right\} \Big/ \begin{array}{c} \text{simultaneous} \\ \text{conjugations} \end{array}.$$

Hence, $H_{d,g} \to U$ is a finite map of degree the number of above orbits.

Sounds like this number is known (always?), but the corresponding numbers of non-simple branches are less well understood. ∎

Corollary 8.9. $\dim H_{d,g} = \dim U = \dim \mathbb{P}^b = b = 2d + 2g - 2.$

Note that things were easier to understand in the presence of extra data.

Example. Say someone gives you an abstract Riemann surface C and asks you how many ways can you deform it? This is also hard to answer, but easy in the context of something like $H_{d,g}$. Here, to get deformations, just move around the branch points p_i. There are b of them, so get a b-dimensional space of deformations (of the map $C \to \mathbb{P}^1$). Every deformation of C will be induced by some deformation of this map, so can get a handle on things this way. △

Now, let's look at $H_{d,g} \to M_g$, $[C \to f] \mapsto [C]$.

Exercise. For $d \gg_g 0$, this is surjective. Hint: embed in projective space and then look at a general projection of the curve onto a line.

What's the fiber dimension of $H_{d,g} \to M_g$? Given a genus g curve, how many ways are there of expressing it as a (simply) branched (d-sheeted) cover of $\mathbb{P}^1$? A map of a curve to $\mathbb{P}^1$ is just a meromorphic function of the curve, so we're look at degree d meromorphic functions on C.[10] To specify a meromorphic function f, we'll first specify its polar divisor D. This is a d parameter family of choices (a degree d divisor is just d points, and be simply branched should be an open condition). Once we've fixed D, the meromorphic functions with polar divisor D is exactly (an open subset in) $\mathscr{L}(D)$. For d large, $\dim \mathscr{L}(D) = d-g+1$ by Riemann-Roch. Altogether, the dimension of the fibers is $d+(d-g+1) = 2d-g+1$. Thus,

$$\dim M_g = \dim H_{d,g} - \dim(fiber) = (2d+2g-2) - (2d-g+1) = 3g-3.$$

Warning 8.10. This answer is wrong when $g \in \{0,1\}$. The point is that curves of genus 0 or 1 have a positive dimensional automorphism group. When looking at the Hurwitz space $H_{d,g}$, we claimed the cover $f : C \to \mathbb{P}^1$ was determined (up to finite choice) by a finite number of branch points on $\mathbb{P}^1$. This is not quite true. If you have an automorphism $\varphi : C \to C$, then $\varphi \circ f$ has the same branch points. When $g = 0$, $\dim \operatorname{Aut} C = 3$ and when $g = 1$, $\dim \operatorname{Aut} C = 1$. So the coarse moduli spaces in these cases really have dimensions $\dim M_0 = (3(0)-3)+3 = 0$ and $\dim M_1 = (3(1)-3)+1 = 1$. If $g(C) \geq 2$, then $\operatorname{Aut}(C)$ is finite, so 0-dimensional. •

$3g-3$ should still be the correct dimension of the associated moduli stack $\mathcal{M}_g$, where one has to be careful about what they mean by $\mathcal{M}_1$ (I think, $\mathcal{M}_1$ with no marked points has always confused me).

Question 8.11. *Is M_g irreducible?*

Given two Riemann surfaces of genus g, can you find a continuously varying family going from one to the other? This is hard to answer, but again becomes easier when using the auxilliary space $H_{d,g}$. Recall the covering space $H_{d,g} \to U$ ($U \subset \mathbb{P}^b$). Hurwitz looked at the monodromy of this cover, and showed that it was transitive on the fibers, so $H_{d,g}$ is connected (hence irreducible) so M_g is irreducible too since $H_{d,g} \twoheadrightarrow M_g$ is dominant for large d.

The star of what we've done has been the diagram

$$\begin{tikzcd} & H_{d,g} \arrow[dl] \arrow[dr] & \\ \operatorname{Sym}^b(\mathbb{P}^1) \setminus \Delta & & M_g. \end{tikzcd}$$

Alternatively, instead of using Hurwitz spaces, we could have introduced the **Severi varieties**

$$V_{d,g} = \left\{(C,f) : C \text{ genus } g \text{ and } f : C \to \mathbb{P}^2 \text{ birational onto image which is nodal of degree } d\right\}.$$

[10] **degree of a meromorphic function** is degree of its polar divisor (or equivalently its zero divisor)

One can understand these $V_{d,g}$'s and then use the map $V_{d,g} \to M_g$ to get at M_g.

Severi didn't like Hurwitz proof of irreducibility of M_g since it was too rooted in topology. He wanted a purely algebraic proof, so he showed $V_{d,g}$ was irreducible algebraically and used this to show that M_g is irreducible. However, his proof of irreducibility for $V_{d,g}$ was completely wrong.

8.2 Curves of genus 1

Let's verify that the family of curves of genus 1 is one-dimensional. Riemann-Roch says that any degree 2 divisor on C (curve of genus 1) has 2 global sections, so gives a map $f : C \xrightarrow{2} \mathbb{P}^1$. This map will have 4 branch points and be determined by them. Via an automorphism of $\mathbb{P}^1$, these branch points can be taken to be $0, 1, \infty, \lambda$ with $\lambda \in \mathbb{P}^1$, so get 1-dimensional family.

Alternatively, a degree 3 line bundle will embed $C \hookrightarrow \mathbb{P}^2$ as a cubic. Look at space of cubics and mod out by PGL_3. A degree 4 line bundle will embed $C \hookrightarrow \mathbb{P}^3$ as the intersection of two quadrics. Look at space of intersection of two quadrics, and mod out by PGL_4.

9 Problem Session (9/24)

9.1 Problem 1

The first problem is about constructing simply branched covers of $\mathbb{P}^1$ of degree 3.

Recall the combinatorics of this setup. We have $C \to \mathbb{P}^1$ of degree d simply branched over $p_1, \dots, p_b \in \mathbb{P}^1$. We take an auxilliary point $p \in \mathbb{P}^1$, and draw arcs $\gamma_1, \dots, \gamma_b$ from p to the p_i's. The complement $\mathbb{P}^1 \setminus \bigcup \gamma_i$ is simply connected, so the covering away from these arcs is simply d disjoint copies. That is, we have

$$\begin{tikzcd} \pi^{-1}\left(\mathbb{P}^1 \setminus \bigcup \gamma_i\right) \arrow[r, equal] \arrow[d] & \bigsqcup_{i=1}^d \left(\mathbb{P}^1 \setminus \bigcup \gamma_i\right) \\ \mathbb{P}^1 \setminus \bigcup \gamma_i & \end{tikzcd}$$

Label the sheets $1, \dots, d$. Because the covering is simply branched, at each point p_i, two of the sheets come together. Similarly, if we take a simple loop around p_i, the monodromy action on the sheets will simply be some transposition $\tau_i = (\alpha_i\, \beta_i) \in S_d$. Thus, we get a b-tuple of transpositions $\tau_1, \dots, \tau_b$ satisfying $\tau_1 \dots \tau_b = 1$ $(\pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_b\}) \simeq \langle a_1, \dots, a_b \mid a_1 a_2 \dots a_b = 1\rangle$. Furthermore, since C is connected, the subgroup $\langle \tau_1, \dots, \tau_b \rangle \leq S_d$ is transitive (i.e. it's possible to get from any sheet to any other). Conversely, specifying any such b-tuple of transpositions will allow you to stitch together $\bigsqcup_{i=1}^d \left(\mathbb{P}^1 \setminus \bigcup \gamma_i\right)$ into a Riemann surface with a degree d simply branched cover of $\mathbb{P}^1$, branched at $p_1, \dots, p_b$. Note that we made a choice earlier; we chose a labelling $1, \dots, d$ of these sheets. Choosing a different labelling conjugates the τ_i's, so this tuple is well-defined only up to simultaneous conjugation.

The upshot is the following.

Proposition 9.1.

$$\#\textit{such covers} = \# \left\{ (\tau_1, \dots, \tau_b) \ \textit{transpositions} \ \middle| \ \begin{matrix} \tau_1, \dots, \tau_b = \mathrm{id} \in S_d \\ \langle \tau_1, \dots, \tau_b \rangle \ \textit{transitive} \end{matrix} \right\} \Big/ \begin{matrix} \textit{simultaneous} \\ \textit{conjugations} \end{matrix}.$$

Remark 9.2. Because the τ_i's generate a transitive subgroup, no element of S_d will leave them unchanged. That is, S_3 acts faithfully on this set of b-tuples of transpositions. ○

Example ($d = 3$)**.** S_3 has 3 transpositions. Assume b is even. The only odd permutation of 3 letters is a transposition, so enough to specify any $b-1$ transpositions $\tau_1, \dots, \tau_{b-1}$. These will be transitive as long as they're not all the same. This gives the count $(3^{b-1} - 3)/6$. △

Joe went on a bit of an aside about further directions with this branching stuff. I didn't take notes on what he said, but he ended with the following question.

Question 9.3. *Can every curve C be expressed as an even branched cover of $\mathbb{P}^1$, i.e. one where the monodromy belongs to the alternating group A_d?*

Unclear if this is known or not?

9.2 Problem 2

Let $C \subset \mathbb{P}^r$ be a smooth, irreducible, non-degenerate curve. Consider the set

$$\{H \in \mathbb{P}^{r*} : H \cap C \text{ contains a point } p \text{ with multiplicity } \geq 3\}$$

of **flex hyperplanes**.

Claim 9.4. *This has codimension 2 in $\mathbb{P}^{r*}$.*

Intuition. For a specified $p \in C$, the space

$$\Sigma_p := \{H \in \mathbb{P}^{r*} : i_p(H \cdot C) \geq 3\}$$

is a linear subspace of $\mathbb{P}^{r*}$. Locally around p, choose a local coordinate t on C so that near t, we have

Use implicit function theorem over $\mathbb{C}$ or (Weierstrass preparation on $\widehat{\mathscr{O}}_{C,p}$ if working algebraically?)

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}^r \\ t & \longmapsto & [v(t)] \end{array}$$

where v is a vector-valued function of t. At p, if $v(p), v'(p), v''(p)$ are linearly independent, then $\Sigma_p \cong \mathbb{P}^{r-3}$ is codimension 3. Now, $\bigcup_{p \in C} \Sigma_p$ is like a 1-parameter family of codim 3 subsets, so should be a codim 2 subset.

Note that v, v' are linearly independent since C smooth. If $v'' \in \operatorname{span}\{v, v'\}$, then we get $\Sigma_p \cong \mathbb{P}^{r-2}$ there. As long as this only happens for finitely many points p, we're still good.

Here's something that could go wrong: what if $v(p), v'(p), v''(p)$ are linearly dependent for all p? That is, what if $v(t) \wedge v'(t) \wedge v''(t) = 0$? We know v, v' are linearly independent, so this is equivalent to $v''(t) \in \operatorname{span}\{v(t), v'(t)\}$. Let's take a derivative of this wedge product (the product rule holds).[11] This gives

$$\pm v(t) \wedge v'(t) \wedge v'''(t) = 0.$$

This says $v'''(t) \in \operatorname{span}\{v(t), v'(t)\}$ and this pattern continues. Thus, $v^{(k)}(0) \in \langle v(0), v'(0)\rangle$. If all derivatives at this point lie in a particular 2-dimensional subspace, then the whole curve lives in this subspace, i.e. C is a line (contradicting non-degeneracy).

Instead of this wedge product stuff, can just take the derivative

Warning 9.5. This argument fails in characteristic p. There are non-constant functions all of whose derivatives vanish, e.g. $f(x) = x^p$. •

In characteristic p, look up **strange curves**.

Hartshorne talks about these in chapter 4 somewhere (section 3 exercises)

9.3 Problem 3

We claimed in class that any curve is expressible as a simply branched cover of $\mathbb{P}^1$. Let's prove this.

Say $C \subset \mathbb{P}^r$. Let $\Lambda \cong \mathbb{P}^{r-2} \subset \mathbb{P}^r$ be a general $(r-2)$-plane. We claim that the projection $\pi_\Lambda : C \to \mathbb{P}^1$ will be simply branched.

Question 9.6. *What can go wrong?*

Either

(1) Two ramification points lined up over the same point in the target; or

(2) There is some point with ramification index ≥ 3

The first issue looks like a line being tangent to two separate points (at least, this is what it looks like in $\mathbb{P}^2$); the second looks like a flex point.

(2) Says that Λ is contained in some hyperplane H so that $H \cap C$ has a point of multiplicity ≥ 3. Now consider

$$\begin{array}{c}\text{locus of such } \Lambda \\ \downarrow \\ \Sigma_C\end{array}$$

Note that $\dim \Sigma_C = r-2$ with fibers $\mathbb{G}(r-2, r-1) \cong \mathbb{P}^{r-1}$. Thus, the locus of such Λ has dimension $(r-2)+(r-1) = 2r-3 < 2(r-1) = 2r-2 = \dim \mathbb{G}(r-2, r)$.

I prolly won't be consistent about this, but let's say $\mathrm{Gr}(k, n)$ is Grassmannian of $\mathbb{C}^k \hookrightarrow \mathbb{C}^n$ and $\mathbb{G}(k,n)$ is Grassmannian of $\mathbb{P}^k \hookrightarrow \mathbb{P}^n$, so $\mathbb{G}(k,n) = \mathrm{Gr}(k+1, n+1)$.

(1) Says that $\Lambda \subset H$ where $H \cap C$ has ≥ 2 double points (i.e. two tangent lines to C). We want to estimate $\dim \{H : H \supset 2 \text{ tangent lines to } C\}$. Set up an incidence correspondence

$$\begin{tikzcd} & \{(p,q,H) : H \supset T_pC, T_qC\} \arrow[dl] \arrow[dr] & \\ C \times C & & \mathbb{P}^{r*} \end{tikzcd}.$$

Pick two points $p, q \in C$. We don't expect any two tangent lines to meet, we expect their tangent lines to be skew (to span a 3-plane). The fibers of the left map will be hypersurfaces containing that 3-plane, so they'll be $\mathbb{P}^{r-4}$'s. Hence, the total space is $2 + (r-4) = r-2$ dimensional. Thus, the image in $\mathbb{P}^{r*}$ is $\leq (r-2)$-dimensional.

Warning 9.7. This has not been completely rigorous. Things to watch out for

[11] $(\alpha(t) \wedge \beta(t))' = \alpha'(t) \wedge \beta(t) \pm \alpha(t) \wedge \beta'(t)$

- Will two general tangent lines be skew? Could there be a (smooth, irreducible, non-degenerate) curve in $\mathbb{P}^3$ such that any two tangent lines intersect?

  Answer. No in char 0, but yes in char p. $\star$

- Other things... $\bullet$

Open Question 9.8. *If $C \subset \mathbb{P}^3$ is smooth, non-degenerate, and irreduible, is it possible that every tangent line to the curve meets the curve again?*

9.4 Problem 5

If $H \subset \mathbb{P}^3$ is a general plane and $C \subset \mathbb{P}^3$ a curve, then $H \cap C$ are in general position, no 3 collinear.

How many tri-secant lines does C have? Look at

$$\left\{(p, q, r) \in C^3 \setminus \Delta : p, q, r \text{ collinear}\right\}.$$

What's the dimension of this space? Project onto first two factors p, q in $C \times C$. Is this map dominant? Look at

$$\begin{array}{c}\left\{(p, q, r) \in C^3 \setminus \Delta : p, q, r \text{ collinear}\right\} \\ \pi \Big\downarrow {\scriptstyle (p,q,r) \mapsto (p,q)} \\ C \times C.\end{array}$$

Question 9.9. *Is π dominant?*

Answer. No, in char 0. $\star$

Thus, the locus of trisecant lines is 1-dimensional. Hence, the locus of hyperplanes containing a trisecant line has dimension 2 (1-parameter family of lines, each line lying in a 1-parameter family of hyperplanes). Hence, the general hyperplane $H \subset \mathbb{P}^3$ won't contain 3 collinear points of C.

Question 9.10 (Audience)**.** *Is it clear that this locus is not* 0*-dimensional instead of* 1*-dimensional?*

Answer. The idea is something like this. We're looking at pairs of points on the curve, and asking whether the line through them meets the curve in another point. So we have a 2-parameter family of lines. The condition of meeting the curve is a single condition, so we should expect the lines that meet it a third time to be codimension 1 in this 2-dimensional family. $\star$

10 Lecture 7 (9/27): Jacobians

Remark 10.1. In genus $0, 1$, all line bundles of a given degree behave the same. $\circ$

In genus 0, this is because there's only one line bundle of a given degree. In genus 1, this is because $\operatorname{Aut} C$ acts transitively on the set of line bundles of a given degree.

This remark no longer holds in genus $g \geq 2$. To talk about this, we'll need a parameter space for line bundles. Actually, we'll see two moduli spaces, for

- effective divisors of degree d; and for

- line bundles of degree d

The first moduli space is simply the d**th symmetric product/power** of C, i.e. the space $C_d := C^d/S_d$.[12] Note that C_d has the desired universal property: for all B, there is a natural bijection

$$\left\{ \begin{array}{c} \text{families of effective} \\ \text{degree } d \text{ divisors on } C \end{array} \right\} \longleftrightarrow \operatorname{Hom}(B, C_d).$$

Recall 10.2. A **family of effective degree d divisors on C over B** is a subscheme $\mathcal{D} \subset B \times C$ flat over B and of degree d. $\odot$

Warning 10.3. Everything said up to now applies if C is an arbitrary variety of arbitrary dimension. However, C_d is smooth only if $\dim C = 1$ (and C smooth). $\bullet$

Example. When $C = \mathbb{P}^1$, $C_d \cong \mathbb{P}^d$ is the (projectiviation of) the space of homogeneous degree d polynomials. $\triangle$

Note 10. I couldn't hear over the AC, but sounded like maybe Joe said this $\mathbb{P}^1$ example is enough to conclude smoothness of C_d for all curves for some reason?

History. Here's a story. It starts w/ calculus, in particular with integrals e.g.

$$\int_{t_0}^{t} \frac{\mathrm{d}x}{\sqrt{x^2+1}}.$$

When Joe was a graduate student, this problem appeared on his quals. He assumed it involved complex analysis and whatnot, and spent like half an hour on it. Later, he learned Barry Mazur put this problem on the quals as a problem in algebraic geometry. Barry was hinting that there's a much more uniform way to approach these sort of integrals.

Think of this integrand as a line integral on the Riemann surface associated to $C : y^2 = x^2 + 1$, i.e. think of it as $\int \mathrm{d}x/y$. Observe that C is rational (genus 0), i.e. $\exists \mathbb{P}^1 \xrightarrow{\sim} C$, $t \mapsto (x(t), y(t))$. Now, we can pull back in order to express the original integral as something like

$$\int R(t)\mathrm{d}t \text{ with } R \text{ rational}$$

(do this by partial fractions). This sort of reasoning applies more generally to integrals of this form (even ones where trig substitution doesn't work so cleanly). However, it only applies when the associated curve is rational.

Next: look at

$$\int_{t_0}^{t} \frac{\mathrm{d}x}{\sqrt{x^3+1}}.$$

This is a much harder problem, and for a long time, it wasn't clear why. We can again think of this as

$$\int \frac{\mathrm{d}x}{y}$$

on the curve $E : y^2 = x^3 + 1$. This curve is not rational, it is genus 1. In particular, E is *not* simply connected, so integrals on E are *not* path independent. Letting $\alpha, \beta \in \mathrm{H}_1(E; \mathbb{Z})$ be generators, the value

[12] Taking quotients of a quasi-projective space by a finite (abstract) group is always possible

of the integral $\int_p^q \omega$ is only well-defined modulo linear combinations of $\int_\alpha \omega, \int_\beta \omega$. Once you see this, you see that this cannot be an elementary function, e.g. since the inverse function is a double periodic function on $\mathbb{C}$ (no elementary function on $\mathbb{C}$ can be doubly periodic). If you want to see more about doubly periodic functions, can look is Ahlfors chapter 7. $\ominus$

What about genus $g \geq 2$? Fix two points $p, q \in C$, and also fix a basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \in \mathrm{H}_1(C; \mathbb{Z})$. Given a holomorphic differential ω on C, we again have that the integral $\int_p^q \omega$ is only well-defined modulo integral linear combinations of $\int_{\alpha_i} \omega, \int_{\beta_i} \omega$.

Remark 10.4. When $g = 1$, you are looking at linear combinations of two complex numbers, so you expect them to generate a lattice. When $g \geq 2$, we have $2g$ complex numbers. "$\mathbb{C}$ modulo linear combinations of $2g$ complex numbers is one of the spaces you scare children with." $\circ$

The way to get around this nastiness is to not consider a single differential form ω, but instead to consider all differential forms (note $\mathrm{H}^0(K_C)$ is g-dimensional). Fix a path γ and consider $\int_\gamma$ as a linear function on $\mathrm{H}^0(K_C)$. In this way, we get a map

$$\mathrm{H}_1(C; \mathbb{Z}) \longrightarrow \mathrm{H}^0(K_C)^*$$

(This map is even injective). Now, given two points p, q, we can think of $\int_p^q$ (without specifying a path) as a well-defined element of the quotient

$$\int_p^q \in \mathrm{H}^0(K_C)^* / \mathrm{H}_1(C; \mathbb{Z}).$$

This space is a g-dimensional complex vector $\mathrm{H}^0(K_C)^*$ space modulo a rank $2g$ lattice $\mathrm{H}_1(C; \mathbb{Z})$. We call this quotient space the **Jacobian** of C, $J(C) := \mathrm{H}^0(K_C)^* / \mathrm{H}_1(C; \mathbb{Z})$. Note that this is a complex torus of (complex) dimension g.

Definition 10.5. Fixing a base point $p_0 \in C$, we get a well-defined map

$$\int_{p_0} : C \xrightarrow{\mu} J(C).$$

This is called the **Abel-Jacobi map**. $\diamond$

We can extend this to symmetric powers. More generally, we have

$$\begin{array}{cccc} \mu_d : & C_d & \longrightarrow & J(C) \\ & D = \sum p_\alpha & \longmapsto & \sum \int_{p_0}^{p_\alpha} \end{array}.$$

Theorem 10.6 (**Abel's Theorem**). *If $D, E \in C_d$ any two divisors of degree d, then*

$$\mu(D) = \mu(E) \iff D \sim E,$$

i.e.

$$\begin{array}{ccc} \left\{ \begin{array}{c} \textit{line bundles of} \\ \textit{degree } d \textit{ on } C \end{array} \right\} & \longrightarrow & J(C) \\ D = \sum n_i p_i & \longmapsto & \sum n_i \int_{p_0}^{p_i} \end{array}$$

is a bijection (note n_i potentially negative above).

Note this (more-or-less) says that the Jacobian is the parameter space for linearly equivalence classes of divisors.

This all seems like complex analysis, and it's not even clear that this $J(C)$ is algebraic (most complex tori are not algebraic). For a long time, there was a campaign to try to define the Jacobian algebraically (in the first half of the 20th century; this helped lead to the notion of an abstract variety).

Let's look at the maps $\mu_d : C_d \to J(C)$. By Abel's theorem, the fibers of μ_d are the complete linear series of degree d, i.e. if $D \in C_d$, then $|D| = \mu_d^{-1}(\mu_d(D))$.

Example. If $d \leq g$, d random points on $C \hookrightarrow \mathbb{P}^{g-1}$ will have no linear relations. By geometric Riemann-Roch, this is saying that for general $D \in C_d$, $r(D) = 0$. Hence, μ_d will be birational onto its image.

Remember: $\mu_d : C_d \to J$ is birational onto its image when $d \leq g$, and is surjective when $d \geq g$.

When $d \geq g$, μ_d will be surjective. The general fiber $\cong \mathbb{P}^{d-g}$ (general D will span all of $\mathbb{P}^{g-1}$ + geometric Riemann-Roch). △

When $d = g$, this says that $\mu_g : C_g \to J(C)$ is birational. This what inspired André Weil to define the notion of abstract variety. C_d is clearly algebraic; Weil proposed to use this map to give coordinate charts on the Jacobian. This map given an isomorphism between an open in C_d and an open $J(C)$, and then you can patch together (translates?) of these opens to describe $J(C)$ abstractly.

Recall 10.7 (Abel, really Abel-Clebsch)**.** $D \sim E \iff \mu(D) = \mu(E)$ ⊙

Proof of $\implies$. (The other direction is substantially harder, and wasn't done by Abel; it was done by Clebsch).

Let $\mathscr{L} = \mathscr{O}(D) \simeq \mathscr{O}(E)$. Then, $D = (\sigma)$ and $E = (\tau)$ for some $\sigma, \tau \in \mathrm{H}^0(\mathscr{L})$. Define $D_\lambda = (\lambda_0\sigma + \lambda_1\tau)$ for $\lambda = [\lambda_0 : \lambda_1] \in \mathbb{P}^1$. This gives a map $\mathbb{P}^1_\lambda \to C_d$. Take the composition

$$\mathbb{P}^1_\lambda \longrightarrow C_d \xrightarrow{\mu_d} J(C).$$

There are no non-constant maps from $\mathbb{P}^1$ to a complex torus! This is because complex tori have lots of global 1-forms (they're Lie groups so cotangent bundle trivial), but $\mathbb{P}^1$ has none. Hence, the differential of this map must be 0 everywhere, so the map is constant. ■

Warning 10.8. We said the Jacobian parameterizes line bundles of a given degree. This however only works if we fix a basepoint. We'd like to talk about things in a way that doesn't require basepoints. •

Notation 10.9. We set

$$\mathrm{Pic}(C) = \{\text{all line bundles on } C\},$$

and

$$\mathrm{Pic}^d(C) = \{\text{line bundles of degree } d \text{ on } C\}.$$

Note we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}(C) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

and that $\mathrm{Pic}(C) = \bigsqcup \mathrm{Pic}^d(C)$. Abel's theorem is saying that $\mathrm{Pic}^0(C) \cong J(C)$.

Recall 10.10 (Last Wednesday)**.** Any line bundle of degree $\geq 2g+1$ on a curve of genus g is very ample. Can we do better? $\odot$

Proposition 10.11. *Let C be any curve of genus g. Let $\mathscr{L}$ be a general line bundle of degree $g+3$. Then, $\varphi_{\mathscr{L}} : C \hookrightarrow \mathbb{P}^3$ is an embedding (in particular, $\mathrm{H}^0(\mathscr{L}) = 4$).*

We'll prove this by dimension count. Let's first set up some notation.

Notation 10.12. Let

$$W_d = \left\{ \begin{array}{c} \text{lin. eq. classes of} \\ \text{eff. divisors of deg. } d \end{array} \right\} = \left\{ \mathscr{L} \in \mathrm{Pic}^d(C) : h^0(\mathscr{L}) \geq 1 \right\} \subset \mathrm{Pic}^d(C).$$

Similarly let

$$W_d^r = \left\{ \mathscr{L} \in \mathrm{Pic}^d(C) : h^0(\mathscr{L}) \geq r+1 \right\}.$$

In other words, we have $\mu_d : C_d \to \mathrm{Pic}^d(C)$ and W_d^r is the locus of the image with fiber dimension $\geq r$ (in particular, $W_d = \operatorname{im} \mu_d$).

Proof of Proposition 10.11. Fix some $L \in \mathrm{Pic}^{g+3}(C)$. When is L not very ample? This is the case iff there exists a divisor $D = p + q$ of degree 2 so that $h^0(L - D) \geq h^0(L) - 1$. Say $h^0(L) = 4$.[13] Then,

$$h^0(L - D) \geq 3 \iff h^0(K - L + D) \geq 1$$

by Riemann-Roch. Let $E = K - L + D$ and note $\deg E = g - 3$. That is, L is not very ample iff $L \in K + W_2 + W_{g-3}$ (i.e. $L = K + D - E$). This locus has dimension $\leq 2 + g - 3 = g - 1$, so a general line bundle of degree $g + 3$ will not be of this form. ∎

11 Problem Session (10/1)

11.1 Problem 1

We have a complex torus $X = V/\Lambda$, so $V \cong \mathbb{C}^n$ and $\Lambda \subset V$ is a full rank lattice. All the tangent spaces of X are identified with the vector space V. Note that X will not in general be an algebraic variety, just a complex manifold. If $n = 1$, it will be, but if $n > 1$, it may not be.

If you choose a random lattice (throw 4 darts at $\mathbb{C}^2$), you will get a compact complex manifold with no nonzero meromorphic functions at all.

Recall 11.1. Jacobians are algebraic varieties. $\odot$

Take $X = J(C) = \mathrm{H}^0(K_C)^* / \mathrm{H}_1(C, \mathbb{Z})$, the Jacobian of a curve C. Choosing a basepoint $p_0 \in C$, we get a map

$$C \longrightarrow X, \ \ p \mapsto \int_{p_0}^{p}.$$

Can also take sums, and so get a map $C_d \to X$. We have this embedding $C \hookrightarrow J(C)$. What is the tangent space to $\mu(C) = W_1 \subset J(C)$ at the point $\mu(p)$?

[13] This is true for a general line bundle of degree $g + 3$. Think back to the geometric Riemann-Roch example where we said the general fiber of μ_d (for $d \geq g$) is $\cong \mathbb{P}^{d-g}$.

Let's differentiate. I didn't quite follow, but something like: we choose a path from p_0 to p, and then let p vary in a small neighborhood. The derivative of the integral $\int_{p_0}^{p} \omega$ w.r.t. the point p is the value of the cotangent vector at p, i.e. is $\omega(p)$. Thus, the tangent space to W_1 at $\mu(p)$ is span $\{\omega_1(p), \ldots, \omega_g(p)\}$ where $\omega_1, \ldots, \omega_g \in \mathrm{H}^0(K_C)$ is a basis.

Note that the tangent space to J at $\mu(p)$ is $\mathrm{H}^0(K_C)^*$. Evaluation at the point p is a nonzero element of this vector space, so it spans a line. That line is the tangent line of $C \hookrightarrow J(C)$ at p.

Let $\mathcal{G} : W_1 \to \mathbb{P}^{g-1}$ be the Gauss map (sending point to its tangent line in $\mathrm{H}^0(K_C)^*$). Then, the composition $C \xrightarrow{\mu} W_1 \xrightarrow{\mathcal{G}} \mathbb{P}^{g-1}$ is simply the canonical map

$$\varphi_K : C \longrightarrow \mathbb{P}\,\mathrm{H}^0(K_C)^*$$

(sending a point $p \in C$ to the hyperplane cut out by evaluation at p).

Thinking of the curve as embedded in its Jacobian, the tangent line at p is the line representing $\varphi_K(p) \in \mathbb{P}^{g-1}$.

Now, for any d with $1 \leq d \leq g-1$, we have

$$W_d = \mu_d(C_d) \xrightarrow{\mathcal{G}} \mathrm{Gr}(d, \mathrm{H}^0(K_C)^*).$$

One similarly gets that the tangent space to W_d at $\mu(D)$ is $\overline{D}$, the span of D in $C \subset \mathbb{P}^{g-1}$. What if the points of the divisor D are linearly dependent (don't span a $\mathbb{P}^{d-1}$)? Then μ_d is not 1-1 near D. The map $C_d \to W_d$ is birational onto its image. To get an embedding, restrict to $C_d \setminus C_d^1 \hookrightarrow W_d$ (with image $W_d \setminus W_d^1$).

Let's now consider $d = g-1$. We have

$$C_{g-1} \longrightarrow W_{g-1} \subset J(C).$$

For a divisor $D = \sum p_i$, the tangent space $T_{\mu(D)}W_{g-1} = \overline{p_1 \ldots p_{g-1}}$. Applying the Gauss map, we get a map

$$C_{g-1} \longrightarrow W_{g-1} \xrightarrow{\mathcal{G}} \left\{\text{hyperplanes in } \mathbb{P}^{g-1}\right\}$$

(this composition is rational. Need divisor not moving in a linear series, i.e. with linearly independent points). In this case, both the source and the target have the same dimension, so it has a degree, the cardinality of the fiber over a general point. Take a general hyperplane H in $\mathbb{P}^{g-1}$. It will intersect C is $2g-2$ distinct points $p_1, p_2, \ldots, p_{2g-2}$. How many divisors of degree $g-1$ have H as the hyperplane they span. We want $g-1$ points among $p_1, \ldots, p_{2g-2}$ which span H. We appeal to the general position lemma: it tells us that if we have a general hyperplane, then any $g-1$ points of the intersection will be linearly independent. Thus, the degree is $\binom{2g-2}{g-1}$.

11.2 Problem 2

Let C be a curve of genus 2, and let $L \in \mathrm{Pic}^5(C)$ be a degree 5 line bundle. The map $\varphi_L : C \hookrightarrow \mathbb{P}^3$ will be an embedding by Riemann-Roch. Ask about the geometry of C. We start with asking about which surfaces contain C. Does it lie on a quadric? Consider

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2))}_{\dim=10} \longrightarrow \mathrm{H}^0(\mathscr{O}_C(2)) = \underbrace{\mathrm{H}^0(2L)}_{\dim=9}.$$

Since $9 < 10$, we conclude that C lies on a(n irreducible) quadric Q. Furthermore, Q is unique by Bézout $(5 \neq 4)$.

Question 11.2. *Is Q smooth or singular?*

Let's first suppose Q is smooth Let's now look at cubics containing C. We have

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(3))}_{\dim=20} \longrightarrow \mathrm{H}^0(\mathscr{O}_C(3)) = \underbrace{\mathrm{H}^0(3L)}_{\dim=14}.$$

Thus, C lies on at least 6 linearly independent cubics. We know 4 of these cubics already (the unions of Q with (coordinate) hyperplanes). We get two legitimately new cubics S, S'. Let's apply Bézout again. We know $Q \cap S$ is a curve of degree 6 containing C, so $Q \cap S = C \cup L$ ($L \cong \mathbb{P}^1$ a line). Since $L \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, it must be a line of one of the two rulings. Thus, C is a curve of type $(2,3)$ (or $(3,2)$) on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. The point is that $C \cup L$ will be of type $(3,3)$.

Could also see this from the genus formula

We can write the line bundle $L = \mathscr{O}_C(1)$ on C as $L = g^1_2 + g^1_3$. The point is that the rulings on Q cut out pencils of degree $2, 3$ on C. There is a unique pencil of degree 2 (coming from the canonical bundle).

Now suppose that Q is singular Here, Q is a cone over a smooth conic curve in the plane. We still get the pair of cubics S, S' as before. $Q \cap S = C \cup L$ is again C union a line by Bézout. The union of C with a line will be the complete intersection of Q with some cubic surface, and so choosing a line again gives rise to a pencil of degree 3 on C. We again get $L = g^1_2 + g^1_3$.

Which is the general case In either case $L = g^1_2 + g^1_3$. The difference is that the g^1_3 has no basepoint when Q is smooth, but does have one when Q is singular (the cone point). If it has a basepoint, then $g^1_3 = p + g^1_2$, i.e. $L = 2K + p$. If it does not have a basepoint, then $L \neq 2K + p$. Hence, the general case is Q smooth.

This is maybe the first example we've looked at in detail where the particular line bundle matters, and not just its degree.

Question 11.3 (Audience)**.** *Can you make an argument work where you send* Pic^5 *to the space of quadrics in* $\mathbb{P}^3$*, and then use this to argue that a general line bundle will give you a smooth quadric?*

Answer (paraphrase)**.** You want to say if you have a family of degree 5 line bundles on C, each lies on a unique quadric, so you can send each member of the family to the quadric it lies on. You need to show that the result is actually a *family* of quadrics in $\mathbb{P}^3$. Then, things work. A generic member of the family of quadrics will be smooth, and the singular ones will correspond to a proper subvariety on the base. However, showing you get a family of quadrics is not so easy; it requires digging through some of chapter III of Hartshorne to justify.

Say we have a line bundle $\mathscr{L}$ on $B \times C$, and let $\pi : B \times C \to B$ be the projection map. We want to look at the map

$$\pi_* \mathscr{O}_{B \times \mathbb{P}^3}(2) \longrightarrow \pi_* \mathscr{O}_{B \times C}(2)$$

of sheaves coming from the restriction map $\mathscr{O}_{B\times\mathbb{P}^3}(2) \to \mathscr{O}_{B\times C}(2)$. The LHS will be a vector bundle of rank 10 while the RHS is a vector bundle of rank 9. The kernel of this map will be a line bundle giving the actual family of quadrics. To make this rigorous, use cohomology and base change. ⋆

11.3 Problem 4

Given a smooth (projective) variety X, and a smooth subvariety $Y \subset X$ of codimension 1, we ask: how many double covers $Z \longrightarrow X$ are there which are branched exactly over Y?

Example. If $X = \mathbb{P}^1$, then Z is unique if $\deg Y$ is even and doesn't exist otherwise. △

Claim 11.4.

$$\left\{ \begin{array}{c} \textit{double covers } Z \longrightarrow X \\ \textit{branched over } Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{line bundles } \mathscr{L} \\ \textit{s.t. } \mathscr{L}^2 \cong \mathscr{O}(Y) \end{array} \right\}$$

We don't have a ton of time, but we can at least see what the correspondences are right now. Verifying that they work is up to you.

We first go $\leftarrow$. Start with a line bundle $\mathscr{L}$ on X with $\mathscr{L}^2 \cong \mathscr{O}(Y)$. We have a section σ of $\mathscr{O}_X(Y)$ that vanishes on Y. We'd like to take the square root of σ. Look at the set of pairs

$$\left\{(p, \tau) : p \in X, \tau \in \mathscr{L}_p, \text{ and } \tau^2 = \sigma(p)\right\}$$

(use identification $\mathscr{L}^2 \cong \mathscr{O}_X(Y)$ to make sense of $\tau^2 = \sigma(p)$). This space is a double cover of X branched over Y.

In the other direction, say we have $Z \xrightarrow{\pi} X$ double cover branched over Y. Look at the sheaf $\pi_*\mathscr{O}_Z$. This will be a vector bundle $\mathscr{E}$ of rank 2. Furthermore, it has an involution coming from the involution $\mu : Z \xrightarrow{\sim} Z$ exchanging the two sheets. Thus, $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathscr{E}$. Look at the eigenspaces. We get a decomposition $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$. The $(+1)$-eigenspace corresponds to functions pulled back from X, so $\mathscr{E} \cong \mathscr{O}_X$. One can check that the line bundle $\mathscr{L} := (\mathscr{E}^-)^*$ has square $\mathscr{O}(Y)$.

Question 11.5. *If C is a genus g curve with evenly many points $p_1, \dots, p_b \in C$, how many double covers are there branched over exactly these points?*

Answer. We want to count the size of $\left\{L \in \mathrm{Pic}(C) : L^2 = \mathscr{O}_C(D)\right\}$. If L, L' are any two elements of this set, then $M = L' \otimes L^{-1}$ is a line bundle so that $M^2 = \mathscr{O}_C$. Thus, we're really counting 2-torsion points on the Jacobian. Hence, the number of such L is exactly 4^g, the number of points of order two on $\mathbb{C}^g/\Lambda$ $(\Lambda \cong \mathbb{Z}^{2g})$. The 2-torsion subgroup here is $\frac{1}{2}\Lambda/\Lambda \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$. ⋆

This set is a torsor for $\mathrm{Pic}[2] = \mathrm{Pic}^0[2]$

12 Lecture 9 (10/4): The Uniform Position Lemma

The uniform position lemma is a key fact governing the hyperplane sections of a curve.

Recall 12.1. The general position lemma says that hyperplane sections of curves are in general position. If $C \subset \mathbb{P}^r$ and H a general hyperplane, then no r points of $H \cap C$ are linearly dependent in $H \cong \mathbb{P}^{r-1}$. ⊙

12.1 Monodromy

Let's start our discussion with a construction. Say X, Y are two varieties of the same dimension n, and let $f : X \to Y$ be a dominant map. We can throw away subvarieties of Y to arrive at an open subset $U \subset Y$ so that

$$V := f^{-1}(U) \xrightarrow{f} U$$

is a topological covering space (recall we're working over $\mathbb{C}$); also can arrange for V, U to be smooth.

To get this, throw away the singular locus of X, and its image in Y. Also throw away the subset of Y where the map has positive-dimensional fibers. At this point, we have a quasi-finite map between smooth varieties. At this point, look at the ramification divisor in X (where the map fails to be a submersion). Throw this away, throw away its image in Y, and throw away the preimage of its image back in X. This is not a covering space.

Say $f : V \to U$ has degree d. Choose a base point $p \in U$. Since f is a covering space, if γ is a loop based at p – i.e. $\gamma : [0,1] \to U$ with $\gamma(0) = p = \gamma(1)$ – and $q \in f^{-1}(p)$, then there is a unique lift $\widetilde{\gamma} : [0,1] \to V$ so that $f \circ \widetilde{\gamma} = \gamma$ and $\widetilde{\gamma}(0) = q$. Thus, the association $q \mapsto \gamma(1) \in f^{-1}(p)$ gives a permutation, associated to γ, of $f^{-1}(p) = \{q_1, \ldots, q_d\}$. This permutation depends only on the homotopy class of the loop γ, so we get a map

$$\pi_1(U, p) \longrightarrow \operatorname{Perm}(f^{-1}(p)) \cong S_d.$$

That is, the fundamental group of the base acts on the fibers above a point.

This does not depend on the choice of U (it's local to p). If $U' \subset U$ is a smaller subset containing p, then $\pi_1(U', p) \twoheadrightarrow \pi_1(U, p)$. Given that U is smooth, you can't block a loop in U by removing a proper subvariety (proper subvarieties have real codimension 2, so can go around them). Thus, the image of the composition

$$\pi_1(U', p) \twoheadrightarrow \pi_1(U, p) \to S_d$$

is always the same. This image M is called the **monodromy group** of the map $f : X \to Y$.

Assumption. Assume throughout that Y is irreducible, but don't assume the same about X. We however, *do* assume that every irreducible component of X dominates Y.

Remark 12.2. X is irreducible $\iff$ the monodromy group is transitive.

If X has multiple components, you can't get from one to the other via loops on U since the points of intersection of the component do not lie over U (they're singular points). ◦

Keep in mind the setup

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "f"] \\ U \arrow[r, "\overset{\text{open}}{\subset}"] & Y \end{tikzcd}$$

Consider the fiber product

$$V \times_U V = \{(p, q_1, q_2) : q_1, q_2 \in f^{-1}(p)\} \longrightarrow U.$$

The diagonal Δ is an irreducible component of $V \times_U V$. Name the complement

Question: Why?

$$V^{(2)} = V \times_U V \setminus \Delta \longrightarrow U.$$

Note the above is dominant with degree $d(d-1)$. The monodromy group M is *twice transitive* (i.e. transitive on pairs of distinct points) $\iff$ $V^{(2)}$ is irreducible. In either case, you're just asking for a loop on U which sends (q_1, q_2) to (q_1', q_2') (for given q_1, q_2, q_1', q_2'). In general, we define

$$V^{(k)} := V \times_U V \times_U \ldots \times_U V \setminus (\text{locus where any two coordinates coincide}).$$

Proposition 12.3. *M is k-transitive $\iff$ $V^{(k)}$ is irreducible.*

Remark 12.4 (An algebraic version of this story). This won't be necessary for us, but worth pointing out regardless. The dominant map $f : X \to Y$ we started with gives an inclusion of function field $K(Y) \hookrightarrow K(X)$. We let $\overline{K(X)}$ be the Galois closure of $K(X)/K(Y)$. In this case, the monodromy group we described is simply $M = \mathrm{Gal}(\overline{K(X)}/K(Y))$. ∘

Let $K(X)$ be the product of the function fields of its components if you really want reducible X.

12.2 Application to curves

Let $C \subset \mathbb{P}^r$ be an irreducible non-degenerate curve (not necessarily smooth). We want to consider a general hyperplane $H \in \mathbb{P}^{r*}$, and say something about the intersection $H \cap C$. If we want to use the hypothesis that H is *general*, it's not enough to consider it by itself (what we want won't be true for an arbitrary hyperplane). Instead, we should consider all hyperplanes at once. To this end, we introduce the **universal hyperplane section**

$$\Phi = \{(H, p) \in \mathbb{P}^{r*} \times C : p \in H\} \longrightarrow \mathbb{P}^{r*}.$$

This fibers of the above map are precisely the hyperplane sections of C, so this map is finite. Let

Question: What is C^* if C is not smooth?

$$\mathcal{U} = \{H \in \mathbb{P}^{r*} : H \pitchfork C\} = \mathbb{P}^{r*} \setminus C^*$$

(H transverse to C, i.e. misses singular points and is not tangent to any point). The induced map

$$\mathcal{V} = f^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$$

is now a covering space. As you move H along a loop of transverse hyperplanes, the intersection points themselves vary unambiguously along with H. When the loop completes, the set of points returns to the original set of points, but they have been reordered.

Theorem 12.5. *If M is the monodromy group of the cover*

$$\begin{array}{ccc} \mathcal{V} & \subset & \Phi \\ \Big| & & \Big| \\ \mathcal{U} & \subset & \mathbb{P}^{r*}, \end{array}$$

then $M = S_d$.

Warning 12.6. This is false in positive characteristic! •

Proof of Theorem 12.5. We start with a simple fact. Given $M \subset S_d$, one has

$$M = S_d \iff M \text{ is twice transitive and contains a transposition}$$

(given the RHS, M contains all transpositions).

To show that the monodromy group M in our situation satisfies these, first consider

$$\mathcal{V}^{(2)} = \{(H, p, q) : p, q \in H \cap C, p \neq q\} \longrightarrow \mathcal{U}.$$

Recall that M twice transitive iff $\mathcal{V}^{(2)}$ is irreducible. For this, we look at the zig-zag

$$\begin{tikzcd} & \mathcal{V}^{(2)} \arrow[dl] \arrow[dr] & \\ \mathcal{U} & & C \times C. \end{tikzcd}$$

The fibers of the right map are $\cong$ opens in $\mathbb{P}^{r-2}$. Since $C \times C$ is irreducible and the fibers of irreducible of the same dimension, we conclude that $\mathcal{V}^{(2)}$ is irreducible as well. Thus, the monodromy is transitive.

Now, we just need to show that M contains a transposition. To show this, we'll start with a hyperplane H_0 which is transverse to C at $d-2$ points, but is simply tangent to C at one point. That is, we claim there exists $H_0 \in \mathbb{P}^{r*}$ so that $H \cap C = 2p + q_1 + \cdots + q_{d-2}$ for some distinct smooth points $p, q_1, \ldots, q_{d-2} \in C$. This is the part that's false in positive characteristic. Note that $H_0 \in \mathbb{P}^{r*} \setminus \mathcal{U}$. We look at neighborhood Δ of $H_0 \subset \mathbb{P}^{r*}$ and the corresponding neighborhoods of $(H, p), (H, q_i) \in \Phi$. The neighborhoods around (H, q_i) will map isomorphically onto Δ, but the one around (H, p) will be a 2-sheeted cover of Δ. Call this neighborhood $\widetilde{\Delta}$. Now, since Φ is smooth, $\widetilde{\Delta}$ cannot be disconnected by removing a proper subvariety. In particular, the locus of $\widetilde{\Delta}$ consisting of transverse hyperplanes is connected. Thus, we can draw an arc γ of transverse hyperplanes with endpoints contained in some fiber. Then the image of this γ is a loop inducing a transposition. ■

Theorem 12.7 (General Position Lemma). *Let $C \subset \mathbb{P}^r$ be an irreducible, non-degenerate curve. Let $H \subset \mathbb{P}^r$ be a general hyperplane. Write $H \cap C = \{q_1, \ldots, q_d\}$. Then, no r points among $\{q_1, \ldots, q_d\}$ are linearly dependent.*

Proof. Consider

$$\begin{tikzcd} \mathcal{V} \arrow[d, no head] & \subset & \Phi \arrow[d, no head] \\ \mathcal{U} & \subset & \mathbb{P}^{r*}, \end{tikzcd}$$

all as before. Look at the rth fiber product $\mathcal{V}^{(r)} = \{(H, p_1, \ldots, p_r) : p_i \text{ distinct} \in H \cap C\}$. Let $\mathcal{W} \subset \mathcal{V}^{(r)}$ be the locus of $(H, p_1, \ldots, p_r)$ such that $p_1, \ldots, p_r$ are linearly dependent. This is the bad locus; we wish to show that it cannot dominate $\mathcal{U}$. The point is that $\mathcal{W}$ is a closed subvariety; furthermore, it is a *proper* subvariety since there exists r-tuples of points which are linearly independent. Since $\mathcal{V}^{(r)}$ is irreducible, we have $\dim \mathcal{W} < \dim \mathcal{V}^{(r)} = \dim \mathcal{U}$, so $\dim \mathcal{W}$ cannot dominate $\mathcal{U}$ and we win. ■

Note 11. Got distracted and missed some stuff Joe was saying about how to interpret this result/proof, I think.

Question 12.8. *When $r = 3$, the general position lemma says that given $C \subset \mathbb{P}^3$ irreducible and $H \subset \mathbb{P}^3$ a general plane, the intersection $H \cap C$ does not contain 3 colinear points.*

Here's a proposed monodromy-less argument for this. General $H \cap C \ni 3$ col. points implies that the locus of trisecant lines is 2-dimensional. Hence, every chord to C meets C again. Writing $H \cap C = \{p_1, \dots, p_d\}$, we get that $p_1, \dots, p_d \in \mathbb{P}^2$ so that the line joining any two contains a third.

So we've arrived at d points in $\mathbb{P}^2$ which are not simultaneously collinear, but for which any line through 2 of them contains a third one of them. Does such a configuration exist?

Apparently there is such a configuration, but only one is known. It's unknown if there are others.

13 Problem Session (10/8)

Note 12. Roughly 7 minutes late

13.1 Arithmetic Genus

Definition 13.1. The **arithmetic genus** of a curve is $p_a(C) = 1 - \chi(\mathscr{O}_C)$ ⋄

This definition works for all curves (smooth, singular, non-reduced, not of pure dimension 1, etc.)

Example. Take an irreducible curve C union an isolated point P. What's the genus of $p_a(C \cup P)$? The H^1 doesn't change, but the H^0 goes up by one, so $p_a(C \cup P) = p_a(C) - 1$. △

This generality will be especially important when we talk about Hilbert schemes.

Consider, for example, twisted cubics $C \subset \mathbb{P}^3$. Such a curve has Hilbert polynomial $p_C(m) = 3m + 1$. Recall the Hilbert polynomial measures the rank of

$$\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(m)) \longrightarrow \mathrm{H}^0(\mathscr{O}_C(m)) = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(3m)).$$

Now, given a Hilbert polynomial, the associated Hilbert scheme will parameterize all subschemes of $\mathbb{P}^3$ with the given polynomial. So the 'family of twisted cubics' is captures by the Hilbert scheme of subsets of $\mathbb{P}^3$ with Hilbert polynomial $p_C(m) = 3m + 1$.

Question 13.2. *What other things are captured by this scheme, do e.g. plane cubics have the same Hilbert polynomial?*

Answer. No, a plane cubic $C \subset \mathbb{P}^2$ has Hilbert polynomial $p_C(m) = 3m$. ⋆

Now, consider this. Take a plane cubic $C \subset \mathbb{P}^2 \subset \mathbb{P}^3$ union with a point $P \in \mathbb{P}^3 \setminus C$. Then, $p_{C \cup P}(m) = 3m + 1$, so even if we don't care about these 'curves', they will appear in the Hilbert scheme.

