Motives at p Notes

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These are my course notes for "Motives at p" at Harvard. Each lecture will get its own 'chapter'. These notes are live-texed and so likely contain many mistakes. Furthermore, they reflect my understanding (or lack thereof) of the material as the lecture was happening, so they are far from mathematically perfect.[1] Despite this, I hope they are not flawed enough to distract from the underlying mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Elden Elmanto, and the course website can be found by clicking this link. The course website includes a set of notes for all the lectures. Before the start of the semester, there was apparently a 3-lecture minicourse on prismatic cohomology. Notes for this are on the course website. Maybe if I feel like it, at some point, I'll read these and type up notes here. Maybe. Finally, I feel like for this class moreso than on average, you should not assume that what I have written down in these notes is correct as is.

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Lecture 1 (9/1/2022)

Note 1. There's a big drawing of some spectral sequence (the motivic spectral sequence $\mathrm{H}^{i-j}_{\mathrm{mot}}(X,\mathbb{Z}(-j)) \implies K_{-i-j}(X)$ on the board with many parts labelled. Maybe I'll take a picture or something after class if it's still there... (Update: I did not)

Various things

- Email: eldenelmantogmail.com
- The time will be moved to 1:30 – 2:45 T/Th (in the same room?).
- OH: TBD (Many times conflict with other Harvard seminars/events)

  Ask anything in office hours (What's an infinity category, what's a Henselian local ring, how's your day been, etc.). Lecture will probably be fast.
- There are notes on the website.

1.1 Intro

A lot of the class will be focused on/around the following theorem.

Theorem 1.1 (E.-Morrow). *Let k be a field, X a qcqs k-scheme. Then, there exists functorial complex $\mathbb{Z}(j)^{mot}(X) \in D(\mathbb{Z})$ for $j \geq 0$ satisfying*

Derived category of $\mathbb{Z}$-modules

(Descent) The functor $X \mapsto \mathbb{Z}(j)^{mot}(X)$ is a Zariski (even Nisnevich) sheaf.

Notation 1.2. We write

$$\mathrm{H}^i_{\mathrm{mot}}(X,\mathbb{Z}(j)) := \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(X)).$$

(AH spectral sequence) There is a spectral sequence

$$E_2^{i,j} = \mathrm{H}^{i-j}_{mot}(X,\mathbb{Z}(-j)) \implies K_{-i-j}(X)$$

(étale comparison) Say $(p, \operatorname{char} k) = 1$ (e.g. $\operatorname{char} k = 0$). Then, there is a natural isomorphism

$$\mathrm{H}^i_{mot}(X;\mathbb{Z}/p\mathbb{Z}(j)) \xrightarrow{\sim} \mathrm{H}^i_{\acute{e}t}(X,\mu_p^{\otimes j}) = \mathrm{H}^i_{\acute{e}t}(X,\mathbb{Z}/p\mathbb{Z}(j)) \text{ for } i \leq j$$

(p-adic comparison) Say $p = 0 \in k$. Then, there is a homotopy Cartesian square

$$\begin{tikzcd} \mathbb{Z}/p\mathbb{Z}(j)^{mot}(X) \arrow[r] \arrow[d] & \mathbb{Z}/p\mathbb{Z}(j)^{syn}(X) \arrow[d] \\ R\Gamma_{cdh}(X,\Omega^j_{log})[-j] \arrow[r] & R\Gamma_{eh}(X,\Omega^j_{log})[-j] \end{tikzcd}$$

Remark 1.3. Apparently only Voevodsky knows what the 'h' stands for, but he's dead. "h stands for Voevodsky" - Chuck, probably ∘

(Hodge comparison) Say $\operatorname{char} k = 0$*. There's a Cartesian square*

$$\begin{tikzcd}
\mathbb{Z}(j)^{mot}(X) \arrow[r] \arrow[d] & R\Gamma_{Zar}\left(X, \widehat{L\Omega}^{\geq j}_{(-)/k}\right) \arrow[d] \\
R\Gamma_{cdh}\left(X, Z^j(-j)[2j]\right) \arrow[r] & R\Gamma_{cdh}\left(X, \widehat{L\Omega}^{\geq j}_{(-)/k}\right)
\end{tikzcd}$$