13.2 Problems 2,3

In problems 2 – 4, we'll see a simpler generalization of genus. Let C be a reduced pure dimensional curve, and let $\nu : \widetilde{C} \to C$ be its normalization.

Question 13.3. *What's the relationship between $p_a(\widetilde{C})$ and $p_a(C)$?*

Definition 13.4. The **geometric genus** of C is $p_a(\widetilde{C})$. $\diamond$

These genii (genera?) are defined in terms of the structure sheaf, so let's start with the relationship between their structure sheaves.

Example (Really, example/remark)**.** Imagine C is a nodal curve (so C looks like α), and $\widetilde{C}$ its normalization (so the node got separated into two points). The normalization map $\nu : \widetilde{C} \to C$ gives an inclusion $0 \to \mathscr{O}_C \to \nu_*\mathscr{O}_{\widetilde{C}}$ of sheaves. Note that, away from the node, this inclusion is an equality (in fact, ν is an isomorphism away from the node). Hence, the cokernel will be supported at the singular points (at the node), so we have

Inclusion just because the map is surjective, I think

$$0 \longrightarrow \mathscr{O}_C \longrightarrow \nu_*\mathscr{O}_{\widetilde{C}} \longrightarrow \mathscr{F} \longrightarrow 0$$

with $\mathscr{F}$ supported at the singular points. $\triangle$

We want to use the above to relate the Euler characteristic of $\mathscr{O}_C$ with that of $\nu_*\mathscr{O}_{\widetilde{C}}$.

Fact. $\chi(\nu_*\mathscr{O}_{\widetilde{C}}) = \chi(\mathscr{O}_{\widetilde{C}})$.

(This comes from looking at the Leray spectral sequence)

The general theorem coming from the Leray spectral sequence is

$$\chi(\mathscr{F}) = \sum (-1)^i \chi(R^i\pi_*\mathscr{F})$$

Say $\pi : X \to Y$ and $\mathscr{F}$ a sheaf on X. Leray gives a spectral sequence

$$E_2^{pq} = \mathrm{H}^p(Y, R^q\pi_*\mathscr{F}) \implies \mathrm{H}^{p+q}(X, \mathscr{F}).$$

Let $\chi_r = \sum_{p,q}(-1)^{p+q}\dim E_r^{pq}$. One can check that this is unchanged as you pass from page to page, and so

$$\chi(\mathscr{F}) = \chi_\infty = \chi_2 = \sum_{p,q}(-1)^{p+q}\dim \mathrm{H}^p(Y, R^q\pi_*\mathscr{F}) = \sum_q (-1)^q\chi(R^q\pi_*\mathscr{F}).$$

In the case of the normalization map $\nu : \widetilde{C} \to C$, one has $R^q\nu_*\mathscr{O}_{\widetilde{C}} = 0$, for $q \geq 1$, since the fibers of ν are 0-dimensional (ν is proper so you can see this e.g. with cohomology and base change, but this is probably overkill. There should be a simpler argument to go from 0-dimensional fibers to no higher direct images[14]).

Remark 13.5 (Audience). To get this fact, you can alternatively use that the normalization map is affine, and so see that the cohomology of $\nu_*\mathscr{O}_{\widetilde{C}}$ agrees with that of $\mathscr{O}_{\widetilde{C}}$ explicitly using Čech cohomology. $\circ$

Corollary 13.6. *In any case, one gets that*

$$\chi(\mathscr{O}_C) = \chi(\nu_*\mathscr{O}_{\widetilde{C}}) - \chi(\mathscr{F}) = \chi(\mathscr{O}_{\widetilde{C}}) - h^0(\mathscr{F}) = h^0(\mathscr{O}_{\widetilde{C}}) - \sum_{p\in C}\dim_{\mathbb{C}}\mathscr{F}_p.$$

Question 13.7 (Audience)**.** *Can you get Riemann-Hurwitz via this sort of analysis?*

[14] Here's one that's slightly better: ν is affine, so the higher direct image presheaves $V \mapsto \mathrm{H}^q(\nu^{-1}(V), \mathscr{F})$ have trivial stalks – since the whole presheaf vanishings on an affine open cover – and hence the sheafification $R^q\nu_*\mathscr{F}$ must be trivial.

Answer. That's something worth thinking about. It certainly works out in the case of an unbranched cover. Less clear off the top of the head of how to make things work when there is ramification. ⋆

Definition 13.8. The **δ-invariant** of a singular point $p \in C$ is $\delta_p := \dim_{\mathbb{C}} \mathscr{F}_p$. ⋄

Example (p a node)**.** Let $C = \alpha$ with p the node, so near p, C looks like X. Above p, the normalization looks like $=$ with q, r the preimages of p. We see immediately that a function on $=$ is the preimage of a function on X iff it takes the same values at q, r. Hence, the stalk of the sheaf is 1-dimensional, so $\delta_p = 1$. △

In case it's not clear, most symbols here are used purely as pictures

Example (p a cusp)**.** Say $C = \prec$ with p the cusp. The normalization looks like $\widetilde{C} = \int$. Since p is a cusp, C looks like $y^2 = x^3$ near it. The cusp has a single preimage $q \in \widetilde{C}$ with local coordinate t for which the normalization map becomes $t \mapsto (t^2, t^3)$.

Let U be a small open around p. When is a function $f \in \mathscr{O}_{\widetilde{C}}(\nu^{-1}(U))$ the pullback of a function $g \in \mathscr{O}_C(U)$? This should give a set of linear conditions on f, and the number of such conditions will be δ_p. Note that f is simply a power series in t. The pullbacks will be the power series in t^2, t^3, so we get all power series with trivial linear component. Thus, the condition is $f'(q) = 0$, so $\delta_p = 1$. △

Example (p a tacnode)**.** Here $C = \asymp$ (except the two pieces simply tangent) with p the point of tangency. Then, $\widetilde{C} = \asymp$ (now an accurate picture) with two preimages q, r. Note that C looks, near p, like $y^2 = x^4$. What are the conditions for f to be the pullback of a function?

We need $f(q) = f(r)$, but this is not enough (imagine f vanishes to order 2 at q, but only order 1 at r). We also need $f'(q) = f'(r)$ (after choosing suitable local coordinates). These gives 2 linear conditions, so $\delta_p = 2$. △

Example (p a planar triple point)**.** Here, $C = *$ is three smooth branches meeting pairwise transversally, with p the singular point. Also, $\widetilde{C} = \equiv$ with preimages q, r, s of p.

We need $f(q) = f(r) = f(s)$ (two linear conditions). This is not enough. If we have a function vanishing to order 2 on two of the branches, then it must vanish to order ≥ 2 on the third branch. The function will be the restriction of a function in the plane who directional derivative is 0 in two directions, so it must be 0 in all directions. Hence, we also need an additional linear relation on the three derivatives $f'(q), f'(r), f'(s)$. Thus, $\delta_p = 3$. △

Example (p a spatial triple point)**.** Say C looks locally like the union of 3 coordinate axes in three space (I don't know a latex symbol that looks like this). This is different from a planar triple point. As before, the normalization looks like three different branches $\equiv$. However, not a function f on $\nu^{-1}(U)$ descends $\iff$ $f(q) = f(r) = f(s)$ (no derivative condition). Thus, $\delta_p = 2$. △

Note 13. There were good audience questions here, but I forget to write them down...

Exercise. You can specialize a node to a cusp, e.g. $y^2 = x^2(x - t)$ is a family of nodal curves that becomes a cuspidal curve when $t = 0$. Can you specialize a cusp to a node?

13.3 Problem 5

Theorem 13.9 (**Marten's theorem**)**.** *Let C be a non-hyperelliptic curve of genus g, and fix some d with $0 < d < 2g - 2$. Then,*

$$\dim W_d(C) \leq d \;\; \textit{and} \;\; \dim W_d^r(C) < d - 2r \;\; \textit{if} \;\; r > 0$$

(variety parameterizing linear series of degree d and dimension $\geq r$, see Notation 10.12).

Marten's theorem is often stated as an inequality, and then one says that if equality holds, the curve C is hyperelliptic.

The argument here is much the same as was used for Clifford's theorem.

Proof. Let $\varphi_K : C \hookrightarrow \mathbb{P}^{g-1}$ be the canonical embedding. Recall that the points of a general hyperplane section are in linear general position (i.e. any $g-1$ points of $H \cap C$ are linearly independent). Hence,

I think we're secretly assuming $d \leq g-1$ (in order to apply GPL), which is ok since $W_d^r \cong W_{2g-2-d}^{r-d-1+g}$ by Riemann-Roch

$$\dim \left\{ H \in \left(\mathbb{P}^{g-1}\right)^* : H \cap C \text{ contains } k \leq g-1 \text{ linearly dependent points} \right\} < g-1.$$

Suppose that $\dim W_d^r \geq d-2r$. Let D be a degree d divisor with $r(D) = r$. By geometric Riemann-Roch, $\overline{D} \cong \mathbb{P}^{d-1-r}$. Thus, we get an r-dimensional family of d-secant $(d-r-1)$-planes. Let's look at hyperplanes containing a member D_0 of this family:

Secretly this will actual be a projective space of dimension $\leq d-1-r$. This is the source of the inequality for $\dim \Phi$ later

$$\Phi = \left\{ (D, H) : H \supset D \text{ and } D \sim D_0 \right\}.$$

Consider the correspondence

$$\begin{tikzcd} & \Phi \arrow[dl] \arrow[dr] & \\ \mathbb{P}^r \cong |D_0| & & (\mathbb{P}^{g-1})^* \end{tikzcd}$$

The fibers of the left map will be $\cong \mathbb{P}^{(g-1)-(d-1-r)-1} = \mathbb{P}^{g-d+r-1}$. Thus,

$$\dim \Phi \geq r + g - d + r - 1 = 2r - d + g - 1.$$

The right map is finite, so by looking at the dimension of the image, we see that

Question: Do we secretly need $r > 0$ to get the inequality (on the right) below?

$$2r - d + g - 1 \leq \dim \Phi \leq g - 2 \implies d > 2r.$$

This is Clifford's theorem (?). The same logic applies when you have a family of linear series; doing this same argument with a family, one will arrive at the conclusion of Marten's theorem. ∎

14 Problem Session (10/15)

Answer: I think so. If $r = 0$, then $W_d = \operatorname{im}\left(C_d \to \operatorname{Pic}^d\right)$ This map is birational onto its image for $d \leq g$, so for such d we

Note 14. Roughly 8 minutes late. Also, non-descript computer issues mean today's notes might be lacking.

Question 14.1. *What is the maximal possible genus of C lying on a cubic surface S?*

(I'm thinking there was discussion preceding this question and that's what I missed. It doesn't seem to be directly related to what came next)

Say we have $\Gamma = H \cap C \subset A \subset \mathbb{P}^2$ with A a cubic curve. What is h_Γ, the Hilbert function of this hyperplane section.

Let's first do the case where $3 \nmid d$, so $d = 3k+1$ or $d = 3k+2$.

First assume $m \le k$. Curves of degree m containing Γ are curves of degree m containing A are curves of degree $3m$. Hence,

Question: What?

$$h_\Gamma(m) = h^0(\mathscr{O}_{\mathbb{P}^2}(m)) - h^0(\mathscr{I}_\Gamma(m)) = h^0(\mathscr{O}_{\mathbb{P}^2}(m)) - h^0(\mathscr{O}_{\mathbb{P}^2}(m-3)) = h^0(\mathscr{O}_{\mathbb{P}^2}(m)) - h^0(\mathscr{I}_A(m)) = \binom{m+2}{2} - \binom{m-1}{2} = 3m.$$

The above applies whether A is smooth or not.

Now say $m \ge k+1$. We claim $h_\Gamma(m) = d$. Here are two approaches

- We want to understand global sections of $\mathscr{I}_{\Gamma/\mathbb{P}^2}(m)$ (not locally free, ideal sheaf of a bunch of points[15]). To understand this, we relate to a line bundle, i.e. consider the exact sequence

  $$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(m-3) \xrightarrow{\cdot F_A} \mathscr{I}_{\Gamma/\mathbb{P}^2}(m) \longrightarrow \mathscr{I}_{\Gamma/A}(m) \longrightarrow 0$$

  (the cokernel above *is* a line bundle), where F_A is the cubic polynomial cutting out A. The corresponding sequence on global sections is exact ($h^1(\mathscr{O}_{\mathbb{P}^2}(m-3)) = 0$), so we see that

  $$h^0(\mathscr{I}_{\Gamma/\mathbb{P}^2}(m)) = h^0(\mathscr{O}_{\mathbb{P}^2}(m)) + h^0(\mathscr{I}_{\Gamma/A}(m)) = h^0(\mathscr{O}_{\mathbb{P}^2}(m)) + h^0(\mathscr{O}_A(mH - \Gamma)).$$

  The LHS above is $h^0(\mathscr{O}_{\mathbb{P}^2}(m)) - h_\Gamma(m) = \binom{m+2}{2} - h_\Gamma(m)$. The RHS above is $\binom{m+2}{2} + h^0(\mathscr{O}_A(mH - \Gamma))$. For this second term, we apply Riemann-Roch, which yields $3m - d$. The upshot is

  $$h_\Gamma(m) = d$$

  when $m \ge k+1$ (and $3 \nmid d$).

-

What if $3 \mid d$? Say $d = 3k$. There are two cases: Γ a complete intersection of A with a curve of degree m, or it's not such a thing. The argument is basically the same. The only difference is that $mH - \Gamma$ is of degree 0 when $m = k$, so it's h^0 is either 0 or 1, depending on if it's trivial. In the end, if $d = 3k$ and Γ is a complete intersection, then

$$h_\Gamma(m) = \begin{cases} 3m & \text{if } m < k \\ d-1 & \text{if } m = k \\ d & \text{if } m > k. \end{cases}$$

If Γ is not a complete intersection, then

$$h_\Gamma(m) = \begin{cases} 3m & \text{if } m < k \\ d & \text{if } m \ge k. \end{cases}$$

In problem 1 or 3, we're asked to give an analogous argument when the curve isn't necessarily smooth. It'll probably be ad hoc. Smooth plane curves are special. It's harder to say something about linear

[15] It is torsion-free of rank 1

series on possibly singular curves. The saving grace here is that our curve is cubic, so if it's singular, it either has a single node or a single cusp.

Here's a lemma that may be useful to prove: Say $\mathcal{D}, \mathcal{E}$ are linear series, and D, E are sets of points. If D imposes independent conditions on $\mathcal{D}$ (i.e. can find an element of series containing all of D save any one point) and E imposes independent conditions on $\mathcal{E}$, then $D \cup E$ imposes independent conditions on $\mathcal{D} + \mathcal{E}$ (something like this. I'm not sure if makes sense the way I've typed it). This isn't quite true, but is up to an off-by-one error.

Now, say $C \subset S \subset \mathbb{P}^3$ with S a smooth cubic surface, and C a smooth, irreducible, non-degenerate curve of degree $d = 3k+1$ in $\mathbb{P}^3$. Let $\Gamma = C \cap H$ by a general hyperplane section. Note that $\Gamma \subset S \cap H$ and $S \cap H$ is a plane cubic curve, so from the above,

$$h_\Gamma(m) \geq \begin{cases} 3m & \text{if } m \leq k \\ d & \text{if } m \geq k+1. \end{cases}$$

Hence, $h^0(\mathscr{O}_C(1)) \geq h^0(\mathscr{O}_C) + h_\Gamma(1) = 1 + 3 = 4$ (this is consistent with C being non-degenerate). We can continue $h^0(\mathscr{O}_C(2)) \geq h^0(\mathscr{O}_C(1)) + h_\Gamma(2) \geq 4 + 6 = 10$ (assuming d large enough). Continuing like this, one gets

$$h^0(\mathscr{O}_C(k)) \geq 3\binom{k+1}{2} \text{ and } h^0(\mathscr{O}_C(k+\ell)) \geq 3\binom{k+1}{2} + \ell d \text{ for all } \ell \geq 0.$$

Now we apply Riemann-Roch. For $\ell \gg 0$, we have

$$(k+\ell)d+1-g = h^0(\mathscr{O}_C(k+\ell)) \geq 3\binom{k+1}{2}+\ell d \implies g \leq (k+\ell)d-3\binom{k+1}{2}-\ell d+1 = k(3k+1)-3\binom{k+1}{2}+1 = 3\binom{k}{2}+k.$$

This gives a bound for genera of curves on a cubic surface. Do there always exists such curves of a given degree and genus satisfying this bound?

Claim 14.2. *If $C \subset S$ is linearly equivalent to $kH + L$ with H the hyperplane class and L any line, then adjunction gives*

$$g(C) = 3\binom{k}{2} + k =: \pi_1(d, 3).$$

Hence, our upper bound is sharp.

(exercise)

Recall 14.3 (from a Wednesday)**.** We worked out Castlenuovo's (spelling?) bound $\pi(d,3)$ and worked out that asymptotically, $\pi(d,3) \sim d^2/4$. ⊙

Note that $k \sim d/3$ (recall $d = 3k+1$), so $\pi_1(d,3) \sim 3(k^2/2) + k \sim d^2/6$.

Claim 14.4. *For genera between these two bounds, any curve with that genus must lie on a quadric surface ($\implies$ $g = (a-1)(b-1)$ for some $a + b = d$)*

Note 15. Had to pop out for a bit

Claim 14.5. *Every genus $g \leq \pi_1(d,3)$ occurs*

Somehow this is related to the fact that if Q is a (smooth) quadric and S is a (smooth) cubic, then $\mathrm{Pic}(Q) = \mathbb{Z}^2$ and $\mathrm{Pic}(S) = \mathbb{Z}^7$ (I guess probably just apply adjunction to get a genus formula and see which values occur?)

Joe is saying more stuff, but I'm having trouble following...

TODO: Take a closer look at Mary's notes on the material you've missed

Remark 14.6 (Keep in mind for **(3b)**). On an integral cubic surface, a general hyperplane section is still integral. ○

15 Lecture 11 (10/18)

Note 16. Roughly 5 minutes late

15.1 Castelnuovo Continued

Today we want to finish Castelnuovo theory. Say $C \subset \mathbb{P}^r$ of degree d.

Recall 15.1. Castelnuovo's idea was to bound $h^0(\mathscr{O}_C(m))$ from below, and then apply Riemann-Roch (using that this line bundle is non-special for $m \gg 0$). Basic steps

- Introduce general hyperplane section $\Gamma = C \cap H$ with $H \cong \mathbb{P}^{r-1} \subset \mathbb{P}^r$. Then,

$$h^0(\mathscr{O}_C(\ell)) - h^0(\mathscr{O}_C(\ell - 1)) \geq h_\Gamma(\ell)$$

  (this is ultimately coming from $0 \to \mathscr{O}_C(\ell-1) \to \mathscr{O}_C(\ell) \to \mathscr{O}_\Gamma(\ell) \to 0$)

- $h_\Gamma(\ell) \geq \min(\ell(r-1)+1, d)$.

An easy induction + Riemann-Roch then let's one see that $g(C) \leq \pi(d, r)$ where

$$\pi(d,r) = \binom{m_0}{2}(r-1) + m_0\varepsilon \text{ with } m_0 := \left\lfloor \frac{d-1}{r-1} \right\rfloor \text{ and } d = m_0(r-1) + r + \varepsilon$$

(note $0 \leq \varepsilon \leq r-2$). The expression for $\pi(d,r)$ is kinda messy because of the minimum in the lower bound. Asymptotically, we have

$$\pi(d,r) \sim \frac{d^2}{2(r-1)}.$$

In particular, for plane curves ($r = 2$), one has $\pi(d,r) \sim d^2/2$ which is indeed the correct asympotic. ⊙

Goal. Verify that the above bound is sharp by exhibiting such curves ("*Castelnuovo curves*"). While we're at it, it'd be nice to classify all such curves.

Recall 15.2. Apparently the lower bound $h_\Gamma(\ell) \geq \min(\ell(r-1)+1, d)$ is sharp with example given by any Γ contained in a rational normal curve in $\mathbb{P}^{r-1}$. ⊙

In fact, the above examples are the only ones.

Lemma 15.3 (**Castelnuovo's Lemma**). *If $d \geq 2r+1$, then*

$$h_\Gamma(2) = 2r - 1 \implies \Gamma \subset r.n.c \subset \mathbb{P}^{r-1}.$$

Note that the rational normal curve containing Γ will exactly be the intersection of all quadrics containing Γ. As a consequence the curve C itself will lie on a surface $S \subset \mathbb{P}^r$ so that $H \cap S$ is a rational normal curve (which forces $\deg S = r - 1$).

Question: Why?

Lemma 15.4. *If $S \subset \mathbb{P}^r$ is an irreducible non-degenerate surface of minimum degree $r-1$, then S is a rational normal scroll or the Vernonese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$.*

Question: What's a rational normal scroll?

Answer: Essentially, more classical terminology for a Hirzebruch surface

Here are the basic facts about S, our rational normal scroll:

- S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$
- $\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}F$ with F a line of the ruling. Furthermore $H^2 = r-1$, $F^2 = 0$ and $FH = 1$.

  If you have a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, an open subset U of that is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1$, so $\mathrm{Pic}(U) = 0$. If you have a line bundle on S, its restriction to U will be trivial, so can pick some nonzero holomorphic section above U, and extend it to a global meromorphic section. It will have zeros/poles only along the fiber F and the hyperplane section H.
- $K_S = -2H + (r-3)F$

  Once we have this, we can apply adjunction to a curve of any class. Say $S \subset S$ has class $C \sim \alpha H + \beta F$. One gets that

  $$d = \deg(C) = CH = (r-1)\alpha + \beta \text{ and } g = g(C) = \binom{\alpha}{2}(r-1) - \alpha - \beta$$

  (adjunction gives genus. Degree computation is more definitional). One can compute that the maximum value of g subject to the constraint $\deg(C) = d$ is obtained by taking $(\alpha, \beta) = (m_0 + 1, r - 2 - \varepsilon)$, giving

  $$g = \binom{m_0}{2}(r-1) + m_0\varepsilon$$

  (allegedly, I didn't do the algebra).

  If $\varepsilon = 0$ (so $d = m_0(r-1)+1$), there's another maximal value where you take $(\alpha, \beta) = (m_0, 1)$ and get the same genus. Hence, when $\varepsilon = 0$, there are two different kinds of **Castelnuovo curves**.

Recall we had set out to ask the question of which genera are possible for degree d curves in $\mathbb{P}^r$. We have now obtained the maximum possible genus, but what about $g < \pi(d, r)$? Which of these occur? The highest genus is obtain by curves lying on a rational normal scroll. For such curves, we know by the above adjunction calculation what all possible genera are. These curves don't cover every possible value of g.

Question 15.5 (Open?)**.** *Assume $\Gamma \subset \mathbb{P}^n$ is any configuration of d points (in linear general position). The smallest possible Hilbert function h_Γ is $h_\Gamma(\ell) = \min(\ell n + 1, d)$ (achieved by Γ lying on a rational normal curve). What is the second smallest possible Hilbert function?*

Question 15.6 (Audience)**.** *Is there a geometric reason that $\pi(r, d)$ is increasing with d but decreasing with r?*

Answer. For decreasing with r, can look at the argument. Getting upper bound on genus from lower bound on Hilbert function of a hyperplane section. This Hilbert function starts $h_\Gamma(1) \geq r+1$ which gets bigger with r, so maybe reasonable that the resulting bound on genus decreases with r. ⋆

Fact (Assuming I heard correctly)**.** Among curves not lying on a quadric in $\mathbb{P}^3$, the maximal genera are obtained by those lying on a cubic.

On the homework, we worked out an upper bound π_1 for the genera of such curves. For values of g with $\pi_1 < g < \pi$, no smooth curves in $\mathbb{P}^3$ of degree d have genus g.

15.2 Curves of low genus ($g = 2$)

Let's look at smooth curves C of genus $g = 2$. Note that the canonical map $\varphi_K : C \xrightarrow{2} \mathbb{P}^1$ is hyperelliptic in this case.

Question 15.7. *Are there degree* 3 *maps* $C \to \mathbb{P}^1$*?*

Say $\mathscr{L} \in \mathrm{Pic}^3(C)$ is a line bundle of degree 3. Riemann-Roch instantly tells us that $h^0(\mathscr{L}) = 2$, so $\varphi_{\mathscr{L}} : C \overset{3}{\dashrightarrow} \mathbb{P}^1$ is a degree 3 rational map to $\mathbb{P}^1$. This won't always extend to a morphism; $\mathscr{L}$ can have basepoints. Specifically, $\mathscr{L}$ has basepoint p iff $\mathscr{L} = \omega_C(p)$ for some $p \in C$.

Question 15.8. *Are there any degree* 3 *line bundles not of the form* $\omega_C(p)$*?*

Answer. Yes. There is a 2-dimensional family $\mathrm{Pic}^3(C)$ of line bundles on a genus 2 curves, but only a 1-dimensional family of those of the form $\omega_C(p)$. ⋆

That was simple enough. How about maps to $\mathbb{P}^2$ (i.e. linear series of projective dimension ≥ 2)?

Say $\mathscr{L} \in \mathrm{Pic}^4(C)$. We know $|\mathscr{L}|$ has no basepoints (since $h^0(\mathscr{L}(-p)) = 2$ always, while $h^0(\mathscr{L}) = 3$, both by RR), so we get a map $C \to \mathbb{P}^2$ with image a quartic curve. Note that the image cannot be smooth since the genus of a smooth quartic curve is 3. Write $L = K_C + D$ where D is an effective divisor of degree 2. Two cases

This is slightly misleading. 4 is actually the degree of the image curve times the degree of the map from C onto its image

- ($D = K_C$, i.e. $\mathscr{L} = \omega_C^{\otimes 2}$) In this case, $C \to \mathbb{P}^2$ is the composition

$$\begin{tikzcd} C \arrow[r, "\varphi_K"'] \arrow[rr, bend left, "\varphi_{\mathscr{L}}"] & \mathbb{P}^1 \arrow[r, "\nu_2"'] & \mathbb{P}^2 \end{tikzcd}$$

  That is, $\varphi_{\mathscr{L}}$ is a degree 2 cover of a conic plane curve.

- ($D \neq K_C$) Then, $h^0(D) = 1$, so $D = p + q$ for a unique pair of points p, q.

  Assumption. Let's assume $p \neq q$.

  Note that $h^0(\mathscr{L}) = 3$ while $h^0(\mathscr{L}(-p-q)) = 2$, so p, q must have the same image under $\varphi_{\mathscr{L}}$, i.e. $\varphi_{\mathscr{L}}(p) = \varphi_{\mathscr{L}}(q)$ is a *node* of $\varphi_{\mathscr{L}}(C)$. We haven't quite shown this yet; need to verify that the tangent lines at p, q map to different tangent lines of $\varphi_{\mathscr{L}}(C)$. This comes from verifying that $\mathscr{L}(-2p) \neq \mathscr{L}(-2q)$, I think. This node will be the only singularity of the image. So ended up with a plane quartic curve with a node.

  Question: (How) does Riemann-Hurwitz extend to singular curves?

  Answer: You can easily get one extension by applying Riemann-Hurwitz to the induced

- ($D \neq K_C, \mathscr{L} = \omega_C(2p)$) In this case, one gets a quartic curve with a cusp.

Now let's look at $\mathscr{L} \in \mathrm{Pic}^5(C)$ so $h^0(\mathscr{L}) = 4$ and $\mathscr{L}$ is base-point free. Consider $\varphi_{\mathscr{L}} : C \hookrightarrow \mathbb{P}^3$. Apparently, we've looked at something like this before (see section 11.2). Keep in mind

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2))}_{\text{dim}=10} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(2))}_{\text{dim}=9}$$

so C lies on a quadric surface. Write $\mathscr{L} = \omega_C(D)$ with D of degree 3.

Recall 15.9. Either $D = K_C + p$ for some $p \in C$, or it doesn't. ⊙

If $D \neq K_C + p$, then it induces an actual morphism, and we can consider the composition

$$\begin{tikzcd} C \arrow[r, "\varphi_K \times \varphi_D"'] \arrow[rr, bend left, "\varphi_{D+K} = \varphi_{\mathscr{L}}"] & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r] & \mathbb{P}^3 \end{tikzcd}$$

(note that φ_K is a double cover and φ_D is a triple cover).

What if $D = K_C + p$? Then, $\varphi_D : C \dashrightarrow \mathbb{P}^1$ is rational.

Claim 15.10. *In this case, $C \subset \mathbb{P}^3$ lies on a unique quadric $Q \subset \mathbb{P}^3$, but Q is a cone. The vertex of the cone will be the image of the point p.*

Note if you project from p (i.e. consider $L(-p) = 2K$), we once again see that $2K_C$ maps $C \xrightarrow{2:1}$ (some plane conic).

Fact. Say C is a hyperelliptic curve and $L \in \mathrm{Pic}^d(C)$ is very ample, so it gives an embedding

$$\varphi_L : C \hookrightarrow \mathbb{P}.$$

Let $\alpha : C \xrightarrow{2} \mathbb{P}^1$ be the hyperelliptic map. Write $\alpha^{-1}(t) =: p_t + q_t$. For each $t \in \mathbb{P}^1$, let $L_t \subset \mathbb{P}$ be the line through $p_t, q_t \in C$. Let $S = \bigcup_{t \in \mathbb{P}^1} L_t$. This will be a ruled surface, and in fact, S is a rational normal scroll.

In the case of a genus 2 curve embedded in $\mathbb{P}^3$ via a degree 5 line bundle, this construction reproduces the quadric containing the curve.

15.3 $g = 3$, Briefly

Say we have a curve C of genus 3.

Assumption. Assume C is non-hyperelliptic.

In this case, the canonical map $\varphi_K : C \hookrightarrow \mathbb{P}^2$ embeds C as a smooth plane quartic.

What sort of maps to $\mathbb{P}^1$ do we have? No degree 2 map by assumption. Are there such maps of degree 3? Yes, look at $K_C(-p)$ for any $p \in C$. This is always base-point free by Riemann-Roch (+ assumption that there's no degree 2 line bundle with 2 sections), so gives a degree 3 map $C \to \mathbb{P}^1$. More concretely, choose any point $p \in C \subset \mathbb{P}^2$ and project $\pi_p : C \to \mathbb{P}^1$ away from that point; this writes it as a degree 3 cover of $\mathbb{P}^1$.

16 Problem Session (10/22)

Note 17. Roughly 15 minutes late

16.1 Problem 1

Missed most of this discussion, but looks like he computed $D^2, D \cdot E$ on the board, and is now computing E^2?

Say $S, T \subset \mathbb{P}^3$ surfaces of desgrees s, t (S smooth), and say $S \cap T = D \cup E$ with D, E (smooth?) curves of degree d, e. Write $g(D) = g$ and $g(E) = h$. One can compute

$$D^2 = (2g-2) - d(s-4), \quad D \cdot E = td + (s-4)d - (2g-2) \text{ and } E^2 = te - (s+t-4)d + 2g - 2$$

(use adjunction a few times?)

Then,

$$2h - 2 = K_S E + E^2 = (s-4)e + te - (s+t-4)d + 2g - 2.$$

Collecting terms gives the **liaison formula**

$$h - g = (e-d) \cdot \frac{s+t-4}{2}.$$

To see that the RHS is an integer, use $e + d = st$ by Bezout.

16.2 Problem 2

Example. Say D a twisted cubic curve. Let S, T be quadric surfaces containing D. $S \cap T = D \cup E$ with E a curve a degree 1 (by Bezout), so E a line. Can check the liaison formula by hand in this case ($h = g$ and $s + t = 4$, so both sides vanish). △

A twisted cubic lies on 3 linearly independent quadrics

Question 16.1 (Beyond the homework)**.** *How many twisted cubics are there?*

Consider the Hilbert scheme parameterizing subschemes of $\mathbb{P}^3$ with Hilbert polynomial $p(m) = 3m+1$. This includes all twisted cubics (they will form an open subscheme?). What is the dimension of this space?

The liaison example relates twisted cubics to lines. We know lines in $\mathbb{P}^3$ are parameterized by a 4-dimensional Grassmannian $\mathbb{G}(1,3)$. To make use of this, we set up an incidence correspondence:

$$\Phi := \left\{ (S, T, D, E) \in \mathbb{P}^9 \times \mathbb{P}^9 \times \mathcal{H}_{3m+1} \times \mathbb{G}(1,3) : S \cap T = D \cup E \right\}$$

($\mathbb{P}^9$ parameterizing quadrics, $\mathcal{H}_{3m+1}$ Hilbert scheme). Project onto third and fourth factors

$$\begin{tikzcd} & \Phi \arrow[dl, "\mathrm{pr}_3"'] \arrow[dr, "\mathrm{pr}_4"] & \\ \mathcal{H}_{3m+1} & & \mathbb{G}(1,3). \end{tikzcd}$$

Look at fibers of right map. Fix a line $L \hookrightarrow \mathbb{P}^3$. It will lie on a 7-dimensional subspace of quadrics; for a general pair of quadrics containing L, the residual curve will be a twisted cubic (degree 3 and genus 0). Hence, the fibers of the right map are, opens in $\mathbb{P}^6 \times \mathbb{P}^6$. Thus, $\dim \Phi = 12 + 4 = 16$.

On the left, the quadrics containing a twisted cubic form a $\mathbb{P}^2$, so the fibers on the left will be open in $\mathbb{P}^2 \times \mathbb{P}^2$. Thus, $\dim \mathcal{H}_{3m+1} = \dim \Phi - 4 = 12$.

Remark 16.2. This is not the simplest way to answer this sort of question (compare: section 7.3) $\circ$

Example. Say D is a rational quartic curve. Dimension counting shows that D lies on a quadric surface S. In fact, S is unique (else by Bezout it'd be the complete intersection of two quadrics, and then adjunction would compute it to have genus $1 \neq 0$). Another dimension count shows that D lies on ($\geq$) 7 independent cubics. Since S lies on 4 independent cubics, we conclude there's some cubic T containing D but not S. Write $S \cap T = D + E$. Then, E must have degree 2. Liaison will tell us that E has genus $p_a(E) = -1$, so E better be reducible.

Think of $D \subset S = \mathbb{P}^1 \times \mathbb{P}^1$. D will be of type $(1,3)$ (or $(3,1)$). At the same time, $S \cap T$ will be type $(3,3)$, so $E \subset S$ must have type $(2,0)$. This is a pair of fibers in one direction, $E \cong \mathbb{P}^1 \sqcup \mathbb{P}^1$ (zero locus of quadratic polynomial in one set of variables), and indeed, $p_a(\mathbb{P}^1 \sqcup \mathbb{P}^1) = -1$. $\triangle$

Let's take a moment to mention a few questions, some of which are still open.

If you have two curves whose union gives a complete intersection, the geometry of these two curves are related to each other, as we have seen. Say two curves C, D are **linked** well, if their union is a complete intersection. This defines an equivalence relation (say on curves in $\mathbb{P}^3$).

Question 16.3. *Is this a non-trivial equivalence relation? Are there any two curves which are not linked?*

There's one equivalence class containing complete intersections (all linked to the empty curve of degree 0). This class contains other curves as well, e.g. the twisted cubic (which is linked to a line).

Answer. Yeah, it's non-trivial. The first example of a curve not linked to a complete intersection is the one from the example we just worked out: a curve of type $(2,0)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. $\star$

Ultimately, Hartshorne and Rao (spelling? I didn't hear too clearly) gave a complete set of invariants for this linkage relationship.

16.3 Problem 3

Say M is an $m \times n$ matrix of linear forms on $\mathbb{P}^r$. To this, we associate the **determinantal variety**

$$\Sigma_k := \{x \in \mathbb{P}^r : \operatorname{rank} M(x) \leq k\}$$

(zero locus of $(k+1) \times (k+1)$ minors of the matrix). What is $\operatorname{codim}(\Sigma_k, \mathbb{P}^r)$? This is a standard quals problem here: $\operatorname{codim}(\Sigma_k, \mathbb{P}^r) \leq (m-k)(n-k)$. For a general matrix of this form, this will be an equality.

If I heard correctly, in general, one says a projective variety is *determinantal* if it is of this form (with expected codimension?). Also, the entries of M don't need to all be linear. They can be of any degree as long as all minors are homogeneous.

Example. A complete intersection is a determinantal variety associated to a $1 \times c$ matrix. $\triangle$

Let's do an example. Say M is a 3×4 matrix of linear forms on $\mathbb{P}^3$, and let

$$D = \left\{X \in \mathbb{P}^3 : \operatorname{rank} M(X) \leq 2\right\}.$$

This has expected codimension $(3-2)(4-2) = 2$, so we expect D to be a curve (and it will be for a general matrix).

A word of what we mean by 'general matrix': such an M defines a linear map $\mathbb{P}^r \hookrightarrow \mathbb{P}\,(m \times n \text{ matrices})$ and inside of the target is the locus M_k of matrices of rank $\le k$.

There is a general formula for the degree of determinantal curves. However, in this case, we can use liaison. This curve D is exactly the zero locus of the 3×3 minors of this matrix. There are 4 such minors, each homogeneous of degree 3. Let's take two of these (say the first three columns and the last three columns). Let Δ_{123} be the cubic surface cut out by the minor of the first 3 columns, and Δ_{234} be the same for the last 3 columns. Write $\Delta_{123} \cap \Delta_{234} = D \cup E$. Note the middle two columns gives a 3×2 matrices. If that matrix has rank 2, the left and right minors vanishing already implies all minors vanish. Hence,

$$E : \operatorname{rank} \begin{pmatrix} M_{12} & M_{13} \\ M_{22} & M_{23} \\ M_{14} & M_{24} \end{pmatrix} \le 1.$$

Note that E above is a twisted cubic (see the textbook's discussion of rational normal curves). This forces $\deg(D) = 6$ and $g(D) = 3$.

Exercise. Say M is a general $n \times (n+1)$ matrix of linear forms on $\mathbb{P}^3$. Let $D_n = \{\operatorname{rank} M(x) \le n-1\}$. Find the degree and genus of D_n. (Hint: use induction)

16.4 Problem 4

Let C be a curve of genus 2, and let $L \in \operatorname{Pic}^5(C)$ be a degree 5 line bundle. By RR, $h^0(L) = 4$ and L is very ample, so we get embedding $\varphi_L : C \hookrightarrow \mathbb{P}^3$. Dimension counting shows that C lies on a quadric surface $Q \subset \mathbb{P}^3$. This Q is unique by Bezout. When is Q smooth?

If $L \cong K_C^2(p)$, then $L(-p) = K_C^2$. We know $\varphi_{K_C} : C \to \mathbb{P}^1$, so $\varphi_{K_C^2}$ is the composition

$$C \xrightarrow{\varphi_K} \mathbb{P}^1 \xrightarrow{\nu_2} \mathbb{P}^2$$

(use RR to see that this map is associated to the complete linear series K_C^2). This composition is $2:1$ onto a conic. Note that $\varphi_{K_C^2}$ is alternatively the composition of

$$C \overset{\varphi_L}{\hookrightarrow} \mathbb{P}^3 \xrightarrow{\pi_p} \mathbb{P}^2$$

of φ_L with projection away from p. The image is a conic, so C must lie on the cone over that conic (with vertex p).

17 Lecture 13 (10/25)

Note 18. Roughly 7 minutes late

Today

- genus 5 and 6, but mostly 5
- Start Brill-Noether theorem

17.1 Genus 5

Assume non-hyperelliptic $\varphi_K : C \hookrightarrow \mathbb{P}^4$, so C an octic curve (degree 8). Looking at the restriction map $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^4}(2)) \to \mathrm{H}^0(\mathscr{O}_C(2))$, shows that C lies on ≥ 3 quadrics Q_1, Q_2, Q_3. Two possibilities: $C = Q_1 \cap Q_2 \cap Q_3$ or $C \subsetneq Q_1 \cap Q_2 \cap Q_3$ (forces this triple intersection to be a surface).

Remark 17.1 (partial converse). If Q_1, Q_2, Q_3 are 3 general quadrics. They will intersect in a smooth curve by Bertini. By adjunction, that smooth curve will be a canonical curve (of genus 5).

By adjunction,

$$\begin{aligned} K_{\mathbb{P}^4} &= \mathscr{O}(-5) & K_{Q_1 \cap Q_2} &= \mathscr{O}(-1) \\ K_{Q_1} &= \mathscr{O}(-3) & K_{Q_1 \cap Q_2 \cap Q_3} &= \mathscr{O}(1) \end{aligned}$$

(assuming the involved varieties are still smooth at each step). ○

So the first of our two possibilities does occur, and is in fact the general case.

17.1.1 Case I

Assumption. First assume that $C = Q_1 \cap Q_2 \cap Q_3$.

Question 17.2. *Is C trigonal (= 3-sheeted cover of $\mathbb{P}^1$)?*

Answer. Let $D = p_1 + p_2 + p_3$ be an effective degree 3 divisor on C. To say that D moves in a pencil says exactly that p_1, p_2, p_3 are collinear. This can't happen! Any quadric containing the curve, intersects the line $L_{p_1 p_2}$ in ≥ 3 points and so must contain this line. Hence, $L_{p_1 p_2} \subset Q_1 \cap Q_2 \cap Q_3$, contradicting that this intersection is exactly C. Thus, C is not trigonal. ⋆

Question 17.3. *Is C a 4-sheeted cover of $\mathbb{P}^1$? i.e. does there exist a divisor $D = p_1 + p_2 + p_3 + p_4$ on C with $r(D) \geq 1$ ($\iff$ $r(D) = 1$, by Clifford).*

Again invoke geometric Riemann-Roch: $r(D) \geq 1 \iff p_1, p_2, p_3, p_4$ lie on a 2-plane Λ. These four points will lie on exactly two conics in Λ, and no more. However, C itself lies on 3 quadrics. This seems fishy. Algebraically, we're looking at

A collection of 4 points in the plane, no three co-linear, is the intersection of two conics

$$\mathrm{H}^0\left(\mathscr{I}_{C/\mathbb{P}^3}(2)\right) \longrightarrow \mathrm{H}^0\left(\mathscr{I}_{D/\mathbb{P}^2}(2)\right).$$

By hypothesis, the source is 3-dimensional, but the target is only 2-dimensional. Thus, there exists a quadric Q with $Q \supset C$ and $Q \supset \Lambda$. A quadric hypersurface in $\mathbb{P}^4$ containing a 2-plane is necessarily singular[16] (either rank 4 or rank 3).

Note 19. Joe drew some pictures of singular quadrics, but I missed their explanation, so I'm not sure what they're doing...

Remark 17.4. Conversely, if Q is singular quadric $\supset C$, get Λ_1, Λ_2 2-planes of opposite rulings (rank $Q = 4$). C meets each of Λ_1, Λ_2 in 4 points. Somehow this gives two different g^1_4's.

I'm kinda confused by what's happening, but sounds like here Q is a cone over a (smooth) quadric $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$. The two degree four maps are something like $C \hookrightarrow Q \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \rightrightarrows \mathbb{P}^1$. ○

[16] Think in terms of (quintic) quadratic forms. A (projective) 2-plane in the associated quadric gives a 3-dimensional isotropic subspace of the quadratic form. A non-degenerate quadratic form can only have isotropic subspaces up to half the total dimension

To conclude that C does have a degree 4 map to $\mathbb{P}^1$, need to verify that C lies on some singular quadric. Note that $\{\text{quadrics in } \mathbb{P}^4\} \cong \mathbb{P}^{14}$. Sitting inside of here is $\mathbb{P}^2 \cong \Gamma = \{Q : Q \supset C\}$ (using the assumption that C is a complete intersection of 3 quadrics). The locus of singular quadrics $X := \{\text{sing quadrics}\} \subset \mathbb{P}^{14}$ is a quintic hypersurface; think of quadrics as locus of quintic quadratic forms, and then the singular ones are those which are degenerate (determinant of associated matrix vanishes). Therefore, the singular quadrics containing C are given by $X \cap \mathbb{P}^2$, a (non-empty) quintic plane curve.

So we get a whole curve of singular quadrics containing C, and so a whole 1-dimensional family of degree 4 maps to $\mathbb{P}^1$.

We've shown we have a hypersurface $X \subset \mathbb{P}^{14}$ of singular quadrics. A general singular quadric will be singular at just 1 point, and so will be rank 4. Can look further at the locus

$$Y := \{\text{quadrics of rank} \le 3\} \subset X \subset \mathbb{P}^{14}.$$

Claim 17.5. $\dim Y = 11$ *(i.e. Y is codim 3 in $\mathbb{P}^{14}$)*

Exercise. Verify this, and also compute its degree if you want.

For a general choice of three quadrics, they'll span a two plane in the same of all quadrics, and so they will miss Y $(11 + 2 < 14)$. Hence, for Q_1, Q_2, Q_3 general, no linear combination has rank ≤ 3 and so $C = Q_1 \cap Q_2 \cap Q_3$ won't lie on a quadric of rank ≤ 3.

Claim 17.6. $X_{\text{sing}} = Y$

So for C a general curve of degree 5, the intersection $\Sigma := \mathbb{P}^2 \cap X$ is a smooth plane quintic curve ($\mathbb{P}^2$ = space of quadrics containing C), and there's is a 2:1 map

$$\left\{g^1_4\text{'s on } C\right\} \xrightarrow{2:1} \Sigma.$$

Remark 17.7. Apparently the Jacobian of the curve is the Prym of this map, whatever that means. ○

Exercise. If C is **bielliptic**, i.e. exists degree two map $C \to E$ with E elliptic,[17] then Σ is reducible, consisting of the union of a line and a quartic plane curve.

17.1.2 Case II

Assumption. Now assume $C \subsetneq Q_1 \cap Q_2 \cap Q_3$.

Claim 17.8. *$Q_1 \cap Q_2 \cap Q_3$ is a surface.*

Proof. If it were a curve, it would be a curve of degree 8, and so would be C. Hence, it must have some surface component S.

Exercise. Show that S contains C.

An irreducible, non-degenerate surface in $\mathbb{P}^4$ can't lie on more that 3 quadrics. Thus, this S must be a rational normal scroll, and we must have $Q_1 \cap Q_2 \cap Q_3 = S$. ■

TODO: Make sense of this argument

Recall 17.9. $\operatorname{Pic}(S) = \mathbb{Z}\langle H, F\rangle$ ⊙

[17] Note this immediately shows that C is expressible as a 4-sheeted cover of $\mathbb{P}^1$ is a 1-parameter family of ways

Canonical curves are Castelnuovo curves (maximal genus given degree), so previous analysis shows that C must have class $3H - F$. This says in particular that $C \cdot F = 3$, so C is trigonal (S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, and the map $C \xrightarrow{3} \mathbb{P}^1$ is just the restriction of $S \to \mathbb{P}^1$).

Corollary 17.10. *A non-hyperelliptic curve of genus* 5 *is trigonal iff it is not a complete intersection.*

17.2 (Genus 6 and) Brill-Noether

Question 17.11. *If C is a general curve of genus g, what is the smallest d so that there exists a map $f : C \to \mathbb{P}^1$ of degree d? Also, how many such maps are there, what's the dimension of the space of all such maps?*

We've answered this already for several small values of g:

g	min. d	#	when
0	1	(1)	
1	2	(∞^1)	Section 8.2
2	2	(1)	Section 15.2
3	3	(∞^1)	Section 15.3
4	3	(finite number)	
5	4	(∞^1)	Section 17.1
6	4?	(finite number?)	Extrapolate

Table 1: Gonality (min d s.t. $\exists C \xrightarrow{d} \mathbb{P}^1$) of general curves of small genus

Say C is a non-hyp curve of genus 6, so $\varphi_K : C \hookrightarrow \mathbb{P}^5$ is a curve of degree 10. One quickly sees that C lies on ≥ 6 quadric hypersurfaces. This doesn't give us much to work with. It's not clear how to do a hands-on analysis here, so we really need a theorem.

This brings us to the start of how discussion of Brill-Noether.

Question 17.12. *If C is a general curve of genus g, what linear series exist on C?*

For any triple (g, d, r), one asks: does a general curve C of genus g possess a g^r_d?

We've repeatedly been asking this question for $r = 1$ and various values of g throughout the course.

Example (Consequence of Brill-Noether)**.** C is expressible as a d-sheeted cover of $\mathbb{P}^1$ $\iff$ $d \geq \lceil \frac{g+2}{2} \rceil$. △

This is the pattern we observed in the table. We can refine our question a little. So far, we've just asked: is $W^r_d(C) \neq \emptyset$, but why not ask

Question 17.13. *What is* $\dim W^r_d(C)$*, for a general curve C of genus g?*

This is what Brill-Noether (aims to?) answer. More on this Wednesday.

Question 17.14 (Audience)**.** *Is is clear that if a curve is expressible as a d-sheeted cover of $\mathbb{P}^1$, then it's also expressible as a $(d+1)$-sheeted cover?*

Answer. Depends on how you think of things. If a curve has a g^1_d, then it automatically has a g^1_{d+1} (e.g. just add a basepoint). However, it can have a basepoint-free g^1_d (and so a map) without have a basepoint-free g^1_{d+1}. For example, sounds like hyperelliptic curves of genus $g \geq 3$ are never trigonal. ⋆

18 Problem Session

Note 20. Roughly 9 minutes late

18.1 Problem 1

Say $C \hookrightarrow \mathbb{P}^4$ genus 5, degree 8. Have $C \subset Q_1, Q_2, Q_3$ 3 quadrics. Say $C \subsetneq Q_1 \cap Q_2 \cap Q_3$. This forces $Q_1 \cap Q_2 \cap Q_3$ to be 2-dimensional (if it were one-dimensional, it'd be pure of dimension 1; no isolated point). We want to conclude that $S := Q_1 \cap Q_2 \cap Q_3$ is a rational normal scroll.

For something like this, need slight strengthening of Bezout.

Theorem 18.1 (First Bézout). *If $X_1, \ldots, X_k \subset \mathbb{P}^n$ are hypersurfaces of degrees $d_1, \ldots, d_k$ which intersect transversely, then* $\deg\left(\bigcap_{i=1}^k X_i\right) = \prod_{i=1}^k d_i$.

Theorem 18.2 (Second Bézout). *If $X_1, \ldots, X_k \subset \mathbb{P}^n$ are hypersurfaces of degrees $d_1, \ldots, d_k$ and $\bigcap X_i$ has dimension $n-k$ – i.e. $\bigcap X_i = \bigcup Z_k$ with Z_k irreducible of dimension $n-k$ – then we can assign to each irreducible component Z_k of $\bigcap X_i$ an intersection multiplicity $m_{Z_k}(X_1, \ldots, X_k)$ so that*

$$\sum_k m_{Z_k}(X_1, \ldots, X_k) \deg Z_k = \prod d_i.$$

Example. Say we have curves $C = V(f)$ and $D = V(g)$ in $\mathbb{P}^2$. For $p \in C \cap D$, one has

$$m_p(C \cdot D) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathbb{P}^2,p}}{(f,g)}. \qquad \triangle$$

Defining intersection multiplicities in complete generality is tricky, it sounds. Something something Serre's Tor formula something something...

Neither of these two versions of Bezout are good enough for us. We'd like to e.g. rule out the possibility that $Q_1 \cap Q_2 \cap Q_3 = C \cup H$ is C union a hyperplane H. In this case, the intersection does not have the expected dimension, and so neither of the above versions tell us anything about the degrees of the components of $Q_1 \cap Q_2 \cap Q_3$.

Theorem 18.3 (Fulton)**.** *Say $X_1, \ldots, X_k \subset \mathbb{P}^n$ hypersurfaces of degree $d_1, \ldots, d_k$, and write*

$$\bigcap X_i = \bigcup Z_\alpha$$

(Z_α irreducible components). Then,

$$\sum \deg Z_\alpha \le \prod d_i.$$

In order to get an equality above, would need to develop Fulton's excess intersection formula.

This rules out the possibility that $Q_1 \cap Q_2 \cap Q_3 = C \cup H$. In fact, it shows that if C is an irreducible component of $Q_1 \cap Q_2 \cap Q_3$, then C must be the whole intersection. Thus, if the intersection is 2-dimensional, then C must be contained in a 2-dimensional component Z of the intersection, and Z is necessary an irreducible, non-degenerate surface.

Remark 18.4. Fulton's intersection theorem is specific to projective space. $\circ$

18.2 Problem 2: Vector bundles on $\mathbb{P}^1$

Theorem 18.5. *If $E \to \mathbb{P}^1$ is a vector bundle of rank r, then E is a direct sum of line bundles.*

Classifying vector bundles on curves of genus $g \geq 1$ is much more non-trivial.

For this problem, we'll prove this, at least when $r = \operatorname{rank} E = 2$.

Definition 18.6. The **degree** of a vector bundle E is $\deg(\det E)$, the degree of the line bundle $\det E := \bigwedge^{\text{top}} E$. ◇

Remark 18.7. We we have an exact sequence

$$0 \longrightarrow \mathscr{O}(a) \longrightarrow E \longrightarrow \mathscr{O}(b) \longrightarrow 0.$$

Then, $\bigwedge^2 E \cong \mathscr{O}(a) \otimes \mathscr{O}(b)$, so $\deg E = a + b$. Also, $\chi(E) = \chi(\mathscr{O}(a)) + \chi(\mathscr{O}(b)) = a + b + 2 = \deg E + 2$.

Note, given any vector bundle of rank 2, can find such a sequence.[18] ○

Claim 18.8. *Say we have*

$$0 \longrightarrow \mathscr{O}(a) \xrightarrow{\alpha} E \xrightarrow{\beta} \mathscr{O}(b) \longrightarrow 0.$$

If $a \geq b - 1$, then this sequence splits, i.e. $E = \mathscr{O}(a) \oplus \mathscr{O}(b)$.

Proof. We have a section $\varphi : \mathscr{O}(b) \to E$ so that $\beta \circ \varphi = \mathrm{id}$. Let $M = \mathscr{O}(b)$ and $L = \mathscr{O}(a)$. Apply the (exact) functor $\operatorname{Hom}(\mathscr{O}(b), -) = M^\vee \otimes (-)$ to get

$$0 \longrightarrow M^\vee \otimes L \longrightarrow M^\vee \otimes E \longrightarrow \mathscr{O} \longrightarrow 0.$$

In $\operatorname{Hom}(M, M) = M^\vee \otimes M = \mathscr{O}$, we have the identity map $\mathrm{id}_M : M \xrightarrow{\equiv} M$, and the original sequence splits iff it is in the image of the map $\operatorname{Hom}(M, E) \to \operatorname{Hom}(M, M)$, i.e. the map $E \otimes M^\vee \to \mathscr{O}$. Taking cohomology gives an exact sequence

$$\begin{tikzcd}
\mathrm{H}^0(\operatorname{Hom}(M,E)) \arrow[r] & \mathrm{H}^0(\operatorname{Hom}(M,M)) \arrow[r] \arrow[d, equal] & \mathrm{H}^1(\operatorname{Hom}(M,L)) \arrow[d, equal] \\
& \mathbb{C} & \mathrm{H}^1(M^\vee \otimes L) \arrow[d, equal] \\
& & \mathrm{H}^1(\mathscr{O}(a-b)) \arrow[d, equal, "\text{since } b-a\geq 1"] \\
& & 0.
\end{tikzcd}$$

Thus, we win by exactness. ■

So, given $E \to \mathbb{P}^1$, we'd like to show there exists a sub-line bundle $L = \mathscr{O}(a) \hookrightarrow E$ with $a = \deg L \geq \frac{1}{2}\deg(E) - 1$. How do we find a sub-line bundle of a vector bundle of rank 2?

Suppose we have a global section $s \in \mathrm{H}^0(E)$ which is nowhere vanishing. This would then span a sub-line bundle (over a point $p \in \mathbb{P}^1$, look at the span of $s(p)$). This sub-line bundle would necessarily

[18] Twist $E^\vee$ to assume it has a section $s : \mathscr{O} \to E^\vee$ (really $s : \mathscr{O} \to E^\vee(a)$ since the original bundle may need to be twisted before its dual has a section). Take the dual of this map $s^\vee : E \to \mathscr{O}$. The kernel will be a line bundle.

be trivial (it has a nonvanishing global section). We claim a similar thing can be done for sections which vanish somewhere.

When we had a non-vanishing section, to define the sub-line bundle (over a trivializing neighborhood U, so $E|_U \cong U \times \mathbb{C}^2$), we were essentially taking a map $U \to \mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$ picking out the line spanned by the value of the section. If our section has some zeros, we only get a rational map $U \dashrightarrow \mathbb{P}^1$, but because U is smooth and 1-dimensional, such a rational map always extends to a morphism $U \to \mathbb{P}^1$, and so can still be used to define a sub-line bundle!

Hence, given E, the goal is to find $\sigma \in \mathrm{H}^0(E)$ w/ as many zeros as possible. For this, we use Riemann-Roch. Pick points $p_1, \dots, p_k \in \mathbb{P}^1$, and look at the vector bundle $E(-p_1 - \cdots - p_k) := E \otimes \mathscr{O}(-p_1 - \cdots - p_k)$. Global sections of this are just global sections of E vanishing at these k-points. Thus, just need H^0 of this to be nonzero. Riemann-Roch says that

$$h^0(E(-p_1 - \cdots - p_k)) \geq \chi(E(-p_1 - \cdots - p_k)) = (\deg E - 2k) + 2.$$

Choosing the maximal k so that this is positive gives a line subbundle of high enough degree to write E as a sum of two line bundles.

Note 21. Missed some remark about normal bundles

Exercise. Take the simplest example of a curve with rank 2 normal bundle – a twisted cubic in $\mathbb{P}^3$ – what is its normal bundle? It will be $\mathscr{O}(a) \oplus \mathscr{O}(b)$ for some a, b. Which ones?

18.3 Problem 3

Recall the construction of scrolls.

Construction 18.9. Start with two disjoint linear subspaces $\mathbb{P}^a, \mathbb{P}^b \subset \mathbb{P}^r$. Choose a rational normal curve C_1 of degree a in $\mathbb{P}^a$ and one C_2 of degree b in $\mathbb{P}^b$. Fix some parameterization $\varphi_\alpha : \mathbb{P}^1 \xrightarrow{\sim} C_\alpha$ of each, and let $X_{a,b} \subset \mathbb{P}^r$ be the surface swept out by lines between corresponding points on the rational normal curves, i.e.

$$X_{a,b} = \bigcup_{t \in \mathbb{P}^1} \overline{\varphi_1(t)\varphi_2(t)} \longrightarrow \mathbb{P}^1_t$$

The lines above are disjoint which is why we get the morphism down to $\mathbb{P}^1_t$ (?).

The fibers of this map are $\mathbb{P}^1$'s, so $X_{a,b}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ (a Hirzebruch surface). This makes it the projectivization of some vector bundle on $\mathbb{P}^1$, and the claim is that

$$X_{a,b} \simeq \mathbb{P}\left(\mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)\right).$$

Warning 18.10. Usually when one has a vector space V, $\mathbb{P}V$ is the space parameterizing lines (i.e. 1-dimensional subspaces) in V. So usually given a vector bundle $E \to X$, $\mathbb{P}E$ is $\{(x, L) : x \in X, L \subset E_x \text{ 1-dim}\}$. However, Grothendieck told us that this is the wrong way. He said we should actually have projectivizations parameterize 1-dimensional quotients instead of 1-dimensional subobjects.

That is, he says $\mathbb{P}V := \{\Lambda \subset V : \Lambda \text{ codim 1 linear subspace}\}$. Not everyone is on board with this (in particular, Joe isn't), so conventions vary. Grothendieck's convention allows one to define the projectivization of any coherent sheaf: for any coherent sheaf $\mathscr{F}$ on any scheme X, can define

$$\mathbb{P}(\mathscr{F}) := \mathbf{Proj}_X\left(\bigoplus_{m\geq 0}\operatorname{Sym}^m\mathscr{F}\right). \qquad \bullet$$

To finish this problem, one invokes the following theorem:

Theorem 18.11. *If $E, E' \longrightarrow X$ are vector bundles, then $\mathbb{P}E \cong \mathbb{P}E'$ (over X') iff $E' \simeq E \otimes \mathscr{L}$ for some line bundle $\mathscr{L}$.*[19]

Corollary 18.12. *$X_{a,b} \simeq X_{a',b'}$ iff $a-b = \pm(a'-b')$.*

18.4 Problem 4

Say C genus 5, non-trigonal. Have canonical embedding $\varphi_K : C \hookrightarrow \mathbb{P}^4$ and $C = Q_1 \cap Q_2 \cap Q_3$. The locus Σ of singular quadrics containing C is a quintic plane curve in $\mathbb{P}^2 \cong \{Q \supset C\}$. Suppose that C is bi-elliptic, i.e. $\exists \pi : C \xrightarrow{2} E$ with $g(E) = 1$. One gets an involution $\iota : C \xrightarrow{\sim} C$ just flipping the two sheets. Can use this to break up the space of differentials into eigenspaces

$$\mathrm{H}^0(K_C) = \mathrm{H}^0(K_C)^+ \oplus \mathrm{H}^0(K_C)^-.$$

Note that $h^0(K_C)^+ = \pi^*\,\mathrm{H}^0(K_E)$ and so is 1-dimensional. Look at the maps associated to these subspaces, e.g.

$$\begin{tikzcd} C \arrow[r, "\varphi_{\mathrm{H}^0(K_C)}"'] \arrow[rr, bend left, dashed, "\varphi_{\mathrm{H}^0(K_C)^-}"] & \mathbb{P}^4 \arrow[r, "\text{project}"'] & \mathbb{P}^3. \end{tikzcd}$$

The map $\varphi_{\mathrm{H}^0(K_C)^-}$ factors through π and so is $2:1$ onto an elliptic curve $E \hookrightarrow \mathbb{P}^3$ embedded as a quartic. This E will be the intersection of two quadrics, and so lies on a pencil of quadrics. Thus, $C \hookrightarrow \mathbb{P}^4$ lies on the cones over those quadrics, giving a 1-parameter family of singular quadrics containing the original canonical curve.

19 Lecture 15 (11/1): Brill-Noether

We want to start the proof (of at least half) of Brill-Noether. This will involve introducing a new construction/notion (inflection points of linear series) which we will discuss on Wednesday. Today, we want to talk about the general setup of proving this theorem. Apparently, there are ≥ 4 known proofs.

Keep in mind that Brill-Noether (BN) describes linear series on a general curve, and also describes the varieties that parameterize them. The basic starting question is

Question 19.1. *Are there any g^r_d's on a curve C?*

Answer. Introduce the **Brill-Noether number**

$$\rho(g,r,d) = g-(r+1)(g-d+r).$$

[19]Look at cohomology sequence induced by $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$

This is the expected dimension of W^r_d (saw this last Wednesday). $\star$

Theorem 19.2 (**Brill-Noether theorem**)**.** *There exists a g^r_d on a general curve C of genus g $\iff$ $\rho(g,r,d) \geq 0$. Moreover,* $\dim W^r_d(C) = \rho$.

Note that we talking about a 'general' curve, so this is implicitly assuming that we have a space parameterizing all curves (and then are saying BN holds for curves in some dense open). We have not, in this class, constructed the moduli space parameterizing genus g curves. This is a highly non-trivial thing to do, so to save time, we'll take the perspective of the 19th century mathematicians who just assumed such a thing existed, and went on with their lives.

Fact. The varieties $W^r_d(C)$, for C any curve of genus g, fit together to form the fibers of a map.

Let $\mathcal{M}_g$ be the moduli space of curves of genus g. Above this is the space

Unclear to me if Joe has in mind the course moduli space or fine moduli stack, but let's just say it's the latter for now?

$$\wp_{d,g} := \left\{(C,L) : C \text{ genus } g \text{ and } L \in \mathrm{Pic}^d(C)\right\}$$

(this is a sub-object of the universal jacobian $\mathrm{Pic}^0_{\mathcal{C}_g/\mathcal{M}_g}$). Inside of here, one has the space $\mathcal{W}^r_d = \left\{(C,L) : h^0(L) \geq r+1\right\}$ (this is $W^r_d(\mathcal{C}_g)$). So the picture is

$$\begin{tikzcd} \wp_{d,g} \arrow[d, dash] & \supset & \mathcal{W}^r_d \arrow[dll] \\ \mathcal{M}_g & & \end{tikzcd}$$

The point with putting these W^r_d's into a family $\mathcal{W}^r_d$ is that they're dimensions are upper semi-continuous. Thus, to show that the dimension is $\geq \rho$, it suffices to find one curve with dimension $= \rho$ (implicitly appealing to some determinantal variety computation from last time showing that $\dim W^r_d(C) \geq \rho$ when $W^r_d(C) \neq \emptyset$ always).

Two halves of BN core

- ('existence') If $\rho \geq 0$, then $W^r_d(C) \neq \emptyset$ and $\dim W^r_d(C) \geq \rho$.
- ('non-existence') If $\rho < 0$, then $W^r_d(C) = \emptyset$ for general C, and $\dim W^r_d(C) = \rho$.

Question: Is this a typo?

To prove the non-existence half, it suffices to exhibit a single curve C of genus g so that $\dim W^r_d(C) = \rho$ (if $\rho \geq 0$) and so that $W^r_d(C) = \emptyset$ (if $\rho < 0$).

This doesn't sound so bad, but this isn't so easy. Most of the ways we have or writing down curves actually violate BN (e.g. hyper-elliptic curves, plane curves, complete intersections, etc. are not general).

There's some remark in the course text about $\mathcal{M}_g$ not being unirational when $g \gg 0$. I wanna say this is related to that?

Exercise. Try (and fail) to verify BN for hyper-elliptic, plane, and complete intersection curves (say, complete intersections in $\mathbb{P}^3$ to keep things reasonable).

Recall 19.3. If I give you an explicit curve, there's an algorithmic way to write down all of its linear series (and this is true over any field). $\odot$

Approach 2 Here's an idea: write down some family of curves (including a general curves), and verify BN over the generic point of this family. Joe called this the second approach (I guess the first was to try to write down a generic curve in isolation?)

Example. If $g = 2$, then $y^2 = x^6 + a_5x^5 + \cdots + a_1x + a_0$ gives a family of curves over $\mathbb{A}^6_{a_0,\dots,a_5}$ which includes every curve of genus 2 (assuming $\operatorname{char} k \neq 2$). △

Example. For $g = 3$, can look at curves given by $\sum_{i+j\le 4} a_{ij}x^iy^j$ which includes a general curve of genus 3 (but not all, e.g. no hyper-elliptics). △

Question 19.4. *Given a genus g, can we find a family of curves of genus g over some open $U \subset \mathbb{A}^n$ (open e.g. because you want to fibers to be smooth) which includes a general curve?*

This was open for a long time, but is now closed.

Answer. Yes! when $g \le 13$ (this was done classically by hand for $g \le 10$ if I heard correctly). When $g \ge 23$, the answer is no. Still open in the middle range. ⋆

Approach 3 New thought: instead of specializing to a general smooth curve, why don't we specialize to a singular curve? This idea goes back to Castelnuovo it seems.

Example (Castelnuovo?)**.** Let C_0 be a g-nodal curve ($p_a(C_0) = g$ and C_0 has g nodes, so $g(\widetilde{C}_0) = 0$). That is, the normalization is $\mathbb{P}^1$ and C_0 is obtained by identifying pairs of points $p_i \leftrightarrow q_i$ (for $i = 1,\dots,g$). The point is that this can be deformed to a smooth curve! That is, there exists a family $\mathcal{C} \to \Delta$ ($\Delta =$ disc) so that C_t is smooth of genus g and C_0 is g-nodal. △

I might be mistaken, but I think this is just because $\overline{\mathcal{M}_g}$ is connected (or possibly just because 'general' means belonging to a dense open?)?

We now need to start worrying a bit about extending the theory we have built up to singular curves.

Definition 19.5. By a **linear series on a singular curve** C_0 we mean a line bundle L on C_0 along with a subspace $V \subset \mathrm{H}^0(L)$. ⋄

Also, the degree of a line bundle on C_0 is defined to be the degree of its pullback to the normalization $\widetilde{C}_0$.

Claim 19.6. *If we have*

$$\{(L_t, V_t) : L_t \text{ line bundle of degree } d \text{ on } C_t \text{ and } V_t \subset \mathrm{H}^0(L_t) \text{ of dim } r+1 : t \neq 0\},$$

then this extends, in the limit, to such a linear series on C_0.

Note 22. Got distracted for a second, and missed something Joe said.

Claim 19.7. *If the g-nodal curve C_0 is general (i.e. the $p_i, q_i \in \mathbb{P}^1$ are general), then $C_0 := \mathbb{P}^1/(p_i \sim q_i)_{i=1}^g$ satisfies Brill-Noether, i.e. $\nexists g^r_d$ on C_0 with $\rho < 0$.*

Proof. Say we have $C_0 = \mathbb{P}^1/(p_i \sim q_i)_{i=1}^g$, and say we have a linear series (L_0, V_0) on C_0. How can we describe this data? Pull back to $\mathbb{P}^1$. Let $\nu : \mathbb{P}^1 \to C_0$ be the normalization map. Let $L := \nu^*L_0 \cong \mathscr{O}_{\mathbb{P}^1}(d)$, so $V := \nu^*L \subset \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(d))$. What condition must V satisfy in order to come from a linear series on C_0, i.e. when does the map $\varphi_{(L,V)} : \mathbb{P}^1 \to \mathbb{P}^r$ factor through C_0?

Observation 19.8. A linear series (L, V) on $\mathbb{P}^1$ is a pullback from C_0 iff for any $\sigma \in V$, $\sigma(p_\alpha) = 0 \iff \sigma(q_\alpha) = 0$ (for all $\alpha = 1,\dots,g$).

Note that a linear series of dimension r and degree d on $\mathbb{P}^1$ exactly corresponds to a plane $\Lambda \cong \mathbb{P}^{d-r-1} \subset \mathbb{P}^d$ (embed $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ by the complete linear series $|\mathscr{O}_{\mathbb{P}^1}(d)|$, and then subseries consist of all divisors containing some given plane Λ. Moreover, the associated map is $\varphi_\Lambda : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d \overset{\pi_\Lambda}{\dashrightarrow} \mathbb{P}^r$). Hence, this space of linear series is parameterized by a Grassmannian $\mathbb{G}(d-r-1,d)$. In this language, what's the condition that $\varphi_\Lambda(p_\alpha) = \varphi_\Lambda(q_\alpha)$? If you think geometrically, this is just saying that a hyperplane containing Λ contains p_α iff it contains q_α. More simply, it means Λ intersects the line $\overline{p_\alpha q_\alpha}$ (all taking place in $\mathbb{P}^d$). Hence, we arrive at the following question

Question 19.9 (Question in Schubert calculus)**.** *Given collection $p_\alpha, q_\alpha \in \mathbb{P}^d$, does there exist $\Lambda \cong \mathbb{P}^{d-r-1} \subset \mathbb{P}^d$ so that $\Lambda \cap \overline{p_\alpha q_\alpha} \neq \emptyset$ for all α?*

Start with a dimension count in $\mathbb{G} := \mathbb{G}(d-r-1,d)$. If $L \subset \mathbb{P}^d$ is any line, set

$$\Sigma_r(L) := \{\Lambda \in \mathbb{G} : \Lambda \cap L \neq \emptyset\}$$

this is a cycle of codimension r in $\mathbb{G}$.[20] So we expect $\exists \Lambda$ w/ $\Lambda \cap \overline{p_\alpha q_\alpha} \neq \emptyset$ for all $\alpha \iff rg \leq \dim \mathbb{G}$ (sum of codimensions at most dimension of Grassmannian), i.e.

$$rg \leq (d-r)(r+1) \iff \underbrace{g-(r+1)(g-d+r)}_{\rho(g,d,r)} \geq 0.$$

Almost done, but not yet. If we took g general lines (in place of $\overline{p_\alpha q_\alpha}$), then all the dimension counts would work out as expected (by a standard Bertini-type argument). However, we're not taking general lines in $\mathbb{P}^d$, but instead are taking general chords to $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$. That is

Proposition 19.10 (Bertini)**.** *If $L_1, \dots, L_g$ are general lines in $\mathbb{P}^d$, then $\dim \bigcap \Sigma_r(L_\alpha) = \rho$ (in particular, is empty if $\rho < 0$).*

We want to show this holds in $L_1, \dots, L_g$ are general chords to a rational normal curve.

Let's ground ourselves, by pausing to look at the first non-trivial case: is a general curve C of genus 3 hyperelliptic? In this case, $C_0 = \mathbb{P}^1/(p_1 \sim q_1, p_2 \sim q_2, p_3 \sim q_3)$. Are there any maps $C_0 \xrightarrow{2} \mathbb{P}^1$? Consider the pullback of such a thing to $\mathbb{P}^1$ embedded as a conic in $\mathbb{P}^2$. Any degree 2 map $\mathbb{P}^1 \to \mathbb{P}^1$ is the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ followed by projection from a point $p \in \mathbb{P}^2$. Such a thing factors through C_0 iff p lies on the chords $\overline{p_\alpha q_\alpha}$. For 3 general pairs of points on the conic, not all three chords will share a point of intersection (imagine a triangle). Note that in this case, every line in $\mathbb{P}^2$ is a chord, so Bertini applies on the nose.

(Got distracted again and missed some stuff)

Are lines $\overline{p_\alpha q_\alpha}$, $\alpha = 1,2,3$ concurrent? Not in general, but they might be (e.g. pick some point in $\mathbb{P}^2$, draw three lines through it, and then make the identifications indicated by those lines). Let's take things a step further. Instead of specializing to three secant lines, what if we took 3 tangent lines? 3 secant lines in the plan can be concurrent, but three tangent lines to a conic in the plane can never be concurrent. To prove this, we'll introduce inflection points next time.

We will finish carrying out this proof over the next couple of lectures... ■

[20] Something like $\Sigma_r(pt)$ has codimension $r+1$, so a 1-parameter family of points will give codimension r

20 Lecture 17 (11/8)

Note 23. Roughly 7 minutes late

Plan for this week

- Today: fun with Weierstrass points + automorphism groups (of curves)
- Wednesday: back to work (proof of Brill-Noether, at least the first half I think)

20.1 Last time

Let C be a smooth projective curve of genus g, and fix a point $p \in C$. Let $\mathcal{D} = (L, V)$ be a degree d linear series on C.

Recall 20.1. We defined the *ramification sequence* of $\mathcal{D}$ at p as follows: first find a basis $\sigma_0, \ldots, \sigma_r \in V$ with

$$\operatorname{ord}_p \sigma_0 < \operatorname{ord}_p \sigma_1 < \cdots < \operatorname{ord}_p \sigma_r.$$

Then, the **vanishing sequence** is

$$a_i(\mathcal{D}, p) := \operatorname{ord}_p \sigma_i$$

(note this is strictly increasing), while the **ramification sequence** is

$$\alpha_i(\mathcal{D}, p) := \operatorname{ord}_p \sigma_i - i.$$

We say that p is an **inflectionary point for $\mathcal{D}$** if $(\alpha_0, \ldots, \alpha_r) \neq (0, \ldots, 0)$. The **weight of p** is

$$w(p) := \sum \alpha_i(\mathcal{D}, p).$$ ⊙

Fact. There are only finitely many inflectionary points for a given $\mathcal{D}$. This is false in positive characteristic.

Note that above definitions make sense for any linear series, even those with base points

Exercise. The map $\varphi_{\mathcal{D}}$ fails to be an immersion near $p \iff \alpha_1(\mathcal{D}, p) > 0$.

Fact. The **Plücker formula** tells us that

$$\sum_{p \in C} w(p) = (r+1)d + r(r+1)(g-1).$$

Exercise. Every point $p \in C$ is an inflectionary point for some linear series (Hint[21])

If we want to use this to pick out only certain points on the curve, we can apply this notion to the canonical series (or to pluricanonical series).

[21] Consider $\mathscr{O}_c(np)$ for $n \gg 0$

20.2 Weierstrass points

Definition 20.2. We say $p \in C$ is a **Weierstrass point** if p is inflectionary for $|K|$, the canonical series. $\diamond$

We can give an equivalent, alternate definition of Weierstrass points.

Observation 20.3. Say D is a general effective divisor of degree d, i.e. $D \in C_d$ general. Geometric Riemann-Roch tells us that

$$h^0(D) = \begin{cases} 1 & \text{if } d \le g \\ d - g + 1 & \text{if } d \ge g \end{cases}$$

(a general effective divisor imposes d independent conditions on the canonical series. A general point will not be a basepoint of $K - p$ or $K - p - q$ or ...).

What if we tried making a similar remark not for general divisors, but for multiples of a general point? We expect

$$h^0(dp) = \begin{cases} 1 & \text{if } d \le g \\ d - g + 1 & \text{if } d \ge g. \end{cases}$$

In other words, we expect that $\nexists f \in \mathcal{M}(C)$ (meromorphic function) so that f is holomorphic on $C \setminus \{p\}$ and $\operatorname{ord}_p f \ge -g$.

Classically, people asked: given $p \in C$, consider

$$\Sigma_p := \{-\operatorname{ord}_p(f) : f \in \mathcal{M}(C) \text{ and } f \text{ holo. on } C \setminus \{p\}\} \subset \mathbb{N},$$

the **Weierstrass semi-group** of p. We expect that, for a general point $p \in C$, this semi-group is simply $\Sigma_p = \mathbb{N}_{\ge g+1}$.

Claim 20.4. *For a general $p \in C$, $\Sigma_p = \{g+1, g+2, g+3, \dots\}$.*

Proof. Note that there exists $f \in \mathcal{M}(C)$ with f holomorphic on $C \setminus \{p\}$ and $-\operatorname{ord}_p(f) = k \iff h^0(kp) > h^0((k-1)p) \iff h^0(kp) = h^0((k-1)p) + 1$. By Riemann-Roch, this is the case iff

$$h^0(K - kp) = h^0(K - (k-1)p),$$

i.e. p is a basepoint of $K - (k-1)p$. This is the case iff $\nexists \omega \in \mathrm{H}^0(K)$ vanishing to order exactly $k-1$ at p. That is $\operatorname{ord}_p \omega \ge k-1 \implies \operatorname{ord}_p \omega \ge k$. The upshot of this is that

$$\mathbb{N} \setminus \Sigma_p = \{a_0(K,p) + 1, \dots, a_{g-1}(K,p) + 1\},$$

the complement of the Weierstrass semi-group at p is the vanishing sequence of the canonical series, shifted by 1. The above sequence of integers is called the **Weierstrass gap sequence**. We see from this that

- $\#(\mathbb{N} \setminus \Sigma_p) = g$
- For general $p \in C$, $\Sigma_p = \{g+1, g+2, g+3, \dots\}$ ■

Last time they showed that for any linear series, a general point is not inflectionary

Corollary 20.5. *$p \in C$ is a Weierstrass point iff $\Sigma_p \neq \{g+1, g+2, \dots\}$ iff $h^0(gp) > 1$.*

Consider the gap sequence $\mathbb{N} \setminus \Sigma_p = \{b_1, \dots, b_g\}$. We define the *weight of p as a Weierstrass point* to be $w(p) = \sum b_i - i$. This is equal to the weight of p as an inflectionary point for $|K|$. By Plücker, the number of Weierstrass points is captured by

$$\sum_{p \in C} w(p) = (r+1)(2g-2) + r(r+1)(g-1) = g^3 - g.$$

(note $r + 1 = g$ above).

"Too many letters on the board, and not enough numbers."

Example ($g = 2$)**.** Say $p \in C$ of genus 2. We know that $h^0(p) = 1$, so two possibilities: $h^0(2p) \in \{1, 2\}$. In other words, the semigroup could be $\{3, 4, 5, \dots\}$ (with gap seq. $(1, 2)$) or $\{2, 4, 5, \dots\}$ (with gap seq. $(1, 3)$ and weight $w(p) = 1$). In particular, every Weierstrass point has weight 1, so there must be exactly 6 Weierstrass points. These will be the ramification points of the canonical map $\varphi_K : C \xrightarrow{2} \mathbb{P}^1$. △

Example (hyperelliptic)**.** A hyperelliptic curve will always be given by an equation of the form

$$y^2 = \prod_{i=1}^{2g-2} (x - \lambda_i).$$

Consider the point $p_i := (\lambda_i, 0)$. There is a meromorphic function with pole of order exactly 2 at p_i: namely, $1/(x_i - \lambda_i)$. Thus, Σ_{p_i} contains all even numbers; furthermore, once it contains an odd number, it contains all numbers past that. Given that it omits g values, this forces

$$\Sigma_{p_i} = \{2, 4, 6, \dots, 2g, 2g+1, 2g+2, 2g+3, \dots\}.$$

Hence, $w(p_i) = g(g-1)/2$. Plučker says that the Weierstrass points have total weight $g^3 - g$. We just found $(2g + 2)$ points each with weight $g(g-1)/2$, so these must have been all of them. That is, the Weierstrass points of a hyperelliptic curve are exactly the ramification points of the hyperelliptic map. △

Example ($g = 3$)**.** If C is hyperelliptic, there are 8 Weierstrass points, each with semigroup $\{2, 4, 6, 7, 8, \dots\}$ (so gap seq. $(1, 3, 5)$) and weight 3. Consider instead the non-hyperelliptic case (so 2 not in the semi-group). There are two possibilities for the semi-group

- $\{3, 5, 6, 7, \dots\}$ (gap $1, 2, 4$) with weight 1. Note $C \hookrightarrow \mathbb{P}^2$ as a smooth plane quartic. The orders of vanishing of holomorphic differentials at p correspond to contact orders of $p \in C$ with lines $L \subset \mathbb{P}^2$. You can get order 0 from a line disjoint from p, order 1 from a line simply passing through p, and order ≥ 2 from a line tangent to p. For p Weierstrass here, we have a line of contact order 3, so p Weierstrass $\iff$ p is a flex point of $|K|$.
- $\{3, 4, 6, 7, \dots\}$ (gap $1, 2, 5$) with weight 2. This corresponds to p being a hyperflex point.

In the non-hyperelliptic case, we get α weight 1 points, β weight 2 points so that $\alpha + 2\beta = 24$.

Fact. All possible combinations occur except $(\alpha, \beta) = (2, 11)$. △

The big general question is

Question 20.6. *Which semigroups $\Sigma \subset \mathbb{N}$ of finite index g (i.e. $\#(\mathbb{N} \setminus \Sigma) = g$) occur as Weierstrass semigroups of points on curves of genus g?*

Sounds like this is answered at least for genus $g \leq 8$. It turns out they all occur, so people naturally conjectured that they always all occur (regardless of g). This is false (due to Buchweitz). Characterizing the ones that do is still open.

Question 20.7 (Audience)**.** *If you have a point that is fixed by the entire automorphism group, then must it be a Weierstrass point?*

Answer. Not sure off the top of my head, but this should be answerable. In the simplest case, consider an automorphism of order 2. If the quotient curve is rational, the fixed points will be Weierstrass by the hyperelliptic case we considered above. If the quotient curve is higher genus, it's less immediately clear how things work out. ⋆

Remark 20.8 (Audience). If the curve has no non-trivial automorphisms, then the answer is no. ∘

20.3 Automorphisms

Theorem 20.9. *Let C be a smooth projective curve of genus $g \geq 2$. Then, $\# \operatorname{Aut}(C) < \infty$.*

Proof. There are two (and a half?) components to the proof. We want to deal with the hyper-elliptic case separately, so let's assume C is non-hyperelliptic. Let $wP(C)$ denote the set of Weierstrass points.

(1) $\# wP(C) > 2g + 2$

One can use Clifford to show that the largest possible weight of a Weierstrass point is $\binom{g}{2}$. In fact, if you have a point of this weight, then your curve must be hyperelliptic, so here $w(p) < \binom{g}{2}$ for all $p \in wP(C)$. Once you have this, $\# wP(C) > 2g + 2$ by Plucker.

(2) If $\varphi : C \xrightarrow{\sim} C$ is any automorphism which fixes $> 2g + 2$ points, then $\varphi = \mathrm{id}$.

This follows from the Lefschetz fixed point formula. It tells us that[22]

$$\# \left\{ \text{fixed pts of } \varphi : C \xrightarrow{\sim} C \right\} = \sum_{i \geq 0} (-1)^i \operatorname{Tr} \left(\varphi^* : \mathrm{H}^i(C) \to \mathrm{H}^i(C) \right).$$

For $i = 0, 2$, φ^* will carry a point to a point and the fundamental class of C to itself, so $\varphi^* = \mathrm{id}$ with $\operatorname{Tr}(\varphi^*) = 1$. For $i = 1$, we claim there exists a Hermitian inner product on $\mathrm{H}^1(C)$ which is preserved by φ, so the eigenvalues of $\varphi^* : \mathrm{H}^1 \to \mathrm{H}^1$ are complex numbers of modulus 1. Thus, $\operatorname{Tr}(\varphi^* | \mathrm{H}^1) \geq -2g$. Thus, Lefschetz says that

$$\# \left\{ \text{fixed pts of } \varphi : C \xrightarrow{\sim} C \right\} \leq 1 - (-2g) + 1 = 2g + 2.$$

Remark 20.10. When applying Lefschetz to C^∞-functions, each fixed point has an index ± 1 you need to assign. For holomorphic functions, everything is orientation preserving, so all the indices will be $+1$. ∘

[22] LHS is really intersection number $\Delta \cdot \Gamma_\varphi$ with $\Delta, \Gamma_\varphi \subset C \times C$ the diagonal and graph of φ

Remark 20.11. Observe that $\varphi^* : \mathrm{H}^0(K) \xrightarrow{\sim} \mathrm{H}^0(K)$ preserves the positive definite Hermitian inner product $H(\omega) = \int_C \omega \wedge \overline{\omega}$. Hence the eigenvalues are complex numbers of modulus 1. At the same time, $\mathrm{H}^1_{dR}(C;\mathbb{C}) = \mathrm{H}^0(K) \oplus \overline{\mathrm{H}^0(K)}$, so get the same conclusion about its eigenvalues. ○

■

21 Problem Session (11/12)

Note 24. Roughly 5 minutes late

21.1 Problem 1

Compare Riemann-Roch for D with Riemann-Roch for $D+p$. Get that $h^0(D+p) = h^0(D)+1$ iff p is a base point of $|K-D|$ (i.e. iff $h^0(K-D) = h^0(K-D-p)$). If D is **special** (i.e. $h^1(D) = h^0(K-D) \neq 0$), for general $p, q \in C$, $h^0(D+p-q) = h^0(D)-1$ (I missed why if he said so). The conclusion from this is that $W^{r+1}_d(C) \subsetneq W^r_d(C)$.

Question 21.1 (Something to think about)**.** *Above really shows $W^{r+1}_d(C)$ is a closed subvariety of $W^r_d(C)$ not containing any irreducible component, so it's codimension ≥ 1. Can it's codimension be > 1?*

21.2 Problem 2

Recall 21.2 (**Clifford's Theorem**)**.** For $p \in C$ any point, $h^0(K - \ell p) \leq g - \frac{\ell}{2}$. ⊙

Thus, the vanishing sequence of $|K_C|$ at p is $\leq (0, 2, 4, \dots, 2g-2)$, so its ramification sequence is $\leq (0, 1, 2, \dots, g-1)$. Thus, it's weight is $\leq \binom{g}{2}$.

Now let's look at pluricanonical series.

Definition 21.3. An m**-fold Weierstrass point** is an inflectionary point of the pluricanonical series $|mK_C|$. ◇

As m increases, the number of distinct m-fold Weierstrass points will also increase. Say $m \geq 2$. Clifford $\implies$ vanishing sequence of $|mK|$ at p is[23]

Secretly what's written below is like a mix of the ramification and vanishing sequences

$$\leq \left(\underbrace{0, 0, \dots, 0}_{(m-1)(2g-2)}, 2, 4, \dots, 2g \right).$$

Thus, the ramification sequence is $\leq (0, \dots, 0, 1, 2, \dots, g)$ and so $w(|mK|, p) \leq \binom{g+1}{2}$. At the same time, the total weight of the m-fold Weierstrass points is growing (quadratically) with m, so as m increases, we can find arbitrarily large subsets of the curve which are fixed by any automorphism.

[23] Note that mK_C is non-special, as is $mK_C - \ell p$ for $\ell \leq (m-1)(2g-2)$

21.3 Problems 3/4

Definition 21.4. For a semigroup $\Sigma \subset \mathbb{N}$, we define its **genus** to be $\#(\mathbb{N} \setminus \Sigma)$ and its **weight** to be[24]

$$w(\Sigma) := \sum_{m \in \mathbb{N}\setminus\Sigma} m - \binom{g+1}{2}. \qquad \diamond$$

We want to count semigroups of weight w and genus $g \gg w$.

The simplest genus g semigroup is $\{g+1, g+2, g+3, \dots\}$. Any genus g semigroup can be written in the form

$$\Sigma = \{g+1-a_1, g+2-a_2, \dots, g+k-a_k, \dots\} \text{ with } \text{ weight } w(\Sigma) = a_1 + \cdots + a_k.$$

Above, we'll have $a_1 \geq a_2 \geq \cdots \geq a_k$. Thus, each semigroup of genus g and weight w gives rise to a partition of w.

Warning 21.5. If you start with a partition $a_1 + \cdots + a_k = w$ of w, the set

$$\Sigma = \{g+1-a_1, g+2-a_2, \dots, g+k-a_k, \dots\}$$

may not be a semigroup. $\bullet$

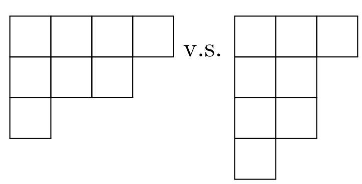
However, if $w < \frac{g}{2}$, then the above warning is irrelevant; any parition of w does give rise to a semigroup.

Question 21.6 (Audience)**.** *Would we have been able to guess beforehand that the answer does not depend on g? (At lest if $g \gg w$)*

Answer (paraphrase)**.** I'm not sure. It's unclear how to see this without going through this argument. $\star$

Question 21.7 (Audience)**.** *The $(b_i - i)$'s also give a partition of w. Is this the same as the partition given by the a_i's?*

Answer. Almost. They (probably) give Young diagrams which are transpose of each other, e.g.



$\star$

Example (weight $w = 2$)**.** There are two possible semigroups:

$$\{g-1, g+2, g+3, g+4, \dots\} \text{ or } \{g, g+1, g+3, g+4, \dots\}.$$

If you take $g = 4$, the two possible ones are

$$\{3, 6, 7, 8, \dots\} \text{ and } \{4, 5, 7, 8, \dots\}$$

[24] If $\mathbb{N} \setminus \Sigma = \{b_1, \dots, b_g\}$. This is $w(\Sigma) = \sum_{i=1}^{g} (b_i - i)$

with gap sequences

$$(1,2,4,5) \text{ and } (1,2,3,6). \qquad \triangle$$

Can we find genus 4 curves with Weierstrass points whose associated semigroups are as above?

For such a $p \in C$, the vanishing sequence of $|K_C|$ at p would be $(0,1,3,4)$ (or $(0,1,2,5)$). In order to construct Weierstrass points with these vanishing sequences, we would like to understand them geometrically.

Let's say $C \hookrightarrow \mathbb{P}^r$ is a smooth curve. Say $p \in C$ is a point and $\Lambda \subset \mathbb{P}^r$ is a linear subspace (of any dimension). We define the **order of contact of Λ with C at p** to be

$$\operatorname{ord}_p(\Lambda \cdot C) := \min_{H \supset \Lambda} \operatorname{ord}_p H$$

(if $H \subset \Lambda$ is a hyperplane, it's defined by a single linear polynomial F_H, and $\operatorname{ord}_p H$ is just the order of vanishing of F_H at p). Furthermore, for any k, among all k-planes $\Lambda \subset \mathbb{P}^r$, there is a unique one w/ maximal order of contact; this is called the **osculating k-plane to C at p**. The osculating 0-plane is the point p, and the osculating 1-plane is the tangent line at p.

Now, say $\varphi_K : C \hookrightarrow \mathbb{P}^3$ is a canonically embedded genus 4 curve. What does it mean to have a point with vanishing sequence $(0,1,3,4)$? This is saying there's no hyperplane with contact order 2 at p, so p must have contact order 3 with its tangent line. Similarly, if p has vanishing sequence $(0,1,2,5)$, it means p has contact of order 5 w/ its osculating 2-plane.

Remark 21.8. In general, the vanishing sequence is telling us the contact orders of p with is osculating k-planes. ∘

Recall 21.9. A canonical curve of genus 4 is the intersection of a quadric and a cubic surface. ⊙

Write $C = Q \cap S \subset \mathbb{P}^3$. To keep life simple, let's look for any example with $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ smooth. We first want a point $p \in C$ with contact of order 3 with its tangent line. Note that if there is a line $L \subset \mathbb{P}^3$ with contact order 3 with the curve, then $L \subset Q$ (else, it would have contact order ≤ 2 by Bezout). Let's fix some line $L \subset Q$ then. Choose coordinates x, y on $\mathbb{P}^1 \times \mathbb{P}^1$ and take $L = \{x = 0\}$, for example. We want a polynomial f on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ (of bidegree $(3,3)$) so that $f|_{x=0}$ vanishes to order 3 in y, i.e.

$$f(x,y) = \alpha y^3 + x(blah).$$

We're almost done. How do we know we can find a smooth curve cut out by an equation of the above form? We apply Bertini. Look at the linear series of all such polynomials; need to show a general member gives a smooth curve.

If I heard correctly, Bertini will tell us that a general member is smooth away from the point p ($= (0,0)$ in local coordinates), so we need to verify that a general member is also smooth at p. For this, it suffices to find one member smooth at p, and for that you can take L union a curve a type $(2,3)$.

Now we want $p \in C = Q \cap S \subset \mathbb{P}^3$ with vanishing sequence $(0,1,2,5)$, i.e. that there is a plane H in $\mathbb{P}^3$ with contact order 5 at p. Set $E = H \cap Q$, a conic plane curve (type $(1,1)$ on Q). We want a polynomial f of bidegree $(3,3)$ on $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ whose restriction $f|_E$ to E vanishes to order 5 at p. One could just write down such a thing, but let's instead see a more general approach to doing things like this...

The space of polynomials of bidegree $(3,3)$ on Q is simply $\mathrm{H}^0(\mathscr{O}_Q(3))$. We're considering the restriction map

$$\mathrm{H}^0(\mathscr{O}_Q(3)) \xrightarrow{\rho} \mathrm{H}^0(\mathscr{O}_E(3)) = \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(6)).$$

We want an element in the image of this map vanishing to order 5 at p. There certainly exists a section σ of $\mathscr{O}_{\mathbb{P}^1}(6)$ vanishing to order 5 at p. Is $\sigma \in \operatorname{im}\rho$? To figure this out, realize the above as coming from the short exact sequence

$$0 \longrightarrow \mathscr{I}_{Q/E}(3) \longrightarrow \mathscr{O}_Q(3) \longrightarrow \mathscr{O}_E(3) \longrightarrow 0.$$

Note that $\mathscr{I}_{Q/E} \cong \mathscr{O}_{\mathbb{P}^3}(-1)|_Q \cong \mathscr{O}_Q(-1)$, so $\mathscr{I}_{Q/E}(3) \cong \mathscr{O}_Q(2)$. We claim that $\mathrm{H}^1(\mathscr{O}_Q(2)) = 0$, so ρ is surjective. To prove the claim, we invoke the exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^3} \longrightarrow \mathscr{O}_{\mathbb{P}^3}(2) \longrightarrow \mathscr{O}_Q(2) \longrightarrow 0$$

which induces

$$0 = \mathrm{H}^1(\mathscr{O}_{\mathbb{P}^3}(2)) \to \mathrm{H}^1(\mathscr{O}_Q(2)) \to \mathrm{H}^2(\mathscr{O}_{\mathbb{P}^3}) = 0$$

from which we conclude that $\mathrm{H}^1(\mathscr{O}_Q(2)) = 0$.

This just leaves making a Bertini-type argument to show that we can find such a polynomial cutting out a smooth curve.

Question 21.10 (Audience)**.** *Could you say something about how one proves Lemma 15.4?*

Answer. The basic idea is that you're looking for a surface $S \subset \mathbb{P}^r$ of minimal degree $r-1$. You want to argue by induction. Take your surface, choose a general point $p \in S$, and then projection away from this point. If the projection map is birational onto its, the image will have degree one less, and so you can inductively apply the theorem to the image of the projection in $\mathbb{P}^{r-1}$.

Want to say if you have a family of lines on the projection, then these came from a family of lines on S. Might worry that you have a bunch of conics on S which get straightened out to lines on the projection, and so the family of lines there is coming from conics on S, not from lines on S. This is what happens in the case of the Veronese surface. One needs to show that the only surface with a ≥ 2 parameter family of conics on it is the Veronese surface.

In general, one can prove a result of the following form:

Theorem 21.11. *If $X^k \subset \mathbb{P}^r$ is non-degenerate of dimension k, then $\deg(X) \geq r-k+1$. Furthermore, if $\deg(X) = r-k+1$, then X is either*

- *a quadric hypersurface*
- *a cone over the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$*
- *a scroll swept out by a 1-parameter family of $(k-1)$-planes.* ⋆

Presumably this is the same thing as a $\mathbb{P}^1$-bundle over $\mathbb{P}^{k-1}$

22 Lecture 19 (11/15)

Note 25. Roughly 6 minutes late

Missed some discussion about historical perspective on classifying curves: curves as embedded in projective space (classified by Hilbert scheme) v.s. abstract iso classes of curves (classified by $\mathcal{M}_g$).

Some class stuff

- Today: Proof of Brill-Noether / 2
- Wednesday: Hilbert schemes (stay on this topic for most of rest of semester?)
- Homework: 2 more, due 11/29 and 12/8

22.1 Brill-Noether

Question 22.1. *For a general curve C of genus g, what is* $\dim W_d^r(C)$*?*

At some point (on a Wednesday), we described C_d^r (divisors of degree d with proj dim $\geq r$) as a determinantal variety; we also showed (or are going to show? I can't remember) that

$$W_d^r(C) \neq \emptyset \text{ if } \rho := g - (r+1)(g-d+r) \geq 0,$$

so $\dim W_d^r(C) \geq \rho$. The second half is to show $\dim W_d^r(C) \leq \rho$ (in particular, if $\rho < 0$, then $W_d^r(C) = \emptyset$).

Goal. If $\rho < 0$, then $W_d^r(C) = \emptyset$.

For this, we need to find a single curve C of genus g so that $\rho < 0$ and $\dim W_d^r(C) = \emptyset$.

The idea behind the original proof goes back to Castelnuovo: specialize to a g-nodal curve of arithmetic genus g, i.e. take $\mathbb{P}^1$, choose g pairs of points p_i, q_i, and then form $C_0 = \mathbb{P}^1/(p_i \sim q_i)$. Let $r_i \in C_0$ denote the image of p_i (equiv. the image of q_i).

Remark 22.2. One should check that this C_0 really exists as an algebraic curve. To do this, just need to specify a topological space along with its structure sheaf. As a topological space, C_0 is exactly $\mathbb{P}^1/(p_\alpha \sim q_\alpha)$, simple enough. The structure sheaf is also not too bad: a regular function on C_0 will just be a regular function on $\mathbb{P}^1$ with the same values on pairs of identified points, i.e.

$$\mathscr{O}_{C_0}(U) := \left\{ f \in \mathscr{O}_{\mathbb{P}^1}(\nu^{-1}(U)) : f(p_\alpha) = f(q_\alpha) \text{ for all } \alpha \text{ such that } r_\alpha \in U \right\}.$$

Let's take a moment to think about this algebraically...

Question 22.3. *In algebraic geometry, can we define the notion of an equivalence relation so that we can take quotients like this?*

Say we have a scheme X, what should we mean by an equivalence relation on X? Set-theoretically, an equivalence relation is a subset of $X \times X$. So, algebro-geometrically, an **equivalence relation** should be a subscheme $\Sigma \subset X \times X$ so that

- $\Sigma \supset \Delta$ (reflexivity)
- $i(\Sigma) = \Sigma$ where $i : X \times X \to X \times X$ the switch (symmetry)

- Considering

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_{23}"] & \\ X \times X & & X \times X, \end{tikzcd}$$

  one has $\pi_{23}\left(\pi_{12}^{-1}(\Sigma) \cap \pi_{23}^{-1}(\Sigma)\right) = \Sigma$ (transitivity)

TODO: Make sure you got this right

Example. If X is a smooth curve and $p, q \in X$, take $\Sigma = \Delta \cup \{(p,q),(q,p)\}$. The quotient X/Σ now has a node. △

Example. If X is a smooth curve and $p \in X$, take $\Sigma = \Delta \cup \{\text{fat point at } (p,p)\}$. The quotient X/Σ now has a cusp. △

Warning 22.4. The quotient of an equivalence relation does not always exist (as a scheme). Not every singularity arises as a quotient by an equivalence relation. •

Missed part of this last bit, but Grothendieck studied these in order to construct the Picard scheme as a quotient by an equivalence relation.

This might be in the Neron Models book or in Kleiman's article in FGA explained?

○

The first part of Castelnuovo's argument was to consider the curve C_0. The second part is the following fact

Fact. There exists a flat family $\mathcal{C} \xrightarrow{\pi} \Delta$ with $C_t = \pi^{-1}(t)$ smooth for $t \neq 0$ and $C_0 = \mathbb{P}^1/(p_\alpha \sim q_\alpha)$. Here, think of Δ as a complex unit disk.

This can be shown algebraically by applying deformation theory. Alternatively, one can show this complex analytically. Topologically, a node looks like two disks intersecting transversally in a single point (picture this as a double cone). Consider the family $\{xy - t = 0\} \to \Delta_t$. This gives a family of annuli specializing to the (local) nodal picture. Analytically, you can take this local picture, and just glue in the rest of the curve to get the desired family of proper curves.

Say we have our family $\mathcal{C} \xrightarrow{\pi} \Delta$ with $\mathcal{C}_0$ a g-nodal curve and $\mathcal{C}_t$ smooth of genus g.