Warning 1.4. The bottom left square above is likely a type, and what should be there seems to be $R\Gamma_{cdh}(X; z^j(-,\bullet)[-2j])$. This doesn't matter too much, I guess, since either way, I don't know what these symbols mean. $\bullet$

(Weight zero) $\mathbb{Z}(0)^{mot}(X) \simeq R\Gamma_{cdh}(X, \mathbb{Z})$

(Cycles comparision)

$$\mathrm{H}^{2j}_{mot}(X, \mathbb{Z}(j)) \simeq \begin{cases} \mathrm{CH}^j(X) & \textit{if } X \textit{ smooth} \\ \mathrm{Pic}(X) & \textit{if } j = 1 \\ \mathrm{CH}_0^{LW}(X) & \textit{if } X \textit{ reduced,notherian surface and } j = 2. \end{cases}$$

(projective bundles) There exists natural classes $c_1(\mathscr{O}(1)) \in \mathrm{H}^2_{mot}(\mathbb{P}^r_X, \mathbb{Z}(1))$ *for any* $r \geq 1$ *such that the induced map*

$$\bigoplus_{k=0}^{r} \mathbb{Z}(j-k)^{mot}[-2k](X) \longrightarrow \mathbb{Z}(j)^{mot}(\mathbb{P}^r_X)$$

(given by pullback followed cup products with c_1^k*) is a quasi-isomorphism.*

(Milnor K-theory) Say A *is a local* k*-algebra (no other assumptions). Then,*

$$K_j^M(A) \cong \mathrm{H}^j_{mot}(A, \mathbb{Z}(j))$$

(Blowup descent) X *noetherian,* $Z \hookrightarrow X$ *give a homotopy Cartesian square*

$$\begin{tikzcd}
\mathbb{Z}(j)^{mot}(X) \arrow[r] \arrow[d] & \left\{\mathbb{Z}(j)^{mot}(rZ)\right\} \arrow[d] \\
\mathbb{Z}(j)^{mot}(\mathrm{Bl}_Z(X)) \arrow[r] & \left\{\mathbb{Z}(j)^{mot}(r\mathcal{E})\right\}
\end{tikzcd}$$

of pro-complexes ($\mathcal{E}$ *is the exceptional divisor of the blowup* $\mathrm{Bl}_Z(X) \to X$*).*

(Weibel vanishing) Say X *noetherian. Then,*

$$\mathrm{H}^i_{mot}(X, \mathbb{Z}(j)) = 0 \textit{ when } i > j + \dim X.$$

History. This is a combination of the work of many people, including Beilinson, Bloch, Cortiñas, etc. (Elden wrote like 12 names on the board).

Below references all in course notes

(1) [BMS0] Beilinson, Macpherson, Schechtman considered the following thought experiment: what if we knew topological K-theory before singular cohomology?

(2) Lichtenbaum "étale motivic cohomology" related to zeta functions

(3) Milne proposed a candidate for "étale motivic cohomology" at p via log de Rham Witt sheaves (we'll see these by the end of the month)

(4) Bloch-Kato proposed a conjecture relating étale cohomology $\mathrm{H}^*_{\text{ét}}(F, \mu_\ell^*)$ to $K_*^M(F)$ (F a field).

Sounds like they use cases of this conjecture which they were able to prove in order to prove a p-adic comparison theorem. Sounds like Faltings proved a generalization using his 'almost mathematics.' Sounds like these two approaches (+ maybe one more?) have now all come together.

(5) Bloch-Lichtenbaum, Bloch's cycle complex.

Sounds like they obtained AHSS for $X = \mathrm{Spec}(\text{field})$.

(6) Suslin-Friedlander. AHSS for $X \in \mathrm{Sm}_F$ (smooth scheme over a field)

(7) Levine revisited Bloch complex, wrote first complete construction of AHSS (late 2000's)

(8) Voevodsky (while at Harvard) constructed his motivic homotopy theory. This gave new construction of AHSS modulo some conjectures (which were later solved by Levine). He also proved Bloch-Kato (étale comparison) and won a fields medal.