Claim 22.5. *For $\rho(d,g,r) < 0$, $\nexists g_d^r$ on C_t for general t.*

The first step is to suppose there does exist a $g_d^r = \left(L \in \operatorname{Pic}^d(C_t), V^{r+1} \subset \mathrm{H}^0(L)\right)$ on C_t for general t. We want to say that the limit of these gives a g_d^r on C_0. Let's take this for granted for now, and circle back to it later...

Given this, we now claim that $\nexists g_d^r$ on C_0 with $\rho < 0$. Recall a g_d^r on C_0 is a line bundle L_0 on C_0 along with a subspace $V_0 \subset \mathrm{H}^0(L_0)$ of dimension $r+1$. To investigate this, we pull it back to $\mathbb{P}^1$, where it becomes $V \subset \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(d))$ satisfying

Maybe recall Claim 19.7

$$\sigma \in V \implies [\sigma(p_\alpha) = 0 \iff \sigma(q_\alpha) = 0 \text{ for all } \alpha].$$

Recall we have a natural embedding $\mathbb{P}^1 \overset{|\mathscr{O}(d)|}{\hookrightarrow} \mathbb{P}^d$ (as a rational normal curve), and any linear series of degree d and dimension r in $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(d))$ corresponds to a linear subspace $\Lambda^{d-r-1} \subset \mathbb{P}^d$ (the subspace you project away from to get the map associated to the linear series). This Λ will correspond to a linear series coming from C_0 iff

$$H \supset \Lambda \implies [H \ni p_\alpha \iff H \ni q_\alpha \text{ for all } \alpha]$$

for hyperplanes H. Equivalently, $\Lambda \cap \overline{p_\alpha q_\alpha} \neq \emptyset$.

Observation 22.6. If $\ell \subset \mathbb{P}^d$ is any line, the locus

$$\Sigma_r(\ell) := \{\Lambda \in \mathbb{G}(d-r-1,d) : \Lambda \cap \ell \neq \emptyset\}$$

is a codimension r cycle in the Grassmannian. If

$$\Sigma_r(p_1q_1) \cap \cdots \cap \Sigma_r(p_gq_g)$$

is a proper intersection (i.e. has expected codimension $r \cdot g$), then them having non-empty intersection would force

$$rg \leq \dim \mathbb{G}(d-r-1,d) = (r+1)(d-r) \iff \rho = g-(r+1)(g-d+r) \geq 0.$$

Thus, we're reduced to verifying that this intersection of r-cycles is indeed proper. That is, for g general chords $\ell_\alpha = \overline{p_\alpha q_\alpha}$ to a rational normal curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^d$, we need to verify that $\Sigma_r(\ell_\alpha)$ intersect properly.

Remark 22.7. This statement is not true for literally every possible collection of g chords, only for general ones. So we still can't write down a particular g-tuple of chords and verify it for them. To get around this, we specialize further. ○

Specialize the chords ℓ_α to g tangent lines. The collection of intersections form a family, so by upper semi-continuity of fiber dimension, if we can show the intersection is proper for tangent lines, then it will be for general lines.

To say that $\Lambda \cap T_r(\mathbb{P}^1) \neq \emptyset$ says that every hyperplane containing Λ contains $T_r(\mathbb{P}^1)$, so $\sigma \in V$ implies that $\sigma(r) = 0 \implies \sigma'(r) = 0$. This is telling us that the ramification sequence of the linear series V on $\mathbb{P}^1$ at r is

$$\geq (0,1,1,\ldots,1)$$

(no 1 in the vanishing sequence). In particular, $w(V,r) \geq r$. Thus, V must have total ramification index $\geq rg$ by considering the points $r_1,\ldots,r_g$. Plücker then tells us that ($g(\mathbb{P}^1)=0$)

$$rg \leq \sum_{p \in \mathbb{P}^1} w(V,p) = (r+1)d + r(r+1)(g-1) = (r+1)d - r(r+1) = (r+1)(d-r).$$

We conclude

Proposition 22.8. *There exists a plane $\Lambda \subset \bigcap_\alpha \Sigma_r(T_{r_\alpha}\mathbb{P}^1)$ only if*

$$\rho = g-(r+1)(g-d+r) \geq 0.$$

History (paraphrase)**.** Back in 19th century, BN was viewed as basically an established fact, even though they knew they hadn't rigorously proved it. Castelnuovo had a further question: in a case where

$\rho = 0$, there should only be finitely many g^r_d's; how many exactly? In wondering this, he had an idea of specializing to a g-nodal curve. He was happy to assume the relevant intersection was transverse, so the number was computable by Schubert calculus. Over time, standards of prove evolved, and people really wanted a rigorous proof (as recently as the 60s). Steve Kleiman had the idea to repurpose Castelnuovo's construction as a proof of Brill-Noether; it boiled down to this statement about general chords have Schubert cycles with proper intersection. He wasn't quite able to work this out, so this remained as a conjecture for a time afterwards until it was finally resolved later on. So this is a proof that was worked over over the course of like 100 years or something. $\ominus$

Let's get back to the hiccup we skipped before. If you have a family of line bundles on (the smooth locus) of a family of curves, it does not necessarily specialize to a line bundle on the singular fiber. However, it will specialize to a torsion-free sheaf at the fiber. Once you have this, you appeal to the fact that a torsion free sheaf (of rank 1) on a nodal curve can only be one of two things at a singular point: either free or the maximal ideal. So really one wants to take the partial normalization of the nodal curve (normalzing points where you get $\mathfrak{m}_p$ instead of $\mathscr{O}_{C,p}$), and work with that.

23 Lecture 21 (11/22): Hilbert schemes of curves in $\mathbb{P}^r$

Notation 23.1. We let $\mathcal{H}_{d,g,r}$ denote the Hilbert scheme w/ Hilbert polynomial $md - g + 1$. We let $\mathcal{H}^\circ_{d,g,r} \subset \mathcal{H}_{d,g,r}$ be the (open) locus of smooth, irreducible, non-degenerate curves.

Question 23.2. *Given d, g, r describe the dimension/irreducibility of $\mathcal{H}^\circ_{d,g,r}$.*

Remark 23.3. If we wanted to do a deeper study of the geometry of this space, we'd want to compactify it first since most of our techniques apply primarily to compact/projective varieties. One compactification would be to consider the whole Hilbert scheme, but that's a bit of a wild space, so a smaller compactification would be more suitable. As far as dimension/irreducibility are concerned, it's enough to look at an open, so we don't worry about compactifying here. $\circ$

We'll limit ourselves today mainly to $\mathbb{P}^3$ instead of general $\mathbb{P}^r$. The point is that in $\mathbb{P}^3$ we have the extra technique of liaison/linkage.

23.1 $d = 3$

The least possible degree of a (non-degenerate, irreducible) curve in $\mathbb{P}^3$ is $d = 3$, in which case it must be a twisted cubic.

Approach 1 Consider the family of maps $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. It will be given by $[F_0, \dots, F_3]$ with $F_i \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(3))$. In order for the image to be non-degenerate, the F_i's need to be linearly independent and so must form a basis for $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(3))$. Finally, simultaneously scaling all the F_i's does not change the map. Thus, this is a family parameterized by an open subset

$$U \subset \mathbb{P}\left(\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(3)^4\right) \cong \mathbb{P}^{15},$$

so we get a map $U \to \mathcal{H}^0_{3,0,3}$ whose fibers are copies of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$. So we get

Proposition 23.4. $\mathcal{H}^\circ_{3,0,3}$ *is irreducible of dimension* 12.

Approach 2 Let's give an alternative means of calculating this. The Hilbert scheme of twisted cubics is a subset of the Grassmannian of 3-dimensional vector spaces of quadrics in $\mathbb{P}^3$ (a twisted cubic is the intersection of 3 quadrics in $\mathbb{P}^3$). What condition is it on the 3-dimensional vector spaces of quadrics that makes them intersect in a twisted cubic? For 3 general quadrics, their intersection will be a point, not a twisted cubic. One way to look at this is to look at pairs of quadrics (which will intersect in a curve) instead of triples; when does that curve contain a twisted cubic curve? If this is the case, the intersection must be the union of a twisted cubic and a line; conversely, if two (general) quadrics jointly contain a line, the residual intersection will be a twisted cubic.

This all suggests that we look at the incidence correspondence ($\mathbb{P}^9$ =space of quadrics in $\mathbb{P}^3$)

$$\Phi = \left\{(C, L, Q_1, Q_2) \in \mathcal{H}^{\circ}_{3,0,3} \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^9 : Q_1 \cap Q_2 = C \cup L\right\}.$$

This has projection maps

$$\begin{tikzcd} & \Phi \arrow[dl] \arrow[dr] & \\ \mathcal{H}^{\circ}_{3,0,3} & & \mathbb{G}(1,3). \end{tikzcd}$$

If we fix a line $L \in \mathbb{G}(1,3)$, what does the fiber look like? They will be open subsets of the pairs of quadrics containing a line, i.e. opens in $\mathbb{P}^6 \times \mathbb{P}^6$. Hence, Φ is irreducible of dimension $\dim \mathbb{G}(1,3) + \dim(\mathbb{P}^6 \times \mathbb{P}^6) = 16$. Now look on the other side; what are the fibers of the map to the Hilbert scheme? The story is the same; two (general) quadrics containing a twisted cubic will have a line as the residue intersection, so the fibers are open subsets in the space of pairs of quadrics containing a twisted cubic, i.e. opens in $\mathbb{P}^2 \times \mathbb{P}^2$. Thus, the Hilbert scheme is irreducible of dimension $\dim \mathcal{H}^{\circ}_{3,0,3} = \dim \Phi - \dim(\mathbb{P}^2 \times \mathbb{P}^2) = 12$.

For the low degree/genus case, there are many methods for estimating the dimension of these Hilbert schemes. In higher degrees, maybe only some of these methods will work. In still higher degrees, we just won't know the answer.

Approach 3 Let's see a third way of doing this computation. This will be specific to rational normal curves in $\mathbb{P}^r$.

Lemma 23.5. *If $p_1, \ldots, p_6 \in \mathbb{P}^3$ are in linear general position (i.e. no 4 coplanar), then there exists a unique twisted cubic $C \ni p_1, \ldots, p_6$.*

Remark 23.6. The subset of the Hilbert scheme corresponding to twisted cubics passing through a specified point is codimension 2. Hence, given the previous computations, we'd expect that given 6 points, there'd only be finitely many twisted cubics passing through them all. ○

Note 26. Missed some of what he said... something about looking at curves with Hilbert scheme of dimension $2n$ passing through n points I think?

Consider

$$\begin{tikzcd} & \Gamma = \left\{(p_1, \ldots, p_6, C) \in (\mathbb{P}^3)^6 \times \mathcal{H}^{\circ}_{3,0,3} : p_i \in C\right\} \arrow[dl] \arrow[dr] & \\ \mathcal{H}^0_{3,0,3} & & (\mathbb{P}^3)^6. \end{tikzcd}$$

The lemma tells us that the right map is an isomorphism onto an open $U \subset (\mathbb{P}^3)^6$ (specifically, the open of points in linear general position). The fibers of the left map are opens in $(\mathbb{P}^1)^6$ (pick any 6 points on $C \cong \mathbb{P}^1$ in linear general position), so $\dim \mathcal{H}^0_{3,0,3} = 18 - 6 = 12$.

23.2 $d = 4$

Remark 23.7. A curve of degree 4 in $\mathbb{P}^3$ must have genus 0 or 1, e.g. by Castelnuovo. ◦

23.2.1 $g = 1$

Let's start with genus 1. As usual, helpful to look at surfaces containing our curve, so consider

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2)}_{\dim=10} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(2))}_{\dim=8},$$

so $C \subset \mathbb{P}^3$ will lie on two linear independent quadrics Q_1, Q_2. Furthermore, we must have $C = Q_1 \cap Q_2$.

If C had genus > 1, it would lie of ≥ 3 quadrics, but this is impossible. A degree 4 curve in $\mathbb{P}^3$ can't lie on > 2 quadrics. This gives a different proof that the only possible genera are $0, 1$.

Corollary 23.8. *A genus 1 quartic curve in $\mathbb{P}^3$ is determined by the 2-dimensional pencil of quadrics containing it.*

(Conversely, the intersection of two quadrics will be degree 4 and genus 1, by adjunction)

Corollary 23.9.

$$\mathcal{H}^\circ_{4,1,3} \overset{open}{\subset} \mathbb{G}(2, 10),$$

so is irreducible of dimension 16.

Fact. If you have 8 general points in $\mathbb{P}^3$, there will be a unique such quartic curve passing through them.

23.2.2 $g = 0$

Now $g = 0$. C will be the image of a map $\mathbb{P}^1 \to \mathbb{P}^3$ given by a 4-tuple of polynomials $[F_0, \ldots, F_3]$ with the $F_i \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(4))$ linearly independent. The condition that this map be an embedding is an open condition, so these maps are parameterized by an open

$$B \subset \mathbb{P}\left(\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(4))^4\right) \cong \mathbb{P}^{19}.$$

We get a map $B \twoheadrightarrow \mathcal{H}^\circ_{4,0,3}$ with fibers $\cong \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$.

Proposition 23.10. $\mathcal{H}^\circ_{4,0,3}$ *is irreducible of dimension* 16.

Exercise. Do this computation using liaison instead (Hint[25]).

Question 23.11 (Audience)**.** *Is it a coincidence that both of these dimensions turned out to be* 16*?*

Answer. That's a good question. Let's do some more examples, and see if we can discover the pattern. ⋆

[25] C will lie on a quadric and a cubic with residual curve the union of two skew lines

23.3 $d = 5$

In this case, the possible genera are $g = 0, 1, 2$. This follows e.g. from Castelnuovo.

23.3.1 $g = 0$

We can do the same thing of looking at maps $[F_0, \ldots, F_3] : \mathbb{P}^1 \to \mathbb{P}^3$ with $F_i \in \mathrm{H}^0(\mathscr{O}_{\mathbb{P}^1}(5))$. The ones giving an embedding will form an open subset of the projectivation of the fourth power of this space. The fibers are PGL_2, and the punchline is

Proposition 23.12. *$\mathcal{H}^\circ_{5,0,3}$ is irreducible of dimension* 20.

At this point, it should be clear that this same strategy works for rational curves of any degree. The conclusion is

Theorem 23.13. *$\mathcal{H}^\circ_{d,0,3}$ is irreducible of dimension $4d$.*

You could try to do a linkage argument here, but things are more subtle than before. The first step in a linkage argument is estimating the number of surfaces our curve lies on. Once $d = 5$, this number depends on the curve; a general rational quintic won't lie on a quadric, but some quintic will. Hence, the incidence correspondence coming from liaison will have more than one component.

23.3.2 $g = 2$

We could do this using liaison. We've done this computation in class before: if $C \subset \mathbb{P}^3$ a (smooth, irreducible, non-deg) curve of degree 5 and genus 2, then C will lie on unique quadric surface Q and on a (non-unique) irreducible cubic surface S not containing Q. The intersection will be degree 6 and so of the form $S \cap Q = C \cup L$ with $L \cong \mathbb{P}^1$ a line. Thus, we consider

$$\Phi = \left\{(C, L, Q, S) \in \mathcal{H}^\circ_{5,2,3} \times \mathbb{G}(1,3) \times \mathbb{P}^9 \times \mathbb{P}^{19} : S \cap Q = C \cup L\right\}$$

with projection maps

$$\begin{tikzcd} & \Phi \arrow[dl] \arrow[dr] & \\ \mathcal{H}^\circ_{5,2,3} & & \mathbb{G}(1,3). \end{tikzcd}$$

The fibers of the right map are opens in $\mathbb{P}^6 \times \mathbb{P}^{15}$ (specify a quadric and a cubic containing the line you start with), so Φ is irreducible of dimension 25. For the left map, if you specify C, there's a unique quadric containing it and a 6-dimensional vector space of cubics containing it. Hence, the fibers of the left map are opens in $\mathbb{P}^5$.

Proposition 23.14. *$\mathcal{H}^\circ_{5,2,3}$ irreducible of dimension* 20.

Another Approach Can we adapt the approach we used for rational curves of arbitrary degree? Instead of thinking of our curves as zero loci of polynomials, we think of them are the images of maps from an abstract genus 2 curve. With this approach in the case of rational curves, we're taking advantage of the facts that there is a unique curve of genus 0 and this curve has a unique line bundle of degree

d. Contrast this with the fact that there are many different curves of genus 2 and each one has many different line bundles of degree 5.

Let M_2 be the moduli space of curves of genus 2, which is irreducible of dimension 3. Choosing a point here corresponds to specifying the source of a map $C \hookrightarrow \mathbb{P}^3$. We next need a line bundle giving this map, so we want a point of

$$\wp_{5,2} = \left\{(C, L) : C \in M_2, L \in \operatorname{Pic}^5(C)\right\}.$$

The fibers of the natural map $\wp_{5,2} \to M_2$ are copies of the Jacobian of the base curve, and so are irreducible of dimension 2. Thus, $\wp_{5,2}$ is irreducible of dimension 5. Since $5 \geq 2(2)+1$, any $L \in \operatorname{Pic}^5(C)$ will be very ample and have $h^0(L) = 5 + 1 - 2 = 4$. Hence, the fibers of $\mathcal{H}^\circ_{5,2,3} \to \wp_{5,2}$ will be open subsets in $\mathbb{P}\left(\mathrm{H}^0(C, L)^4\right)$, and so are irreducible of dimension 15.

Proposition 23.15. *$\mathcal{H}^\circ_{5,2,3}$ is irreducible of dimension* 20.

23.4 A general statement

Let's see how much of the most recent argument generalizes to curves of any degree and genus. For $g \geq 2$, we have M_g irreducible of dimension $3g-3$. In order for $\wp_{d,g} \to M_g$ to be simple, we need d large so that all line bundles of degree d will behave similarly.

Assumption. Assume $d > 2g-2$ and $d \geq g+3$ (so L has at least 4 sections).

Keep in mind the maps

$$\mathcal{H}^\circ_{d,g,3} \to \wp_{d,g} \to M_g.$$

We know $\dim M_g = 3g-3$ and $\dim \wp_{d,g} = 4g-3$ (fibers are Jacobians). Define

$$\mathcal{G}^3_{d,g} := \left\{(C, L, V^4 \subset \mathrm{H}^0(L))\right\} \longrightarrow \wp_{d,g}.$$

This has fibers given by opens in $\mathbb{G}(4, d-g+1)$. Now we consider

$$\mathcal{H}^\circ_{d,g,3} \longrightarrow \mathcal{G}^3_{d,g}$$

with fibers $\cong \mathrm{PGL}_4$. Adding everything up

Proposition 23.16. $\dim \mathcal{H}^\circ_{d,g,3}$ *is irreducible of dimension* $4d$*, when* $d > 2g-2$ *and* $d \geq g+3$. Also need to ensure L is very ample, so probably also want $d \geq 2g+1$ to be safe

More on the significance this next time.

23.5 Announcement

We'll only do one more homework assignment, due the week of December 6th. Still feel free to play around with these things on your own though.

24 Lecture 22 (11/29)

Note 27. Roughly 13 minutes late

24.1 Class stuff

- Last class on Wednesday
- Last assignment due 12/8

  There will be section on Friday

24.2 Hilbert scheme

We're studying the restricted Hilbert scheme

$$\mathcal{H}^\circ_{d,g,r} := \{C \subset \mathbb{P}^r : C \text{ irred., smooth, non-degerate}\}.$$

We observed that in $\mathbb{P}^3$, for $d \le 5$, $\mathcal{H}^0_{d,g,3}$ is irreducible of dimension $4d$.

Let's do the analogous analysis for $\mathbb{P}^r$ in place of $\mathbb{P}^3$. Assume $d \ge g+r$ and $d \ge 2g+1$. Then we have maps

$$\begin{tikzcd}
\mathcal{H}^\circ_{d,g,r} \arrow[d, "\text{fibers } \cong \mathrm{PGL}_{r+1}"] & \\
\mathcal{G}^r_d \arrow[r, equal] \arrow[d, "\text{fibers open in } \mathrm{Gr}(r+1,d-g+1)"] & \big\{(C,L,V) : V^{r+1} \subset \mathrm{H}^0(L) : \varphi_V \text{ embedding}\big\} \\
\mathcal{P}^r_{d,g} \arrow[r, equal] \arrow[d, "\text{fibers are } W^r_d(C)\text{'s}"] & \Big\{(C,L) : L \in \mathrm{Pic}^d(C) : h^0(L) \ge r+1\Big\} \\
\mathcal{M}_g \arrow[r, equal] & \{C : \text{genus } g\}
\end{tikzcd}$$

One stares at this and analyzes the fibers (i.e. observes that they're irreducible and computes their dimensions), and ultimately concludes

Theorem 24.1. *$\mathcal{H}^\circ_{d,g,r}$ is irreducible of dimension*

$$h(d,g,r) := 4g - 3 + (r+1)(d-g+1) - 1$$

(this is $4d$ if $r = 3$), when $d \ge \max(g+r, 2g+1)$.

Question 24.2. *Can we carry out this kind of analysis without the extra assumptions on d?*

Note 28. Missed some stuff he said cause I was busy catching up with what was on the board...

In the case $d \le g+r$ (but where $\rho(g,d,r) := g - (r+1)(g-d+r) > 0$), Brill-Noether tells us that the fiber of $\mathcal{P}^r_{d,g}$ over a *general* $C \in \mathcal{M}_g$ is irreducible of dimension $\rho(d,g,r)$. This tells us that is a unique irreducible component of $\mathcal{H}^0_{d,g,r}$ which dominates $\mathcal{M}_g$, and it will have dimension $= (3g-3) + \rho(d,g,r) + (r^2+2r) = h(d,g,r)$.

The previous sentence shows this with $\mathcal{P}^r_{d,g}$ in place of $\mathcal{H}^0_{d,g,r}$. To get $\mathcal{H}^0_{d,g,r}$ here, also need to know that a general member of

Warning 24.3. If I heard correctly, it sounds like when $\rho(d,g,r) \le 0$, the Hilbert scheme does not dominate $\mathcal{M}_g$, and this Brill-Noether stuff tells us nothing about its dimension. •

Remark 24.4. It sounds like Joe and Eisenbud cranked out the case $\rho = -1$ and proved the pattern persists; there's a unique irreducible component of expected dimension $h(d,g,r)$. Another person (missed the name[26]) cranked out the case $\rho = -2$ and showed the pattern persists there. Sounds like its known the pattern fails for $\rho \ll 0$. $\circ$

Remark 24.5. Above, we need $\rho > 0$ since $W^r_d(C)$ is irreducible in this case. When $\rho = 0$, $W^r_d(C)$ is not irreducible, but one can analyze the monodromy of the universal family, show it acts transitively, and use this to conclude that there's a unique irreducible component of expected dimension dominating $\mathcal{M}_g$. $\circ$

24.3 Back to $\mathbb{P}^3$

Recall 24.6. For all $d \leq 5$, we have seen that $\mathcal{H}^0_{d,g,3}$ is irreducible of dimension $4d$. $\odot$

Question 24.7. *What about $d = 6$?*

In this case, the parametric approach (i.e. the tower on the previous page) works for $g \leq 3$. Note that the maximal possible genus of a degree 6 curve in $\mathbb{P}^3$ is 4, so this leaves only $g = 4$.

Remark 24.8. A curve of degree 6 and genus 4 in $\mathbb{P}^3$ is a canonical curve. $\circ$

One can show (and we have shown previously) that in this case $C = Q \cap S$ is the complete intersection of a quadric Q and a cubic S. We have a map $\mathcal{H}^0_{6,4,3} \to \{\text{Quadrics}\}$. The space of quadrics has dimension $9 = h^0(\mathscr{O}_{\mathbb{P}^3}(2)) - 1$ and the fibers of this map are opens in $\mathbb{P}^{15}$, so $\dim \mathcal{H}^0_{6,4,3}$ is irreducible of dimension 24.

I think on a Wednesday?

24.3.1 More examples

Example $(d = 8, g = 9)$**.** Look at the restriction map $\mathscr{O}_{\mathbb{P}^3}(2) \to \mathscr{O}_C(2)$. The dimension of the global sections of the source is 10 while the dimension of global sections of the target is

$$16 - 9 + 1 + \begin{cases} 1 & \text{if } \mathscr{O}_C(2) = K_C \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} 9 & \text{if } \mathscr{O}_C(2) = K_C \\ 8 & \text{otherwise.} \end{cases}$$

We claim that in this case, $\mathscr{O}_C(2) = K_C$. If not, C would lie on ≥ 2 quadrics, violating Bezout. Thus, C lies on a unique quadric Q. Note C will not lie on a cubic by Bezout. What about quartics? Look at

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(4))}_{\dim=35} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(4))}_{\dim=24}.$$

Hence, C lies on at least 11 quartic surfaces. There is a 10-dimensional vector space of quartic polynomials containing Q. Hence, there is a quartic $S \not\supset Q$ which contains C. Thus, $C = Q \cap S$ is the intersection of a quadric and a quartic surface. Now we can just look at the map $\mathcal{H}^0_{8,9,3} \to \mathbb{P}^9 = \{Q\}$ whose fibers are opens in $\mathbb{P}(\text{quartic polys/polys vanishing on } Q) \cong \mathbb{P}^{35-10-1} \cong \mathbb{P}^{24}$. Thus, $\mathcal{H}^\circ_{8,9,3}$ is irreducible of dimension $33 \neq 4(8)$. $\triangle$

So the dimension estimate really doesn't always hold. Still, it's been irreducible in every case we've looked at, so let's fix that.

[26]Dan something?

Example $(d = 9, g = 10)$**.** As always, we ask, "what sort of equations define C?" Does C lie on a quadric?

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(2))}_{\dim=10} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(3))}_{\dim=\begin{cases}10 & \text{if } \mathscr{O}_C(2) = K_C \\ 9 & \text{otherwise.}\end{cases}}.$$

We have a genuine ambiguity here (neither possibility leads to a contradiction).

- Let's first say C does not lie on a quadric ($\implies \mathscr{O}_C(2) = K_C$)

  Look at cubics:

  $$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(3))}_{\dim=20} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(2))}_{\dim=18}.$$

  Hence, C lies on a pencil of cubic surfaces. Since C does not lie on a quadric, all of these cubics are irreducible. Thus, $C = S \cap S'$ is the intersection of any two of these cubics[27] (equivalently, it's the base locus of the pencil). Thus, such C are parameterized by an open subset of $\mathrm{Gr}(2, 20)$. Hence, we get a component of $\mathcal{H}^\circ_{9,10,3}$ which is irreducible of dimension $2(20 - 2) = 36 = 4(9)$.

- Now say C lies on a quadric Q.

  Q must be irreducible. Let's assume for the moment that Q is smooth.

  Exercise. For such a C, Q must be smooth.

  Note that C will have type (a, b) where

  $$a + b = 9 \text{ and } (a - 1)(b - 1) = 10.$$

  This forces $(a, b) = (3, 6)$ (or $(6, 3)$) on Q. Hence, get a map $\{C \text{ of this type}\} \to \mathbb{P}^9 = \{\text{quadrics}\}$ whose fibers are open in $\mathbb{P}(\text{bihom poly of bidegree } (3, 6)) \cong \mathbb{P}^{27}$ (here $27 = (3+1)(6+1)-1$). Thus, we have another irreducible component of dimension 36. △

You can play around and find more examples with multiple components of different dimensions.

24.4 Sneak Peek for Wednesday

We want to look at another way of estimating the dimension of the Hilbert scheme. The parametric approach gave us the expected dimension $h(d, g, r)$. Instead of looking at the number of parameters needed to specify all the data, we can go back to thinking of curves as zero loci of polynomials. This is directly so easy. If someone gives you a collection of polynomials and asks you about the dimension of their common zero locus, there's no easy way to answer this. Here's one way of getting an approximate (and approximately wrong) answer: compute the dimension of the Zariski tangent space at a point. If you have a single common zero of all the polynomials you're given, you can evaluate the derivatives of all the polynomials at that point, and then use linear algebra to estimate the dimension of the Zariski tangent space.

[27] Conversely, can check that if you intersect two general irreducible cubics, you get a smooth curve of degree 9 and genus 10

Warning 24.9. Singular points exist, and everywhere non-reduced schemes all exist. •

So this approach won't always give a perfectly correct answer, but it's something.

Goal. Estimate $\dim \mathcal{H}^0$ by finding the dimension of its Zariski tangent space at a point $[C]$, i.e. compute $\dim T_{[C]}\mathcal{H}^0$.

We can't write down the equations defining the Hilbert scheme, but we can still get at its geometry using its universal property: maps into it are in natural bijection with (flat) families of curves over the source scheme. To do this, we use the following characterization of T_pX (say X a k-scheme): a tangent vector of X at p is simply a map $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to X$ sending the unique point of $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2)$ to $p \in X$. That is,

$$T_pX = \left\{ \varphi : \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to X \,\middle|\, p = \left(\operatorname{Spec} k \to \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \xrightarrow{\varphi} X \right) \in X(k) \right\}.$$

Exercise. Describe the natural addition and scalar multiplication laws on the above set.

Because we understand maps to the Hilbert scheme, we can understand $T_{[C]}\mathcal{H}^0$ using this perspective. A map $\varphi : \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to \mathcal{H}^0$ carrying $\operatorname{Spec} k \to [C]$ is exactly a family

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathbb{P}^r_{k[\varepsilon]/(\varepsilon^2)} \arrow[dl] \\ & \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) & \end{tikzcd}$$

with $\mathcal{C}_0 \cong C$. This is a 1st order deformation of C.

25 Lecture 23 (12/1): Last lecture of Fall

Note 29. Roughly 16 minutes late

Looks like Joe recapped what was done last time... Today we'll try to estimate $\dim \mathcal{H}^0_{d,g,r}$ by the dimension of its tangent space.

25.1 Tangent Space

In general, for any scheme X/k and point $p \in X(k)$, one has

$$T_pX = \left\{ \begin{array}{ccc} \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) & \longrightarrow & X \\ \cup & & \Cup \\ \operatorname{Spec} k & \longmapsto & p \end{array} \right\}$$

We can apply to this $[C] \in \mathcal{H}^0_{d,g,r}$. Using the universal property of $\mathcal{H}^0_{d,g,r}$ (maps into $\mathcal{H}^0$ are flat families of projectively embedded curves) this gives

$$T_{[C]}\mathcal{H}^0 = \left\{ \begin{array}{c} \mathcal{C} \subset \mathbb{P}^r_{k[\varepsilon]/\varepsilon^2} \text{ flat over } \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \\ \text{such that } \mathcal{C}_0 \simeq C \end{array} \right\}.$$

Above, $\mathcal{C}_0 \hookrightarrow \mathbb{P}^r_k$ is the fiber of $\mathcal{C}$ over $\operatorname{Spec} k$.

Remark 25.1. Flatness over a reduced base is fairly trivial. Say B is a smooth curve, and we have a closed subscheme $\mathcal{C} \subset \mathbb{P}^r_B$. Flatness is then the following: given $b \in B$

$$\overline{\mathcal{C} \setminus \mathcal{C}_b} = \mathcal{C},$$

i.e. if you remove the fiber over b and then take the closure, you recover the original family. Think: every fiber is the limit of the nearby fibers. ○

Note 30. I'm distracted today, so these notes will be missing some of what was said

Remark 25.2. I kinda lost the thread that we're on, but say $C = V(f_1, \dots, f_k)$ is the zero set of some polynomials. Then we can write $\mathcal{C} = V(f_1 + \varepsilon g_1, \dots, f_k + \varepsilon g_k)$, and the question is: what do we need the g's to satisfy in order for this to be a flat family? Now, you can apply the algebraic definition of flatness (probably use the 'local criterion for flatness'?) to see that $\mathcal{C}$ is flat iff for all $f \in I_C$, there exists $g \in \mathscr{O}_U$ s.t. $f + \varepsilon g \in I_{\mathcal{C}}$ and g is unique mod I_C. So we get a homomorphism of sheaves $\mathscr{I}_{C/\mathbb{P}^r} \to \mathscr{O}_{\mathbb{P}^r}/\mathscr{I}_{C,\mathbb{P}^r} = \mathscr{O}_C$ which descends to give a map

$$\mathscr{I}_{C/\mathbb{P}^r}/\mathscr{I}^2_{C/\mathbb{P}^r} \longrightarrow \mathscr{O}_C$$

(this homomorphism spits out g, given f). Note that (by taking duals), the above homomorphism is the same thing as a section of the normal bundle of $C \hookrightarrow \mathbb{P}^r$.

$\mathscr{I}/\mathscr{I}^2$ being the conormal bundle is definitional in AG. I guess if you're working analytically, then you usually first define the *normal bundle* $N_{C/\mathbb{P}^r}$ via the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^r}|_C \longrightarrow N_{C/\mathbb{P}^r} \longrightarrow 0.$$

When one dualizes this sequence, they get $0 \to N^\vee_{C/\mathbb{P}^r} \to T^\vee_{\mathbb{P}^r}|_C \to T^\vee_C \to 0$. The kernel of this map can be identified by hand, and one ends up deducing that $N^\vee_{C/\mathbb{P}^r} = \mathscr{I}_C/\mathscr{I}^2_C$. ○

Corollary 25.3. *For $C \in \mathcal{H}^0_{d,g,r}$, one has $T_{[C]}\mathcal{H}^0 = \mathrm{H}^0(N_{C/\mathbb{P}^r})$.*

What's the dimension of the normal bundle? For C a smooth curve in $\mathbb{P}^r$, the normal bundle $N_{C/\mathbb{P}^r}$ will be a vector bundle of rank $(r-1)$. The normal bundle could (probably) be just about any vector bundle since C is just an arbitrary smooth curve, so how do we estimate the size of its space of global sections? We Riemann-Roch.

Definition 25.4. Given any rank r vector bundle $E \to C$ on a smooth curve, we define $\deg(E) := \deg\left(\bigwedge^r E\right)$. ◇

Observe that if $0 \to E \to F \to G \to 0$ is a short exact sequence, then $\deg(F) + \deg(E) + \deg(G)$.

Theorem 25.5 (Riemann-Roch for Vector Bundles on Smooth Curves). *Given a vector bundle $E \to C$ on a smooth curve, one has*

$$\chi(E) = \deg(E) + \operatorname{rank}(E)\chi(\mathscr{O}_C) = \deg(E) - r(g-1).$$

(Prove this by induction on the rank by putting E in an exact sequence $0 \to F \to E \to G \to 0$ with $\operatorname{rank}(F), \operatorname{rank}(G) < \operatorname{rank}(E)$)

Exercise. If E is any vector bundle of rank $r > 1$ on a smooth curve C, then E has a sub line bundle.

Given this, we just need to know $\deg(N_{C/\mathbb{P}^r})$ and then we can estiamte its h^0 by its χ.

Recall 25.6. There is an exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^r}|_C \longrightarrow N_{C/\mathbb{P}^r} \longrightarrow 0. \qquad \odot$$

The degree of $T_C = \omega_C^{-1}$ is $2-2g$. The degree of $T_{\mathbb{P}^r}$ is $r+1$ (it's top exterior power is $\omega_{\mathbb{P}^r}^{-1}$), so $\deg(T_{\mathbb{P}^r}|_C) = (r+1)d$. Thus,

$$\deg N_{C/\mathbb{P}^r} = (r+1)d + (2g-2).$$

Hence,

$$\dim \mathcal{H}^0_{d,g,r} \leq h^0(N_{C/\mathbb{P}^r}) \geq \chi(N_{C/\mathbb{P}^r}) = (r+1)d + (2g-2) - (r-1)(g-1) = (r+1)d - (r-3)(g-1).$$

Note the inequalities go in opposite directions, so we can't formally conclude a bound on $\dim \mathcal{H}^0_{d,g,r}$ from this; it's moreso just a heuristic.

Recall 25.7. Last time we obtained the following formula for the expected dimension:

$$h(d,g,r) = 4g - 3 - (r+1)(g-d+1) - 1.$$

This looks like a different estimate, but they're actually the same (check this). $\odot$

Remark 25.8. When $r = 3$, today's heuristic becomes

$$(3+1)d - (3-3)(g-1) = 4d$$

recovering the pattern we had observed before. $\circ$

We'll say one last thing, and then end early today.

Note 31. Missed most of this last thing. Whoops... Something about this estimate

$$h(d,g,r) = (r+1)d - (r-3)(g-1)$$

failing spectacularly for large g because of the existence of Castelnuovo curves.

Question 25.9. *Could there be any component of the Hilbert scheme of dimension* 0*?*

No. Given any curve you can move it around using automorphisms of projective space, so any component will have dimension $\geq \dim \mathrm{PGL}_{r+1} - 3$ (the -3 is because of projective automorphisms carrying the curve to itself).

Question 25.10. *Are there components of dimension* $= \dim \mathrm{PGL}_{r+1} - 3$*?*

Yes, for a rational normal curve.

Definition 25.11. Say $C \subset \mathbb{P}^r$ is **rigid** if every deformation of C is a translate of C under PGL_{r+1}. $\diamond$

If I heard correctly, this means there's an open nbhd of $[C] \in \mathcal{H}^0_{d,g,r}$ on which PGL_{r+1} acts transitively

Open Question 25.12. *Are there rigid curves which are not rational normal curves?*

Sounds like something like this would give a component of the Hilbert scheme mapping to a point in M_g.

26 Problem Session (12/3)

Note 32. Roughly 13 minutes late

26.1 Problem 1 Probably

Theorem 26.1. *Given $p_1, \ldots, p_{n+3} \in \mathbb{P}^n$ in linear general position, there is a unique rational normal curve $C \subset \mathbb{P}^n$ with $C \ni p_1, \ldots, p_{n+3}$.*

Proof Sketch. (Existence) First normalize to assume $p_1 = [1 : 0 : \ldots 0 :], \ldots, p_{n+1} = [0 : \cdots : 0 : 1]$, $p_{n+2} = [1 : 1 : \cdots : 1]$, and $p_{n+3} = [\lambda_0 : \cdots : \lambda_n]$. Consider map

$$\begin{array}{rccc} f: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^n \\ & t & \longmapsto & \left[\dfrac{a_0}{t-b_0}, \ldots, \dfrac{a_n}{t-b_n}\right]. \end{array}$$

Then, $f(b_i) = [0, \ldots, 0, 1, 0, \ldots, 0]$, $f(0) = [a_0/b_0, \ldots, a_n/b_n]$ (so can take $a_i = b_i$), and $f(\infty) = [a_0, \ldots, a_n]$.

(Uniqueness) If C, C' are r.c.n$\subset \mathbb{P}^n$ with $\#(C \cap C') \geq n+3$, then $C = C'$.

Use induction. The base case $n = 2$ holds by Bezout (two quadrics with 5 points in common). For $n > 2$, choose any $p \in C \cap C'$ and projective: $\pi_p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$. I missed why but looking at this projection map is the key. ■

Now consider the Hilbert scheme

$$\mathcal{H} = \mathcal{H}^0_{n,0,n} = \{\text{rational normal curves} \subset \mathbb{P}^n\}.$$

Consider the maps

$$\begin{tikzcd} & \Phi := \{(p_1, \ldots, p_{n+3}, C) : p_1, \ldots, p_{n+3} \in \mathbb{P}^n, C \in \mathcal{H}, \text{ and } p_1, \ldots, p_{n+3} \in C\} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & U \overset{\text{open}}{\subset} (\mathbb{P}^n)^{n+3} \end{tikzcd}$$

The right map is dominant and $1:1$ by the above, so Φ is irreducible of dimension $n(n+3)$. The fibers of the left map are opens in $(\mathbb{P}^1)^{n+3}$, so $\mathcal{H}$ is irreducible of dimension $n(n+3) - (n+3) = (n-1)(n+3)$.

Remark 26.2. For plane conics this gives $(2-1)(2+3) = 5$ which is the right answer ($\mathbb{P}\Gamma(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(2))$). For cubic rnc's in $\mathbb{P}^3$, it gives $(3-1)(3+3) = 12 = 4(3)$ which we previously computed. ○

Question 26.3 (Audience)**.** *There was a claim in lecture before about requiring curves to contain a point being a codimension 2 condition on the Hilbert scheme; can you explain this?*

Answer. Let $\mathcal{H}$ be any component of restricted Hilbert scheme. Let $p \in \mathbb{P}^3$ be a point. Let $\mathcal{H}_p = \{C \in \mathcal{H} : C \ni p\} \subset \mathcal{H}$. Look at

$$\begin{tikzcd} & \Phi = \{(C, p) : p \in C\} \arrow[dl] \arrow[dr] & \\ \mathcal{H} & & \mathbb{P}^3 \end{tikzcd}$$

Looking at fibers at counting dimensions shows that $\mathcal{H}_p \subset \mathcal{H}$ is codimension 2. $\star$

Question 26.4. *If* $\dim \mathcal{H} = 2m$ *is even, then for a general choice of* m *points* $p_1, \dots, p_m \in \mathbb{P}^3$*, there will exists a finite number of* $C \in \mathcal{H}$ *which contain those points. How many?*

Note 33. Mostly missed what Joe said about this. Whoops

26.2 Problem 2

Let $C \subset \mathbb{P}^3$ be a quintic rational curve. Such a curve won't have to lie on a quadric surface, but it will lie on some cubics:

$$\underbrace{\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(3))}_{\dim=20} \longrightarrow \underbrace{\mathrm{H}^0(\mathscr{O}_C(3))}_{\dim=16}$$

$\implies$ C lies on ≥ 4 cubics. Say $S, S' \subset \mathbb{P}^3$ are two cubics containing C. Then,

$$S \cap S' = C \cup D$$

with D a curve of degree 4 and genus -1 (by liaison). This will force D to either be the union of two disjoint conics, or of a line and a twisted cubic. In this case, it will be the line and the twisted cubic.

If you have a cubic surface S with a conic curve $E \subset S$, then E will be contained in a plane H. The intersection $H \cap S$ is now degree 3, so will be the union of E and a line L. Say E' is a second, disjoint from E, conic on S. Then, $H \sim E + L \sim E' + L'$ as divisor classes on S (L' some line). Now, $3 = H^3 = (E+L)(E'+L') = EE' + EL + E'L + LL' \leq 2 + LL'$ which forces $L = L'$. Thus, to get two disjoint conics, you need two hyperplanes meeting in a line L, and then to take the residual intersections of those hyperplanes with S. The upshot is that if $E, E' \subset S$ are disjoint conics, then any cubic surface $S' \supset Q, Q'$ must contain the line L. In particular $S \cap S'$ won't be irreducible.

TODO: Make this paragraph make sense

Question: Why are $EL', E'L \leq 1$?

Thus, $D = L \cup B$ is a line and a twisted cubic. Now look at

$$\begin{tikzcd}
& \Phi = \{(A, L, B, S, S') : S \cap S' = E \cup L \cup B\} \arrow[dl] \arrow[dr] & \\
\mathcal{H}^0_{5,0,3} & & \mathbb{G}(1,3) \times \mathcal{H}^0_{3,0,3}
\end{tikzcd}$$

The right space has dimension 16. For the fibers of the right map, how many cubic surfaces do a given line L and twisted cubic B lie on? It's 4 conditions to contain a line and 10 to contain a twisted cubic, so fibers should be opens in $\mathbb{P}^5 \times \mathbb{P}^5$. Thus, $\dim \Phi = 26$ and it's irreducible. One checks that C lies on exactly 4 cubics so the fibers on the left are opens in $\mathbb{P}^3 \times \mathbb{P}^3$ and so $\dim \mathcal{H}^0_{5,0,3} = 20$ (and it's irreducible).

26.3 Problem 3

This one is kind of a slog. We'll skip it for now

26.4 Problem 4

Say $C \subset \mathbb{P}^n$ is a smooth curve. Consider the exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^n}|_C \longrightarrow N_{C/\mathbb{P}^n} \longrightarrow 0.$$

Note that $N_{C/\mathbb{P}^n}$ is a vector bundle of rank $n-1$. If C is rational, all vector bundles are sums of line bundles; what line bundles give the normal bundle?

Setup 26.5. Say $C \subset \mathbb{P}^3$ is a twisted cubic ($C \cong \mathbb{P}^1$). We want to determine $N_{C/\mathbb{P}^3}$

We will have $N_{C/\mathbb{P}^3} \cong \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b)$ for some $a, b \in \mathbb{Z}$. We can easily determine $a + b = \deg N_{C/\mathbb{P}^3}$ since the sequence we started with gives

$$\deg N_{C/\mathbb{P}^3} = \deg(T_{\mathbb{P}^3}|_C) - \deg(T_C) = (\deg C)(\deg \omega_{\mathbb{P}^3}^\vee) - \deg \omega_{\mathbb{P}^1}^\vee = 3(4) - 2 = 10.$$

Now, here's the trick: look for sub-line bundles (of largest possible degree). One way to get a sub line bundle of $N_{C/\mathbb{P}^3}$ is to take a smooth surface $\supset C$ and then look at the normal bundle of that smooth surface. Let $Q \subset \mathbb{P}^3$ be a smooth quadric containing our twisted cubic C. Then, we get an exact sequence

$$0 \longrightarrow N_{C/Q} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{Q/\mathbb{P}^3}|_C \longrightarrow 0.$$

Note that $N_{Q/\mathbb{P}^3} = \mathscr{O}_Q(2)$, so its restriction to C is $\mathscr{O}_{\mathbb{P}^1}(6)$. Consequently, $N_{C/Q}$ will have degree 4.

Recall 26.6. To split a vector bundle $\mathscr{E}$ of rank 2, you really would like a sub line bundle of degree $\geq \frac{1}{2} \deg \mathscr{E}$. $\odot$

At this point, we at least know $N_{C/\mathbb{P}^3}$ is either $\mathscr{O}(4) \oplus \mathscr{O}(6)$ or $\mathscr{O}(5)^{\oplus 2}$.

To figure out which one, we take a completely different approach. Pick a point $p \in C$. Define sub-line-bundle $L \subset N_{C/\mathbb{P}^3}$ by requiring that at each $q \in C$, the fiber L_q at q is the subspace of $(N_{C/\mathbb{P}^3})_q$ spanned by the line $\overline{pq}$.

Question 26.7 (Audience)**.** *Do we know there's no tangent line that hits the curve at 2 distinct points?*

Answer. This line would intersect C in 3 points (with multiplicity). Thus, any quadric surface containing the twisted cubic would contain this line. But the twisted cubic is the intersection of all quadric surfaces containing it.

Any collection of points on a twisted cubic will be in linear general position. $\star$

There's another issue: we haven't defined a sub-line-bundle at the point p. However (since we're over a smooth curve), there will be a natural extension of to an entire sub-line bundle L. This L will have degree 5, and so $N_{C/\mathbb{P}^3} \cong \mathscr{O}(5)^{\oplus 2}$. How do we calculate the degree of L? Observe that this construction gives a sub-line-bundle $L^p = L$ attached to any point p on C; what would happen if we chose a different point p'? We claim the lines $\overline{pq}$ and $\overline{p'q}$ are independent in the fiber of $N_{C/\mathbb{P}^3}$ above q (otherwise, the plane spanned by p, p', q would be tangent to the curve at q and so intersect C in $4 > 3$ points). Thus, $N_{C/\mathbb{P}^3} = L^p \oplus L^{p'}$ for any distinct $p, p' \in C$, so they better have degree 5.

The last part asks an analogous question for rational normal curves in $\mathbb{P}^n$. The answer there will be very similar. Again, for any point $p \in C$, define the sub line bundle $L^p \subset N_{C/\mathbb{P}^n}$ as before.

Claim 26.8. *For any $p_1, \ldots, p_{n-1}$ distinct on C, $N_{C/\mathbb{P}^n} = \bigoplus L_{p_i}$.*

Have a linear relationship would give a hyperplane H meeting the points $p_1, \ldots, p_{n-1}$ and which is tangent at q; this is a no-no since $\deg C = n$. Now one just computes degrees. The L_{p_i}'s all have the same degree since they very continuously with the choice of points. One concludes $\deg L_{p_i} = \frac{n(n+1)-2}{n-1} = n+2$.

You can ask this same sort of question for smooth rational curves which are not necessarily rational normal curves. What splitting types for their normal bundles can occur? It sounds like this was answered relatively recently for rational curves in $\mathbb{P}^3$, but is still open in higher dimensions.

Question 26.9 (Audience)**.** *What's the plan for next semester?*

Answer. Still working out what exactly is feasible, but it'd be good to talk about finer properties of these moduli spaces. Instead of just irreduciblicity and dimension, can ask e.g. 'is the moduli space of curves unirational?' There should be a syllabus up in January with more details. ⋆

27 Lecture 1 (1/25/2022) – Second Semester Start

Note 34. Roughly 6 minutes late

27.1 Class stuff

- Classes Tu/Th 12 – 1:15pm
- section: Fridays 12 – 1:15pm (same room), starting 2/1
- Homeworks: due Mondays, starting 2/4

Topics

- plane curves
- moduli spaces (bulk of class)

Joe said more about these two things, but I was busy catching up

27.2 Plane curves

Let's start with a problem.

Recall 27.1. Given a curve C (smooth + projective) and a divisor $D = \sum_{p \in C} m_p p$, we write

$$\mathscr{L}(D) = \{f \in \mathcal{M}(C) : \operatorname{ord}_p(f) \geq m_p \text{ for all } p\} = \{f \in \mathcal{M}(C) : (f) + D \geq 0\}. \qquad \odot$$

Problem 27.1. *Given C and D,*

(1) *find* $\mathrm{H}^0(K) = \{$*holomorphic* 1-*forms*$\}$.

(2) *find* $\mathscr{L}(D) = \mathrm{H}^0(\mathscr{O}_C(D))$.

(Above, 'find' = 'write down a basis')

In general, it's not clear how to do this. How can one go from a random collection of polynomials in $\mathbb{P}^r$ to an actual description of its space of 1-forms? Our strategy will be to project the curve down to the plane, and then work with the plane curve.

Warning 27.2. So far, we've been working mainly with smooth, projective curves. However, most of these are not embeddable in the plane $\mathbb{P}^2$ (e.g. by the genus formula $g = \binom{d-1}{2}$). Instead, we'll need to start working with singular plane models of curves. $\bullet$

Fact. Every smooth, projective curve C will have a birational embedding $\pi : C \overset{\sim}{\dashrightarrow} C_0 \hookrightarrow \mathbb{P}^2$ onto a singular plane curve.

Above, C will be the normalization of C_0.

There are two ways we can think about/approach this.

- Given an arbitrary smooth, projective curve, one can argue that a general projection down to the plane will map C birationally onto a curve with at most nodes as singularities.

  Hence, it would be sufficient to work only with plane nodal curves. However, there are situations where the most natural projection has other singularities as well. Hence, it will be more convenient to be able to work with arbitrary singularities.

- Alternatively, we can do things in 3 steps

  (1) First consider $C \subset \mathbb{P}^2$ a smooth plane curve, and answer the problem in this special case.

  (2) Broaden scope to nodal curves $C_0 \subset \mathbb{P}^2$.

  At this point, we will have, in theory, a complete solution. Since every curve can be realized as the normalization of a nodal plane curve.

  (3) However, we'll go a step further and answer things for $C_0 \subset \mathbb{P}^2$ with arbitrary singularities.

Note 35. Got distracted and missed some comments by Joe. Something about 'Gorenstein'

Let's get started.

27.2.1 $C \subset \mathbb{P}^2$ smooth plane curve of degree d

Maybe compare start of this section to Section 4.2

More explicitly, say $C = V(F)$ where $F(X, Y, Z)$ is homogeneous of degree d.

Notation 27.3. We'll write $f(x, y) := F(x, y, 1)$.

Recall 27.4. We want to find $\mathrm{H}^0(K_C)$ and $\mathrm{H}^0(D)$ for any divisor $D \in \mathrm{Div}(C)$. ⊙

"I'm gonna do something that I have had occasion to tell many of you not to do, which is to introduce coordinates and do everything in coordinates"

We can choose coordinates $[X, Y, Z]$ on $\mathbb{P}^2$ satisfying

- The point $[0, 1, 0] \notin C$
- The line an infinity $L_\infty = V(Z)$ intersects the curve transversely, i.e.

$$L_\infty \cap C = \{p_1, \ldots, p_d\}$$

  with the p_i distinct.

(Joe drew a picture. Maybe one day I'll start bringing my ipad to lecture and drawing these pictures as well, but certainly not today)

Let's try and find $\mathrm{H}^0(K_C)$. Note that the space of *meromorphic* 1-forms is 1-dimensional over $\mathcal{M}(C)$, so if we write down any meromorphic 1-form, all others will be multiples of it by some rational/meromorphic function. Let's start with $\mathrm{d}x$. Here $x = X/Z$ and $y = Y/Z$ so $U = \mathbb{P}^2 \setminus L_\infty \cong \mathbb{A}^2_{x,y}$. Thus, $\mathrm{d}x$ is holomorphic on $\widetilde{C} := C \cap U$.

Note that this $\mathrm{d}x$ is the pullback of the differential $\mathrm{d}x$ on $\mathbb{P}^1$ under the map projecting C to the x-axis. Consequentially, $\mathrm{d}x$ (on C) will have double poles at the points $p_1, \ldots, p_d$ above $\infty \in \mathbb{P}^1$ (this projection $C \to \mathbb{P}^1$ we're pulling back along is unramified at $\infty \in \mathbb{P}^1$). Now, we need to find rational functions to multiply $\mathrm{d}x$ by which will kill off the poles at the points p_i (w/o introducing new poles).

A random polynomial of degree n on $U \cong \mathbb{A}^2_{x,y}$ will have a pole of order n at each of the p_i's. Hence, we can kill the poles of $\mathrm{d}x$ by dividing by any polynomial $h(x,y)$ of degree ≥ 2. This will introduce new poles at the points (in U) at which h vanishes. Thus, we need to choose h to have zeros only at points where $\mathrm{d}x$ vanishes.

Fact. The differential $\mathrm{d}x$ will vanish at the ramification points of the projection $C \to \mathbb{P}^1$ onto the x-axis, i.e. at the zeros of $\frac{\partial f}{\partial y}$.

On $\widetilde{C} = \{f = 0\} \subset \mathbb{A}^2_{x,y}$, we can write

$$0 = \mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y.$$

Since C is smooth, we know that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ have no common zeros. Consequentially,

$$\operatorname{ord}_p(\mathrm{d}x) = \operatorname{ord}_p\left(\frac{\partial f}{\partial y}\right) \quad \text{since} \quad \frac{\partial f}{\partial x}\mathrm{d}x = -\frac{\partial f}{\partial y}\mathrm{d}y.$$

Thus,

$$\omega_0 = \frac{\mathrm{d}x}{\frac{\partial d}{\partial y}}$$

will be holomorphic on $\widetilde{C}$. Note that $\deg(\partial f/\partial y) = d-1$, so ω_0 will have zeros of order $(d-1)-2 = d-3$ at p_i. Thus, if $d \geq 3$, we get a holomorphic differential.[28] Letting $H = \sum p_i$, we have computed $(\omega_0) = (d-3)H$.

Thus, we can multiply ω_0 by a polynomial $g(x,y)$ of degree $\leq d-3$ to get another holomorphic differential. As such, we consider the space

$$\left\{ g(x,y)\frac{\mathrm{d}x}{\partial f/\partial y} : \deg g \leq d-3 \right\} \subset \mathrm{H}^0(K_C).$$

Question 27.5. *Are these all the holomorphic differentials on C?*

Note that $(\omega_0) = (d-3)H$ has degree $d(d-3)$, and so we conclude from this that

$$2g - 2 = d(d-3) \implies g = \binom{d-1}{2}.$$

By counting, we conclude that in fact

$$\left\{ g(x,y)\frac{\mathrm{d}x}{\partial f/\partial y} : \deg g \leq d-3 \right\} = \mathrm{H}^0(K_C)$$

since the LHS has dimension $\binom{(d-3)+2}{2} = \binom{d-1}{2}$.

Before looking at arbitrary divisors, let's see some consequences.

Recall 27.6. If $D = q_1 + \cdots + q_n$, then geometric Riemann-Roch says that

$$r(D) = n - g + h^0(K-D) = n - \left(h^0(K) - h^0(K-D)\right).$$

[28] If $d \leq 2$, then $g(C) = 0$ and there are no holomorphic differentials

Note that the expression in the parentheses is the number of conditions D imposes on the canonical series. $\odot$

Question 27.7. *What's the smallest degree of a divisor D on C w/ a non-trivial linear series, i.e. $r(D) > 0$? Equivalently, what's the smallest n s.t. C is expressible as an n-sheeted cover of $\mathbb{P}^1$?*

We're really asking, what's the smallest number of points on the curve that may fail to impose linearly independent conditions on the canonical series? We've seen that the canonical series on a plane curve is spanned by polynomials of degree $\leq d-3$. Now, any $n+1$ points in the plane will impose linearly independent conditions on polynomials of degree n (assuming I heard correctly). Thus,

Proposition 27.8. *If C is a smooth plane curve of degree d, and $C \to \mathbb{P}^1$ is any map of degree n, then $n \geq d-1$.*

This is sharp since projection from a point on the curve will realize it as a $(d-1)$-sheeted cover of $\mathbb{P}^1$.

Let's go back a second. Say we have $p_1, \ldots, p_m \in \mathbb{P}^2$.

Claim 27.9 (Homework?)**.** *If $m \leq n+1$, then the points p_i impose independent conditions on the space of polynomials of degree n. If $m = n+2$, then the points $p_1, \ldots, p_m$ fail to impose independent conditions iff they are colinear.*

(Look for polynomials vanishing on all but one of the points by taking unions of lines through them, or something like this)

Onto the second problem: given a smooth plane curve $C \hookrightarrow \mathbb{P}^2$ and a divisor D on C, we'd like to find $\mathrm{H}^0(\mathscr{O}_C(D))$.

Write $D = E - F$ with $E, F \geq 0$ both effective. Next, choose polynomial $G(X,Y,Z)$ of degree m so that G vanishes on E, but $G \not\equiv 0$ on C. Now we write $(G|_C) = E + A$ for A some divisor on C. Note that $E + A \sim mH$ (with $H = p_1 + \cdots + p_d$ the hyperplane divisor from before). We also want to choose a polynomial J of the *same degree* m s.t. J vanishes on $A+F$, but $J \not\equiv 0$ on C. If no such J exists, then $\mathrm{H}^0(\mathscr{O}_C(D)) = 0$. Write $(J|_C) = A + F + D' \sim mH$. Thus,

$$D' \sim mH - A - F \sim E - F = D$$

is an effective divisor linearly equivalent to D.

Claim 27.10. *This process produces all effective divisors $\sim D$.*

Proof. Conversely, say $D' \sim D$ with $D' \geq 0$. We can write $D' + A + F \sim mH$. We claim there will exist a polynomial J of degree m s.t. $(J) = D' + A + F$, i.e. that

$$\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(m)) \to \mathrm{H}^0(\mathscr{O}_C(m))$$

is surjective. Note that the cokernel of the above map embeds into $\mathrm{H}^1(\mathscr{O}_{\mathbb{P}^2}(m-d)) = 0$, so the claim holds. $\blacksquare$

Next time, we'll want to consider the analogous construction for nodal plane curves in order to answer these two questions for their normalizations.

Question 27.11 (Audience). *Can you say more about what we'll do in this course?*

Answer (Paraphrase). There's a number of things we could try to cover. We want to stick, as much as possible, with things we can actually prove within this class. We'd like to talk about constructing $\overline{M}_g$, the compactified moduli space of (stable) genus g curves. Importantly, this $\overline{M}_g$ is a modular compactification of M_g. Once we have that, we can try and study its geometry. The hope is that we can give a reasonable argument that $\overline{M}_g$ is not unirational. Unfortunately, there's a lot of technical detail involved, so it's not entirely clear how much we'll be able to say vs. what we'll just have to accept. $\star$

28 Lecture 2 (1/27)

Note 36. Only like 3 minutes late

Joe spent some time in the beginning going over, out loud, what we did/talked about last time.

Recall 28.1. Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree d. Let $D = E - F$ be a divisor on C (E, F both effective). To find $|D|$, we do the following 2-step process

(1) Find a polynomial G of degree m s.t. G vanishes on E, but $G \not\equiv 0$ on C. In other words, G defines a plane curve which passes through C in the points of E (plus other points). We write $(G) = E + A$ (the intersection of C and the plane curve cut out by G), and note that $A \sim mH - E$.

(2) Look for polynomials J of degree m s.t. J vanishes on $A + F$ (but not on all of C). Then we can write $J = A + F + D'$ so now

$$D' \sim mH - A - F \sim D$$

is an effective divisor linearly equivalent to D.

All effective divisors $D' \in |D|$ arise in this way. This follows from the fact that the polynomials of degree m in $\mathbb{P}^2$ cut out a complete linear series $|mH|$ on C, i.e. $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(m)) \to \mathrm{H}^0(\mathscr{O}_C(m))$ is surjective (since its cokernel embeds into $\mathrm{H}^1(\mathscr{O}_{\mathbb{P}^2}(m-d)) = 0$). $\odot$

Example (Group law on an elliptic curve). Take $d = 3$, so C is an elliptic curve, at least after we choose an "origin" $o \in C$. We define a group law on C with origin o via $p +_C q := r$ where r is the unique point of C so that

$$r \sim p + q - o.$$

Concretely, for the first step, take $m = 1$ and draw the line $\overline{pq}$ (this is a poly vanishing at $p + q$). Write $\overline{pq} \cap C = p + q + s$. For the second step, we take the line $\overline{os}$ through s and the origin, and then write $\overline{os} \cap C = s + o + r$. This r is the sum $p +_C q$ in the group law on C. $\triangle$

Remark 28.2. We can use this process to estimate $|D|$ ($D = E - F$ as usual). First choose G s.t. $(G) = E + A$, so $\deg A = md - e$. Next we find J s.t. $(J) = A + F + D'$. How many choices do we have for J?

Question 28.3. *How many polynomials J are there so that $J = 0$ on $A + F$, modulo polynomials vanishing on C?*

The space of all degree m polynomials modulo those vanishing on C has dimension

This is $h^0(\mathscr{O}_C(m))$, assuming $h^1(\mathscr{O}_C(m)) = 0$ (e.g. assuming $m \gg 0$), but I think Joe has in mind

$$\binom{m+2}{2} - \binom{m-d+2}{2} = md - \binom{d-1}{2} + 1.$$

Thus, the "expected" dimension $h^0(\mathscr{O}_C(D))$ is

$$h^0(\mathscr{O}_C(D)) = \deg(D) - \binom{d-1}{2} + 1$$

(at least, when the points of $A + F$ impose independent conditions on polynomials of degree m). $\circ$

When we talk about nodal plane curves, something like the above will be telling us something new.

Question: Will it not still just amount to Riemann-Roch for singular (geometrically integral) curves?

28.1 Nodal curves

Setup 28.4. Let C be a smooth, projective curve (not necessarily planar). Assume we have a regular, birational $\nu : C \to C_0 \subset \mathbb{P}^2$ with C_0 nodal of degree d (so $C = \widetilde{C_0}$ is the normalization).

Such a map will always exist (proof later).

Note 37. Let $q_1, \ldots, q_\delta \in C_0$ be its nodes (δ in total), and write $\nu^{-1}(q_i) = \{r_i, s_i\}$. Finally, let $\Delta = \sum_i (r_i + s_i) \in \mathrm{Div}(C)$.

Simplifying Assumption. Assume there are no vertical tangents at the nodes. This will be true for a general choice of coordinate system.

Goal. Find $\mathrm{H}^0(K_C)$

Recall 28.5. In the smooth case, we chose coordinates, and then started w/ the meromorphic differential $\mathrm{d}x$. We saw it was holomorphic in the (finite) plane, but had double poles at its points at infinity. To cancel these out, we looked at $\omega_0 = \mathrm{d}x/(\partial f/\partial y)$. We then used

$$0 \equiv \mathrm{d}f \implies \frac{\partial f}{\partial x}\mathrm{d}x = -\frac{\partial f}{\partial y}\mathrm{d}y$$

and that C_0 was *smooth* (so one partial derivative is always nonzero) so conclude that ω_0 is holomorphic in the finite plane, and so $(\omega_0) = (d-3)H$. $\odot$

In the nonsmooth case, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ can have common zeros. In fact, they *do* have common zeros, exactly at the nodes. At them, they will vanish to order 1, so (using that there are no vertical tangents to avoid needing to consider the order of vanishing of $\mathrm{d}x$), ω_0 as above will have simple poles at the nodes. That is,

Question: Why?

$$\omega_0 = \frac{\mathrm{d}x}{\partial f/\partial y} \implies (\omega_0) = (d-3)H - \Delta \in \mathrm{Div}(C).$$

Recall 28.6 (when C_0 smooth)**.** In the smooth case, we said ω_0 was holomorphic and that we could get other holomorphic differentials by multiplying it by any polynomial of degree $\leq (d-3)$, i.e.

$$\mathrm{H}^0(K_C) = \left\{ \frac{g(x,y)\mathrm{d}x}{\partial d/\partial y} : \deg g \leq d-3 \right\}. \quad \odot$$

In the present case, we also need to require that g vanishes at the nodes, i.e.

$$\mathrm{H}^0(K_C) = \left\{ \frac{g(x,y)\mathrm{d}x}{\partial d/\partial y} \;\middle|\; \begin{matrix} \deg g \le d-3 \\ g(q_i) = 0 \end{matrix} \right\} \tag{28.1}$$

To see that this really is the hold space, observe that

$$2g - 2 = \deg(\omega_0) = d(d-3) - \delta \implies g = \binom{d-1}{2} - \delta$$

is the dimension of the RHS of (28.1).[29]

Recall 28.7 (in the smooth case)**.** We saw that C was *not* expressible as an m-sheeted cover of $\mathbb{P}^1$ for any $m \le d-2$. ⊙

Example ($\delta = 1$)**.** First note that if you project from the node of C_0, this will express the curve as a $(d-2)$-sheeted cover of $\mathbb{P}^1$. Can it be expressed as a branched cover of lower degree?

To say that a divisor D moves in a linear series is to say that D together with the node fail to impose independent conditions on the canonical series (on polynomials of degree $\le d-3$?). This can only happen if $\deg D \ge d-2$. If $\deg D = d-2$, this can only happen if its points are collinear with the node. △

Exercise. Explore what happens for higher δ.

Onto the second problem: given $C \to C_0 \subset \mathbb{P}^2$ as before along with a divisor $D = E - F$ on C.

Goal. Find $|D|$.

Here's the solution

(1) Find a polynomial G of degree m s.t.

- $G(q_i) = 0$
- G vanishes on E
- $G \not\equiv 0$ on C_0

Write $(G) = E + \Delta + A$

if $\operatorname{supp} \Delta \cap \operatorname{supp} E \neq \emptyset$, we want G to vanish to the appropriate degree, i.e. this is the correct expression (with A effective) for all E

(2) Find J of degree m s.t.

- $J(q_i) = 0$
- $J(F) = 0$
- $J(A) = 0$

Write $(J) = \Delta + F + A + D'$. Then, $D' \sim mH - \Delta - F - A \sim D$.

Claim 28.8. *We get, in this way, all effective divisors $D' \sim D$.*

Note 38. Got distracted and missed what he was saying. Something about how to prove this claim...

Consider the blowup $S = \mathrm{Bl}_{\{q_1,\dots,q_\delta\}}(\mathbb{P}^2) \xrightarrow{\pi} \mathbb{P}^2$.

[29] A priori, the dimension of the RHS is $\ge \binom{d-1}{2} - \delta$, but the above genus computation shows that this must be an equality.

Notation 28.9. Let $E_i \subset S$ be the exceptional divisor over $q_i \in \mathbb{P}^2$. Let $H \in \operatorname{Div}(S)$ be the pullback of the hyperplane class in $\mathbb{P}^2$, and let $E = E_1 + \cdots + E_\delta \in \operatorname{Div}(S)$ be the sum of the exceptional divisors.

Note that C is (isomorphic to) the proper transform of C_0 in S. This has class

$$C \sim dH - 2E \in \operatorname{Div}(S).$$

Furthermore, $K_S = -3H + E$.[30]

Claim 28.10. $\mathrm{H}^0(\mathscr{O}_S(mH - E)) \to \mathrm{H}^0(\mathscr{O}_C(mH - \Delta))$ *is surjective*

Proof. Look at the exact sequence of sheaves:

$$0 \longrightarrow \mathscr{O}_S(\underbrace{mH - E - (dH - 2E)}_{(m-d)H+E}) \longrightarrow \mathscr{O}_S(mH - E) \longrightarrow \mathscr{O}_C(mH - E) \longrightarrow 0.$$

We claim that $\mathrm{H}^1(\mathscr{O}_S((m-d)H + E)) = 0$. By Serre duality, we know

$$\mathrm{H}^1(\mathscr{O}_S((m-d)H + E)) \simeq \mathrm{H}^1\left(\mathscr{O}_S((d-m-3)H)\right)^\vee.$$

The point is that this is a line bundle pulled back from the plane, and so the Leray spectral sequence will tell us that

TODO: Add an aside carrying out this computation

$$\mathrm{H}^1(\mathscr{O}_S((d-m-3)H)) \cong \mathrm{H}^1(\mathscr{O}_{\mathbb{P}^2}(d-m-3)) = 0.$$ ■

28.2 Every smooth curve is birational to a nodal plane curve

We've made the assertion titling[31] this subsection a couple times now, so we should probably prove it at some point.

"I thought this was pretty obvious/trivial, but it's false in characteristic p– well, it true in characteristic p, but not in the one I'm gonna prove it to you. Bare that in mind."

Proof of (sub)section heading. Embed $C \hookrightarrow \mathbb{P}^n$ (working in char 0). We claim that if $\Lambda \subset \mathbb{P}^n$ is a general $(n-3)$-plane, then $\pi_\Lambda : C \to \mathbb{P}^2$ will project C birationally onto a nodal curve C_0. For this, we introduce the secant variety of C in $\mathbb{P}^n$. We have a map

I think this argument is in chapter 4 of Hartshorne (section 3?)

$$\begin{array}{ccc} C_2 & \longrightarrow & \mathbb{G}(1,n) \\ p+q & \longmapsto & \overline{pq} \end{array}$$

(which sends $2p$ to the tangent line). The **secant variety** is the union

$$S := \bigcup_{D \in C_2} \overline{D}$$

of these lines. This the union of a 2-parameter family of lines, so $\dim S = 3$. The point is that if $p \in \mathbb{P}^n \setminus S$ is not on the secant variety, then $\pi_p : C \to \mathbb{P}^{n-1}$ is in fact an embedding. Thus, we may reduce to the case that $n = 3$.

[30] Follows from general computation of canonical class of a blowup of a surface at a point or from thinking about taking a meromorphic 2-form on $\mathbb{P}^2$ and pulling it back to S

[31] That's a strange looking word

We need to know that, for a general point $p \in \mathbb{P}^3$, p won't lie on any tangent line[32] to C (so projection from p will be an immersion) and that p does not lie on any 3-secant line. More on this next time...

Suffices to show that the locus of trisecants is a proper subvariety of the secant variety, i.e. that not every secant is a trisecant. In characteristic 0, one uses uniform position lemma to conclude this. In characteristic p, it can be the case that every secant is a trisecant. ■

29 Lecture 3 (2/1)

Note 39. In some sense, a few minutes late

29.1 From Last Time

Let $C \subset \mathbb{P}^n$ be a smooth projective curve, and let $\Lambda \subset \mathbb{P}^n$ be a general $(n-3)$-plane. Then, the projection $\pi_\Lambda : C \to C_0 \subset \mathbb{P}^2$ is birational onto its image, and C_0 is nodal.

Recall 29.1. We reduced this to the case $n = 3$. ⊙

Remark 29.2. π_p is birational precisely when p (the point/0-plane in $\mathbb{P}^3$ we project from) lie on only finitely many secant lines to C. ○

Consider the incidence correspondence

$$\begin{tikzcd} & \Phi = \left\{(D,p) \in C_2 \times \mathbb{P}^3 : p \in \overline{D}\right\} \arrow[dl] \arrow[dr] & \\ C_2 = \mathrm{Sym}^2(C) & & \mathbb{P}^3. \end{tikzcd}$$

Stare at this (and count dimensions) until you're convinced that a general p will only lie on finitely many secants.

Remark 29.3. π_p is an immersion iff p does not lie on any tangent lines to C ○

This is also immediate (1-parameter family of tangent lines gives 2-dimensional bad locus, but $\dim \mathbb{P}^3 = 3 > 2$).

Remark 29.4. For C_0 to be nodal, we need π_p to be nowhere $3:1$. That is, we need to know that the trisecant lines don't fill up $\mathbb{P}^3$. ○

Since there's a 3-dimensional family of secant lines, to show that the family of trisecants has dimension ≤ 2, it suffices to say that not every secant line is trisecant, i.e. it is sufficient to exhibit a single secant line which is not trisecant.

Warning 29.5. This is false in positive characteristic, there are curves in $\mathbb{P}^3$ for which every secant meets the curve a third time. •

Proof not all secants are trisecants in characteristic 0. Choose a general plane $H \subset \mathbb{P}^3$. The uniform position lemma (Lemma 12.7 suffices) tells us that the points of $H \cap C$ are in linear general position, i.e. no 3 collinear. Thus, taking 2 of these, the secant through them won't hit the curve a third time, i.e. any secant contained in a general H won't be trisecant. ■

I think, but am not 100% sure, that general position and uniform position are two different lemmas

[32] There's a 1 parameter family of tangent lines, so their union cuts out a 2-dimensional variety, so not all of $\mathbb{P}^3$

Note 40. Joe started saying something else, but I was distracted and so missed it...

We still need to show that the image $C_0 = \pi_p(C)$ is nodal (maybe it has something like $\asymp$).

Remark 29.6. If C_0 has a point with preimages $q, r \in C$ s.t. the tangent lines at q, r map to the same tangent line in C_0, then those tangent lines (of q, r) must lie in the same 2-plane. Hence, they must intersect. ○

To show C_0 is nodal, need to show that not every pair of tangent lines to C intersect. If they do all meet, then

$$\pi_{T_pC} : C \to \mathbb{P}^1$$

would be a dominant morphism with derivative 0 everywhere. This can't happen in characteristic 0.

Question 29.7. *What do you do in characteristic p?*

Answer (One option)**.** Ignore it, and just always work in characteristic 0. ⋆

That's not always the most satisfying option.

Remark 29.8. To say that every pair of tangent lines to C meet is to say that for all $p, q \in C$, the divisor $D = 2p + 2q$ fails to impose independent conditions on $|\mathscr{O}_C(1)|$ (linear series cut out by hyperplanes). Note there's an exact sequence

$$\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^3}(1)) \longrightarrow \mathrm{H}^0(\mathscr{O}_D(1)) \longrightarrow \mathrm{H}^1(\mathscr{I}_{D/\mathbb{P}^3}(1)) \longrightarrow 0 = \mathrm{H}^1(\mathscr{O}_{\mathbb{P}^3}(1)),$$

so to say D doesn't impose $4 = h^0(\mathscr{O}_D(1))$ linear conditions is exactly to say that $\mathrm{H}^1(\mathscr{I}_{D/\mathbb{P}^3}(1)) \neq 0$. By Serre vanishing, you can fix this by taking some large n-uple embedding. So after re-embedding C, you can take appropriate projections. ○

29.1.1 Recall Process for finding Linear series

Recall 29.9. The situation is we have $C \xrightarrow{\pi} C_0 \subset \mathbb{P}^2$ with C_0 a degree d nodal plane curve. Say C_0 has nodes $q_1, \dots, q_\delta$ and write $\pi^{-1}(q_i) = \{r_i, s_i\}$. We also set $\Delta = \sum(r_i + s_i)$. ⊙

Say we are given $D = E - F$ on C. Choose a polynomial G of degree m s.t. $G(q_i) = 0$ and $G \equiv 0$ (E). That is, $(G) = E + \Delta + A$ with A effective.

Simplifying Assumption (If I understand, this changes literally nothing (it's purely psychological))**.** Assume $\operatorname{supp} E \cap \operatorname{supp} \Delta = \emptyset$.

Not choose a poly H of degree m s.t. $H(q_i) = 0$ and $H \equiv 0 \pmod{F + A}$. Write $(H) = \Delta + F + A + D'$. Then, $D' \sim D$.

We want to verify that this process gives the complete linear series $|D|$.

Claim 29.10. *We get, in this way, all effective divisors $D' \sim D$.*

As before, consider the blowup $S = \mathrm{Bl}_{q_1,\dots,q_\delta}(\mathbb{P}^2)$. The proper transform $C \subset S$ of $C_0 \subset \mathbb{P}^2$ is the normalization of C_0 (since C_0 nodal or else I think you need multiple blowups). Let $E_1, \dots, E_\delta \subset S$ be the exceptional divisors, and set

$$E := E_1 + \cdots + E_\delta.$$

Then, $C \sim dH - 2E$ where H is (the pullback of) the hyperplane class. By general theory, also $K_S \sim -3H + E$.

Claim 29.11. *Plane curves of any degree m passing through the nodes of C_0 cut out a complete linear series $|mH - \Delta|$ on C. That is, the map*

$$\mathrm{H}^0(\mathscr{O}_S(mH - E)) \longrightarrow \mathrm{H}^0(\mathscr{O}_C(mH - E))$$

is surjective.

Proof. The cokernel of this map embeds into $\mathrm{H}^1(\mathscr{O}_S(mH - E - C))$. Since $C \sim dH - 2E$, we see that $(mH - E - C) \sim ((m-d)H + E)$. Now, we apply Serre duality to see that

$$\mathrm{H}^1(\mathscr{O}_S((m-d)H + E)) = \mathrm{H}^1(\mathscr{O}_S(K_S - [(m-d)H + E]))^\vee = \mathrm{H}^1(\mathscr{O}_S((-3 + d - m)H))^\vee.$$

Now, the Leray spectral sequence tells us that

$$\mathrm{H}^1(\mathscr{O}_S(-3 + d - m)H) = \mathrm{H}^1(\mathscr{O}_{\mathbb{P}^2}(d - m - 3)) = 0$$

and so we win. ■

On Thursday, we'll talk about doing this for plane curves with arbitrary singularities (buzz phrase: 'completeness of the adjoint series').

29.2 Genera of curves

History. Today, we know of the genus of the curve as the number of holes of a Riemann surface. This picture only came about with the work of Riemann, but curves had been studied already for 100 years by this point. So how did earlier mathematicians think of the genus? Essentially, they anticipated Riemann-Roch. The earliest curves to be studied were rational curves. On such curves, if you look at the space of functions with at most d poles, it has dimension $d+1$. They then this failed for other curves. For a curve C and a divisor of large degree d, it doesn't have $d+1$ sections, but instead it has $d+1-p$ sections for a value p they called the **deficiency** of the curve. It was only after Riemann that this p started being written g and called the 'genus'. ⊖

Let C be a smooth curve of degree d in $\mathbb{P}^2$, and say we want to compute its deficiency. Start with some high degree divisor, and then compute its space of global sections. Let's look at the linear series $|\mathscr{O}_C(m)|$ for $m \gg 0$. This is cut out by polynomials of degree m in the plane, so we should have

Remember: Computing $h^0(\mathscr{O}_C(m))$ is a nicer way to get the genus of a plane curve than the usual adjunction argument

$$h^0(\mathscr{O}_C(m)) = \binom{m+2}{2} - \binom{m-d+2}{2} = md - \binom{d-1}{2} + 1,$$

the dimension of the space of degree m plane curves minus the dimension of the space of those curves which contain C. Thus, the deficiency is precisely $\binom{d-1}{2}$.

Now, suppose C_0 is a nodal plane curve of degree d with normalization $C \to C_0$. How do we compute the deficiency of C? We would try the same thing of looking at plane curves of degree m. This won't work e.g. since it'll give the same answer and e.g. since it won't give a complete linear series. Instead, we want to look at the linear series cut on C by plane curves of degree m passing through the nodes of C_0, i.e. at $\mathscr{O}_C(mH-\Delta)$. We saw a moment ago that these plane curves passing through the nodes do cut out a complete linear series, so we now just need to compute its dimension and degree. It's degree is $md-2\delta$. It's dimension is

$$h^0(\mathscr{O}_C(mH-\Delta)) = md - \binom{d-1}{2} - \delta + 1$$

(plane curves of degree m passing through the nodes modulo those which vanish identically on C_0). Thus, the deficiency of C must be $\binom{d-1}{2} - \delta$. Note, having to pass through a node is a single condition with decreases the degree by 2, and so has the effect of dropping the genus by 1.

29.2.1 Other singularities

Example (Planar triple point)**.** Suppose that C_0 has a triple point[33] $p \in C_0$ and let $\{q, r, s\} \subset C$ be its preimage in the normalization. To compute the genus of C, let's compute the difference between the degree and dimension of a large *complete* linear series. Look at curves of degree m in $\mathbb{P}^2$ passing through p. This has degree $md-3$. Passing through p is a single linear condition on the degree m curve, so it has dimension $\left(md - \binom{d-1}{2} + 1\right) - 1$. This suggests that the genus drops by 2; however, that's not quite right (recall 'planar triple point' example from section 13.2). We can do better.

Take plane curves of degree m which pass through p w/ multiplicity ≥ 2. These have degree $md-6$ but dimension $md - \binom{d-1}{2} + 1 - 3$. This suggests a genus drop of 3, which is the correct value. △

Example (tacnode)**.** Suppose C_0 has a tacnode[34]. Looking at plane curves through the tacnode would be "a two for one deal" (suggest a genus drop of 1), but we can do better. If we instead look a plane curves passing through the tacnode with the same tangent line, we get "a four for two deal" which suggests the correct genus drop: 2. △

Exercise. Work out deficiency calculations for other types of singularities. Ask yourself, "What's the best deal you can get?"

29.3 Something else, I missed what

Let C_0 be an arbitrary reduced and irreducible curve (not assumed planar). Let $\nu : C \to C_0$ be its normalization. We want to compare the genus of C to the (arithmetic) genus of C_0, i.e. $1 - \chi(\mathscr{O}_{C_0})$. To compare their genera, we compare their structure sheaves. The normalization includes a map $\mathscr{O}_{C_0} \hookrightarrow \nu_*\mathscr{O}_C$. Consider the cokernel

$$0 \longrightarrow \mathscr{O}_{C_0} \longrightarrow \nu_*\mathscr{O}_C \longrightarrow \mathscr{F} \longrightarrow 0,$$

and note that it's a skyscraper sheaf $\mathscr{F}$ supported at the singular points of C_0.

Recall 29.12 (From section 13.2)**.** $\chi(\nu_*\mathscr{O}_C) = \chi(\mathscr{O}_C)$ and so $g(C) = p_a(C_0) - h^0(\mathscr{F})$. ⊙

We'll continue on Thursday...

[33] Looks like a tri-branched node
[34] Looks like two branches meeting tangentially

30 Lecture 4 (2/3)

A few announcements

- Problem #2 on homework is gone

  "We don't mind, but why?" - some student. Problem 1 was the assertion from class that m points in $\mathbb{P}^n$ impose independent conditions on polynomials of degree $\leq n-1$. Problem 2 was extending this from reduced schemes of dimension 0 to arbitrary 0-dimensional subschemes. This is supposed to not be much harder, but Joe is blanking on the proof, so he decided to take it off the homework.

- There's one problem on the second homework assignment which will be covered by the material from today, so the problem is already on canvas if you want an early start.

- There will be sections/office hours on Fridays $12-1{:}15$. The location is partially TBD. For the time being, we'll do them in Joe's office SC 339.

Plan for the next week or two

- Today: carry out analysis of computing complete linear series for plane curves with arbitrary singularities

- Look at Severi varieties parametrizing plane curves of given degree and *geometric* genus.

- After this, we want to move on to the main topic of the semester: the geometry of the moduli space of curves (probably start in a week or two)

Remark 30.1 (References). The stuff up to now (geometry of individual plane curves) is described in chapter 14 of the "book" w/ Eisenbud (personality of curves).

Once we start talking about Severi varieties, our main reference will be Harris-Morrison's 'Moduli of Curves'. The Severi variety discussion relevant to us will start in chapter 1. ○

30.1 Singularities

Remark 30.2. There are many interesting problems involving the behavior of singularities of plane curves in families. We won't get to touch on this too much. ○

For now, we'll mainly be focused on some invariants of singularities. Let $C_0 \subset \mathbb{P}^2$ be a reduced and irreducible plane curve of degree d. Let $\nu : C \to C_0$ be its normalization. Recall the exact sequence

$$0 \longrightarrow \mathscr{O}_{C_0} \longrightarrow \nu_* \mathscr{O}_C \longrightarrow \mathscr{F} \longrightarrow 0.$$

Slogan. The difference between the (arithmetic) genera of C_0 and C is captured by the question, "What's a function on C_0 v.s. what's a function on C?"

Specifically, the exact sequence (+ ν being finite) tells us that

$$g(C) = p_a(C_0) - h^0(\mathscr{F}) = \binom{d-1}{2} - \sum_{p \in C_{0,sing}} \dim_{\mathbb{C}}(\mathscr{F}_p).$$

Definition 30.3. The value $\dim_{\mathbb{C}}(\mathscr{F}_p)$ is called the **δ-invariant** of the singularity $p \in C_0$. ◇

Note 41. For some examples of computing these δ-invariants, see Section 13.2. Joe is doing a few examples on the board right now, but I don't know if I feel like typing them up again...

Fact. $\delta_{\text{node}} = 1$, $\delta_{\text{cusp}} = 1$, $\delta_{\text{tacnode}} = 2$, $\delta_{\text{planar triple point}} = 3$, $\delta_{\text{spatial triple point}} = 2$.

We can introduce an ideal $\mathscr{I}_p \subset \mathscr{O}_{C_0,p}$ defined by $\mathscr{I}_p = \operatorname{Ann}(\mathscr{F}_p)$, so any function upstairs multiplied by an element of $\mathscr{I}_p$ must come from a function downstairs. This is called the **conductor ideal**.

Fact (Key fact)**.** $\dim \mathscr{O}/\mathscr{I}_p = \delta_p = \dim(\mathscr{F}_p)$, i.e. the codimension of the conductor ideal is the delta invariant.

Warning 30.4. The key fact is special to planar curves. It says that the singularity is **Gorenstein**. The above fact is a consequence of every local complete intersection being Gorenstein. •

Example. $\mathscr{I}_{\text{planar triple point}} = \mathfrak{m}_p^2$ while $\mathscr{I}_{\text{spatial triple point}} = \mathfrak{m}_p$. In the latter case, the conductor ideal has codimension 1 but δ-invariant 2, so spatial triple points are not Gorenstein. △

Recall 30.5. We have

$$C \xrightarrow{\nu} C_0 \subset \mathbb{P}^2$$

with C_0 a degree d plane curve. We want to find $\mathrm{H}^0(\omega_C)$ and also $|D|$ for any divisor D. ⊙

Let's find $\mathrm{H}^0(\omega_C)$. Start with a differential $\mathrm{d}x$ (really, it's pullback to C). This will have some poles, so we want to divide by a polynomial to kill them. The way to do this, in the smooth case, was to form

$$\omega_0 = \frac{\mathrm{d}x}{\partial f/\partial y}$$

where $C_0 = \{f = 0\} \subset \mathbb{P}^2$. We observe that ω_0 has zeros of order $d-3$ along the line L_∞ at ∞. In the smooth case, ω_0 has no poles, so we could take any $g\omega_0$ where $\deg g \leq d-3$, and these gave all holomorphic differentials. In the nodal case, ω_0 had a simple pole at each point lying over a node, so we could only allow g (of degree $\leq d-3$) for which $g(q_i) = 0$ for all nodes $q_i \in C_0$.

In general, if $\nu^{-1}(p) = \{q_1, \ldots, q_m\}$, then ω_0 will have poles at the points q_i, so let's say $\operatorname{ord}_{q_i}(\omega_0) = -n_i$. In this case, what we need is that $\operatorname{ord}_{q_i}(g) \geq n_i$ for each i, so g kills the poles. This leads us to another ideal.

Definition 30.6. The **adjoint ideal** $\mathcal{A}$ on C_0 is

This is not the 'A' that Joe wrote. I'm not sure which fond he was going for, it looked like a big lowercase roman a

$$\mathcal{A} = \{g : \operatorname{ord}_{q_i}(g) \geq n_i\},$$

i.e. $g\omega_0$ if holomorphic $\iff g \in \mathcal{A}$. ◇

Fact. The adjoint ideal is the conductor ideal.

The upshot is that the space of differentials in get in this way is the number of polynomials g of degree $\leq d-3$ minus the number of conditions of g to be in the adjoint ideal (= number of conditions of g to be in the conductor ideal = δ-invariant), and so we in fact get all holomorphic differentials, i.e.

$$\mathrm{H}^0(K_C) = \left\{\nu^* \frac{g(x,y)\mathrm{d}x}{\partial f/\partial y} : g \in \mathcal{A}_d\right\}$$

and this is called 'completeness of the adjoint series'.

30.1.1 Brief Higher Dimensional Analogue

Say $S \subset \mathbb{P}^3$ is a surface of degree d, and say $p \in S$ is an isolated singular point. Let $\widetilde{S} \to S$ be the desingularization. S

TODO: Add in rest of notes from today

31 Lecture 5 (2/8): Severi Varieties

Note 42. 9ish minutes late

Today: Severi varieties. Thursday/next week: moduli spaces

The basic objects are ($N = \binom{d+2}{2} - 1$)

$$\begin{array}{ccl}
\mathbb{P}^N & = & \{\text{plane curves of degree } d\} \\
\cup & & \\
V_{d,g} & = & \left\{\begin{array}{c}\text{integral curves} \\ \text{degree } d, \text{ genus } g\end{array}\right\} \\
\cup & & \\
U_{d,g} & = & \left\{\begin{array}{c}\text{integral curves of degree } d \\ \text{with } \delta = \binom{d-1}{2} - g \text{ nodes}\end{array}\right\} = U^{d,\delta}
\end{array}$$

Basic facts

- $U_{d,g} = U^{d,\delta}$ is *smooth* of codimension δ in $\mathbb{P}^N$, i.e.

$$N - \delta = \binom{d+2}{2} - 1 - \left[\binom{d-1}{2} - g\right] = 3d + g - 1.$$

  We'll prove this today in class.

- $U_{d,g}$ is dense in $V_{d,g}$

  This is not obvious, but is also not deep.

- $U_{d,g}$ (and hence $V_{d,g}$ and $\overline{V}_{d,g}$) are irreducible.

  We will not show this.

We have been looking (since the Fall) at families of curves in projective space. The most straightforward way of describing these is as maps of curves to projective space. Let

$$\mathcal{H}_{d,g,n} = \left\{ (C, f) \,\middle|\, \begin{array}{l} C \text{ smooth projective curve of genus } g \\ f : C \to \mathbb{P}^n \text{ non-deg map of degree } d \end{array} \right\}$$

(Note: if f is required to be an embedding, then this would be a Hilbert scheme).

One could impose smoothness conditions, e.g.

Want conditions that still allow $\mathcal{H}_{d,g,n}$ to dominate M_g

- For $n \geq 3$, could require f to be an embedding

- For $n = 2$, could require f birational onto nodal image
- For $n = 1$, could require that f is simply branched

Recall 31.1. In the Fall, we derived an "expected dimension"

$$h(d, g, n) = (n+1)d - (n-3)(g-1)$$

when looking at Hilbert schemes. $\odot$

Remark 31.2. Note that this works for $n = 1$ since the Hurwitz scheme is a covering space of $\mathrm{Sym}^b \mathbb{P}^1$ where b is the number of branch points. When $n = 2$, the first bullet point in our basic facts says that

$$\dim U_{d,g} = h(d, g, 2) = 3d + g - 1$$

also always has the expected dimension. $\circ$

For $n \geq 3$, the situation with Hilbert schemes can be quite messy. They certainly don't always have the expected dimension. They can even have multiple components of different dimensions.

Let's think a bit more about the second bullet point, that $U_{d,g}$ is dense in $V_{d,g}$. On one hand, this is saying that given any singularity of a plane curve, you can deform it into a collection of nodes without changing the total δ-invariant, w/o changing the arithmetic genus.

Example. If you start with a tacnode ($\asymp$ meeting transversally), you can move the branches vertically to get two nodes. $\triangle$

Example. If you have a triple point $*$, you can move one branch away from the other two to get something with 3 nodes. $\triangle$

It's less clear how to do this with other complicated singularities. Imagine e.g. a cusp $\prec$ or worse ($y^p = x^q$). It's not so obvious how to deform this into a collection of nodes.

Finally, let's say something about $U_{d,g}$ being irreducible. This question was proposed by Severi himself who had the goal of proving that M_g (moduli of genus g curves) is irreducible. Note that we have a map

$$\mathcal{H}_{d,g,n} \longrightarrow M_g$$

which is dominant if $d \gg g, n$. This gives the possibility of proving global theorems about M_g by looking at these spaces. For example, M_g irreducible $\Longleftarrow$ $\mathcal{H}_{d,g,1}$ is irreducible for all d, g.

History. This was done in the late 19th century (by Clebsch, Hurwitz, etc.). They did this by analyzing the monodromy of the cover $\mathcal{H}_{d,g,1} \to U \subset \mathbb{P}^b = \mathrm{Sym}^b(\mathbb{P}^1)$; they proved that the monodromy acts transitively on the fibers, so $\mathcal{H}_{d,g,1}$ inherits irreducibility from U. Severi did not like this proof. His main issue was that it was not purely algebraic; it relies, in an essential way, of working over $\mathbb{C}$ and applying the classical topology. Severi wanted something that would work over arbitrary fields/characteristics if possible. He suggested trying to prove that $\mathcal{H}_{d,g,2}$ is irreducible. Having suggested this, he then proceeded to give a false proof of the assertion that these Severi varieties are irreducible. Severi's student Zariski tried to give a correct proof, but it sounds like it took until Zariski's student Mumford before this was finally done.[35] $\ominus$

[35] Not sure if I heard this last assertion correctly

Anywho, let's get back to work...

Goal. We want to prove that $U_{d,g} = U^{d,\delta}$ is smooth of codimension δ in $\mathbb{P}^N$ (recall $N = \binom{d+2}{2} - 1$).

Let's start with the case $\delta = 1$.

Remark 31.3. When talking about Hilbert schemes of curves in higher dimensional projective spaces, we did not know how to deform the curves. Obviously, the curve is defined by some set of equations and we could just vary those coefficients; however, if you vary those coefficients randomly, you may not end up with a curve! It's not so clear what conditions are needed for a procedure like this to stay within the realm of curves. However, plane curves are simply. They are all defined by a single equation, and varying its equation any which way will still produced a plane curve! So let's use this fact. ◦

Introduce the incidence correspondence

$$\Sigma := \left\{(C,p) : C \times \mathbb{P}^2 \text{ degree } d \text{ and } p \in C_{sing}\right\} \subset \mathbb{P}^N \times \mathbb{P}^2$$

fitting into

$$\begin{tikzcd} & \Sigma \arrow[dl] \arrow[dr] & \\ \mathbb{P}^N & & \mathbb{P}^2. \end{tikzcd}$$

Remark 31.4. Σ is smooth since it's a projective bundle over $\mathbb{P}^2$. We'll see its smoothness again in a bit. ◦

We can write out equations defining Σ. Note that

$$\Sigma = \left\{(f,p) : f(p) = \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = 0\right\} \text{ where } f(x,y) = \sum_{i+j\leq d} a_{ij}x^iy^j.$$

Say p is the point $p = (x,y)$. In these coordinates (the a_{ij}'s and x, y), Σ is the zero locus of 3 polynomials:

$$F(a,x,y) = \sum a_{ij}x^iy^j, \;\; G(a,x,y) = \sum ia_{ij}x^{i-1}y^j, \;\text{ and }\; H(a,x,y) = \sum ja_{ij}x^iy^{j-1}.$$

Remark 31.5. One could explicitly prove that Σ is smooth by looking at the partial derivatives of these three defining equations w.r.t. the coordinates a_{ij}, x, y. ◦

Example. Let's compute these partial derivatives w.r.t to x, y, a_{00}. The results are in Table 2. △

	F	G	H
x	0	a_{20}	a_{11}
y	0	a_{11}	a_{02}
a_{00}	1	0	0

Table 2: Table of derivatives of defining equations of Σ w.r.t x, y, a_{00}

Note that to say that $(0,0)$ is a *node* of f is to say that its matrix of second derivatives is nonsingular. This implies that Σ is smooth at $(f, 0)$. Consider the projection $\pi : \Sigma \to \mathbb{P}^N$. Looking at the kernel of matrix shows that $\mathrm{d}\pi$ is injective and furthermore

Question: What?

$$\operatorname{Im}(\mathrm{d}\pi) = \{f : f(0,0) = 0\} = \{(C,p) : C \ni p_0\}\,.$$

This is saying that $U^{d,1} \subset \mathbb{P}^N$ is locally closed of codimension 1, and (if I'm not too lost) that the tangent space at a point $[C] \in U^{d,1}$ is the set of equations vanishing at the node of C.

Now, suppose that C is a plane curve with δ nodes (and no other singularities). Look at $\pi^{-1}(C) \subset \Sigma$. We have δ points of Σ lying over C. Each of these points has a neighborhood where the projection map is an immersion. Now, we have a collection of δ such hypersurfaces, and we're asking how they intersect. Let's say our δ nodes are $p_1, \dots, p_\delta$. Each of these has a corresponding point $(C, p_i) \in \Sigma$ mapping to $U^{d,g}$. The tangent space of (C, p_i) consists of curves of degree d containing p_i. Recall that the nodes of a curve impose independent conditions on curves of degree $\geq (d-3)$ (part of computation of canonical series of curves). Thus, these tangent spaces must be linearly independent. That is, each point (C, p_i) has a neighborhood mapping isomorphically onto a smooth hypersurface in $\mathbb{P}^N$, and these δ hypersurfaces have linearly independent tangent hyperplanes. Thus, the intersection of these hypersurfaces (which gives a neighborhood of $U^{d,\delta}$ around $[C]$) is smooth of codimension δ. This is exactly what we wanted to show.

Exercise. Go home and go over this argument until it makes sense.

Example ($d = 2$)**.** This is looking at $\{\text{conics } C \subset \mathbb{P}^2\} \cong \mathbb{P}^5$. Inside where, can consider locus $T = \{\text{singular conics}\}$ and $S = \{\text{double lines}\} \subset T$. Note every plane conic is smooth or a pair of lines or a double line. What does S look like as a subset of $\mathbb{P}^5$? Well you get a double line by specifying a line and then doubling it, so we get a map

$$\begin{array}{ccc} \mathbb{P}^{2*} & \longrightarrow & S \subset \mathbb{P}^5 \\ L & \longmapsto & 2L. \end{array}$$

Equivalently, this map sends $\{AX + BY + CZ\} \mapsto \{(AX + BY + CZ)^2\}$. In coordinates, this is

$$[A, B, C] \mapsto [A^2, B^2, C^2, 2AB, 2AC, 2BC].$$

Away from characteristic 2, this is just the quadratic Veronese map, i.e. S is the Veronese surface in $\mathbb{P}^5$. What does T look like? Think of the space of conics as the projectivization of the space of symmetric 3×3 matrices. Then, $T = \mathbb{P}(\text{rank} \leq 2)$ and $S = \mathbb{P}(\text{rank} = 1)$. This means that T is a cubic hypersurface, cut out by the vanishing of the derivative.

Note, a matrix of rank ≤ 2 can be expressed as the sum of 2 rank 1 matrices (e.g. diagonalize it). Conversely, the sum of any 2 rank 1 matrices has rank ≤ 2. In the above terms, this says that T is the secant variety of the Veronese surface S. This is a bit unusual. A surface in $\mathbb{P}^5$ has a $(4 = 2 + 2)$-dimensional family of secant lines, so you'd expect the secant variety to be 5-dimensional and hence be all of $\mathbb{P}^5$; however, that doesn't happen for the Veronese surface. In fact, this is the unique non-degenerate surface in $\mathbb{P}^5$ whose secant variety is properly contained in $\mathbb{P}^5$.

Exercise (challenge)*.* Prove this fact. $\triangle$

We're low on time, so let's just set up some questions about cubic plane curves for you to have fun with.

First of all, consider the loci

$$\begin{array}{cccccc}
 & & & \mathbb{P}^9 & & \\
 & & & \| & & \\
9 & & & \{\text{plane cubic curves}\} & & \\
 & & & \cup & & \\
8 & & & \{\text{singular cubics}\} & & \\
 & & \subset & & \supset & \\
7 & \{\text{cuspidal cubics}\} & & & & \{\text{conics} + \text{lines}\} \\
 & \cup & & \supset & & \cup \\
6 & \{\text{triangles}\} & & & & \{\text{lines} + \text{tangent conic}\} \\
 & & \supset & & \subset & \\
5 & & & \{\text{asterisk}\} & & \\
 & & & \cup & & \\
4 & & & \{\text{double line} + \text{line}\} & & \\
 & & & \cup & & \\
2 & & & \{\text{triple lines}\} & &
\end{array}$$

This is a complete stratification. Each type of cubic determines a subvariety (closure of indicated locus) of $\mathbb{P}^9$. These subvarieties have dimension equal to the number to the left of them.

32 Lecture 6 (2/10)

32.1 Announcements

Homework #2 will be due 2/21. This is because the full asignment is not yet complete/uploaded to canvas.

32.2 Moduli Spaces of Curves

Today we start our discussion of moduli spaces of curves. This will occupy us for the rest of the semester. We will try to prove real theorems about these moduli spaces. Admittedly, we will have to build up everything from the ground up, so we will be limited in how much we can cover in complete detail. Especially in the beginning, this will involve some amount of repeating what was said already last semester. With that out of the way, let's get into it...

Definition 32.1. A **moduli problem** consists of two things

- A class of objects
- a notion of family

  Note that if you have a family $\mathcal{C}$ over B and a morphism $f : B' \to B$, then you should be able to pull $\mathcal{C}$ back to a family $f^*\mathcal{C}$ over B' ⋄

Example (moduli of curves)**.** One can try to parameterize "isom classes of smooth projectuve genus g curves" where a family of such objects is a "smooth, projective morphism $\mathcal{C} \to B$ whose fibers are curves of genus g" △

Goal. Given a moduli problem, we would like to get the set of objects the structure of a scheme in a *natural* way.