Remark 1.5. There are stories of Voevodsky sleeping on the roof of the science center. ○

(9) Geisser-Levine obtained a version of p-adic comparsion for smooth schemes

(This will be the subject of a large chunck of this class)

(10) Corinñas, Haesemeyer, Weibel++ related things to Hodge theory.

⊖

1.2 What is a "motive"?

Elden suggests not worrying too much about what a "motive" is. You should instead think of motives as just being indicative of a lifestyle.

Question 1.6 (Audience)**.** *Ultimately, isn't there supposed to be a category of motives?*

Answer (Elden)**.** I don't know, I don't care. ⋆

Visually, the subject involves "bigradings" (a weight and a cohomological degree).

Here's an early example of a "motivic theorem."

Theorem 1.7. *Let X be a smooth, projective $\mathbb{C}$-variety. Then, there is a natural isomorphism*

$$\mathrm{H}^n_{\text{sing}}(X, \mathbb{C}) \simeq \bigoplus_{i+j=n} \mathrm{H}^i(X, \Omega^j).$$

This is the **Hodge decomposition** theorem.

Remark 1.8. This tells you that that singular cohomology which seems to have only one grading really has two gradings. ○

Question 1.9 (Audience). *Are these gradings or filtrations?*

Answer. In the end you want to upgrade things to filtrations, but for now, let's think gradings. ⋆

Hodge decomposition holds more generally for compact Kähler manifolds (think $X \hookrightarrow \mathbb{CP}^N$).

Corollary 1.10 (of Hodge decomp). *Let $f : X \to Y$ be a morphism of smooth, projective $\mathbb{C}$-varieties which induces an isomorphism of singular cohomology. Then, there is an induced isomorphism on their Hodge cohomology groups $\mathrm{H}^i(-, \Omega^j)$.*

It's a priori not obvious that an isomorphism on singular cohomology should induce e.g. an isomorphism on $\mathrm{H}^i(-, \mathscr{O})$.

Slogan. The object being graded has an influence on the graded pieces (and, of course, vice versa).

Our first goal in this class will be to prove this theorem (Hodge decomposition?), or to at least do this modulo analysis. What we'll talk about will follow work of Deligne and Illusie using characteristic $p > 0$ methods.

1.3 Frölicher/Hodge-to-de Rham spectral sequence

In the next class, we'll discuss universal properties of the de Rham complex $\Omega^\bullet_{X/S}$.

Construction 1.11. Let $f : X \to S$ be a smooth morphism. Then, there is a cochain complex of $\mathscr{O}_X$-modules (w/ differentials which are not $\mathscr{O}_X$-linear) given by

I think they might be $f^{-1}\mathscr{O}_S$-linear though

$$\Omega^\bullet_{X/S} = \left[\cdots \to 0 \to 0 \to \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_{X/S} \xrightarrow{\mathrm{d}} \Omega^2_{X/S} \to \dots \right]$$

using homological grading, i.e. $\Omega^i_{X/S}$ is in homological degree $-i$ (= cohomological degree i) above, called the **relative de Rham complex**.

Remark 1.12. Elden recommends getting used to switching between homological and cohomological gradings, not sticking with always using one. ○

This has a decreasing filtration, called **stupid truncation**, given by

$$\Omega^{\geq j}_{X/S} = \left[0 \to \Omega^j_{X/S} \xrightarrow{\mathrm{d}} \Omega^{j+1}_{X/S} \to \dots \right].$$

We will write this as

$$\mathrm{Fil}^{\geq *}_{\mathrm{Hdg}} \Omega^\bullet_{X/S} \longrightarrow \Omega^\bullet_{X/S},$$

and call it the **Hodge filtration**. This is a filtration in chaim complexes $K(\mathscr{O}_S)$.

Notation 1.13. We'll write decreasing filtrations with the index in the superscript, and increasing filtrations with the index in the subscript.

Notation 1.14.

$$R\Gamma_{dR}(X/S) := R\Gamma(X, \Omega^\bullet_{X/S})$$

This is a complex computing hypercohomology of the de Rham complex. We write

$$\mathrm{Fil}^{\geq *}_{\mathrm{Hdg}} R\Gamma_{dR}(X) := R\Gamma\left(X, \mathrm{Fil}^{\geq *}_{\mathrm{Hdg}} \Omega^{\bullet}_{X/S}\right).$$

This gives a filtered object in $D(\mathscr{O}_S) : \mathrm{Fil}^{\geq *}_{Hdg} R\Gamma_{dR}(X/S) \to R\Gamma_{dR}(X/S)$.

What is the associated graded of this filtration? They are

$$\mathrm{gr}^j_{Hdg} R\Gamma_{dR}(X/S) := \mathrm{cofiber}\left(\mathrm{Fil}^{\geq j+1}_{Hdg} R\Gamma_{dR}(X/S) \to \mathrm{Fil}^{\geq j} R\Gamma_{dR}(X/S)\right)$$

which is quasi-isomorphic to

$$R\Gamma\left(X; \Omega^j_{X/S}\right)[-j].$$

Exercise (easy). Convince yourself of the above.

Hence, we get a spectral sequence

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^i_{X/S}) \implies \mathrm{H}^{i+j}(R\Gamma_{dR}(X/S)) =: \mathrm{H}^{i+j}_{\mathrm{dR}}(X/S),$$

the **Hodge-to-de Rham spectral sequence**.

(Note that the cohomological degree comes second in the indexing instead of first).

Warning 1.15. If I heard correctly, the above is different to the hypercohomology spectral sequence associated to a sheaf of complexes. The above is more specific to de Rham cohomology. •

Theorem 1.16 (**Hodge-to-de Rham degeneration**). *Let K be a field of characteristic zero, X/K a smooth, projective K-scheme. Then, the Hodge-to-de Rham spectral sequence collapses at the E_1-page, and hence there is a filtration on $\mathrm{H}^*_{dR}(X/K)$ whose graded pieces are the Hodge cohomology groups $\mathrm{H}^*(X, \Omega^*)$.*

Warning 1.17. No claim above about a splitting of the filtration. It will split since this is a filtered vector space. However, there's no claim about a canonical splitting. You won't get one in general. There is one over $\mathbb{C}$, but this requires analysis. •

Let's end with the beginning of the proof of Theorem 1.16. We will do some "spreading out." This is encoded by a diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & \widetilde{Y} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "p_{\mathfrak{X}}"] & X \arrow[l] \arrow[d, "p_X"] \\
s = \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_2(k) \arrow[r] & S & \operatorname{Spec} K \arrow[l]
\end{tikzcd}$$

(if $k = \mathbb{F}_p$, then $W_2(k) = \mathbb{Z}/p^2\mathbb{Z}$) such that

(1) $p_{\mathfrak{X}}$ is smooth and projective

(2) S is smooth over $\mathbb{Z}$

(3) there is a d such that the dimension of the fibers of $p_{\mathfrak{X}}$ are all at most d

(follows from **(1)**+**(2)**)

(4) $s \hookrightarrow S$ is a closed immersion and k is a perfect field of characteristic $p > d$.

(5) $\operatorname{Spec} W_2(k) \to S$ map is determined by universal property of W_2

(6) All squares are pullbacks

2 Lecture 2 (9/6)

2.1 Administrative stuff

Office hours I ('Stable categories after dark') Thursday 5pm at SC231

2.2 Material

Recall 2.1. Let $f : X \to S$ be a smooth morphism. To this, one associates $R\Gamma_{dR}(X/S)$ the relative de Rham complex. This is equipped w/ a Hodge filtration

$$\mathrm{Fil}^{\geq *}_{Hdg}\, R\Gamma_{dR}(X/S) := R\Gamma(X, \Omega^{\geq j}_{X/S})$$