By 'natural', we mean that for all families $\mathcal{C} \xrightarrow{\pi} B$, we want an associated map $\varphi_\pi : B \to M$ which is functorial in B. That is, given $f : B' \to B$, the map associated to the family $\mathcal{C}' := \mathcal{C} \times_B B' \xrightarrow{\pi'} B'$ is $\varphi_{\pi'} = \varphi_\pi \circ f$.

We can rephrase the discussion so far functorially. Consider the two functors

$$\begin{array}{ccc} \mathrm{Sch} & \longrightarrow & \mathrm{Set} \\ B & \longmapsto & \{\text{families}/B\} \\ B & \longmapsto & \mathrm{Mor}(B, M) \end{array}$$

The assignment $(\mathcal{C} \xrightarrow{\pi} B) \rightsquigarrow (\varphi_\pi : B \to M)$ above is simply a natural transformation between these two functors. To call M a (fine) moduli space, we want this to furthermore be a natural isomorphism.

Remark 32.2. If we have such a natural isomorphism, then $M(\mathbb{C}) \simeq \{\text{families}/\mathbb{C}\} \simeq \{\text{objects}\}$. Thus, such an M will succeed in the goal of giving the set of objects a scheme structure. ○

Definition 32.3. A **fine moduli space** for a functor $F : \mathrm{Sch} \to \mathrm{Set}$ is a scheme M with a natural isomorphism $F \xrightarrow{\sim} \mathrm{Mor}(-, M)$ of functors. Such an M, if it exists, is determined uniquely by F. ⋄

Example. The Hilbert scheme is a fine moduli space for the functor parameterizing smooth projective curves $C \subset \mathbb{P}^n$ of degree d and genus g. △

Non-example. The functor parameterizing isom classes of curves of genus 0 does not admit a fine moduli space.

Indeed, every genus 0 curve over $\mathbb{C}$ is $\simeq \mathbb{P}^1$, so such a moduli space would have to be a single point $*$. However, a family of genus 0 curves is a projective bundle, so if there were a fine moduli space, then one could conclude that every $\mathbb{P}^1$-bundle over any scheme is trivial. This is false. ▽

Non-example. The functor parameterizing isom classes of curves of genus g does not admit a fine moduli space.

Let's look more closely at the case of $g = 1$. Say C is a smooth projective curve of genus 1. Then, there exists a degree 2 map $C \to \mathbb{P}^1$ which will be branched over 4 points in $\mathbb{P}^1$. We can compose with an automorphism of $\mathbb{P}^1$ to take 3 of these to be $0, 1, \infty$, so we may represent C as $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. This gives a family $\mathcal{C} \to \mathbb{A}^1_\lambda \setminus \{0, 1\}$ of smooth curves of genus 1 over the λ-line. Note that different fibers may be isomorphic (had to choose 3 points to send to $0, 1, \infty$ in some order). In fact,

$$\mathcal{C}_\lambda \cong \mathcal{C}_{\lambda'} \iff \lambda' \in \left\{\lambda, 1-\lambda, \frac{1}{\lambda}, 1-\frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}\right\}.$$

Thus, S_3 acts on $\mathbb{P}^1_\lambda$ and we would hope that the quotient $\mathbb{P}^1_\lambda/S_3$ is (a compactification of) our moduli space. Note that any quotient of $\mathbb{P}^1$ by a finite group is still $\mathbb{P}^1$, i.e. $K(\mathbb{P}^1_\lambda)^{S_3} = \mathbb{C}(j)$ for some function

Unclear to me if what follows is strictly correct or if it secretly only works for $M_{1,1}$ (moduli of pointed genus 1 curves). At the very least, I can't see where things would

$j(\lambda)$. Traditionally, one takes

$$j(\lambda) = 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

Note that $j(\lambda)=\infty \iff \lambda\in\{0,1,\infty\}$. The upshot is that we have a bijection

$$\left\{\begin{matrix}\text{isom classes of}\\ \text{curves of genus 1}\end{matrix}\right\} \longleftrightarrow \text{points of } \mathbb{A}^1_j.$$

However, this $\mathbb{A}^1_j$ is not a fine moduli space of curves of genus 1. One can show that the above discussion does extend to giving a natural transformation

$$\left\{\begin{matrix}\text{familes of genus 1}\\ \text{curves}/B\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{regular functions}\\ j \text{ on } B\end{matrix}\right\},$$

but this map is neither injective nor surjective.

To see that it's not surjective, stare at the expression for j and observe that every 0 of the j-function of an actual family of genus 1 curves has order divisible by 3.[36]

To see that it's not injective, we need to know that two non-isomorphic families can have the same j-function. We'll do this by exhibiting a non-trivial (i.e. not a product) family with constant j-function. Choose $B'\to B$ any unramified double cover. Fix a curve E of genus 1, let $\sigma: B'\to B'$ be the involution exchanging sheets, and let $\tau: E\to E$ be multiplication by -1. Let

$$\mathcal{C} := \frac{B'\times E}{\langle(\sigma,\tau)\rangle} \longrightarrow B.$$

This will be, in general, a non-trivial family, i.e. $\mathcal{C}\not\cong B\times E$ even though every fiber is isomorphic to E. $\triangledown$

There are (at least) two ways of dealing with the nonexistence of a fine moduli space.

- Can define a **coarse moduli space**[37] for a functor F as a space M with a natural transformation $F\to \mathrm{Hom}(-,M)$ such that
  - for all maps $B\to M$, there exists a finite cover $B'\to B$ such that $B'\to B\to M$ is the map associated to some family over B' (i.e. to some element of $F(B')$).
  - For all pairs of families $\mathcal{C},\mathcal{D}\in F(B)$ with the same associated $B\to M$, there exists a finite cover $f: B'\to B$ such that the pullbacks $f^*\mathcal{C}\simeq f^*\mathcal{D}$ are isomorphic.

  It's not clear to me that this definition uniquely characterizes M

  Fact. A coarse moduli space for smooth curves of genus g exists.

  (Proven by Deligne-Mumford in 1969)

  In this paper if I'm not mistaken

- Even we can't find a fine moduli space, then this just means your category isn't big enough. An alternate approach is to enlarge your category, e.g. from the category of schemes to the category of (algebraic) stacks.

[36] In particular, this shows there's no family of curves of genus 1 associated to the identity $\mathbb{A}^1_j \xrightarrow{=} \mathbb{A}^1_j$

[37] Another popular definition of coarse moduli space is that it's a scheme (or algebraic space) M with a natural transformation $F\to\mathrm{Hom}(-,M)$ inducing a bijection on $\mathbb{C}$-points and which is initial for natural transformations from F to schemes

Fact. A fine moduli space for smooth curves of genus g exists as an algebraic stack (and even as a DM stack)

History. Joe spent some time talking about how stacks are quite technical, but somehow really the right place to do AG. At least, DM stacks are close enough to schemes to extend much of the theory. "One could imagine DM stacks replacing schemes as the basic objects, but this hasn't happened." He then went into a bit of history. In the 19th century, people didn't worry to much. Then in the 20th century, people wanted to really give rigorous foundations for AG and make sense of varieties over more general bases (e.g. non-algebraically closed fields). Then in the 60's, Grothendieck introduced his notion of schemes and argued (quite persuasively) that these were the right objects to consider. This led to a (successful) rewrite of the subject from the ground up in the language of schemes. Now, Deligne and Mumford's paper introducing (DM) stacks appeared in 1969. This was poor timing since algebraic geometers had just concluded a 20 year period where they had rewritten the subject in the language of schemes, so there wasn't enough will to rewrite things in the language of stacks. And to this day, stacks are not yet seen as the basic objects of AG. $\ominus$

If you are interested in learning about stacks, and want an introduction to the ideas behind them, Joe recommends an earlier paper by Mumford ('65): 'Picard groups of moduli problems'

Most of algebraic geometry deals with projective varieties over algebraically closed fields. Often, if you have e.g. a quasi-projective variety, one thinks of it as a projective variety with some subvariety removed (then they can use projective techniques). Similarly, things are simpler over algebraically closed fields than non-closed ones. For example, the topology of degree d curves over $\mathbb{R}$ can vary wildly depending on the particular equation cutting them out; however, every degree d curve over $\mathbb{C}$ is homeomorphic.

The point of the above is to motivate the desire to compactify M_g, to view it as an open subset of a compact/projective variety $\overline{M}_g$.

So here's some things coming up

- How do you see that M_g is not already compact?
- Once we have a way of proving that it's not compact, this will give us a way of finding a compactification, so we'll introduce the Deligne-Mumford compactification of M_g.

33 Lecture 7 (2/15)

33.1 Announcements

Two talks

- Eric Larson: Interpolation (AG Seminar, today at 3:00 in room 507)

  The simplest example of interpolation is the fact that through any two points in $\mathbb{P}^2$ there is a unique line. Similarly, five points in $\mathbb{P}^2$ (in general position) will be passed through by a unique conic. Generally, can ask, "given a family $\mathcal{C} \subset B \times \mathbb{P}^r$ of curves in projective space (with base B of dimension $\dim B = h$), for any given m general points $p_1, \ldots, p_m \in \mathbb{P}^r$, does there exist a $b \in B$

so that $C_b \ni p_1, \ldots, p_m$?" We don't want to repeat curves, so generally assume that the induced map $B \to \mathcal{H}$ to the Hilbert scheme is an embedding (in fact, most of the time, one takes B to be a component of the Hilbert scheme). To study such a question, it's natural to introduce an incidence correspondence

$$\begin{tikzcd}
& \Phi = \{(b, p_1, \ldots, p_m) \in B \times (\mathbb{P}^r)^m : C_b \ni p_1, \ldots, p_m\} \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
B & & (\mathbb{P}^r)^m
\end{tikzcd}$$

The question then becomes, "Is β dominant?" Dimension counting gives $\dim \Phi = h + m$, so we expect/naively guess that $\exists C_b$ through m general points $\iff h \geq m(r-1)$. Intuitively, it's $r-1$ conditions for a curve in $\mathbb{P}^r$ to pass through a given point, so $m(r-1)$ conditions for it to pass through m general points. We say **interpolation holds** for the family if the expectation holds.

Fact (Exercise)**.** Interpolation always holds if $r = 2$.

Non-example. In $\mathbb{P}^3$, take the family of canonical curves of genus 4 ($C = Q_2 \cap S_3$ is the intersection of a quadric and a cubic). The family of these (i.e. the relevant component of Hilbert scheme) has dimension 24. Now, may naively expect that you can find a curve of this family through 12 points. However, 12 general points in $\mathbb{P}^3$ won't lie on a quadric surface[38], so interpolation fails in this example. $\triangledown$

Recently, Eric Larson and Isabel Vogt found a nice proof of the current state of the art for figuring out when interpolation holds.

Theorem 33.1. *Interpolation holds for any component of the Hilbert scheme $\mathcal{H}$ dominating moduli $\mathcal{M}_g$, with exactly 4 counterexamples (including the one above).*

Recall 33.2. When the Brill-Noether number is negative, there are no components of the Hilbert scheme dominating moduli. When it is nonnegative, there is a unique such component. $\odot$

Open Question 33.3. *When $h = m(r-1)$, we expect finitely many $b \in B$ such that C_b passes through m general points. Larson-Vogt prove this in the course of proving the above theorem. However, determining this number on the nose is still open in general.*

(Sounds like Ravi Vakil answered some case of this in his thesis)

Remark 33.4. Can ask the same sorts of questions for families of surfaces or higher dimensional varieites. Sounds like Aaron Landesman worked on this for his Harvard senior thesis. $\circ$

- Hannah Larson, Lines

  Open neighborhood seminar, Wednesday at 4:30 in room 507

Remark 33.5. I think the big takeaway from the above is that Stanford algebraic geometers are out here doing things.[39] $\circ$

There's a new version of homework 2 on the course website. Unclear if we'll get far enough in lecture this week to justify it being due on Monday.

[38] Can't put a quadric through more than 9 general points

[39] Most names above are no longer currently associated with Stanford, but they all were at one point or another while I was there

33.2 Compactifying M_g

Question 33.6. *Can we write down a general curve of genus g?*

(We want to answer this by the end of the semester)

What does this have to do with M_g and why would having a compactification be useful? Before answering this directly, let's look at some low genus examples.

Example ($g = 2$)**.** Here we can write down $y^2 = x^5 + a_4x^4 + \cdots + a_1x + a_0$. This is a generically smooth family of genus 2 curves over $\mathbb{A}^5$. Thus, it gives a rational map $\mathbb{A}^5 \dashrightarrow M_2$ (i.e. a map $\mathbb{A}^5 \supset U \to M_2$) which is dominant and in fact even surjective.

The point is we have a family of curves in free parameters which includes a general curve of genus 2. △

Example ($g = 3$)**.** A general curve of genus 3 is non-hyperelliptic and so it a (canonically embedded) plane quartic. Thus, the family $\sum_{i+j\le 4} a_{ij}x^iy^j$ gives rise to a dominant rational map $\mathbb{A}^{15} \dashrightarrow M_3$ (not surjective since it misses the hyperelliptic ones). △

Question 33.7 (Question 33.6 rephrased)**.** *Can we find an open subset $U \subset \mathbb{A}^N$ of affine space and a family of curves $\mathcal{C} \to U$ inducing a dominant map $U \to M_g$?*

The key of this rephrasing is the following observation

Remark 33.8. If such a family exists for a given g, then M_g is *unirational.* ○

Definition 33.9. In general, we say X is **unirational** if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ for some n. ◇

Rational varieties are obviously unirational.

Fact. If $\dim X \le 2$, then X is unirational $\iff$ it is rational. The case $\dim X = 1$ is **Luroth's theorem** (this is e.g. a consequence of Riemann-Hurwitz + Riemann-Roch[40]). The case $\dim X = 2$ is due to Castelnuovo-Enriques.

Example. A smooth cubic threefold is unirational, but not rational. △

Note that an easy way of showing something is not unirational is to show that it has holomorphic top forms (see e.g. one of the footnotes). However, this only works when talking about compact varieties (e.g. $\mathbb{A}^n$ has tons of holomorphic n-forms, but is still unirational), so to show that M_g is not unirational, it'd be useful to first compactify it.

The first step in compactifying M_g is answering the following question:

Question 33.10. *Is M_g compact?*

Remark 33.11. We should be saying 'proper' instead of 'compact', but 'proper' doesn't have a nice verb form analogous to 'compactify' so meh ○

[40]Alternatively and even simpler, if $g(C) > 0$, then it would have a (nonzero) holomorphic 1-form. You could pull that back along a dominant map $\mathbb{P}^1 \to C$ to get a holomorphic 1-form on $\mathbb{P}^1$, but no such thing exists.

Recall 33.12 (**Valuative criterion for properness**, more-or-less)**.** Say X is a quasi-projective variety.[41] X is projective iff for all meromorphic maps $\Delta^* \to X$ from the punctured disk to X, there exists a (unique) extension $\Delta \to X$. ⊙

Taking $X = M_g$ above, we see that to say that M_g is proper is to say that for any family of smooth curves $\mathcal{C}^* \to \Delta^*$, after a finite base change[42], the family extends to one over the whole disk Δ, i.e. we want a diagram like

$$\begin{tikzcd} \mathcal{C}^* \arrow[d] & \mathcal{C}_2^* \arrow[l] \arrow[r, hook] \arrow[d] & \mathcal{C}_2 \arrow[d] \\ \Delta^* & \Delta^* \arrow[l, "t^m \leftarrow t"'] \arrow[r, hook] & \Delta \end{tikzcd}$$

Claim 33.13. *Let $\mathcal{C} \to \Delta$ be a family of curves with C_t smooth for $t \neq 0$, but C_0 singular with a single node as its only singularity. Furthermore, assume that $\mathcal{C}$ has a smooth total surface. Even after a finite chance, we can't fill in with a smooth curve.*

(in other words, M_g is not proper)

How do we prove something like this? There's a lot we can do; we can take finite base changes, we can blowup the total space, we can blowdown the total space, etc. How do we show that no combination of such changes will give a family with the same curves away from the origin, but with a smooth curve at the origin? The answer is that we will calculate the monodromy of this family, and show that it is not of finite order.

Interlude on monodromy Consider $\mathcal{C} \to B$ a family of smooth curves (in the above case, we'll take $B = \Delta^*$). We want to form a covering space of B:

$$\Sigma = \{(b, \gamma) : b \in B \text{ and } \gamma \in \mathrm{H}_1(\mathcal{C}_b, \mathbb{Z})\} \longrightarrow B.$$

The point is that $\mathcal{C} \to B$ is topologically (analytic topology over $\mathbb{C}$) a fiber bundle, and so locally looks like a product $C_b \times U$ (with $U \subset B$ a small neighborhood around b). This let's us locally identify the homology of the fibers and so this to give Σ a topology so that $\Sigma \to B$ is a covering space (with countable fibers $\mathrm{H}_1(C_b, \mathbb{Z})$). Thus, we get a monodromy map

$$\pi_1(B, b) \longrightarrow \mathrm{Aut}(\mathrm{H}_1(C_b, \mathbb{Z})).$$

Remark 33.14. When $B = \Delta^*$, $\pi_1(B, b) \simeq \mathbb{Z}$, so we can identify the monodromy with a single matrix, the image of a generator of the fundamental group. ∘

Note that if $B = \Delta$, then B is simply connected, so the monodromy will be trivial.

Back to the task a hand In the situation of Claim 33.13, the monodromy will have infinite order. Thus, it cannot be removed after finite base change.

History. This monodromy calculation was somehow 'understood' by Picard, but was not really written down and proven until Lefschetz. Thus, it bears both of their names. ⊖

[41] Ask yourself, after embedding X is projective space, are there any holes? Does it contain all limit points?

[42] Secretly, this finite base chance is hinting that we're really using the valuative criterion of properness for algebraic stacks instead of for schemes

Let $\mathcal{C} \to \Delta$ be a family with $\mathcal{C}_t$ smooth for $t \neq 0$ and $\mathcal{C}_0$ nodal with a single singularity. Let $p \in \mathcal{C}_0$ be the node. In a neighborhood of p, the family is given simply by $y^2 = x^2 - t$. Let's take a look at the topology of this chunk.

Remark 33.15. Over $\mathbb{R}$, it looks like a punch of hyperbolas specializing to a union of two lines. ◦

Remark 33.16. Over $\mathbb{C}$, it looks like a 2-sheeted cover of the disk branched over $\pm\sqrt{t}$.

TODO: Add drawing

To get a picture, note that we can disconnect the total space by removing a little arc between $\pm\sqrt{t}$. In fact, we can fatten up the ark to a small wedge. We can then recover the whole space by identifying points on the upper edge of the removed wedge on one sheet with the lower edge of the removed wedge on the other sheet. Thus, the picture is of two disks connected by a twisted/orientation-reversing cylinder between them.

As t goes around the origin, one circle gets twisted halfway around and the other gets twisted halfway around the other direction.

Note 43. I'm starting to get lost. Maybe I'll take a look at Voisin II's section on Dehn twists and Picard-Lefschetz and whatnot... ◦

Theorem 33.17 (**Picard-Lefschetz**). $\gamma \mapsto \gamma + (\gamma \cdot \delta)\delta$ *where* δ *is the vanishing cycle (a simple cycle around the cylinder)*

This transformation is not of finite order since $\delta \in \mathbb{Z}^{2g}$ is non-torsion. We may run into trouble if the vanishing cycle is trivial in homology. In such cases, the special fiber is reducible (imagine a g-holed torus with a homologically trivial cycle that gets pinched to a point).

On Thursday, we'll start talking about how to actually compactify M_g.

34 Lecture 8 (2/17)

Remark 34.1. Joe considers the compactification of M_g by Deligne and Mumford to be one of the biggest results in the theory of algebraic curves in the 20th century. ◦

In general, to compactify a variety X is to express it as an open subset of a compact variety $\overline{X}$; it does not necessarily include any further conditions on $\overline{X}$. In the present case, $X = M_g$ is not just a random variety, but is in fact a moduli space. Hence, it would naturally be desirable to get a compactification $\overline{M}_g$ which is also itself a moduli space. A priori, there's no reason any given compactification of a moduli space must also be a moduli space (i.e. must also parameterize a reasonable set of objects), but we will be able to show the existence of such a thing in this case.

Goal. Find/describe a *modular compactification* of M_g.

In other words, we'll take the class of smooth projective genus g curves and embed this in a larger class of curves (to be specified) s.t. the larger class admits a projective (in particular, proper) moduli space.

Recall 34.2. The valuative criterion tells us that the larger class will have a compact moduli space if: for all families $\mathcal{C}^* \to \Delta^*$ of smooth curves, (possibly after a base change) we can fix this in to a family $\mathcal{C} \to \Delta$ w/ fiber $\mathcal{C}_0$ in the larger class (moreover, there should be a unique possibility for $\mathcal{C}_0$). ⊙

Question: Is it easy to see that the stacky valuative criterion for properness

Example (plane cubics)**.** Recall that the moduli space here is given by the j-line $\mathbb{A}^1_j$ and so it's natural to expect that a reasonable compactification would be given by $\mathbb{P}^1$. Can we give a moduli interpretation to the point at infinity in $\mathbb{P}^1$?

- First attempt: add the cuspidal cubic curve $y^2 = x^3$.

  Consider the following family. First say $C : y^2 = x^3 + ax + b$ is a *smooth* cubic (so a, b fixed). Now consider the family
  $$\mathcal{C} = \left\{y^2 = x^3 + at^2x + bt^3\right\} \longrightarrow \Delta_t.$$
  This is a family of smooth curves $\mathcal{C}_t$ ($t \neq 0$) degenerating to a cuspidal curve $\mathcal{C}_0 : y^2 = x^3$ when $t = 0$. However, note that $\mathcal{C}_t \simeq C$ when $t \neq 0$. Thus, a moduli space M parameterizing smooth plane cubics + the cuspidal cubic would be non-separated (the family $\mathcal{C}_t$, $t \neq 0$, would not have a unique limit). In fact, it would be widely unseparated: the point "$\infty \in M$" corresponding to the cuspidal cubic would necessarily lie in the closure of any other point of M.

  Remark 34.3. This is the sort of problem that can arise when trying to compactify your moduli. For the case of moduli spaces of curves, Deligne and Mumford essentially solved this problem. Sounds like there are now other ways of compactifying M_g which are known though. ○

△

Here's the solution due to Deligne and Mumford ('69).

Definition 34.4. Say a projective curve C of arithmetic genus $g \geq 2$ is **stable** if

- It's only singularities are nodes
- $\#\operatorname{Aut}(C) < \infty$ ◇

Example. If I heard correctly, the moduli space of all nodal curves is also compact. However, to get a separated moduli space, it's not enough to just ask for nodal curves. Indeed, if C is a smooth curve, consider the constant family $\mathcal{C} := C \times \Delta \to \Delta$, and fix a point $p \in \mathcal{C}_0$. Then, $\mathcal{C}' := \operatorname{Bl}_p \mathcal{C}$ is isomorphic to $\mathcal{C}$ away from $0 \in \Delta$, but $\mathcal{C}'_0 \not\simeq \mathcal{C}_0$. However, both of these limits are nodal, so the moduli space of nodal curves is not separated. Hence, we want to avoid something like $\mathcal{C}'_0$ (which is a nodal union of C and $\mathbb{P}^1$). △

Note 44. Missed most of some discussion about alternate modular compactifications of M_g. Sounds like one of Mumford's students described a second compactification, and then later someone else wrote a paper describing all possible modular compactifications of M_g.

Theorem 34.5 (Deligne-Mumford)**.** *There exists a proper moduli space $\overline{M}_g$ of stable curves.*

(This $\overline{M}_g$ is in fact even projective)
What's up with this finite automorphism condition?

Recall 34.6. A *smooth* curve of genus $g \geq 2$ has finite automorphism group. ⊙

For more general curves C (say, C reducible), any automorphism will permute its irreducible components. Does, there's a finite index subgroup $\operatorname{Aut}^0(C) \subset \operatorname{Aut}(C)$ consisting of the automorphisms which carry each irreducible component C_α of C to itself. Such an automorphism will permute the intersection points of any two components. Hence, insider there, there's another finite index subgroup $\mathrm{Aut}^1(C) \subset \mathrm{Aut}^0(C)$ which further fix each point of $C_\alpha \cap C_\beta$ for all α, β. Now, we make the following observations

- Any component of genus ≥ 2 has finite automorphism group
- Any component of genus 1 will necessarily meet a second component (since $g(C) \geq 2$), and so contribute finitely many automorphisms to $\mathrm{Aut}^1(C)$, i.e. $\#\,\mathrm{Aut}(E,p) < \infty$ for E of genus 1 and $p \in E$
- $\mathbb{P}^1$ has infinitely many automorphisms fixing any two points, but only finitely many automorphisms fixing any three or more points.

Corollary 34.7. *Stability is equivalent to saying that C is nodal and*

- *every smooth rational component of C meets the rest of C at least 3 times.*

Claim 34.8 (DM, **stable reduction**). *Given any family $\mathcal{C} \to \Delta$ of curves with $\mathcal{C}_t$ smooth for $t \neq 0$ and $\mathcal{C}_0$ arbitrary, there exists a base change $\Delta' \cong \Delta \xrightarrow{t \mapsto t^m} \Delta$ so that there exists a family $\widetilde{\mathcal{C}} \to \Delta'$ (birational to $\mathcal{C}' := \mathcal{C} \times_\Delta \Delta'$) so that $\widetilde{\mathcal{C}}_0$ is stable. Furthermore, is $\widetilde{\mathcal{D}} \to \Delta'$ is a second such family, then $\widetilde{\mathcal{C}}_0 \simeq \widetilde{\mathcal{D}}_0$.*

The birational map in the above theorem will be some sequence of blowups and blowdowns in practice. Generally, stable reduction is something which one can carry out concretely in practice.

Example. say $\pi : \mathcal{C} \to \Delta$ is a family of curves with $\mathcal{C}_t$ smooth for $t \neq 0$ and $\mathcal{C}_0$ smooth except for a single cusp $p \in \mathcal{C}_0$. To keep life easy, we'll further assume that the total space $\mathcal{C}$ is smooth. According to Deligne and Mumford, it should be possible to replace the stable fiber $\mathcal{C}_0$ (which looks like $\prec$) by a stable curve. Let's try and do this...

This is probably gonna be hard to follow without me being able to draw pictures here. Oh well, look at the relevant section of Harris-Morrison's 'Moduli of Curves'

- First step is to get rid of the cusp, and we do this by blowing it up, i.e. replace $\mathcal{C}$ by $\mathrm{Bl}_p\,\mathcal{C}$. Doing this normalizes the cuspidal curve, so $\mathcal{C}_0$ now looks like $\widetilde{C}_0 + 2E_1$ where $\widetilde{C}_0$ is the proper transform/normalization of the original curve $\mathcal{C}_0$ and $E_1 \simeq \mathbb{P}^1$ is the exceptional divisor. Furthermore, E_1 meets $\widetilde{C}_0$ tangentially. So we have a non-reduced curve whose reduction has a tacnode $\asymp$; that sounds worse.
- Blowup again. This gives $\mathcal{C}_0 = \widetilde{C}_0 + 2E_1 + 3E_2$ meeting at a triple point $*$.
- One more blowup. Now get $\mathcal{C}_0 = \widetilde{C}_0 + 2E_1 + 3E_2 + 6E_3$ whose reduction is a set theoretic normal crossing, i.e. each of $E_1, E_2, \widetilde{C}_0$ meets E_3 in a different point (so looks like E with E_3 as the 'backbone')
- If the above exceptional divisor was reduced, it'd be nodal and we'd be happy. However, it's not, so we start making base changes, first one of order 2. That is, base change along $\Delta_s \to \Delta_t$ where $s^2 = t$.

  We'll actually do two things, so let's motivate the second: at a point of E_1 (away from E_3), we can take local coordinates z, w on $\mathcal{C}$ so that $E_1 = (w)$ and $t = w^2$ (this is what it means for $2E_1$ to appear in $\mathcal{C}_0$). After base change, this becomes $s^2 = w^2$ which is two components meeting in

In this process, the multiplicity of the newest exceptional divisor is the order to which t vanishes at the a point. To get rid of this, we normalize. Thus, the real operation here is base change and then normalize. To effect of this is to take the double cover of the surface branched along the union of the components appearing in the special fiber with odd multiplicity.

To compute the new $\mathcal{C}_0$, first observe that $\widetilde{C}_0$ and E_2 are contained in the branched locus, so they remain after the operation. On the other hand, $E_1 \simeq \mathbb{P}^1$ does not meet the branch locus, so we take an unramified double cover of E_1. Thus, E_1 will be replaced by two rational curves $E_1', E_1'' \simeq \mathbb{P}^1$. Finally, E_3 meets the branch locus in 2 points, so we take a double cover of $E_3 \simeq \mathbb{P}^1$ branched at two points. This is again a $\mathbb{P}^1$ (e.g. by Riemann-Hurwtiz), so we end up with

$$\mathcal{C}_0 = \widetilde{C}_0 + 3E_2 + E_1' + E_1'' + 3E_3$$

(E_3 meets everything, but nothing else meets anything else; picture looks like an E with extra teeth)

- Now, we make a base change of order 3 and pass to the normalization. This corresponds to taking the triple cover branched along the union of the components of multiplicity not divisible by 3, i.e. along $\widetilde{C}_0 + E_1' + E_2''$. Note that E_2 does not meet this branch locus, so we get an unramified cover of E_2, i.e. 3 different $\mathbb{P}^1$'s. Similarly, get a (cyclic) triple cover of $E_3 \simeq \mathbb{P}^1$ branched at 3 points, so it becomes a curve of genus 1.[43] Thus, we end up with

$$\mathcal{C}_0 = E_3 + E_1' + E_1'' + E_2' + E_2'' + E_2''' + \widetilde{C}_0$$

now reduced (note $g(E_3) = 1$ but $g(E_{\text{anything else}}) = 0$)

- The $\mathcal{C}_0$ above is nodal, but not stable. We have smooth rational curves (e.g. E') meeting the rest of the fiber in one point. Each of them has intersection 0 with the total fiber and intersection 1 with the rest of the fiber, so self-intersection -1. Thus, we can blow them down to finally arrive simply at a nodal union of E_3 and $\widetilde{C}_0$ with $g(E_3) = 1$ and $g(\widetilde{C}_0) = g - 1$. $\triangle$

35 Lecture 9 (2/22)

Today: we want to talk more about stable curves and their moduli.

Recall 35.1. A *stable curve* is a nodal curve with finite automorphism group. $\odot$

The moduli space of stable curves gives a modular compactification of the space of smooth genus g curves. This was proven by Deligne-Mumford and then later again by Mumford-Knudsen. At some point, we'd like to talk a little about both of their approaches to proving this, but not today.

35.1 Pointed curves

Let's consider a variant of $\overline{M}_g$.

[43] We can even say which genus 1 curve this is. Since E_3 has an automorphism of order 3 (as its a cyclic triple cover of $\mathbb{P}^1$), it must be the curve with j-invariant 0

Definition 35.2. A **stable n-pointed curve** of genus g is a tuple $(C, p_1, \ldots, p_n)$ where C is a nodal curve with arithmetic genus g, $p_1, \ldots, p_n$ are distinct *smooth* points of C, and we require that $\# \operatorname{Aut}(C, p_1, \ldots, p_n) < \infty$. The p_i's are called **marked points**. ⋄

Remark 35.3. $\operatorname{Aut}(C, p_1, \ldots, p_n)$ is the group of automorphisms which fix the points $p_1, \ldots, p_n$. Geometrically, this corresponds to asking that every smooth rational component $\mathbb{P}^1 \subset C$ contains at least 3 *special points*, where a point is 'special' if its a node or one of the p_i's. ○

Theorem 35.4 (DM '69)**.** *There exists a coarse moduli space $\overline{M}_{g,n}$ for stable n-pointed genus g curves. Furthermore, $\overline{M}_{g,n}$ is proper.*

Remark 35.5. $M_{g,n}$ will be a fine moduli space when $n \gg_g 0$; briefly, a non-trivial automorphism can only fix so many points, so when n is sufficiently large, you kill off all automorphisms. However, the same is not true for $\overline{M}_{g,n}$; even when n is large, you only get a coarse space, because non-trivial automorphisms can persist (e.g. permute components or imagine $C \cup \mathbb{P}^1$ with all marked points in the $\mathbb{P}^1$). ○

Here's one potential source of worry: what happens when two points come together? That is, if you have C with marked points $p, q \in C$, you can imagine letting p approach q. Since $\overline{M}_{g,2}$ is compact, there must be some (unique) stable 2-pointed limit when p approaches q. What is it? Similarly, what if C is nodal and $p \in C$ is market; then, what's the limit if you let p approach the node of C? (Recall marked points are required to be smooth)

Example ($p \to q$)**.** Say we have a fixed curve C with $p, q \in C$. Imagine that q is fixed, but p varies. Start with the constant family $\Delta \times C \to \Delta$ (Δ is the unit disk as usual). We have the horizontal section $\Delta \times q$ along with some other section σ (corresponding to p) which actually varies. Say $\sigma(t) \neq q$ for $t \neq 0$, but $\sigma(0) = q$. We want to 2-pointed stable limit as $t \to 0$.

With all these examples, probably best to sit down and try to draw what's going on. I could go back and add images to these notes, but I'm lazy, so I won't

The naive limit is not stable (since $\sigma(0) = q$ are not distinct), so to fix this, we blow up $\Delta \times C$ at the point $(0, q)$. In the new total space, the central fiber is a nodal union $E \cup C$ (E the exceptional divisor) with the strict transform of the sections $\sigma, \Delta \times q$ both meeting E, but now in two distinct points. Thus, the stable limit is a nodal union $C \cup \mathbb{P}^1$ (with node at $q \in C$) with two marked points on the $\mathbb{P}^1$. △

Note that, in the above situation, it does not matter how quickly p approaches q. The stable limit always looks the same since $\operatorname{Aut} \mathbb{P}^1$ acts 3-transitively on $\mathbb{P}^1$.

However, if there were 3 sections coming together, then we'd get a nodal union of C and $\mathbb{P}^1$ with 3 marked points on the $\mathbb{P}^1$. These four special points on the $\mathbb{P}^1$ (the 3 marked points + the nodal point) have some j-invariant (or cross ratio), and this j-invariant can take any possible value, depending on the relative speed/angle of approach of the sections coming together.

Example ($p \to$ node)**.** Say C is a nodal curve with marked (smooth) point $p \in C$. What happens when p approaches the node? What can try a similar approach starting with total space $\Delta \times C$, but this surface is singular. Hence, it's a little delicate to blow up. To avoid dealing with these, we replace C with its (pointed) normalization $(\widetilde{C}, q, r)$ (here, q, r lie over the node of C).

So consider $\Delta \times \widetilde{C} \to \Delta$ with constant sections q, r. Let σ be the section corresponding to the marked point p, so $\sigma(t) \in \widetilde{C} \setminus \{q, r\}$ for $t \neq 0$, but $\sigma(0) \in \{q, r\}$. Say, $\sigma(0) = r$. We blow up the total space at this point of intersection, and arrive at a central fiber which is a nodal union of $\widetilde{C}$ and the exceptional divisor $E \simeq \mathbb{P}^1$, with σ, r passing through E at two distinct (smooth) points and q passing through $\widetilde{C}$ at some smooth point. Now, we can take this blown up surface, and re-identify the (disjoint) sections r, q to see that the stable limit is a copy of $\widetilde{C}$ with a rational bridge $\mathbb{P}^1$ passing through two points of $\widetilde{C}$ (the points r, q), and having a single marked point p. $\triangle$

Note 45. Missed a couple small thinks Joe said, not sure about what...

Say C is a fixed (irreducible) nodal curve. Let $p, p' \in C$ be a pair of distinct smooth points. Thus, (C, p, p') is a stable pointed curve.

Question 35.6. *What is the limit as both p, p' approach the node? In particular, does the limit depend on the particular family realizing this approach?*

As before, we pass to the pointed normalization $(\widetilde{C}, q, r)$ of C and start with the constant family $\Delta \times \widetilde{C} \to \Delta$. We have the horizontal sections q, r of this family. In addition, we have two more sections σ, σ' corresponding to the points p, p'. Suppose, $\sigma(t), \sigma'(t)$ both avoid q, r for $t \neq 0$, but both hit one of these when $t = 0$.

- First suppose that $\sigma(0) = q \neq r = \sigma'(0)$.

  We do the usual thing and blow up the total space at both these points of intersection. This gives a central fiber consisting of a nodal chain $E - \widetilde{C} - E'$ (so $E, E' \simeq \mathbb{P}^1$ both meeting $\widetilde{C}$ in a single point, but not meeting each other). We have q, σ meeting E, and r, σ' meeting E' in the central fiber. We now re-identify q, r in the central fiber, and the resulting picture is a little hard to describe in just words, so I encourage you the draw this out.

  Here's my attempt: we have three components $\widetilde{C}, E, E'$. $\widetilde{C}$ has no marked points, but two nodal points (where it meets E, E'). Furthermore, E, E' meet each other at a single point (coming from identifying q, r) and they each have a single marked point.

  In any case, the result here does not depend on the particular approach (since every $\mathbb{P}^1$ in the result has exactly 3 special point).

- Now suppose that $\sigma(0) = r = \sigma'(0)$.

  Again, we blow up, but now at a single point of intersection. We get a nodal union of $\widetilde{C}$ and $E \simeq \mathbb{P}^1$ with q passing through $\widetilde{C}$, but r, σ, σ' passing through E. Now, when we re-glue q to r, we get $\widetilde{C}$ with a rational bridge having two marked points (p, p').

  Here, we have a non-trivial cross ratio (coming from having 4 special points on the $\mathbb{P}^1$), and so the resulting answer will depend on how p, p' both approach the node (along the same branch).

35.2 The boundary of $\overline{M}_g$

We have a space $\overline{M}_g$ which contains M_g as an open subset. We'd like to understand the boundary of $\overline{M}_g$, i.e. the complement $\overline{M}_g \setminus M_g$.

Let C be stable genus g curve. It's normalization $\widetilde{C}_\alpha = \bigsqcup C_\alpha$ is the disjoint union of the normalizations of its components. Say C_α smooth of genus g_α. For each node $p_i \in C$, we get two points $q_i, r_i \in \widetilde{C}$ above it.

We want to describe this data combinatorially.

Definition 35.7. The **dual graph of a stable curve** C has one vertex for each irreducible component of C, and one edge for each node (joining to the vertices corresponding to the components containing its preimages in the normalization). Furthermore, one labels each vertex with the genus of the corresponding component. $\diamond$

Considering these dual graphs will give us a **stratification** of $\overline{M}_g$, i.e. a collection of locally closed subsets s.t. the closure of any one is a union of itself and other strata. For a given graph Γ, we consider

$$M_\Gamma = \{\text{stable curves with dual graph } \Gamma\} \subset \overline{M}_g.$$

Question 35.8 (Audience)**.** *Is it each to see that these M_Γ are locally closed?*

Answer. It would be using some deformation theory, but we haven't yet really introduced deformation theory, and it's unclear if we want to. Without using it, it's less immediately clear. $\star$

Example (Stratification in genus 2)**.** Possible dual graphs

(1) A single vertex with weight 2, i.e. a smooth genus 2 curve.

(2) A single vertex with a self loop and weight 1, i.e. a nodal geometric genus 1 curve

(3) Two vertices with an edge between them and both weight 1, i.e. a nodal union of two elliptic curves

(4) A single vertex with 2 self loops and weight 0, i.e. a rational curve with 2 nodes

(5) A weight one vertex joined with a weight 0 vertex (and the weight 0 vertex has a self-loop), i.e. a nodal union of an elliptic curve and a rational curve with a single node

(6) A pair of weight 0 vertices with three edges between them, i.e. two rational curves with 3 points identified between them

(7) A pair of weight 0 vertices each with a self-loop and with one edge between them, i.e. a nodal union of two rational curves w/ a single node

Note, to be stable, a graph needs ≥ 3 edge ends connected to each rational vertex. $\triangle$

We'd like to determine $\dim M_\Gamma$ in terms of the graph Γ. We'll start now, and pick this up next time since we only have a few minutes left.

Example (Case **(7)** above)**.** In this case, we have exactly 3 marked points on each $\mathbb{P}^1$ (each normalization of a rational component), so all such curves are isomorphic, i.e. $\dim M_{\Gamma(\mathbf{7})} = 0$ (the moduli is a single point). $\triangle$

Example (Case **(5)** above)**.** Get a one-parameter family of these, by letting the elliptic curve on the left vary. In the closure, the j-invariant of the elliptic curve goes to ∞, and so you get something of type **(7)**. $\triangle$

Example (Case **(4)** above)**.** These come from taking a single $\mathbb{P}^1$ and identifying 2 pairs of points, so we're in effect marking 4 points on $\mathbb{P}^1$. These have a cross ratio, so we get a 1-parameter family. $\triangle$

Fact. Cases **(2)**,**(3)** have dimension 2, cases **(4)**,**(5)** have dimension 1, and cases **(6)**,**(7)** have dimension 0. In general, the codimension of each strata is the number of edges in the dual graph (= the number of nodes of the curves).

We'll show this on Thursday. Note that this will give us a start towards the divisor class theory of $\overline{M}_g$.

Note 46 (added some time after lecture). If I am recalling lecture correctly, Joe remarked at the end that the strata M_Γ are images of maps from products of moduli spaces to M_g, e.g. note that (continuing with the genus 2 example) $M_{\mathbf{(3)}}$ is the image of a map $M_{1,1} \times M_{1,1} \to \overline{M}_2$ which sends a pair of marked elliptic curves to the (unmarked) stable curve obtained by gluing them together at their marked points. On the one hand, since $\dim M_{1,1} = 1$ ($M_{1,1} = \mathbb{A}^1_j$ is the j-line), this makes it believable that $\dim M_{\mathbf{(3)}} = 2$. On another hand, more generally, this sort of reasoning shows that M_Γ is constructible (i.e. a union of locally closed subsets) since it's the image of a morphism of schemes; if you further accept that $M_{g,n}$ is irreducible, then I guess this in fact gives a way to see that M_Γ is locally closed.

36 Lecture 10 (2/24)

Last time, we asserted that we can compactify M_g by throwing in isomorphism classes of stable curves (those with only nodal singularities and with finite automorphism groups). Now, we embark on a relatively large scale project, even accepting that we've obtained a modular compactification $\overline{M}_g$ of M_g. To really make use of the fact that $\overline{M}_g$ is a modular compactification, we'll need to take the theory of smooth curves we've developed and extend it over to the theory of stable curves. This will allow us to treat the objects $\overline{M}_g$ parameterizes essentially uniformly.

The first thing we'll want to do is finish up our discussion of dual graphs which we started last time.

36.1 The boundary of $\overline{M}_g$, continued

Let C be a stable curve of genus g with δ nodes.

Recall 36.1. We associate to this the dual graph Γ_C which is a weighted graph with one vertex for each component C_α of C which is given weight/label/marking $g_\alpha = g(C_\alpha)$; furthermore, this is one edge for each node, connecting the two (not necessarily distinct) irreducible components containing this node. ⊙

This association gives us a stratification of $\overline{M}_g$, a way of writing it as a disjoint union of locally closed subschemes.

Example ($g = 3$)**.** The possible dual graphs here are

- A single vertex with weight 3

  This is the open $M_3 \subset \overline{M}_3$ of smooth curves

- A single vertex with weight 2 and one self-loop

  A genus 2 curve with a node

- A weight 2 vertex attached to a weight 1 vertex

  A nodal union of an elliptic curve and a genus 2 curve

- A single vertex of weight 1 with two self-loops

  An elliptic curve with two nodes

- Two weight 1 vertices attached by an edge, and one of them also has a self loop

  A nodal union of an elliptic curve and an elliptic curve with a node

- Same as above, except the vertex with self-loop has weight 0 and the other has weight 2

  A nodal union of a genus 2 curve and a rational curve with a node

- Two weight 1 vertices with two edges between them

  A double nodal union of two elliptic curves

- A length 3 path of weight 1 vertices

  A length 3 chain of elliptic curves △

Let's consider some numbers. Let C be a stable curve with normalization $\widetilde{C} = \bigsqcup C_\alpha$. Write $g_\alpha = g(C_\alpha)$ and say C_α contains n_α points lying over nodes of C (so $\sum n_\alpha = 2\delta$). We want to estimate the dimension of the locus of curves with this dual graph Γ. First observe that $(C_\alpha, p_1, \ldots, p_{n_\alpha}) \in M_{g_\alpha, n_\alpha}$ is a stable pointed curve, so this data varies in a family of dimension $\dim M_{g_\alpha,n_\alpha} = 3g_\alpha - 3 + n_\alpha$ (this expression is correct even when $g_\alpha \in \{0,1\}$, by the condition of stability). The upshot is that $\left\{C \in \overline{M}_g : \Gamma_C = \Gamma\right\}$ is the image of a map

$$\prod_\alpha M_{g_\alpha,n_\alpha} \to \overline{M}_g.$$

Fact. The above map is finite (which should suffice to conclude that the image is locally closed).

Thus,

$$\dim \underbrace{\left\{C \in \overline{M}_g : \Gamma_C = \Gamma\right\}}_{\Sigma_\Gamma} = \sum_\alpha (3g_\alpha - 3 + n_\alpha).$$

Now, it's not hard to show the **genus formula**

$$g = p_a(C) = g(\widetilde{C}) + \delta = \sum (g_\alpha - 1) + 1 + \delta = \sum g_\alpha - \nu + 1 + \delta$$

where ν is the number of irreducible components. Now, note that

$$\dim \Sigma_\Gamma = \sum (3g_\alpha - 3 + n_\alpha) = \sum 3(g_\alpha - 1) + 2\delta = 3g - 3 - \delta \text{ so } \operatorname{codim}(\Sigma_\Gamma) = \#\text{nodes}.$$

Let's look at some extreme cases

- $\delta = 1$ (codimension 1 strata)

  These are curves with one node, i.e. dual graphs with exactly one edge. This means the dual graph either has 1 vertex (with weight $g-1$) or two vertices (with weights α and $g-\alpha$). This gives approximately $\frac{g}{2}$ strata with codimension 1, i.e. $\frac{g}{2}$ divisors on $\overline{M}_g$.

  This is significant since one of the main ways we study projective varieties is by looking at their divisors and line bundles. Thus, it's natural to try and understand the Picard group of $\overline{M}_g$.

Remark 36.2. We also see that the boundary of $\overline{M}_g$ is the union of these $\sim \frac{g}{2}$ divisors. ○

- What's the largest possible δ? Well, $\operatorname{codim} \Sigma_\Gamma = \delta$, so δ certainly can't exceed $3g-3$. In fact, there will be finitely many stable curves of genus g with $3g-3$ nodes. In this case, we must have
  - $g_\alpha = 0$ for all α.

    If you had a component of positive genus, you could vary its moduli. This would contradict $\dim \Sigma_\Gamma = 0$.
  - each component meets the other components exactly 3 times.[44]

    If there were 4 or more points of intersection, you could vary the cross ratio of those 4 points in order to move the curve in a family, contradicting $\dim \Sigma_\Gamma = 0$.
  - There are $2g-2$ components

    Given previous two bullet points, this follows from the genus formula.

Thus, the dual graph Γ of such a C is a *trivalent graph* (i.e. every vertex is contained in 3 edges (w/ multiplicity)).

Example ($g = 2$)**.** The two possibilities here are two vertices with 3 edges between them or two vertices with one edge between them and each having a self-loop. △

Example ($g = 3$)**.** Can take $\Gamma = K_4$, complete graph on 4 vertices. The curve here is 4 lines in $\mathbb{P}^2$.

Could also take a vertex with valence 3 so that each of its neighbors has a self-loop.

Could take something that looks like $\bullet = \bullet - \bullet = \bullet$ with one more edge connecting the other two vertices.

The last possibility is $C. - . <: ||$ where each . is a vertex (4 in total) and everything else denotes edges (6 in total). △

Definition 36.3. A stable curve with the maximum number $\delta = 3g-3$ of nodes is called a **terminal curve**. ◇

Open Question 36.4. *How many terminal curves of genus g are there?*

(e.g. 2 is genus 2 and 5 in genus 3)

Question 36.5 (Audience)**.** *Is it easy from looking at these dual graphs to tell which strata specialize to which others?*

Answer. Let's reverse the question: given a strata, which others is it in the closure of? This corresponds to removing/smoothening a node of the curve (turning $\times$ into $\asymp$). Let Γ be a graph and let e be one of its edges. We can define another dual graph $\widetilde{\Gamma}$ by deleting an edge and identifying the two vertices at which the edge is attached, i.e. by contracting the edge e. The new single vertex has genus equal to the sum of the genera of the original vertices (1 fewer node and 1 fewer component, so arithmetic genus unchanged this way). If you contract a self-loop, you instead increase the genus by 1 (1 fewer node, but same number of irreducible component) ⋆

TODO: Add in a figure giving an example of this

[44] Including the possibility that it contains both branches of a node and one other point of intersection

36.2 Theory building for stable curves

The first thing we did when talking about smooth curves was define the notion of a divisor (as a finite formal linear combination of points). This definition does not work for singular curves. The first thing we'll want to do here is see the correct definition of divisors for stable curves.

Observation 36.6. If C is a smooth curve, then an effective divisor on C is the same thing as a codimension 1 subscheme of C.

Definition 36.7. Let C be a possibly singular curve. An **effective (Cartier) divisor** on C is defined to be a locally principal[45] subscheme $D \subset C$ of dimension 0. $\diamond$

Example. Say $C \subset \mathbb{P}^2$ (so $C = \alpha$) is a nodal cubic, with node p. First note that p is not a Cartier divisor. Any function vanishing at the point p will do so to order ≥ 2 and so not cut out a reduced subscheme.[46] At the same time, there exists infinitely many Cartier divisors of degree 2 with support at p, parameterized by lines passing through p (except you get a degree 3 divisor whenever your line is tangent to one of the branches). $\triangle$

Construction 36.8. Given an effective Cartier divisor $D \subset C$, w/ local defining equation f, we define the invertible sheaf/line bundle $\mathscr{O}_C(D)$ by setting

$$\mathscr{O}_C(D)(U) = \{\text{rational functions } g \text{ on } U : fg \text{ is regular}\}.$$

Note that this is simply dual to the ideal sheaf of D (which is invertible by definition of a Cartier divisor).

Definition 36.9. A **Cartier divisor** is a formal difference of effective Cartier divisors. $\diamond$

Definition 36.10. If f is a meromorphic/rational function on C, we write f locally as g/h and define the **divisor of** f to be $\operatorname{div}(f) = V(g) - V(h)$. We say two Cartier divisors D, E are **linearly equivalent** if there exists a rational function f with $(f) = D - E$. $\diamond$

Fact.

$$\operatorname{Pic} C \simeq \frac{\{\text{Cartier divisors}\}}{\text{linear equivalence}}$$

What's coming up next week?