whose gradied pieces are

$$\mathrm{gr}^{*}_{Hdg} R\Gamma_{dR}(X/S) \simeq R\Gamma(X, \Omega^{j})[-j].$$

Have in mind the picture

$$\begin{tikzcd} \mathrm{Fil}^{\geq j+1}_{Hdg} \arrow[r] & \mathrm{Fil}^{\geq j}_{Hdg} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{Fil}^{\geq 0}_{Hdg} & \simeq & R\Gamma_{dR}(X/S) \\ & R\Gamma(X, \Omega^{j}_{X/S})[-j] & & & & \end{tikzcd}$$

taking class in $D(\mathscr{O}_S)$, a stable ∞-category. This filitration gives rise to the Hodge-to-de Rham spectral sequence

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^i_{X/S}) \implies \mathrm{H}^{i+j}_{\mathrm{dR}}(X/S)$$

We want to prove the following theorem

Theorem 2.2. *Let $S = \operatorname{Spec} K$ and say f is a smooth, proper morphism. Assume $\operatorname{char} K = 0$. Then, this spectral sequence collapses (on the E_1-page)*

We will prove this using char p methods after spreading out. By 'spreading out' we mean to obtain a diagram of the form (all squares Cartesian)

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \widetilde{Y} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "p_X"] & X \arrow[l] \arrow[d, "p_X"] \\ s = \operatorname{Spec} \kappa \arrow[r] & \operatorname{Spec} W_2(\kappa) \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} K \arrow[l] \end{tikzcd}$$

satisfying

(1) $p_{\mathcal{X}}$ is smooth and proper

(2) $\operatorname{Spec} A$ is smooth over $\mathbb{Z}$

($\operatorname{Spec} A$ smooth over $\mathbb{Z}_p$ ($p = \operatorname{char} \kappa$) should be enough)

(3) the dimension of any fiber of $p_{\mathcal{X}}$ is bounded by some number d

(4) the map $\operatorname{Spec} W_2(\kappa) \to S$ is determined by the closed immersion $\operatorname{Spec} \kappa \hookrightarrow S$

(5) $\operatorname{char} \kappa > d$

Let's sketch how one constructs such a thing. First write

$$K = \bigcup_{\alpha} A_{\alpha}$$

with each A_α a subalgebra of K which is f.type over $\mathbb{Z}$. Now, claim, for large enough α in this directed system, we can find a smooth proper A_α-scheme $\mathcal{X}_\alpha$ fitting into a Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X}_\alpha \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} A_\alpha. \end{tikzcd}$$

Remark 2.3. Just finding a scheme only requires that $X \to \operatorname{Spec} K$ is locally finitely presented. Locally finitely presented essentially means that if $\operatorname{Spec} K = \varprojlim \operatorname{Spec} A_\alpha$, then X spreads out to some $\mathcal{X}_\alpha / A_\alpha$. $\circ$

It categorical terminology, this is saying you have a compact object (or something like that)

Now we make $\mathcal{X}_\alpha$ better by a "game" called descent.

Example. Smoothness is descendable in the sense that for $\beta \gg \alpha$, can find $\mathcal{X}_\beta \to \operatorname{Spec} A_\beta$ smooth. You can prove this using something like the Jacobian criterion for smoothness (tells us that smoothness can be defined using finitely many equations). For properness, one first shows that projectivity is descendable, and then uses Chow's lemma to conclude the same for properness. $\triangle$

Now, we want a bigger index $\gamma \gg \beta$ such that $\operatorname{Spec} A_\gamma$ is smooth over $\mathbb{Z}$. We can do this because if $f : T \to S$ is locally of f.pres, then being smooth is an open condition on T. Even more concretely, if you have a finite presentation for A_γ over $\mathbb{Z}$, then you only need to invert its Jacobian to obtain A smooth over $\mathbb{Z}$.