- We need an appropriate analogue of the notion of a canonical bundle. The cotangent sheaf won't work for stable curves. We'll need to extend Riemann-Roch to stable curves.
- We also need an appropriate analogue of the Jacoabian of a stable curve. We can ask if there's still a variety parameterizing line bundles on stable curves, and if so, how its geometry is affected by the geometry of the stable curve it comes from.

[45] i.e. in a neighborhood of any point, D is cut out by a single equation

[46] Alternatively, the tangent space at p has dimension 2 and so any principal subscheme will have tangent space of dimension ≥ 1 at p

37 Lecture 11 (3/1)

37.1 Administrative Stuff

Harvard has updated its mask guidance. As of Thursday, assuming the class room is suitable (as ours is), Joe will be able to lecture maskless. Let him know if you'd rather he kept it on for any reason. Also, potentially, the students can be unmasked if we're sat far enough apart. By default, the plan would be for Joe to be unmasked and the students to remain masked.

Plan for today

- The (degree 0) Picard group $\mathrm{Pic}^0(C)$ of a nodal curve
- The dualizing sheaf of a nodal curve

We've developed these notions for smooth curves. We now would like to generalize this story to singular curves. We focus on nodal curves since we have seen these suffice to compactify M_g. One could also imagine generalizing them to arbitrary singular curves, and this works to varying degrees. For the purposes of this coarse, we stick with the nodal case.

Let's begin with a formula: say C is a nodal curve with δ nodes and k irreducible components C_α. Let $\widetilde{C}_\alpha$ be the normalization of C_α, and let $g_\alpha = g(\widetilde{C}_\alpha)$ be the geometric genus of C_α. Note that the normalization of C is simply $\widetilde{C} = \bigsqcup \widetilde{C}_\alpha$. Note that the dual graph of C has k vertices and δ edges.

Notation 37.1. We'll always use $g(-)$ to refer to the arithmetic genus of $-$. If we need its geometric genus, we'll write $g(\widetilde{-})$.

Given the above setup, one has the **genus formula for nodal curves**

$$g(C) - \delta = g(\widetilde{C}) = \sum g_\alpha - k + 1 \implies g(C) = \sum g(\widetilde{C}_\alpha) - k + 1 + \delta.$$

This follows from examining the cohomology of the exact sequence $0 \to \mathscr{O}_C \to \nu_* \mathscr{O}_{\widetilde{C}} \to \mathscr{F} \to 0$.

Remark 37.2. $g(C) = \sum g_\alpha \iff \delta = k - 1$, i.e. iff the dual graph is a tree (connected graph with one fewer edge than vertex). In general, $\sum g(\widetilde{C}_\alpha) = g - h^1(\Gamma_C)$ where $h^1(\Gamma_C)$ is the rank of first singular cohomology of the dual graph.[47] ○

37.2 $\mathrm{Pic}^0(C)$ of a nodal curve

Let's first consider the case where C is irreducible w/ a single node $p \in C$. Let $q, r \in \widetilde{C}$ be the points of the normalization lying above the node. Pullback along $\widetilde{C} \xrightarrow{\nu} C$ gives a natural (surjective) map

$$\mathrm{Pic}^0(C) \xrightarrow{\nu^*} \mathrm{Pic}^0(\widetilde{C}) \to 0.$$

Remark 37.3. A line bundle L on C is the same thing as a line bundle $\widetilde{L}$ on $\widetilde{C}$ together with a choice of isomorphism $\widetilde{L}_q \simeq \widetilde{L}_r$. Such an identification corresponds to multiplication by a nonzero scalar, so the kernel of the map is $\mathbb{C}^\times$, i.e. we have

$$\mathbb{C}^\times \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^0(\widetilde{C}) \longrightarrow 0.$$

[47] If $T \subset \Gamma_C$ is a spanning tree, then T is contractible so Γ_C is homotopy equivalent to Γ_C/T with is a bouquet of $k - \delta - 1$ circles

$\circ$

Question 37.4 (Audience)**.** *Is it obvious that this data always determines a line bundle L?*

Answer. Here's one way to see it. Given $\widetilde{L}$ on $\widetilde{C}$ with an identification $\varphi : \widetilde{L}_q \xrightarrow{\sim} \widetilde{L}_r$, we form the sheaf L on C with sections

$$L(U) = \left\{\sigma \in \widetilde{L}(\nu^{-1}(U)) : \varphi(\sigma(q)) = \sigma(r)\right\}$$

One then verifies that this is locally free. $\star$

Is the map $\mathbb{C}^\times \longrightarrow \mathrm{Pic}^0(C)$ in the previous remark injective? If not, you'd have a line bundle L on C with an automorphism acting differently on the fibers above q, r. This can't happen because automorphisms of a line bundle are (nonvanishing) global functions, and a global holomorphic function on an irreducible curve must be constant. Thus, we have $0 \to \mathbb{C}^\times \to \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\widetilde{C}) \to 0$.

Exercise. If C has two smooth irreducible components meeting at a node, then you get an isomorphism $\mathrm{Pic}^0(C) \xrightarrow{\sim} \mathrm{Pic}^0(\widetilde{C})$ (the latter group is the product of the Jacobians of the two components).

Remark 37.5. In either case (connected curve with 1 node and no other singularities), $\mathrm{Pic}^0(C)$ has dimension g. When C irreducible, it is not compact, but when C is reducible, it is compact. $\circ$

Let's now consider the more general setting. Say C is any nodal curve and write

$$\widetilde{C} = \bigsqcup \widetilde{C}_\alpha \xrightarrow{\nu} C.$$

We again get a map $\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(\widetilde{C}) = \prod \mathrm{Pic}^0(\widetilde{C}_\alpha)$ which is surjective.

Warning 37.6. $\mathrm{Pic}^0(C)$ usually refers to degree 0 line bundles. If C is reducible, a line bundle has a degree on each component, and one usually takes its degree to be the sum of these. However, when we write $\mathrm{Pic}^0(\widetilde{C})$, we really want line bundles which are degree 0 individually on each component (and not just those with total degree 0). $\bullet$

To descend a line bundle on $\widetilde{C}$ to one on C, we need to make δ identifications, so we get

$$(\mathbb{C}^\times)^\delta \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \prod \mathrm{Pic}^0(\widetilde{C}_\alpha) \longrightarrow 0.$$

However, the left map may not be injective if there are nodes lying on two components. What one actually gets is an exact sequence

$$0 \longrightarrow (\mathbb{C}^\times)^a \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \prod \mathrm{Pic}^0(\widetilde{C}_\alpha) \longrightarrow 0 \text{ where } a = h^1(\Gamma_C).$$

In particular, $\mathrm{Pic}^0(C)$ is a g-dimensional group which is compact iff Γ is a tree (in this case, we say that C is of **compact type**).

Question 37.7. *Can we compactify* Pic^0*?*

(Sounds like finding a good compactification of an arbitrary singular curve is still open)

As before, start with the example of C irreducible with a single node $p \in C$. Recall this gives

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{Pic}^0(C) \longrightarrow \mathrm{Pic}^0(\widetilde{C}) \longrightarrow 0.$$

To figure out what the compactification should be, let's look at degree 0 line bundles on C as they approach the hole in $\mathbb{C}^\times$. Let $q, r \in \widetilde{C}$ be the points over the node. Let $s, t \in C$ be two different smooth points. Consider the line bundle $\mathscr{O}_C(s-t)$ as t varies. What happens when $t \to p$? The naturally limiting sheaf is $\mathscr{I}_p(s)$, the sheaf of meromorphic functions on C with a simple pole allowed at s and which vanish at p. Now, the maximal ideal at the node point p is not free (it requires two generators, but is only rank 1). Thus, what we get is not a line, but only a torsion-free sheaf (of rank 1).

The upshot is that we can compactify $\operatorname{Pic}^0(C)$ by taking

$$\overline{\operatorname{Pic}^0(C)} = \left\{ \begin{matrix} \text{torsion-free sheaves} \\ \text{of rank 1 on } C \end{matrix} \right\}.$$

(implicitly, we only want the degree 0 such sheaves above. We'll say what the degree of a rank 1 torsion-free sheaf is in a moment).

Question 37.8. *What does the result of forming this compactification look like?*

Keeping with the same example of C irreducible with a single note, note that $\operatorname{Pic}^0(\widetilde{C})$ is a $(g-1)$-dimensional abelian variety, and that $\operatorname{Pic}^0(C) \to \operatorname{Pic}^0(\widetilde{C})$ is a $\mathbb{C}^\times$-bundle. We can compactify this to a $\mathbb{P}^1$-bundle over $\operatorname{Pic}^0(\widetilde{C})$ by introducing a cross-section at 0 and one at ∞. By considering $\lim_{t \to p} \mathscr{O}(s-t)$ as t approaches p along the two branches, one sees that we want to identify the 0 cross-section above $\mathscr{O}(s-q)$ to the ∞-cross section above $\mathscr{O}(s-r)$. The quotient of the $\mathbb{P}^1$-bundle by this identification is $\overline{\operatorname{Pic}}^0(C)$.

TODO: Come back and make sense of this description

Warning 37.9 (Assuming I heard correctly)**.** There's not a natural map $\overline{\operatorname{Pic}}^0(C) \to \operatorname{Pic}^0(\widetilde{C})$ •

Example (Cuspidal curves)**.** Say C irreducible with a single cusp $p \in C$ with preimage $q \in \widetilde{C}$. We again get a map $\operatorname{Pic}^0(C) \to \operatorname{Pic}^0(\widetilde{C}) \to 0$. What is the kernel? What data do you need to specify to descend a line bundle from $\widetilde{C}$ to C? In this case, you get

$$0 \longrightarrow \mathbb{C} \longrightarrow \operatorname{Pic}^0(C) \longrightarrow \operatorname{Pic}^0(\widetilde{C}) \longrightarrow 0.$$

In general, for C an irreducible curve, you'll get a similar exact sequence where the kernel is some product of $\mathbb{C}$'s and $\mathbb{C}^\times$'s whose dimension is the sum of the δ-invariants of the singularities.[48] △

37.3 Dualizing sheaves

In the case of smooth curves, the canonical bundle played a special role due to its appearance in Serre duality and Riemann-Roch. We want a suitable analogue for nodal curves. This will be given by the dualizing sheaf (on a possibly singular curve).

The definition of the dualizing sheaf of a nodal curve will be relatively straight forward, but may seem a bit mysterious. To motivate it, let's first review a proof of Riemann-Roch.

Proof sketch of RR. Consider a smooth curve C with a divisor $D = p_1 + \cdots + p_d$ (assume p_i distinct to keep life simple). We would like to find $\dim \mathscr{L}(D)$ where

$$\mathscr{L}(D) = \left\{ f \in \mathcal{M}(C) \;\middle|\; \begin{matrix} f \text{ has at worse a simple pole at each } p_i \\ \text{and } f \text{ is holomorphic elsewhere} \end{matrix} \right\}.$$

[48]The tangent space at the identity of $\operatorname{Pic}^0(C)$ is $\mathrm{H}^1(C, \mathscr{O}_C)$ which has dimension $g(C)$ (assuming C irreducible)

Fix local coordinates z_i around each p_i. For $f \in \mathscr{L}(D)$, we can write $f(z_i) = a_i/z_i + g(z_i)$ with g holomorphic. The residues $(a_1, \ldots, a_d)$ determines f up to the addition of a constant (two functions with the same residues would have a global holomorphic difference). Thus, $\dim \mathscr{L}(D) \leq d+1$.

However, we can do better (this is the key step). If ω is any holomorphic 1-form on C, and $f \in \mathscr{L}(D)$, then $f\omega$ will be a meromorphic 1-form ($\implies$ the sum of its residues is 0). If we write $\omega(z_i) = b_i \mathrm{d}z_i +$(stuff vanishing at p_i), then,

$$0 = \sum \operatorname{Res}(f\omega) = \sum a_i b_i.$$

Thus, for every holomorphic differential ω, we get a linear relation on the a_i's. This gives $(g - h^0(K-D))$ linearly relations on the a_i's (e.g. $b_i = 0$ for all i, the associated relation is trivial). Thus,

$$h^0(D) \leq d + 1 - (g - h^0(K-D)).$$

The same relation applied to $K - D$ gives $h^0(K-D) = (2g-2-d) + 1 - (g - h^0(D))$. Adding these two inequalities up gives $h^0(D) + h^0(K-D) \leq h^0(K-D) + h^0(D)$. Since two inequalities added up to an equality, they must have both been an equality to start. ■

The key was that to a meromorphic section of the canonical bundle, we can associate residues, and the sum of these residues is 0. We also needed it to have degree $2g-2$ and to have g global sections.

On Thursday, we'll see the construction of a line bundle on a nodal curve satisfying these properties.

38 Lecture 12 (3/3): Dualizing sheaves

Recall 38.1. To prove RR, we needed 3 facts about K_C

- $\deg K_C = 2g - 2$
- $h^0(K_C) = g$
- For all meromorphic sections ω of K_C,

$$\sum_{p \in C} \operatorname{Res}_p \omega = 0.$$ ⊙

We would like to find, on a nodal curve C, a sheaf with analogous properties.

Example. Say C irreducible with a single node $p \in C$ lying under $q, r \in \widetilde{C}$, where $\nu : \widetilde{C} \to C$ is the normalization.

- 1st attempt: take

$$K_C(U) = \left\{ \begin{matrix} \text{holomorphic 1-forms} \\ \text{on } \nu^{-1}(U) \end{matrix} \right\},$$

  i.e. $K_C = \nu_* K_{\widetilde{C}}$. This fails on all accounts
  - $\deg K_C = \deg(K_{\widetilde{C}}) = 2(g-1) - 2 = 2g - 4$
  - $\mathrm{H}^0(K_C) = \mathrm{H}^0(K_{\widetilde{C}})$ has dimension $g - 1$

Given that a holomorphic 1-form on $\widetilde{C}$ only has $2g-4$ zeros, we want to allow (simple) poles at q, r.

- 2nd attempt: take the *dualizing sheaf* ω_C to be defined by

$$\omega_C(U) = \left\{\omega \in K_{\widetilde{C}}(q+r)(\nu^{-1}(U)) : \mathrm{Res}_q(\omega) + \mathrm{Res}_r(\omega) = 0\right\}$$

  This will give the correct sheaf. △

Let's give the general definition.

Definition 38.2. Let C be any reduced curve (arbitrary singularities) and let $\nu : \widetilde{C} \to C$ be its normalization. The **dualizing sheaf** of C is given by

$$\omega_C(U) = \left\{ \begin{array}{c} \text{meromorphic differentials } \omega \\ \text{on } \nu^{-1}(U) \end{array} \;\middle|\; \begin{array}{c} \forall f \in \mathscr{O}_C(U), \forall p \in U \\ \displaystyle\sum_{q \in \nu^{-1}(p)} \mathrm{Res}_q(f\omega) = 0 \end{array} \right\} \quad \diamond$$

Why does this specialize to what we said before in the nodal case?

Note 47. I was too busy fiddling with the previous displayed equation to hear the answer to this question, so... exercise: figure out the answer.[49]

Observation 38.3. In the nodal case (C only has nodal singularities),

- ω_C is locally free (of rank 1).

  If you choose local coordinates x near q and y near r, then a generator for ω_C near p is given by the pair $\left(\frac{\mathrm{d}x}{x}, -\frac{\mathrm{d}y}{y}\right)$.

  Warning 38.4. ω_C is not always locally free if you allow C to have arbitrary singularities. •

- $h^0(\omega_C) = g$.

  Note that $h^0(K_{\widetilde{C}}(q+r)) = g$ by Riemann-Roch, and that $\mathrm{H}^0(\omega_C) = \mathrm{H}^0(K_{\widetilde{C}}(q+r))$ since global meromorphic 1-forms on smooth curves always have residues adding up to 0.

- $\deg(\omega_C) = 2g-2$

  The total number of zeros of a section minus the total number of poles is $2g-4$. There are two poles at q, r, so there better be $2g-2$ zeroes.

I'm pretty sure degree of a line bundle on C is the degree of its pullback to the normalization

Using these, one can obtain Riemann-Roch for nodal curves.

Above, we've had in mind that C is irreducible. In general, we want to be able to handle also reducible (but still connected) C.

Example (reducible nodal)**.** Say C is a nodal union of a genus α curve A and a genus $g-\alpha$ curve B, so $\widetilde{C} = A \sqcup B$. What can we say about the dualizing sheaf ω_C (same definition) in this case?

- It has basepoints

TODO: Come understand this example

[49] Sounds like the main point is that you can find functions vanishing identically along one branch at p but to order $n-1$ along the other branch. Thus, if you have a pole or order ≥ 2 at q, you can multiply a function to get a meromorphic differential with a simple pole at q but which vanishes identically near r, and such a things residues can't add to 0

You can't find a non-holomorphic differential with a simple pole at a single point on either component A, B of the normalization (residues must sum to 0 on each component). If you look at sections near the node p, they'll be the generator $(\mathrm{d}x/x, -\mathrm{d}y/y)$ multiplied by functions vanishing at p. Thus, ω_C won't be globally generated in this case. △

Example (cusp)**.** Say C has a cusp at $p \in C$ and no other singularities (so C irreducible). Let $\nu : \widetilde{C} \to C$ be the normalization. Let $q \in \widetilde{C}$ lie over the cusp. We have a local coordinate t around q, so that $\nu : t \mapsto (t^2, t^3)$.

Remark 38.5. C has the same underlying topological space as $\widetilde{C}$, but its structure sheaf consists of functions with derivative vanishing at q, i.e.

$$\mathscr{O}_C(U) = \left\{ f \in \mathscr{O}_{\widetilde{C}}(\nu^{-1}(U)) : f'(q) = 0 \right\}. \qquad \circ$$

Note that $\omega_C(U)$ consists of meromorphic differentials ω on $\widetilde{C}$ s.t. the residue of $f\omega$ at q vanishes for all $f \in \mathscr{O}_C(U)$. This allows double poles at q whose residue vanishes. However, it does not allow for higher order poles (or for simple poles). That is,

$$\omega_C(U) = \left\{ \omega \in K_{\widetilde{C}}(2q) : \mathrm{Res}_q(\omega) = 0 \right\}.$$

Note that this satisfies the three conditions we identified earlier. Furthermore, ω_C is locally free, with generator at p given by $\mathrm{d}t/t^2$. △

Example (tacnode)**.** Say C is irreducible with a tacnode at $p \in C$ and no other singularities. Let $\nu : \widetilde{C} \to C$ be the normalization, and let $r, q \in \widetilde{C}$ lie over the tacnode. Sections of the dualizing sheaf can't have poles of order ≥ 3 at r, q. This is because, locally near $p \in C$, you can find a function which vanishes identically along one branch, but only to order 2 at the point lying over the other branch. Multiplying such a thing by a differential with a triple pole at one point will give a 1-form with a simple pole at r (say) but vanishing near q whose residue then don't add to 0. On the other hand, one can allow double poles at q, r over p. The dualizing sheaf here will be a line bundle, and it will satisfies the three conditions we identified earlier. △

The simplest example where the dualizing sheaf is not locally free is given by a spatial triple point.

Example (spatial triple point)**.** A spatial triple point (locally) looks like the union of the three coordinate axes in $\mathbb{A}^3$. Let C be irreducible with a spatial triple point $p \in C$ and no other singularities. Let $\nu : \widetilde{C} \to C$ be the normalization. In this case, a section of the dualizing sheaf will look like a meromorphic differential on $\widetilde{C}$ with at worse simple poles at the points $q, r, s \in \widetilde{C}$ above p (and whose residues add to 0). Thus, near p, one gets sections of the form

$$\left(\frac{\mathrm{d}x}{x}, -\frac{\mathrm{d}y}{y}, 0 \right), \left(\frac{\mathrm{d}x}{x}, 0, -\frac{\mathrm{d}z}{z} \right), \left(0, \frac{\mathrm{d}y}{y}, -\frac{\mathrm{d}z}{z} \right).$$

There's only one relation among them, so the dualizing sheaf here is not locally free. △

Question 38.6. *When is ω_C locally free?*

Theorem 38.7. *Let C be a reduced curve. Then, ω_C is locally free iff C is Gorenstein.*

One day I'll need to actually learn what this word (and Cohen-Macaulay)

In particular, this holds when C is planar, or more generally, if C is a local complete intersection.

38.1 Another characterization of dualizing sheaves

We've talked about the canonical bundles of smooth curves since the beginning. We've now defined the dualizing sheaf and claimed it's the right analogue in the case of singular curves.

Question 38.8. *Say we have a family $\pi : \mathcal{C} \to \Delta$ of curves where C_t is smooth for $t \neq 0$ and C_0 is nodal. To keep things simple, also assume $\mathcal{C}$ is smooth. Does the canonical bundle of the general fiber relate to the dualizing sheaf of C_0?*

Here's a simpler question: ignoring the singular fiber, so the canonical bundles of the fibers fit together to a form a line bundle on $\mathcal{C}$?

Yes. Let $\Delta^* = \Delta \setminus \{0\}$ and let $\mathcal{C}^* = \pi^{-1}(\Delta^*)$. Form the **relative cotangent bundle**

$$T^*_{\mathcal{C}^*/\Delta^*} := T^*_{\mathcal{C}^*}/\mathrm{d}\pi(T^*_{\Delta^*}),$$

i.e. it's defined by the exact sequence

$$T^*_{\Delta^*} \longrightarrow T^*_{\mathcal{C}^*} \longrightarrow T^*_{\mathcal{C}^*/\Delta^*} \longrightarrow 0$$

of cotangent bundles. This relative cotangent bundle will restrict to the canonical bundle (= cotangent bundle) on each fiber. Now, does this extend to a line bundle on all of $\mathcal{C}$?

Fact. Yes. It extends to the *relative dualizing sheaf* $\omega_{\mathcal{C}/\Delta}$. This is a line bundle on $\mathcal{C}$ s.t. (among other things), $\omega_{\mathcal{C}/\Delta}|_{C_0} = \omega_{C_0}$.

This uniquely characterizes the dualizing sheaf. If you have a smooth surface and a line bundle away from a single point, then there's a unique way to extend it to a line bundle on the whole surface.

38.2 Plane curves revisited

We started this semester by talking about plane curves. At the time, we were thinking of them in relation to their normalizations. Now that we've generalized some notions from smooth curves to arbitrary curves, let's revisit this material with a new eye.

Recall 38.9. We started by looking at smooth $C : f(x, y) = 0$. We write that the holomorphic differential $\omega_0 = \mathrm{d}x/\partial f/\partial y$ and said that we could get more by multiplying this by polynomials (of degree $\leq d-3$).

We then looked at nodal C and saw that ω_0 had simple poles at the points of $\widetilde{C}$ lying over the nodes of C. ⊙

In either case, the ω_0 you write down gives a holomorphic section of the dualizing sheaf.

39 Lecture 13,14 (3/8,10): Out of town

40 Lecture 15 (3/22)

Note 48. Roughly 12 minutes late. Oops

Today

- Construction of M_g
- divisor class theory of $\overline{M}_g$ (continued from last week)

Next week: guest lecture by Prof. deMarco on Teichmuller theory (3/29)

Constructions of $M_g/\overline{M}_g$

- DM approach ('69)

  gluing together deformation space

- Mumford-Knudsen ('76)

  via geometric invariant theory (GIT)

- Teichmuller theory

40.1 Deformation theory

Note 49. Missed some motivation/intuition Joe explained out loud

Let X be a given curve of genus g.

Definition 40.1. A **deformation** of X is a germ of a family of curves over a pointed scheme (B, b_0) $\diamond$

That is, we're considered families $\mathcal{C} \to B$ of curves over B equipped with an isomorphism central fiber $\varphi : X \xrightarrow{\sim} C_0 := C_{b_0}$. By 'germs' of such families, we mean that two are considered equivalent if they agree around opens of their points.

Theorem 40.2. *There exists a "**versal deformation space**" of X, i.e. a deformation $\mathcal{C} \to \Delta$ (with $0 \in \Delta$ and $X \cong C_0$) so that every deformation of X is a pullback of this family.*

Warning 40.3. There's no uniqueness claim above. 'versal' is 'universal' without uniqueness. Consequenctly, this condition does not uniquely characterize the family $\mathcal{C} \to \Delta$, e.g. if you have one such family, you can replace the base with $\Delta \times S$ for whatever random scheme S you want. $\bullet$

To get uniqueness, one asks for a versal deformation space which is *minimal* w.r.t. this property (such a space also exists).

Remark 40.4. Moduli spaces may or may not exist, but versal deformation almost always exist. $\circ$

Note 50. Got distracted and missed some stuff Joe was saying about inductively extending deformations over $\operatorname{Spec} k[\varepsilon]/(\varepsilon^n)$

Suppose the curve X has automorphisms. Since the data of a deformation $\mathcal{C} \to B$ includes a choice of identification $\varphi : X \xrightarrow{\sim} C_0$, we see that $\operatorname{Aut}(X)$ acts on the set of deformations over a given base. In particular, it will act on a versal deformation space. From this, you should expect that a given fiber of a versal deformation $\mathcal{C} \to \Delta$ appears $\operatorname{Aut}(X)$ times, so this versal deformation space will not give a neighborhood of the point $[X]$ in the moduli space. What one would like to do is take the quotient of Δ by $\operatorname{Aut}(X)$ and hope that this now gives a (germ of a) neighborhood of the point $[X]$ in the moduli space. Roughly speaking, this is what Deligne and Mumford do. They want to glue together quotients of deformation spaces, but in order to actually carry this out, one needs to pass from the world of schemes to the world of stacks.

Question 40.5 (Audience)**.** *This makes is clear that we want stable curves to have finite automorphisms. How do we see that if we throw in weirder singularities then the resulting space is non-separated or otherwise bad?*

Answer. Secretly, there's nothing stopping us trying to do this with other types of curve singularities allowed. DM's choice of stable curves was indeed a choice. People later wondered if there were other classes of curves for which this construction will produce a compact moduli space. For example, David Schubert's thesis considers curves with nodes/cusps as singularities and no elliptic tails. More recently, David Smythe (spelling?) found a way of generating infinitely many classes of curve which produce compact moduli. ⋆

40.2 Knudsen-Mumford approach

Their approach was to use Geometric Invariant Theory (GIT). The basic idea is that it may not be clear how to construct a moduli space of curves directly, but it's a little easier to construct a moduli space of curves equipped with some extra structure.

Example. Curves of genus g equipped with a closed embedding into $\mathbb{P}^r$ (of given degree d) have a moduli space, given by the Hilbert scheme $\mathcal{H}$. Note that $\mathrm{PGL}_{r+1} = \mathrm{Aut}(\mathbb{P}^r)$ will act on the Hilbert scheme $\mathcal{H}$, and so our candidate for M_g is the quotient $\mathcal{H}/\operatorname{PGL}_{r+1}$.

The problem with this is that quotients don't always exist in algebraic geometry; they do for finite groups, but not for positive-dimensional groups (consider e.g. trying to form the quotient of $\mathbb{C}^\times \curvearrowright \mathbb{C}$. There will be two orbits, with one lying in the closure of the other). So, how do you overcome this? △

Let's spend a bit of time on this issue of quotients not always existing.

Example. Say we want a moduli space of curves of genus 1. A (smooth) curve of genus 1 can always be embedded in the plane as a cubic curve, and the space of plane cubics is a $\mathbb{P}^9$. The group $\mathrm{PGL}_3 = \mathrm{Aut}(\mathbb{P}^2)$ acts on this $\mathbb{P}^9$, and we'd like to say that $M_1 = \mathbb{P}^9/\operatorname{PGL}_3$ is the quotient of this action. Note first that the orbit containing $y^2 = x^3$ lies in the closure of every orbit of smooth curves, e.g. if $E : y^2 = x^3 + ax + b$ is smooth, we can consider the family

$$E_t : y^2 = x^3 + t^2ax + t^3b$$

for which $E \cong E_t$ if $t \neq 0$ but $E_0 \simeq (y^2 = x^3)$. Thus, a point in the quotient corresponding to the cuspidal curve would lie in the closure of every other point, so the quotient cannot be a variety. △

Let's now give an overview of the setup/steps of GIT.

Setup 40.6. We have a reductive algebraic group G acting on a projective variety X, and we want to form the quotient X/G.

The formalism of GIT classifies the orbits of $G \curvearrowright X$ into three categories

- The stable orbits

  These behave well when you want to take a quotient. These will sweep out an open subset $X^s \overset{\text{open}}{\subset} X$ such that a good, geometric quotient X^s/G exists. However, X^s/G if often not compact.

Example. In the $\mathrm{PGL}_3 \curvearrowright \mathbb{P}^9$ plane cubic example from before, nodal curves will not be stable. However, you can have a family of smooth curves specializing to a nodal one, so the stable quotient won't be proper in this case. △

- Strictly semistable orbits.

  The stable and semistable locus X^{ss} also admits a quotient X^{ss}/G, which is now (usually?) compact.

  Warning 40.7. The points of the quotient X^{ss}/G no longer correspond 1-1 to semistable orbits. In the plane cubic example, the closure of the orbit of nodal cubics contains the orbit of a 'lines + conics' and also the orbit of 'triangles'. •

- Unstable orbits.

  These will lie in the closure of a general orbit, so cannot be included if you want a nice quotient.

Part of the power of GIT is that it gives you a method of determining which orbits are of each type, as well as telling you how to construct the corresponding quotient (in the cases they exist).

I think Mumford's 'Geometric Invariant Theory' book is the standard reference, but Mukai's 'An Introduction to Moduli and Invariants' gives a gentler introduction to the subject in a few of its middle chapters

The idea of Knudsen and Mumford to construct M_g is to apply GIT to the Hilbert scheme $\mathcal{H}$ of tricanonical curves $\varphi_{3K} : C \hookrightarrow \mathbb{P}^{5g-6}$. One wants to form the quotient $\mathcal{H}/\mathrm{PGL}_{5g-5}$ and show that it is M_g.

Remark 40.8. Knudsen and Mumford found that in this case of $\mathrm{PGL}_{5g-5} \curvearrowright \mathcal{H}$, there are no strictly semistable orbits. Furthermore, remarkably, the stable orbits are exactly those corresponding to 'stable curves' in the sense of DM. ○

40.3 Teichmüller theory

Fact. Teichmüller was a literal nazi.

Recall 40.9. Guess lecture next Wednesday by Prof. DeMarco on Teichmuller theory. ⊙

This approach is very much complex analytic. It's hard to imagine it's possible to apply it to ground fields other than $\mathbb{C}$.

At a high level, you once again start with curves equipped with extra structure, and then remove the dependence on the extra structure. What's the extra structure in this case?

Start with pairs (X, σ) with X a compact genus g Riemann surface and σ a '*pair of pants decomposition*' of X. If I'm understanding correctly, this is a collection of loops on X which expresses it as a union of 'pairs of pants,' pictured as the obvious cobordism between S^1 and $S^1 \sqcup S^1$ (up to homotopy equivalence, this is a disk in the plane minus two sub disks).

It turns out the family of pairs of pants has 3 real parameters. If you have a Riemann surface D that looks like a disk minus two subdisks, its universal cover is a disk. A disk has a unique (Poincaré) metric. This metric will descend to D (the covering group will preserve it), and so in th descended metric, you can ask for the lengths of the three boundary circles of D. These lengths determine D up to biholomorphism.

The upshot is that we can decompose X into $2g-2$ pairs of pants, each specified by specifying the lengths of 3 boundary loops. Since these loops are identified in pairs, this gives $3(2g-2)/2 = 3g-3$ real parameters. Since to glue these to form X, we need to identify circles, we observe that this identification is determined up to rotation, so we get an additional $3g-3$ angular parameters. Combined with the real parameters, we find $3g-3$ complex parameters, as we'd hope.

Fact. The set of possible structures forms an open ball in $\mathbb{R}^{6g-6}$, called "Teichmuller space"

This gives a moduli space for curves equipped with a pants decomposition. What's the relevant group action to quotient by to get a moduli space of curves unqualified?

Say X is a compact genus g Riemann surface (thought of for the moment as the underlying real surface). Consider the group $\Gamma_g = \text{Diff}(X)/(\text{isotopy})$ (the **mapping class group**) of diffeomorphisms of X, considered up to isotopy. It turns out that the data of a pair of pants decomposition is the same as an element of this group Γ_g.[50] Thus, one gets $M_g = T_g/\Gamma_g$ (note that Γ_g is discrete).

Remark 40.10. The different constructions of M_g each have their own virtues.

- The Teichmuller approach gives you chance of understanding the topology on M_g. The Teichmuller space T_g is contractible (and the Γ_g-action is free and transitive, apparently), so the (co)homology of M_g will be the group cohomology of Γ_g. ◦

41 Lecture 16 (3/24)

Note 51. 10 minutes late

Today: divisor classes/line bundles on $\overline{M}_g$.

Sounds like we want to give two ways of describing divisor classes on $\overline{M}_g$.

- divisors on variety $\overline{M}_g$

  Remark 41.1. Coming from the DM construction of $\overline{M}_g$ (from gluing together quotients of deformation spaces by finite groups), we see that $\overline{M}_g$ only has finite quotient singularities. A consequence of this is that every divisor on it is $\mathbb{Q}$-Cartier. Hence, we will work with $\text{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ ◦

- line bundles on the moduli functor/stack

  This is a gadget which associates to any family $\mathcal{C} \to B$ of stable curves (with $\dim B = 1$ if you like) a line bundle on the base B. This association must furthermore be compatible with pullback.

  Remark 41.2. We can simplify life further. It suffices to have such an association only for bases B which are curves. Furthermore, this is in fact no continuous part of $\text{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ (see Theorem 41.7), so a line bundle on it is determined by the (a priori coarser) information of simply the degree$\in \mathbb{Q}$ of the corresponding line bundle on base curves B. ◦

Example. Let $\Delta = \{\text{singular curves}\} \subset \overline{M}_g$ (the 'boundary divisor' I believe). This is a divisor on $\overline{M}_g$ and so determines a (rational) divisor class. Equivalently, this associates to a given family $\mathcal{C} \to B$ the number of singular fibers. Really, this is too simple.

Warning 41.3. A family with two nodal fibers can specialize to a family with a single nodal fiber that has two nodes. The degree of Δ on both of these must be the same.

As another complication, say you have a family $\mathcal{C} \to B$ with a single singular fiber (with one node) over $0 \in B$. Now suppose $B' \to B$ is a double cover branched over 0. Then, $\mathcal{C}' = \mathcal{C} \times_B B'$ has a single

[50] If I'm understanding correctly, really, the set of pairs of pants decompositions is a Γ_g-torsor

singular fiber with only one node, but Δ should have degree 2 on B' (we expect degrees to scale under pullback). $\bullet$

Hence, we'll need to associate multiplicities to nodes in the fibers and then have Δ be the divisor summing up these multiplicities. $\triangle$

Fact. If $\mathcal{C} \to B$ is a one-parameter family of stable curves and $p \in C_0 \subset \mathcal{C}$ is a node of C_0 ($0 \in B$ some point), then $\mathcal{C} \to B$ has a local equation near p of the form $xy - t^m$ for some m (here t is a local coordinate on B). This m is our desired *multiplicity.*

Note that if $m > 1$, then the surface $\mathcal{C}$ will be singular at the node point (and I think this is an iff).

Question 41.4 (Audience)**.** *What number do you assign to a family consisting entirely of singular curves?*

Answer. We'll answer this next week, after introducing a bit of deformation theory. $\star$

Recall 41.5. The locus of irreducible curves with one node forms an irreducible divisor Δ_0. Reducible curves with a single node look like a nodal union of a genus α curve and a genus $g - \alpha$ curve; this locus is called Δ_α ($\alpha = 1, \dots, \lfloor g/2 \rfloor$). $\odot$

Hence, our divisor class Δ is

$$\Delta = \Delta_0 + \Delta_1 + \cdots + \Delta_{\lfloor g/2 \rfloor}.$$

Let's see some other (rational) line bundles on $\overline{M}_g$. These will not be associated to divisors on $\overline{M}_g$, but will instead be described as line bundles on the functor.

Example (**Hodge class**)**.** The "Hodge class" λ is the assignment

$$\lambda\left(\mathcal{C} \xrightarrow{\pi} B\right) \rightsquigarrow \det\left(\pi_* \omega_{\mathcal{C}/B}\right).$$

Note that $\pi_*\omega_{\mathcal{C}/B}$ is a rank g vector bundle on B with fiber over $b \in B$ equal to $\mathrm{H}^0(K_{C_b})$. The **Hodge bundle** is the rank g vector bundle $(\mathcal{C} \to B) \rightsquigarrow \pi_*\omega_{\mathcal{C}/B}$. $\triangle$

Remark 41.6. If $\overline{M}_g$ were a fine moduli space, it would support a corresponding universal family of curves $\pi : \mathcal{C}_g \to \overline{M}_g$. In this case, we would directly get a rank g vector bundle over $\overline{M}_g$ using this same construction: $\pi_*\omega_{\mathcal{C}_g/\overline{M}_g}$. $\circ$

$\overline{M}_g$ is not a fine moduli space, so we can't make the above construction. However,

Fact. There exists a family $\mathcal{C} \to M$ of curves such that the induced map $M \xrightarrow{\varphi} \overline{M}_g$ is finite, surjective!

(This is the key to our two descriptions of rational line bundles on $\overline{M}_g$ being the same)

Say φ above has degree m. Then, the Hodge class λ can be described as

$$\lambda = \frac{1}{m}\varphi_* \det\left(\pi_*\omega_{\mathcal{C}/M}\right) \in \mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}.$$

Example (**kappa class**)**.** We also define the rational divisor class κ via

$$\kappa(\mathcal{C} \xrightarrow{\pi} B) = [\omega_{\mathcal{C}/B}]^2$$

(Above, B a curve, so $\mathcal{C}$ a surface, so can take self-intersections of line bundles) $\triangle$

Theorem 41.7 (**Harer**)**.** $\mathrm{H}^1(M_g, \mathbb{Q}) = 0$ *and* $\mathrm{H}^2(M_g, \mathbb{Q}) = \mathbb{Q}$

Question: Is it possible (and easy) to go from this to $\mathrm{H}^1(\overline{M}_g, \mathbb{Q}) = 0$

Remark 41.8. Harer is a topologist and thinks of these cohomology groups as the group cohomology of the mapping class group Γ_g. Harer constructs a contractible simplicial complex on which Γ_g acts, and then computes the cohomology of the quotient. ○

What's the significance of this, at least for our purposes? The first statement tells us that a line bundle on $\overline{M}_g$ is determined by the degrees of its associated line bundles (look at the exponential exact sequence over $\mathbb{C}$). The second tells us that $\mathrm{Pic}(M_g) \otimes \mathbb{Q}$ is rank 1, so generated by either λ (or κ). Hence, $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ is generated by $\lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$.

The above implies that we have a relation among the classes $\lambda, \kappa, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$.

Fact (**Mumford relation**)**.**

$$\lambda = \frac{\kappa + \delta}{12}.$$

Question 41.9 (Guiding questions)**.** *Which divisor classes are effective? ample? big?*

If someone gives us a divisor class, can we express it in terms of these natural classes?

Example (General pencil of plane quartic curves)**.** Let $F, G \in \mathbb{C}[X, Y, Z]$ be general homogeneous polynomials of degree 4. Consider the family

$$\begin{tikzcd} \mathcal{C} = \left\{(t,p) \in \mathbb{P}^1 \times \mathbb{P}^2 : t_0 F(p) + t_1 G(p) = 0\right\} \arrow[d] \\ \mathbb{P}^1 \end{tikzcd}$$

The first thing to do is verify that this is a family of stable curves.

Remark 41.10. Here's an equivalent description of the family. Let $\mathbb{P}^{14}$ be the space of all quartic plane curves. Then, $\mathcal{C} \to \mathbb{P}^1$ corresponds to a general line in $\mathbb{P}^{14}$. ○

In $\mathbb{P}^{14}$, the locus Δ of singular curves is irreducible of dimension 13, and a general $[C] \in \Delta$ is irreducible w/ 1 node. Thus, the locus of curves which are not stable is codimension ≥ 2, so a general line will miss it. In fact, a general line will give a family where every fiber is smooth or irreducible w/ 1 node (e.g. gives an arc in $\overline{M}_3$ meeting Δ_0 but not Δ_1).

Proof of above claims. Consider

$$\begin{tikzcd} & \Sigma = \left\{(C,p) \in \mathbb{P}^{14} \times \mathbb{P}^2 : p \in C_{sing}\right\} \arrow[dl] \arrow[dr] & \\ \mathbb{P}^{14} & & \mathbb{P}^2. \end{tikzcd}$$

If you fix $p \in \mathbb{P}^2$, the condition that a curve is singular at it is 3 linear conditions, so the fibers of the right map are $\cong \mathbb{P}^{11}$. Hence, Σ is smooth, irreducible of dimension 13. It's image under the left map is precisely Δ. One needs to show that a general curve of Σ is irreducible with 1 node, and this follows from Bertini allegedly. ■

Back to our family $\mathcal{C} \to \mathbb{P}^1$. We know it's a family of stable curves.

Claim 41.11. *The total space $\mathcal{C}$ is smooth.*

Proof. Consider the map $\mathcal{C} \to \mathbb{P}^2$. This map will be generically one-to-one. It fails to be so if both F, G vanish at $p \in \mathbb{P}^2$ (in which case the fiber is a $\mathbb{P}^1$). Hence, $\mathcal{C} \to \mathbb{P}^2$ is in fact the blowup of $\mathbb{P}^2$ at the 16 points where $F = G = 0$. Hence, $\mathcal{C}$ is smooth. ■

This tells us that the multiplicities of the nodes in the family are all $= 1$. Thus, $\deg \Delta$ on this family is simple the number of singular fibers (each singular fiber has 1 node).

Goal. Compute degrees of $\lambda, \kappa, \delta_0, \delta_1$ on this family.

- $\deg(\delta_1) = 0$ since all fibers irreducible
- $\deg(\delta_0) = \#$singular fibers

  Remark 41.12. In the $\mathbb{P}^{14}$, the locus of singular curves is a hyperplane. We want to degree of this hyperplane (we've intersected it with a general line, and are asking how many points of intersection there are). ○

  We'll compute this using a sort of **generalized Riemann-Hurwitz**.

  Let's start with a basic fact about (analytic) varieties/$\mathbb{C}$. Say X is a smooth variety, and $Y \subset X$ is a closed subvariety. Then, $\chi(X) = \chi(Y) + \chi(X \setminus Y)$ (topological Euler characteristic), e.g. by taking a tubular neighborhood of Y and then applying Mayer-Vietoris. Now suppose X is smooth, and we have a map $X \to B$ with B a smooth curve. Say π has singular fibers over the points $b_1, \ldots, b_\delta$.

  "Since we assumed X is smooth, not every fiber can be singular, since thank god we're in characteristic 0"

  Let $Y = \bigsqcup X_{b_i}$, so the complement $X \setminus Y$ is now a fiber bundle over $B \setminus \{b_1, \ldots, b_\delta\}$. Now we're in business. Note that (X_η general smooth fiber)

$$\chi(X \setminus Y) = \chi(B \setminus \{b_1, \ldots, b_\alpha\})\chi(X_\eta) = (\chi(B) - \delta)\chi(X_\eta) \text{ and } \chi(Y) = \sum_{i=1}^{\delta} \chi(X_{b_i}).$$

  Thus,

$$\chi(X) = \chi(B)\chi(X_\eta) + \sum_{b \in B} (\chi(X_b) - \chi(X_\eta)) \tag{41.1}$$

  (note $\chi(X_b) = \chi(X_\eta)$ if X_b is smooth).

  Let's apply (41.1) to our situation. We have X a general pencil of plane quartics, fibered over $\mathbb{P}^1$. Note that $\chi(X_\eta) = -4$, $\chi(\mathbb{P}^1) = 2$, and each singular fiber is a genus 2 curve with two points identified, so has Euler characteristic $-2 - 1 = -3$. Thus,

$$\chi(X) = 2(-4) + \#\text{sing. fibers}.$$

  At the same time, we say earlier that the projection map $X \to \mathbb{P}^2$ represents it as the blowup of $\mathbb{P}^2$ at 16 points, so $\chi(X) = \chi(\mathbb{P}^2) + 16 = 19$. Thus,

$$\deg(\delta_0) = 27.$$

  △

Since we did not get through the remaining parts of this calculation, we will differ the homework for a week.

42 Lecture 17,18 (3/29,31): Out of town

43 Lecture 19 (4/5)

Three more weeks in the semester apparently. With the time left, we want to develop machinery that will allow us to prove theorems about the geometry of $\overline{M}_g$ even though we do not have a clear picture of what this space looks like for $g \gg 0$.

Example. When $g = 1$, $M_1 \simeq \mathbb{A}^1_j$ is the j-line. When $g = 2$, every genus 2 curve is hyperelliptic, uniquely expressible has a double cover of $\mathbb{P}^1$ branched at 6 points, which we can take to be $0, 1, \infty, \lambda_1, \lambda_2, \lambda_3$. Hence, M_2 is the quotient of $(\mathbb{P}^1)^3$ (the choice of λ_i's) by the action of the symmetric group S_6. △

Last time, we obtained

$$\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} = \mathbb{Q}\left\langle \lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor} \right\rangle$$

using Harer's theorem. We also described a curve $B \cong \mathbb{P}^1 \hookrightarrow \overline{M}_3$ (a general pencil of plane quartics), and calculated the degrees on B of $\lambda, \delta_0, \delta_1$. We had two reasons for doing this

- We wanted to make the divisor classes a little more concrete, show by example that it is possible in practice to compute these things.
- We also want to develop techniques for finding the class of a given divisor (e.g. the canonical/dualizing divisor on $\overline{M}_g$)

  $\overline{M}_g$ is itself mysterious, but it has a lot of subvarieites which are of interest to us.

  Example. When we talked about Brill-noether, we described what linear series a general curve of genus g has. What about the linear series a general curve does not have? Such linear series will generally live on curves coming from some subvariety of $\overline{M}_g$.

  e.g. in $\overline{M}_3$, one has the **hyperelliptic divisor**

  $$H = \overline{\{C \in M_3 : C \text{ hyperelliptic}\}} \subset \overline{M}_3.$$

  Fact. The dimension of the hyperelliptic locus in M_g is $2g - 1$.

  (so H is codimension $\dim M_3 - \dim H = 1$ in $\overline{M}_3$)

  What is the class of H, expressed as a linear combination $H \sim a\lambda_0 - b_0\delta_0 - b_1\delta_1$? One gets information e.g. from the family $B \subset \overline{M}_3$ from last time. Recall

  $$\deg_B(\lambda) = 3, \quad \deg_B(\delta_0) = 27, \quad \text{and} \quad \deg_B(\delta_1) = 0.$$

  Recall 43.1. A curve of genus 3 is either hyperelliptic or a plane quartic ⊙

Hence, $\deg_B(H) = 0$ since B is disjoint from the hyperelliptic locus.[51] Thus, $3a - 27b_0 = 0$. The strategy is now clear; one computes degrees for various subvarieites of $\overline{M}_g$ in order to obtain enough linear relations to pin H down exactly. △

Let's mention some other potential divisors on $\overline{M}_g$:

- Curves with a Weierstrass point of weight 2

  Recall 43.2. Every curve of genus g has $g^3 - g$ Weierstrass points, counted with weights. For a general curve, each of these points will be simple (i.e. have weight 1). ⊙

  Hence, the space of Weierstrass points is a degree $g^3 - g$ cover of M_g. This suggests looking at the branch divisor, i.e. the curves which have a Weierstrass point of weight 2.

- curves w/ "**vanishing Θ-null**," i.e. a line bundle $L \in \operatorname{Pic}^{g-1}(C)$ so that $L^2 \cong K_C$ and $h^0(L) \geq 2$, i.e. curves with a "**semicanonical pencil**."

  This is a divisor, but that shouldn't be immediately obvious.

- Brill-Noether tells us about linear series on general curves. If we look for linear series with negative Brill-Noether number, we'll get a corresponding subvariety of $\overline{M}_g$ (those curves with a g^r_d so that $\rho(g, r, d) = blah$ for some $blah < 0$).

  Fact. If g, r, d are given so that $\rho(g, r, d) = -1$ (e.g. $g = 3, r = 1, d = 2$). Then,

  $$\overline{\{C : C \text{ has a } g^r_d\}} \subset \overline{M}_g$$

  is codimension 1. The resulting divisors are called **Brill-Noether divisors**.

Goal. Understand $\operatorname{Pic}(\overline{M}_g) \otimes \mathbb{Q}$

To keep things simple, let's restrict ourselves to the subspace generated by λ, δ ($\delta = \delta_0 + \cdots + \delta_{\lfloor g/2 \rfloor}$), i.e. only look at classes of the form $a\lambda - b\delta$.

Question 43.3.

- *What is the ample cone?*
- *What is the effective cone?*
- *Where in this picture*[52] *is* $K_{\overline{M}_g}$*?*

  (We'll see that the canonical class $K_{\overline{M}_g}$ *is included in span of* λ, δ*).*
- *Where in this picture are the various divisor classes considered above?*

Answer (The punch line)**.** To help guide us in the next lecture, let's go ahead and state without justification the answer to some of these questions.

[51] This is some ignored subtlty here coming from H really being the *closure* of the hyperelliptic locus. One should understand what singular curves this encompasses in order to justify the claim before this footnote.

[52] Joe drew a plane on the board

- The ample cone is
$$\{a\lambda - b\delta : a > 11b > 0\}.$$
- The effective cone obviously contains all $a\lambda - b\delta$ with $a \geq 0$ and $b \leq 0$ since these are non-negative combinations of an effective divisor ($\delta = -(-\delta)$) and an ample divisor (λ). Hence, there will exist some S_g so that the effective cone is
$$\{a\lambda - b\delta : a \geq 0 \text{ and } a > S_g \cdot b\}.$$
- $K_{\overline{M}_g} = 13\lambda - 2\delta$.

Note that $K_{\overline{M}_g}$ is never ample. If $S_g > 6\frac{1}{2}$, then $K_{\overline{M}_g}$ is outside the effective cone (as are all its powers, so $\overline{M}_g$ will have negative Kodaira dimension). If $S_g < 6\frac{1}{2}$, then $K_{\overline{M}_g}$ will be big, and so $\overline{M}_g$ will be of general type. This is worth repeating

Fact. $\overline{M}_g$ is of general type if $S_g < 6\frac{1}{2}$.

($\overline{M}_g$ can have intermediate Kodaira dimension only if $S_g = 6\frac{1}{2}$) ⋆

How do we get a handle on S_g? You can compute explicit effective divisor classes. Doing so will give an upper bound for S_g since it must be small enough for the class you've computed to end up in the effective cone.

Goal. Calculate classes of effective divisors in $\overline{M}_g$.

Most of the divisors we deal with won't be in this cone, but one can always take the smallest coefficient of one of the boundary divisors δ_i, and use that to bound S_g. This leaves one important question: where are we gonna the necessary test curves (curves in $\overline{M}_g$ used to compute divisor classes à la the plane quartic example from before).

Warning 43.4. On a variety of general type, through a general point, there are no rational curves. Hence, it should not be easy to write down a 1-parameter family of curves of a given large genus $g \gg 0$. •

To get around this, we will work with families of singular curves.