At this point, we find some $p > d > \dim(\text{fibers of } p_{\mathcal{X}})$ such that $\operatorname{Spec} A$ has a point $x = \operatorname{Spec} \kappa$ with residue characteristic p. This gives the following diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d] & \operatorname{Spec} \kappa \arrow[l, hook'] \arrow[d, "\substack{\text{nilpotent}\\ \text{extension}}"] \\ \operatorname{Spec} \mathbb{Z} & \operatorname{Spec} W_2(\kappa) \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

Since $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ is smooth, the infinitesimal lifting criterion gives the existence of the dashed arrow above. Thus, we have obtained the desired

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \widetilde{Y} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "p_{\mathfrak{X}}"] & X \arrow[l] \arrow[d, "p_X"] \\ s = \operatorname{Spec}\kappa \arrow[r] & \operatorname{Spec} W_2(\kappa) \arrow[r] & \operatorname{Spec} A & \operatorname{Spec} K. \arrow[l] \end{tikzcd}$$

We want to say something about the Hodge and de Rham cohomologies of X. We want to relate them to the same over Y, so we need a base change result. In particular, we'll use flat base change.

Lemma 2.4. *Let S be an affine, noetherian, integral scheme. Let $f : X \to S$ be smooth and proper. Then, for any Cartesian square*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd},$$

we have

(1) If $R^i f_* \Omega^j_{X/S}$ is locally free of constant rank h^{ji}, then so is the sheaf $R^i f'_* \Omega^j_{X'/S'}$.

(2) Ditto for $R^n f_* \Omega^\bullet_{X/S}$

With this result, we are reduced to the following theorem.

Theorem 2.5 (Deligne-Illusie)**.** *Let κ be a field of characteristic $p > 0$. Let X be a smooth, proper κ-scheme of dimension $\dim X < p$ which furthermore lifts to $W_2(\kappa)$. Then, the Hdg-dR spectral sequence degenerates at E_1.*

Proof of Theorem 1.16, assuming Theorem 2.5. Write $h^{ij} := \dim_K \mathrm{H}^j(X, \Omega^i)$ and $h^n := \dim_K \mathrm{H}^n_{\mathrm{dR}}(X/K)$. Then, the claim is equivalent to

$$\sum_{i+j=n} h^{ij} = h^n$$

for all n. Flat base change + Theorem 2.5 verifies this equality. ■

This is the end of 'Lecture 0'. In the next few classes, we'll prove Theorem 2.5.

2.3 'Lecture 1': The one in which we see some miracles at p

Our goal is to prove the Deligne-Illusie theorem. We'll begin by just making some comments about spectral sequences, and how you should think of them.

Notation 2.6. We'll try to write something like M for objects of the derived category, but something like M^* (with a star there) for complexes.

Remark 2.7. Say A is a commutative ring, and let $D(A)$ be its derived category. You should really think of this as an ∞-category, but for now, can just think of it as a triangulated category if you want. This derived category has a t-structure (think: notion of positivity and negativity). Can consider

$$D(A)^{\geq 0} = \left\{ M : \mathrm{H}^i(M) = 0 \text{ if } i < 0 \right\} \hookrightarrow D(A)$$

(using cohomological convention) as well as $D(A)^{\le 0} \hookrightarrow D(A)$. There are also **truncation functors**

$$\tau^{\ge 0} : D(A) \to D(A)^{\ge 0}$$

such that

$$\mathrm{H}^i(\tau^{\ge 0} M) = \begin{cases} \mathrm{H}^i(M) & \text{if } M \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$

Can similarly build $\tau^{\le 0}, \tau^{[j-1,j]}$, etc. Note that there is an exact triangle

$$\mathrm{H}^{i-1} M[-i+1] \longrightarrow \tau^{[i-1,i]} M \longrightarrow \mathrm{H}^i M[-i] \xrightarrow{\delta} \mathrm{H}^{i-1} M[-i+2]$$

(note **exact triangles** in this category are of the form $A \to B \to C \xrightarrow{\delta} A[1]$). The δ above is a class in (square brackets on the left used for $\mathrm{Hom}_{D(A)}$)

$$\left[\mathrm{H}^i M, \mathrm{H}^i M[2]\right] = \mathrm{Ext}^2\left(\mathrm{H}^i M, \mathrm{H}^{i-1} M\right).$$