Example. Fix a curve C of genus α and another curve D of genus $g - \alpha$. One gets a stable curve of genus g (assuming $\alpha, g - \alpha \neq 0$) by identifying one point on C with one point on D. Varying these points gives a 2-parameter family of stable curves. We only want a 1-parameter family, so fix a point $q \in D$, and then consider the one parameter family
$$\{C_p = (C \sqcup D)/(p \sim q)\}_{p \in C}.$$
Concretely, start with $C \times D$ and $C \times C$, both thought of as schemes over C. Take the cross section $\Gamma := C \times \{q\} \subset C \times D$ as well as the diagonal $\Delta \subset C \times C$. Identifying $\Gamma \subset C \times D$ with $\Delta \subset C \times C$, we get
$$\mathcal{C} := (C \times D \sqcup C \times C)/(\Gamma \sim \Delta) \longrightarrow C,$$
and this is the total space of the family. △

Example. Fix a curve C of genus $g-1$, and fix a point $q \in C$. Consider the family of curves $C_p = C/(p \sim q)$. Note that we take C_q to be the stable limit of the nearby curves; that is, C_q will look like the curve C with a $\mathbb{P}^1$ with a single node attached. △

Example. Let $\{C_\lambda\}_{\lambda \in \mathbb{P}^1}$ be a pencil of plane cubics. Note that two plane cubics intersect in 9 points, so the total space $\mathcal{C} = \left\{(t,p) \in \mathbb{P}^1 \times \mathbb{P}^2 : p \in C_t\right\}$ (the blowup of $\mathbb{P}^2$ at the 9 points of intersection) will have 9 exceptional divisors E_i ($i = 1, \ldots, 9$). It will also have exactly 12 singular fibers.

Fix some curve D of genus $g-1$ along with a point $q \in D$. Consider the family

$$\{C_\lambda \sqcup D/p \sim q : p \in E_i \cap C_\lambda\}$$

for a fixed i. △

Next time we'll talk about computing the degrees of the boundary components restricted to such singular families.

44 Lecture 20 (4/7)

Goal. For $D \subset \overline{M}_g$ an effective divisor, find the class of D in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q} = \left\langle \lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor} \right\rangle$.

Remark 44.1. Essentially any property of a curve that is not shared by all curves will give a subvariety of the moduli space. Many of these will be codimension 1, and so give divisors. ○

Example. There's the Weierstrass divisor (curves w/ Weierstrass point of weight 2), the theta divisors (curves w/ a semicanonical pencil), and the Brill-Noether divisors (curves w/ a g^r_d where $\rho(g,r,d) = -1$) △

Our proposed method for computing these divisor classes is to make use of test curves $B \subset \overline{M}_g$. Given such a thing, one computes $\deg_B \lambda, \deg_B \delta_\alpha, \deg_B D$ and so deduces a relation of the coefficients a, b_α of the expression $D = a\lambda - \sum_\alpha b_\alpha \delta_\alpha$, i.e. one obtains

$$\deg_B D = a \deg_B \lambda - \sum b_\alpha \deg_B(\delta_\alpha).$$

Enough such relations will let us determine the class of D.

Example ($g = 3$)**.** Consider $H \subset \overline{M}_3$, the (closure of the) locus of hyperelliptic curves.

Warning 44.2. It's not a priori clear which stable curves are 'hyperelliptic' in the sense of belonging to this divisor H. •

Recall the test curve $B \subset \overline{M}_3$ given by a (general) pencil of plane quartics. This had

$$\deg_B \lambda = 3, \quad \deg_B \delta_0 = 27, \quad \text{and} \quad \deg_B \delta_1 = 0.$$

Note that a smooth plane quartic is never hyperelliptic. Note that a general pencil of plane quartics will meet the boundary in general points, and general points of the boundary cannot be contained in the closure of a divisor, so $\deg_B H = 0$. One deduces $a = 9b_0$. △

I'm not sure how to make this rigorous. I think this is just saying that H meets

Example. Fix a curve C_0 of genus α and a curve C_1 of genus $g-\alpha$ with fixed point $r \in C_1$. Consider the family of curves $\{C_q = (C_0 \sqcup C_1)/(q \sim r) : q \in C_0\}$. These are all stable curves (assuming $\alpha \neq 0 \neq g-\alpha$). The total space of the family is

$$\mathcal{C} := \{(C_0 \times C_0 \sqcup C_0 \times C_1)/\Delta \sim \Gamma\} \longrightarrow C_0 \text{ where } \Gamma = C_0 \times \{r\}.$$

$\triangle$

Remark 44.3. Let X be a smooth projective variety, and let $D \subset X$ be a smooth (effective) divisor. Let $C \subset X$ be a curve. If C, D meet transversely, then

$$\deg \mathscr{O}_X(D)|_C =: (C \cdot D) = \#C \cap D.$$

More generally, if $C \cap D$ is finite, then

$$C \cdot D = \sum_{p \in C \cap D} m_p(C \cdot D).$$

What if $C \subset D$? First observe that $\mathscr{O}_X(D)|_D = N_{D/X}$. Hence,

$$(C \cdot D) = \deg \left(N_{D/X}|_C\right).$$

$\circ$

Fact. Say $C \in \overline{M}_g$ has exactly one node, and let $\nu : \widetilde{C} \to C$ be its normalization. Write $\nu^{-1}(p) = \{q, r\}$. Let $\Delta \subset \overline{M}_g$ denote the boundary. Then,

$$\Delta \text{ is smooth at } [C] \text{ and } \left(N_{\Delta/M_g}\right)_{[C]} = T_q\widetilde{C} \otimes T_r\widetilde{C}.$$

(Try to think about why this would be true. We'll prove it next week)

We will need to be able to deal with families which include curves with multiple nodes.

Definition 44.4. If C is a stable curve with node $p \in C$, we define its **index** to be

$$\delta(p) = \begin{cases} 0 & \text{if } C \setminus \{p\} \text{ connected} \\ \alpha & \text{if } C \setminus \{p\} \text{ has two connected components of arithmetic genera } \alpha, g-\alpha. \end{cases}$$

Note that C lives in the boundary component $\delta_{\delta(p)}$. $\diamond$

In general, say C has nodes $p_1, \ldots, p_\delta$. In a neighborhood of $[C] \in \overline{M}_g$, Δ has normal crossings (as a divisor in $\overline{M}_g$) with one branch for each node.

Example. Go back to the example $\mathcal{C} \to C_0$ of gluing a variable $q \in C_0$ (genus α) to a fixed $r \in C_1$ (genus $g-\alpha$). Let $B \cong C_0 \to \overline{M}_g$ be the corresponding test curve. I did not follow why, but the above discussion tells us that

$$\left(N_{\Delta/M_g}\right)|_B = N_{C_0 \times \{r\}/C_0 \times C_1} \otimes N_{\Delta/C_0 \times C_0}.$$

Note that $N_{C_0 \times \{r\}/C_0 \times C_1}$ is the trivial bundle (whose fiber at any point is T_rC_1). The degree of $N_{\Delta/C_0 \times C_0}$

restricted to C_0 is the topological Euler characteristic $\chi(C_0) = 2 - 2\alpha$.[53] Thus,

$$\deg_B \delta_0 = 0, \quad \deg_B \delta_\alpha = 2 - 2\alpha, \quad \text{and} \quad \deg_B \delta_\beta = 0 \text{ for any } \beta \neq \alpha.$$

What is $\deg_B \lambda$? Note that a section of the dualizing of a curve C_q is a meromorphic differential on the normalization with possible a single pole at the points lying over the node. A meromorphic differential on an integral curve can't have a single pole (residues sum to 0), so get space $\mathrm{H}^0(C_0, \omega) \oplus \mathrm{H}^0(C_1, \omega)$ (note this has dimension g). Thus, $\pi_* \omega_{\mathcal{C}/B}$ is the trivial bundle over B with fibers $\mathrm{H}^0(K_{C_0}) \oplus \mathrm{H}^0(K_{C_1})$, and thus $\deg_B \lambda = 0$. △

See Section 38

Note these test curves will pick out one coefficient (unless $\alpha = 1$, i.e. unless you vary a point on a genus 1 curve)

Example (Above when $g = 3$)**.** In a slightly more concrete case, get $B \subset \overline{M}_3$ coming from identifying a variable point q on a genus 2 curve C_0 with a fixed point r on a genus 1 curve C_1. This satisfies

$$\deg_B \lambda = 0, \quad \deg_B \delta_0 = 0, \quad \text{and} \quad \deg_B \delta_1 = -2.$$

What is $\deg_B H$? How many curves in this family are limits of hyperelliptic curves?

Exercise. Think about this. △

Let's see another test curve (useful for picking out δ_0)

Example. Fix a smooth curve C_0 of genus $g - 1$. Fix some $r \in C_0$ and let $q \in C_0$ be variable. Consider the family $\{C_q := C_0/(q \sim r) : q \in C_0\}$ (where C_r is interpreted as the stable limit of nearby curves, C_0 with a nodal tail attached at r). This family has total space

$$\mathcal{C} := \mathrm{Bl}_{(r,r)}(C_0 \times C_0)/(\widetilde{\Delta} \sim \widetilde{\Gamma}) \longrightarrow C_0.$$

Above, $\widetilde{\Delta}$ is the strict transform of the diagonal and $\widetilde{\Gamma}$ is the strict transform of the cross section $C_0 \times \{r\}$. We're running out of time, so let's just state the answer, and see how to get it next time...

Let $B \cong C_0 \to \overline{M}_g$ denote the corresponding test curve. Then,

$$\deg_B \lambda = 0, \quad \deg_B \delta_0 = -1, \quad \deg_B \delta_1 = 1, \quad \deg_B H = 1.$$ △

45 Lecture 21 (4/12)

I think most (but not all) of this lecture was contained in the previous one

This theorem about $\overline{M}_g$ being general type has been around for 40 years, but this is Joe's first time teaching it in a class. It has turned out that there are more technical ingredients needed to work out the argument than he originally imagined, so there will be a few things we take for granted and/or cover briefly for sake of time.

Coming up...

- Grothendieck-Riemann-Roch formula

  This will be used to determine the canonical divisor class on $\overline{M}_g$.

- A bit of deformation theory

[53] The conormal bundle of the diagonal, pulled back to C_0, is the cotangent space Ω^1

- Admissible covers and limit linear series

Recall 45.1.

$$\operatorname{Pic}(\overline{M}_g) \otimes \mathbb{Q} = \mathbb{Q}\left\langle \lambda, \delta_0, \dots, \delta_{\lfloor g/2 \rfloor} \right\rangle. \qquad \odot$$

Given an effective divisor $D \subset \overline{M}_g$, we'd like to be able to compute $a, b_0, \dots, b_{\lfloor g/2 \rfloor} \in \mathbb{Z}$ so that $D \simeq a\lambda - \sum_i b_i \delta_i$.

Example. Consider the hyperelliptic locus $H \subset \overline{M}_3$.

Here's one way one might try to compute the class of this divisor. Use **Porteous' formula**: given a map $\varphi : E^m \to F^n$ of vector bundles on a variety X, what is the class of the locus

$$M_k = \{x \in X : \operatorname{rank} \varphi_x \le k\}?$$

Note that M_k has expected codimension $(n-k)(m-k)$

Porteous expresses the class of M_k in terms of the chern classes of E, F. In the present case, look in $\overline{C}_3 = \overline{M}_{3,1}$ (stable curves (C,p) with a marked point $p \in C$). Note that $\overline{M}_{3,1} \to \overline{M}_3$ "is the universal curve" (literally true for stacks, not literally true for schemes). We want a map of bundles on $\overline{C}_3$ which detects the hyperelliptic locus. Let E be the line bundle on $\overline{C}$ with fibers $E_{(C,p)} = \mathrm{H}^0(K_C)$ (i.e. E is the pullback of the Hodge bundle) and let F be the line bundle on $\overline{C}_3$ with fibers $F_{(C,p)} = \mathrm{H}^0(K_C/K_C(-2p))$. Note that E is rank 3, F is rank 2, and there's a natural map $E \to F$. This map has rank 1 exactly along the hyperelliptic locus. This will indeed have codimension 2 in $\overline{C}_3$ as one expects.[54] Hence, Porteous could be used to compute the class of H, at least in M_3. It would show that H's class in M_3 is a multiple of λ, and so let one deduced (without appeal to Harer) that is class in $\overline{M}_3$ is in the span of λ and the boundary components.

Remark 45.2 (Audience question + Joe's response). For a non-hyperelliptic curve, the multiplication map $\mathrm{H}^0(K_C) \otimes \mathrm{H}^0(K_C) \to \mathrm{H}^0(2K_C)$ is surjective (due to Noether), and the converse holds as well. Can you apply Porteous using this for other values of g? Not quite since Porteous only applies when M_k has expected codimension. The hyperelliptic locus will have codimension $g-2$ which is much smaller than expected here. ○

Instead of a Porteous argument, we will compute H using the method of test curves. That is, we'll describe some curves $B_i \subset \overline{M}_g$ and then compute $\deg_{B_i}(\lambda), \deg_{B_i} \delta_\alpha$, and $\deg_{B_i}(D)$. This gives some system of linear equations which, assuming we choose appropriate B_i's, will determine the coefficients a, b_α of D. On that note, let's leave this example block and talk more about the test curves we'll be using... △

We will use the following test curves

- Fix a curve C_0 of genus α and a curve C_1 of genus $g - \alpha$. Let $q \in C_1$ be a fixed point. For $p \in C_0$, we consider the curve $C_p = C_0 \sqcup C_1/(q \sim p)$. This gives a family

$$B_\alpha = \{C_p\} \subset \Delta_\alpha \subset \overline{M}_g.$$

[54] It's codimension 1 in M_3, and each hyperelliptic curve has 8 points where the rank drops

- Fix a curve C_0 of genus $g-1$ along with a point $q \in C_0$. Then, we get the family

$$\{C_p = C_0/(p \sim q)\}_{p \in C} \subset \Delta_0 \subset \overline{M}_g.$$

(with C_q interpreted as the stable limit of nearby curves)

- Let $\{C_t\}$ be a pencil of plane cubics, say interpolating between $\{F=0\}$ and $\{G=0\}$. Consider the total space

$$\mathcal{E} := \left\{(t,p) \in \mathbb{P}^1 \times \mathbb{P}^2 : t_0F(p) + t_1G(p) = 0\right\},$$

and note that $\mathcal{E} \cong \mathrm{Bl}_{F=G=0}(\mathbb{P}^2)$ is the blowup of $\mathbb{P}^2$ at 9 points. Fix a curve C_0 of genus $g-1$ along with a point $q \in C_0$. Consider the curves $C_t := C_0 \sqcup E_t/(q \sim p)$ (with $t \in \mathbb{P}^1$). Here, p is the point in the fiber E_t of $\mathcal{E} \to \mathbb{P}^1$ meeting a fixed exceptional divisor of $\mathcal{E}$. So we have a family with total space

$$\mathcal{C} := \left(\mathbb{P}^1 \times C \sqcup \mathcal{E}\right)/(\mathbb{P}^1 \times \{q\} \sim E) \longrightarrow \mathbb{P}^1$$

(where $E \subset \mathscr{E} = \mathrm{Bl}_{F=G=0}(\mathbb{P}^2)$ is some fixed exceptional divisor).

Remark 45.3. Say X and $D \subset X$ are both smooth (D a divisor). Say $C \subset X$ is a curve. If $C \not\subset D$, then the intersection number $C \cdot D$ is $\deg_C(D)$ which is the number of points of intersection of $C \cap D$, counted with multiplicity. If $C \subset D$, we still set $C \cdot D := \deg \mathscr{O}_X(D)|_C$. Note that

$$\mathscr{O}_X(D)|_C = \left(\mathscr{O}_X(D)|_D\right)|_C = N_{D/X}|_C$$

is the normal bundle of D in X, restricted to C. ○

Remark 45.4 (The normal bundle to Δ at a point $[C]$). Say C is a stable curve w/ exactly one node $p \in C$. Let $\nu : \widetilde{C} \to C$ be the normalization, and write $\nu^{-1}(p) = \{q, r\}$.

Suppose for the moment that $C \subset S$ is contained in a smooth surface S. The normal bundle of C in S is $\mathscr{O}_C(C) = \mathscr{O}_S(C)/\mathscr{O}_S$. Define

$$N_\nu := \operatorname{coker}\left(\nu_* : T\widetilde{C} \longrightarrow \nu^*TS\right)$$

A section of N_ν near $x \in \widetilde{C}$ looks like a normal vector field to the curve $C \subset S$ along the branch corresponding to the point x. I'm not really sure what Joe's saying, but apperently one can convince themself that the sections of the normal bundle $\mathscr{O}_C(C)$ of the singular curve are rational sections of N_ν with possibly simple poles at q, r. This in turn says

$$(N_{C/S})_p/(N_\nu)_q = T_r(\widetilde{C}) \otimes \mathscr{O}_{\widetilde{C}}(q)/\mathscr{O}_{\widetilde{C}} = T_r(\widetilde{C}) \otimes T_q(\widetilde{C})$$

The left hand side is $(N_{\Delta/M_g})_{[C]}$, deformations that preserve C modulo those perserving the node.

I think this whole discussion was not meant to be close to a rigorous computation, but more of a hint at why $(N_{\Delta/M_g})_{[C]} = T_r(\widetilde{C}) \otimes T_q(\widetilde{C})$. ○

Example. Consider the test curve B_α from before. Convince yourself that

$$\left(N_{\Delta/\overline{M}_g}\right)|_{B_\alpha} = N_{\Delta/C_0 \times C_0} \otimes N_{\Gamma/C_0 \times C_1} \text{ where } \Gamma = C_0 \times \{q\}.$$

Note that $\Gamma \cdot \Gamma = 0$ (it's a fiber) while $\deg N_{\Delta/C_0 \times C_0} = 2 - 2\alpha$. Hence,

$$\deg_{B_\alpha}(\delta_\alpha) = 2 - 2\alpha \text{ while } \deg_{B_\alpha}(\delta_\beta) = 0 \text{ for all } \beta \neq \alpha.$$

One can also show that $\deg_{B_\alpha}(\lambda) = 0$. △

46 Lecture 22 (4/14)

Let's keep in mind a summary of (some of) the test curves we've seen and their degrees

Example.

- There's B_α coming from gluing a variable point of a curve C_0 of genus α to a fixed point of a curve C_1 of genus $g - \alpha$. This has

$$\deg_{B_\alpha}(\delta_\beta) = 0 \text{ if } \beta \neq \alpha, \quad \deg_{B_\alpha}(\delta_\alpha) = 2 - 2\alpha, \text{ and } \deg_{B_\alpha}(\lambda) = 0.$$

- There's B_1 coming from gluing a fixed point q of a curve C_0 of genus $g - 1$ to a variable point p of the same curve. This has[55]

$$\deg_{B_1}(\delta_\alpha) = 0 \text{ if } \alpha \geq 2, \quad \deg_{B_1}(\delta_1) = 1, , \quad \deg_{B_1}(\delta_0) = \widetilde{\Delta}^2 + \widetilde{\Gamma}^2 = 2 - 2g, \text{ and } \deg_{B_1}(\lambda) = 0.$$

Potential notation clash with B_α, except $B_{\alpha=1}$ it not useful (all degrees are 0)

  For the Hodge bundle computation, differentials on a fiber look like meromorphic differentials on C_0 (the normalization) with simple poles at p, q. Get a trivial rank $g - 1$ subbundle by considering holomorphic differentials on C_0. The quotient is determined by the residue at either point p, q. Since q is fixed, the quotient will be a trivial line bundle.

- To get a test curve with non-trivial Hodge class, we start with a pencil of plane cubics E_t (one of the simplest families with non-trivial Hodge class). Let B_0 come from gluing a fixed point of a curve C_0 of genus $g - 1$ to an element of a general pencil E_t of plane cubics. This has

$$\deg_{B_0}(\delta_\alpha) = 0 \text{ if } \alpha \geq 2, \quad \deg_{B_0}(\delta_1) = \Gamma^2 = -1, \quad \deg_{B_0}(\delta_0) = 12, \text{ and } \deg_{B_0}(\lambda) = 1.$$

  Note that the total space of the family E_t is the blowup of $\mathbb{P}^2$ at 9 points, and we get a cross section Γ by using one of the exceptional divisors (Γ gives a choice of basepoint in each fiber which we use for gluing).

These provide enough test curves to compute the class of any effective divisor $D \subset \overline{M}_g$, assuming we can compute the degrees of D restricted to each of these curves. △

Remark 46.1. For carrying out our plan of computing divisor classes like this, we don't need to use test *curves* per se. We could use higher dimensional subvarieties as long as we know enough about their Chow rings. ○

[55]For δ_1 computation, apparently if you have a family which is generically smooth whose total space is smooth, then when you acquire a node, it appears with multiplicity one. Maybe something like this came up earlier? Also, blowing up a point will decrease self-intersection by 1

Example. Get a 4-dimensional subvariety of $\overline{M}_g$ by fixing a pointed curve $q \in C_0$ of genus $g-2$ and gluing it to any pointed curve $p \in C$ of genus 2 ($\dim \overline{M}_{2,1} = (3 \cdot 2 - 3) + 1 = 4$). One could use a family such as this instead of test curves. We won't. △

With that covered, we want to talk today about Chern classes and Grothendieck-Riemann-Roch.

46.1 Chern classes

One can develop a well-behaved theory of Chern classes in multiple settings (CW complexes, (complex) manifolds, algebraic varieties, etc.). We'll discuss an approach that works for complex manifolds/varieties.

Question 46.2. *Say we're given a rank n complex vector bundle E over a space X. When is E trivial?*

Answer. E is trivial $\iff$ there exists sections $\sigma_1, \ldots, \sigma_n$ of E which are everywhere independent. ⋆

Instead of looking for n linearly independent sections, can start with the simplest case of looking for a single section which is every linearly independent (i.e. nowhere vanishing).

Idea. Choose a section σ of E and associate to it the class $[V(\sigma)]$ of its 0-locus in the Chow ring (specifically, in $A^n(X)$). This class will be independent of the choice of section (assuming e.g. the image of σ is transverse to the zero locus).

If τ is another section, can consider $\sigma_t := t_0\sigma + t_1\tau$ and show that $[V(\sigma_t)]$ is independent of t (and so of σ).

We call $c_n(E) := [V(\sigma)]$ the **top Chern class of** E.

Definition 46.3. In general, if $\sigma_1, \ldots, \sigma_k$ are sections of E, they fail to be linearly independent exactly along $V(\sigma_1 \wedge \cdots \wedge \sigma_k)$. It's class

$$c_{n-k+1}(E) := [V(\sigma_1 \wedge \cdots \wedge \sigma_k)] \in A^{n-k+1}(X)$$

is called a **Chern class** of E. ◇

Remark 46.4. If $k = n$ above, the locus where the section fail to be linearly independent is given by the vanishing of the determinant of an $n \times n$ matrix. This will have codimension 1 and cut a representative of the first Chern class. ○

Definition 46.5. The **total Chern class** of E is

$$c(E) := 1 + c_1(E) + c_2(E) + \cdots + e_n(E) \in A^*(X). \quad \diamond$$

Fact.

- Given $f : X \to Y$ and E a vector bundle on Y,

$$c(f^*E) = f^*c(E).$$

- If $E = \mathscr{O}_X(D)$ is a line bundle, then $c_1(E) = [D] \in A^1(X)$

- **Whitney product formula**
$$c(E \oplus F) = c(E)c(F)$$
for any vector bundles E^m, F^n on X, e.g. $c_1(E \oplus F) = c_1(E) + c_1(F)$ and $c_{n+m}(E \oplus F) = c_m(E)c_n(F)$.

  Exercise. Without appealing to the product formula, try justifying the equality
$$c_{n+m-1}(E \oplus F) = c_m(E)c_{n-1}(F) + c_{m-1}(E)c_n(F).$$

Sounds like this and more is worked out in chapter 5 of '3264 and all that'

Fact. On a smooth variety X, can extend the definition of Chern classes to arbitrary coherent sheaves.

Start with a sheave $\mathscr{F}$, and then find a resolution by vector bundles
$$0 \longrightarrow \mathscr{E}_k \longrightarrow \cdots \longrightarrow \mathscr{E}_1 \longrightarrow \mathscr{E}_0 \longrightarrow \mathscr{F} \longrightarrow 0$$
(will be finite length by the Hilbert syzygy (spelling?) theorem). The Chern class of $\mathscr{F}$ is then defined by what is must be to satisfy Whitney, an alternating product of Chern classes of the $\mathscr{E}_i$'s.

46.2 Riemann-Roch

Example (Riemann-Roch for curves). Let C be a smooth curve of genus g.

- Then, $\chi(\mathscr{O}_C) = 1 - g$. Equivalently, $\chi(\mathscr{O}_C) = \frac{1}{2}c_1(T_C)$.
- For $\mathscr{O}_C(D)$ (say D effective), get sequence $0 \to \mathscr{O}_C \to \mathscr{O}_C(D) \to \mathscr{O}_C(D)|_D \to 0$ from which one concludes
$$\chi(\mathscr{L}) = c_1(\mathscr{L}) + \frac{1}{2}c_1(T_C).$$
- Now say $\mathscr{E}$ is a vector bundle of rank 2. Can find an exact sequence $0 \to L \to E \to M \to 0$ and so get
$$\chi(E) = \chi(L) + \chi(M) = c_1(E) + (\operatorname{rank} E)c_1(T_C)/2.$$
  This will hold for arbitrarily ranked vector bundles on C.
- In fact, for any coherent sheaf $\mathscr{F}$ on the smooth curve C, one has
$$\chi(\mathscr{F}) = c_1(\mathscr{F}) + \operatorname{rank}(\mathscr{F}) \cdot \frac{c_1(T_C)}{2}.$$

The first bullet point is non-trivial, but once you have it, everything else follows without too much effort. △

Example (Riemann-Roch for surfaces). Let S be a smooth projective surface. We start with **Noether's formula**
$$\chi(\mathscr{O}_X) = \frac{c_1^2(T_S) + c_2(T_S)}{12}$$
(again, this is not easy). Once you have it though, you can obtain a Riemann-Roch for an arbitrary coherent sheaf on S.

- Let $L = \mathscr{O}_S(D)$, say with D effective. Consider $0 \to \mathscr{O} \to L \to L|_D \to 0$. One concludes that

$$\chi(L) = \frac{c_1^2(L) + c_1(L)c_1(T_S)}{2} + \frac{c_1^2(T_S) + c_2(T_S)}{12}.$$

- Can now look at higher rank vector bundles. It is no longer the case that you can always get a line subbundle with locally free quotient, but we can have Chern classes for arbitrary coherent sheaves. One shows you can always fit a coherent sheaf into a short exact sequence with lower rank coherent sheaves. This allows one to do the induction, and so conclude that for any arbitrary coherent sheaf $\mathscr{F}$ on S, one has

$$\chi(\mathscr{F}) = \frac{1}{2}\left[c_1^2(\mathscr{F}) - 2c_2(\mathscr{F}) + c_1(\mathscr{F})c_1(T_s)\right] + \operatorname{rank}(\mathscr{F})\chi(\mathscr{O}_S). \qquad \triangle$$

We'd like to extend this to varieties of arbitrary dimension. As the formulas get more hairy when you increase dimensional, we'll need to introduce more abstraction/notation in order to be able to make a succinct statement.

Say $\mathscr{F}$ is a coherent sheaf on X of rank r. Write

$$c(\mathscr{F}) = \prod_{i=1}^{r}(1+\alpha_i)$$

(for now, think of the α_i's as formal symbols satisfying $\alpha_1 + \cdots + \alpha_r = c_1(\mathscr{F})$, yadda yadda, $\alpha_1 \dots \alpha_r = c_r(\mathscr{F})$). Define the **Chern character**

$$\operatorname{ch}(\mathscr{F}) := \sum_i e^{\alpha_i} = \sum_i \left(1 + \alpha_i + \frac{\alpha_i^2}{2} + \frac{\alpha_i^3}{6} + \dots\right).$$

Remark 46.6. The coefficients of $\operatorname{ch}(\mathscr{F})$ will be symmetric polynomials in the α_i's, and so can be written in terms of the elementary symmetric polynomials of the α_i's, i.e. in terms of the Chern classes. ◦

Definition 46.7. The **Todd class** of $\mathscr{F}$ is

$$\operatorname{Td}(\mathscr{F}) := \prod_i \frac{\alpha_i}{1-e^{-\alpha_i}}$$

This will given another formal power series in $\alpha_1, \dots, \alpha_r$ whose coefficients are symmetric (so expressible in terms of Chern classes). ⋄

Example.

$$\operatorname{Td}(\mathscr{F}) = 1 + \frac{1}{2}c_1(\mathscr{F}) + \frac{c_1^2(\mathscr{F}) + c_2(\mathscr{F})}{12} + \frac{c_1(\mathscr{F})c_2(\mathscr{F})}{24} + \dots \qquad \triangle$$

The Chern character and Todd classes are ways of repackaging the information of the Chern classes.

Theorem 46.8 (**Atiyah-Singer-Bott-Hirzebruch-Riemann-Roch**, up to spelling)**.** *Let $\mathscr{F}$ be a coherent sheaf on a smooth projective variety X. Then,*

$$\chi(\mathscr{F}) = \left\{\operatorname{ch}(\mathscr{F}) \cdot \operatorname{Td}(T_X)\right\}_n,$$

the nth graded piece of this product (evaluated against the class of a point to get a number instead of a Chow class).

Where does Grothendieck fit into all of this? Briefly, he made things relative. We'll say more on Tuesday.

47 Lecture 23 (4/19)

Last time we talked about Chern classes and various Riemann-Roch formulae. We were just about to state Grothendieck Riemann-Roch when we ran out of time. Today, we'll finish up this discussion, and then see two applications of Grothendieck-Riemann-Roch: deriving Mumford's relation, and computing the canonical class of $\overline{M}_g$.

Recall 47.1. Let $E \to X$ be a rank n complex vector bundle. Let $\sigma_1, \dots, \sigma_{n-k+1}$ be sections of E. Then, the locus $V(\sigma_1 \wedge \cdots \wedge \sigma_{n-k+1})$ where these sections fail to be independent represents the k**th Chern class**

$$c_k(E) = [V(\sigma_1, \wedge \cdots \wedge \sigma_{n-k+1})] \in A^k(E).$$

Think of these as obstructions to finding everywhere linearly independent sections. ⊙

History. Let's talk a bit about how Chern classes were first defined, before Chern. This was done by Whitney, and we'll see in particular the definition of the top Chern class $c_n(E)$. Say X is a simplicial complex. We want to find an everywhere nonzero section σ of E, a rank n topological vector bundle over X. Say $n = 1$ to keep things simple (E a line bundle). Whitney's idea was to attempt to do so inductively over successive skeleta of X.

- First choose value of σ on the 0-skeleton X_0. Just pick nonzero points in each relevant fiber.
- Want to extend to X_1 next. Consider a 1-simplex I of X. We have already fixed the values of σ on the endpoints of I. Since $\mathbb{C}^\times$ is path-connected, we know we extend extend a map $\partial I \to \mathbb{C}^\times$ to a map $I \to \mathbb{C}^\times$. Hence, we're in good shape.
- Now imagine a 2-simplex Δ of X. We already have a map $\partial\Delta \to \mathbb{C}^\times$, and we want to extend this to a map $\Delta \to \mathbb{C}^\times$. This is possible iff the loop $\partial\Delta \to \mathbb{C}^\times$ is contractible, is trivial in $\pi_1(\mathbb{C}^\times) \simeq \mathbb{Z}$.

  Looking at the value of these various 2-simplicies, we get a 2-cocycle valued in $\pi_1(\mathbb{C}^\times) \simeq \mathbb{Z}$ of X. One can check that this is in fact a 2-cochain. Making different choices earlier on would give rise to a different 2-cochain, but one that differs by a coboundary. Thus, we end up with a well-defined obstruction class in $\mathrm{H}^2(X;\mathbb{Z})$. This is the first Chern class.

For a rank n vector bundle, you are trying to extend maps to $\mathbb{C}^n \setminus 0 \simeq S^{2n-1}$. Thus, this will be possible until you reach the $2n$-simplices, and then you get a well-defined (simplicial) cohomology class in $\mathrm{H}^{2n}(X;\mathbb{Z})$; this will be the top Chern class $c_n(E)$. ⊖

Exercise. Think about why this definition agrees with the earlier one given in terms of the vanishing locus of a nonzero section (in the case that X is a complex manifold).

Remark 47.2. The definition we've been working with says we can think of Chern classes as classes of degeneracy loci of bundle maps $\mathscr{O}^k \to E$ (where this map fails to be injective). ∘

Why look at these particular bundle maps?

Question 47.3. *Above we take $k \le n =: \operatorname{rank} E$. What if we took $k \ge n$ and asked where $\mathscr{O}^k \to E$ fails to be surjective?*

Answer. This gives the Segre classes $s_k(E) = \left[\text{deg loc of } \mathscr{O}^{n+k-1} \to E\right]$. These turn out to be expressible in terms of the Chern classes (and vice versa). If one sets $s(E) = 1 + s_1(E) + \dots$ the toal Segre class, then one has

$$c(E)s(E^*) = 1.$$

⋆

More generally, if $\varphi : E \to F$ is any (suitably transverse) map of vector bundles, we can associate to it the locus

$$M_k(\varphi) := \{x \in X : \operatorname{rank} \varphi_x \le k\},$$

and then consider the class $[M_k(\varphi)]$.

Fact. $[M_k(\varphi)]$ is expressible in terms of $c(E), c(F)$.[56]

Now, let's consider the *Chern character*

Note the nth order term is $\frac{1}{n!}\left(\sum \alpha_i^n\right)$

$$\operatorname{ch}(E) := \sum e^{\alpha_i} = \operatorname{rank}(E) + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \dots,$$

where the elementary symmetric polynomials in the α_i's are the Chern classes of E. Also consider the *Todd class*

$$\operatorname{Td}(E) := \prod \frac{\alpha_i}{1 - e^{-\alpha_i}} = 1 + \frac{c_1(E)}{2} + \frac{c_1(E)^2 + c_2(E)}{12} + \dots.$$

Todd arrived at the definition of his class from considering the Riemann-Roch formula.

Recall 47.4 (Riemann-Roch)**.** Let X be a smooth projective n-dimensional complex variety. For a coherent sheaf $\mathscr{F}$ on X, one has

$$\chi(\mathscr{F}) = \{\operatorname{ch}(\mathscr{F}) \operatorname{Td}(T_X)\}_n.$$

In particular, $\chi(\mathscr{O}_X) = \{\operatorname{Td}(T_X)\}_n$. ⊙

The goal for Todd was defining a class so that $\chi(\mathscr{O}_X) = \{\operatorname{Td}(T_X)\}_n$.

History (what Todd did)**.** He's looking for a polynomial of degree n in the Chern classes (that will compute $\chi(\mathscr{O}_X)$). The number of such polynomials is the number $p(n)$ of partitions of n. So figuring out this polynomial is basically a matter of linear algebra. For X any product of projective spaces, we know $\chi(\mathscr{O}_X) = 1$ ($h^0 = 1$ and no higher cohomology). Each such X gives a relation on the coefficients of what we'll become the nth graded piece of the Todd class. There are $p(n)$ such products of projective spaces of dimension n. Todd went ahead and solved the resulting system of linear equations, and then afterwards saw that the solutions fit together into the expression $\operatorname{Td}(X) = \prod \frac{\alpha_i}{1 - e^{-\alpha_i}}$. ⊖

With that, let's transition from our discussion of classical Riemann-Roch to Grothendieck's more modern version.

Remark 47.5. Think of Riemann-Roch as a formula for computing $h^0(\mathscr{F})$, with error terms coming from the higher cohomology of $\mathscr{F}$. ∘

[56] I imagine this follows e.g. from computing the cohomology of $BU(n)$

Suppose now that we have a morphism $\pi : X \to B$ and a sheaf $\mathscr{F}$ on X. We think of this as a family of sheaf $\mathscr{F}_b$ on X_b on varieties parameterized by B. We would like an object which measures how the spaces $\mathrm{H}^0(X_b, \mathscr{F}_b)$ of fibral global sections changes as we move along B. The best choice for such an object is the direct image sheaf $\pi_*\mathscr{F}$ on B.

Warning 47.6. $\pi_*\mathscr{F}$ won't be a vector bundle with fibers $\mathrm{H}^0(X_b, \mathscr{F}_b)$ in general. e.g. this won't be the case is the dimensions of these h^0's jumps as you vary $b \in B$. •

Theorem 47.7 (**Grothendieck-Riemann-Roch**).

$$\sum_{i\ge 0}(-1)^i \operatorname{ch}\left(R^i\pi_*\mathscr{F}\right) = \pi_*\left(\operatorname{ch}(\mathscr{F})\cdot \operatorname{Td}\left(T_{X/B}\right)\right).$$

$T_{X/B}$ the relative tangent bundle, not the dual of the dualizing sheaf

Equivalently,

$$\operatorname{Td}(T_B)\sum_{i\ge 0}(-1)^i \operatorname{ch}\left(R^i\pi_*\mathscr{F}\right) = \pi_*\left(\operatorname{ch}(\mathscr{F})\operatorname{Td}(T_X)\right)$$

For my own benefit, here's an exposition I like.

Let's see some applications.

Example (**Mumford's relation**). Let $\mathcal{C} \xrightarrow{\pi} B$ be a family of smooth curves. Recall the classes λ, κ, δ from before. In particular, recall $\lambda = c_1(\pi_*\omega_{\mathcal{C}/B})$. Note that, by duality, one has $R^1\pi_*\omega_{\mathcal{C}/B} = (\pi_*\mathscr{O}_{\mathcal{C}})^\vee = \mathscr{O}_B$. Grothendieck-Riemann-Roch gives

$$\lambda = \left\{\pi_*\left(\operatorname{ch}(\omega_{\mathcal{C}/B})\cdot \operatorname{Td}(T_{\mathcal{C}/B})\right)\right\}_1$$

Set $\gamma = c_1(\omega_{\mathcal{C}/B})$. The above reads (note $T_{\mathcal{C}/B} = \omega_{\mathcal{C}/B}^\vee$ since $\mathcal{C} \to B$ smooth)

$$\begin{aligned}\lambda &= \left\{\pi_*\left(\left(1+\gamma+\frac{\gamma^2}{2}+\dots\right)\left(1-\frac{\gamma}{2}+\frac{\gamma^2}{12}+\dots\right)\right)\right\}_1\\ &= \pi_*\left(\frac{\gamma^2}{2}-\frac{\gamma^2}{2}+\frac{\gamma^2}{12}\right) = \frac{\kappa}{12}\end{aligned}$$

What if the family has isolated singular fibers? Away from the singular locus, the relative tangent bundle is still the dual of the relative dualizing sheaf. However, at the points where there are singular fibers, $T_{\mathcal{C}/B}$ is no longer locally free. This will mean that when we take its second Chern class, we'll pick up the fundamental class of the singular locus of π. Hence, one will end up with $\lambda = (\kappa+\delta)/12$. △

Example (Computing $K_{\overline{M}_g}$). GRR applies to pushforwards. How can be describe the cotangent space to $\overline{M}_g$ in terms of sheaves on the (universal) curve itself? For this, we'll quote some deformation theory.

Fact. Let X be a smooth variety (or one with only "mild singularities"). Then,

$$\{\text{1st order deformations of } X\} \simeq \mathrm{H}^1(X, T_X),$$

where a **1st order deformation of** X is a family $\mathcal{X}/\operatorname{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ whose fiber X_0 over the reduced point $\operatorname{Spec}\mathbb{C} \hookrightarrow \operatorname{Spec}\mathbb{C}[\varepsilon]/(\varepsilon^2)$ is isomorphic to X.

Recall 47.8. If $X \subset Z$, the first order deformations of X as a subscheme of Z are given by $\mathrm{H}^0(X, N_{X/Z})$. ⊙

Note that we have an exact sequence $0 \to T_X \to T_Z|_X \to N_{X/Z} \to 0$, and the induced map $\mathrm{H}^0(X, N_{X/Z}) \to \mathrm{H}^1(X, T_X)$ is exactly the map sending an embedded deformation in Z to the underlying abstract deformation of X.

Imagine we have a universal curve $\mathcal{C}_g \to M_g$. The upshot of the above fact is that then

$$T_{M_g} = R^1\pi_* T_{\mathcal{C}_g/M_g}.$$

Similarly, the dual of the space of 1st order deformations is $\mathrm{H}^0(X, T_X^* \otimes \omega_X)$, so $T_{M_g}^* = \pi_*\left(T_{\mathcal{C}_g/M_g}^* \otimes \omega_{\mathcal{C}_g/M_g}\right)$. We want to use GRR to compute the first Chern class of this relative dualizing sheaf.

We only have one minute left, so we will not carry this out today. $\triangle$

48 Lecture 24 (4/21)

Today: finish calculation of $K_{\overline{M}_g}$ and introduce admissible curves.

Recall 48.1 (GRR)**.** Say $\pi : X \to B$ and $\mathscr{F}$ on a sheaf on X. Then,

$$\sum (-1)^i \operatorname{ch}\left(R^i\pi_*\mathscr{F}\right) = \pi_*\left(\operatorname{ch}(\mathscr{F}) \operatorname{Td}(T_{X/B})\right). \qquad \odot$$

Recall 48.2 (fact from deformation theory)**.**

$$\{\text{1st order deformations of } X\} \simeq \mathrm{H}^1(X, T_X).$$

(apparently originally due to Kodaira and Spencer, in the setting of complex compact manifolds) $\odot$

Note that above is saying that

$$T_{M_g} = R^1\pi_*(T_{\mathcal{C}_g/M_g}),$$

with $\mathcal{C}_g \to M_g$ the (imagined) universal curve. By duality theory, this is equivalently saying

$$T_{M_g}^* = \pi_*\left(T_{\mathcal{C}_g/M_g}^* \otimes \otimes \omega_{\mathcal{C}_g/M_g}\right)$$

(note that, on each fiber, $T_{\mathcal{C}_g/M_g}^* \otimes \omega_{\mathcal{C}_g/M_g}$ restricts to a torsion-free rank 1 sheaf of degree $4g-4$ so its higher cohomology will vanish). Now,

Maybe need $g \gg 0$. I haven't done the computation

$$K_{\overline{M}_g} = c_1(T_{M_g}^*) = \left\{\pi_*\left(\operatorname{ch}(\Omega \otimes \omega) \cdot \operatorname{Td}(T_{\mathcal{C}/M})\right)\right\}_1 \text{ where } \Omega := T_{\mathcal{C}/M}^*.$$

Introduce the notation

$$\gamma = c_1(\omega_{\mathcal{C}/M}) \in A^1(\mathcal{C}) \text{ and } \eta = [\text{nodes}] \in A^2(\mathcal{C}).$$

With these defined we have,

$$\begin{aligned}\operatorname{Td}(T_{\mathcal{C}/M}) &= 1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12} + \dots \\ \operatorname{ch}(\Omega \otimes \omega) &= 1 + 2\gamma + (2\gamma^2 - \eta) + \dots\end{aligned}$$

(note $T_{\mathcal{C}/M}, \Omega \otimes \omega$ are line bundles away from the nodes of the fibers. This is apparently useful in figuring out the right expressions above in the presences of nodes, but I'm not 100% sure how one computes these things)

We now compute

$$\pi_* \left[\left(1 - \frac{\gamma}{2} + \frac{\gamma^2 + \eta}{12}\right) \left(1 + 2\gamma + (2\gamma^2 - \eta)\right) \right] = (3g-3) + \pi_* \left[\frac{13}{12}\gamma^2 - \frac{11}{12}\eta \right] = (3g-3) + \frac{13}{12}\kappa - \frac{11}{12}\delta.$$

Thus, using the Mumford relation $\lambda = (\kappa + \delta)/12$, we conclude that

$$K_{\overline{M}_g} = c_1(T_M^*) = \frac{13}{12}\kappa - \frac{11}{12}\delta = 13\lambda - 2\delta.$$

Recall 48.3. In the subgroup $\{a\lambda - b\delta\} \subset \operatorname{Pic}(\overline{M}_g) \otimes \mathbb{Q}$, the **ample cone** is

$$\{a\lambda - b\delta : a > 11b > 0\}.$$

The **effective cone** will look like everything to the left of a line of some slope S_g (and in the first or second quadrant), i.e. it is

$$\{a\lambda - b\delta : a > S_g b > 0\},$$

assuming I'm following.

Remark 48.4. Anything effective is a sum of something ample and something effective, and so is big. ○

From our computation of $K_{\overline{M}_g}$, we conclude that $K_{\overline{M}_g}$ will be big if $S_g < 6.5$. ⊙

Warning 48.5. The singular locus of M_g is codimension 2, so it does not affect divisor class calculations. However, while $K_{\overline{M}_g}$ being big will give a lot of sections of the (rational) canonical divisor class, when these get pulled back to a desingularization of $\overline{M}_g$, they may no longer be (different) regular differential forms on the desingularization. If they are, one says that the singularities of $\overline{M}_g$ are canonical. We will take for granted that this is the case. •

Open Question 48.6. *What is the value of S_g in general?*

To show that $S_g < 6\frac{1}{2}$ for $g \gg 0$, we will compute the classes of effective divisors, and use these results to bound S_g from above. Typically, these classes won't lie in the span of δ, λ; they will look like $a\delta - \sum_i b_i\delta_i$ (usually $b_0 < b_1, \dots, b_{\lfloor g/2 \rfloor}$ if I heard correctly). However, we can still define the **slope** of D to be

$$\operatorname{slope}(D) := \frac{a}{\min(b_i)} =: s(D),$$

and then define

$$S_g := \min_{D \subset \overline{M}_g} s(D).$$

Recall 48.7. If we find some D with $s(D) < 6.5$, then this means we can write M_g as a sum of an effective divisor and a non-negative linear combination of boundary components. This suffices to show that it is big. ⊙

Remark 48.8 (Assuming I heard correctly). For any effective divisor anyone's actually computed, it always works out that b_0 is the smallest coefficient among the b_i's. No one's proven this must always be the case. ○

The plan now is to

- find a lot of effective divisors
- calculate their classes
- find one of minimal slope

Example (divisors on M_g)**.**

- curves w/ a Weierstrass point of weight ≥ 2. There are two kinds of these.

  These gives divisors with slope $\to 9$ as $g \to \infty$.

- curves w/ semicanonical pencil.

  Example ($g = 4$)**.** The canonical model of genus 4 curve is the complete intersection of a quadric and a cubic in $\mathbb{P}^3$. It has exactly two g^1_3's if the quadric is smooth. In this case, they are distinct, so there's no semicanonical pencil. However, when the quadric is singular, the g^1_3 cut out by the ruling of that singular cone is a semicanonical pencil. △

  This gives a divisor with slope $\to 8$ as $g \to \infty$.

- curves w/ a g^r_d where $\rho(g, r, d) = -1$.

  This gives a divisor with slope $\to 6$ as $g \to \infty$. Specifically, it will have slope $6 + \frac{12}{g+1}$ which is $< 6\frac{1}{2}$ when $g > 23$ (and is $= 6\frac{1}{2}$ when $g = 23$).

Hence, the Brill-Noether divisors will be the only ones we need to look at for our purposes. △

Warning 48.9. $\rho(g, r, d) = g - (r + 1)(g - d + r)$ cannot be equal to -1 for all values of g. In particular, $\rho(g, r, d) = -1 \implies g + 1$ is composite. When $g + 1$ is prime, there will be another divisor class (we haven't yet introduced) that one can look at instead. •

Fact. Brill-Noether divisors have class

$$c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right)$$

with c a constant given by an explicit but complicated expression. In particular, every Brill-Noether divisor class on $\overline{M}_g$ has the same class, up to scaling.[57]

Recall 48.10. Given $D \subset \overline{M}_g$, $D \sim a\lambda - b_0\delta_0 - b_1\delta_1 - \dots$, can compute a, b_i by calculating $\deg_{B_\alpha}(D)$ for all our test curves B_α. See Section 46 for a reminder of what these curves are. ⊙

Question 48.11. *How do we calculate $\deg_{B_\alpha}(D)$? In particular, which points of $\overline{M}_g$ are in the closures of our divisors $D \subset M_g$, i.e. which stable curves are limits of smooth curves $[C] \in D$?*

[57]Note e.g. that if $g = 23$, then $g + 1 = 24$ has multiple factorizations so you get several different Brill Noether divisors. Nevertheless, their divisor classes all lie in the same 1-dimensional subspace.

Example. In $\overline{M}_3$, consider $\overline{H} := \overline{\{\text{hyperelliptic curves}\}}$? What is $\deg_{B_1}(\overline{H})$? Recall elements of B_1 are nodal unions of curves of genera $1, 2$.

To answer this, need a way to describe limits of linear series on smooth curves C_t as they specialize to a singular curve C_0. It's unknown how to do this in general, but there are two ways which each apply in certain circumstances

- admissible curves (applies only to 1-dimensional linear series, pencils)
- limit linear series (applies only on curves of compact type, i.e. those whose dual graphs are trees) △

48.1 Admissible Covers

Recall 48.12 (small Hurwitz scheme). This was

$$\mathcal{H}_{d,g} = \left\{(C, f : C \to \mathbb{P}^1) : C \text{ smooth of genus } g \text{ and } C \xrightarrow{f} \mathbb{P}^1 \text{ simply branched degree } d \text{ cover}\right\} \qquad \odot$$

This fits into a natural diagram

$$\begin{tikzcd} & \mathcal{H}_{d,g} \arrow[dl] \arrow[dr] & \\ M_g & & (\mathbb{P}^1)^b \setminus \Delta, \end{tikzcd}$$

where $b = 2d + 2g - 2$ is the number of branched points, and Δ is the "big diagonal" (locus where two points coincide). The right map above is a covering space, so $\mathcal{H}_{d,g}$ has the structure of a b-dimensional variety. This diagram can be used to compute $\dim M_g$ and even, with some more work, to show that it is irreducible.

Question 48.13. *Can we describe the closure of the image of the left map?*

We will do so by compactifying $\mathcal{H}_{d,g} \overset{\text{open}}{\subset} \overline{\mathcal{H}}_{d,g}$ so that the map $\mathcal{H}_{d,g} \to M_g$ extends to a regular map $\overline{\mathcal{H}}_{d,g} \to \overline{M}_g$. Then, the closure of the image of $\mathcal{H}_{d,g}$ is just the image of $\overline{\mathcal{H}}_{d,g}$.

Remark 48.14. Instead of letting branch points come together (compactifying $(\mathbb{P}^1)^b \setminus \Delta$ to $(\mathbb{P}^1)^b$), we'll construct $\overline{\mathcal{H}}_{d,g}$ so that it lives over $\overline{M}_{0,b} \supset (\mathbb{P}^1)^b \setminus \Delta$. One way of thinking of this is that when points come together, we separate them by blowing up. ○

49 Lecture 25 (4/26): Last Lecture, but I'm out of town (whoops)

I don't know what all was in this last lecture, but you can check out this paper for the 'admissible covers' compactifying $\mathcal{H}_{d,g}$ and the details of finishing the computation

50 List of Marginal Comments

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