Note that if $\delta = 0$, the our earlier sequence splits, i.e.

$$\tau^{[i-1,i]} M \cong \mathrm{H}^{i-1} M[-i+1] \oplus \mathrm{H}^i M[-i].$$

Can play this same sort of game e.g. with $\tau^{[a,b]} M$ for $b \ge a$. Will produce a filtered object

$$\mathrm{H}^b M[-b] \to \cdots \to \tau^{[a+1,b]} M \to \tau^{[a,b]} M.$$

In general, there are obstruction elements living in groups like $\mathrm{Ext}^3(\mathrm{H}^i M, \mathrm{H}^{i-2} M)$ which obstruct the splitting of this filtration. These obstruction elements can be interpreted as differentials in a spectral sequence. Go to office hours to say more.

To say that a spectral sequence collapses is to say that all (higher) differentials vanish. This is to say that these obstruction elements vanish, i.e. to say that some n-step filtrations split. ○

Definition 2.8. We say that M is **decomposable** if there is an equivalence

$$M \simeq \bigoplus \mathrm{H}^i(M)[-i]$$

such that the induced map on cohomology is the identity.[2] ◇

It's usually easier to construct a map

$$\bigoplus \mathrm{H}^i(M)[-i] \to M$$

and show that it is an isomorphism than to directly show that a spectral sequence collapses.

Theorem 2.9 (Deligne-Illusie, strong form). *Let S be a scheme over a perfect field k of char $p > 0$. Let X be a smooth, proper S-scheme. Let $F_{X/S} : X \to X^{(1)}$ be the relative Frobenius. Fix a lift $\widetilde{S}$ of S to W_2 which is flat over W_2. Then,*

[2]Sounds like the natural map (in the ∞-category) goes from cohomology to M

(1) *If $\widetilde{X}^{(1)}$ lifts to $\widetilde{S}$, then*

$$\tau^{[0,p]} F_{X/S,*}\Omega^{\bullet}_{X/S}$$

is decomposable.

(2) *The collection of all such lifts are in bijection with all possible splittings.*

See Figure 1 for a reminder of the definition of relative Frobenius.

2.4 The de Rham complex

Fix a commutative ring A. What is the nature of the functor

$$B \longmapsto \Omega^{\bullet}_{B/A}$$

(B an A-algebra)? Let's start with a simpler question: what is the nature of the functor

$$B \longmapsto \Omega^1_{B/A}?$$

To start, the object $\Omega^1_{B/A}$ is naturally a B-module. Though, as B varies, it's strange to try to think of this as a functor from A-algebras to B-modules. Let's say more.

Definition 2.10. If B is an A-algebra, an **A-derivation** of B is the datum of M, a B-module, and an A-linear map $D : B \to M$ such that

$$D(fg) = fDg + gDf$$

(and also $D(a) = 0$ for $a \in A$). ◇

Fact. $\Omega^1_{B/A}$ is the **universal A-derivation** in the sense that $\exists! \mathrm{d} : B \to \Omega^1_{B/A}$ such that

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \cong \mathrm{Der}_A(B, M).$$

So want to think of $\Omega^1_{B/A}$ as some kind of adjoint. We'd prefer to replace Der_A above with a Hom.

$$\begin{tikzcd}
X \arrow[dr, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\
& X^{(1)} \arrow[r, "W"] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

$$x^p \longmapsfrom x$$

Figure 1: Diagram of Frobenii. F_X and F_S above are absolute Frobenius, while $F_{X/S}$ is relative Frobenius

Remark 2.11. Let M be a B-module. We can form $M \oplus B$, the **square-zero extension of B by M**. This is a B-algebra where

$$(m, b) \cdot (m', b') = (mb' + m'b, bb').$$

An A-derivation is simply an A-algebra section of the projection $M \oplus B \to B$. ◦

Recall that $\Omega^j_{B/A} := \bigwedge^j_B \Omega^1_{B/A}$. We see that

$$\bigoplus_{j \geq 0} \Omega^j_{B/A}$$

is a *graded* algebra over B. However, it's got more going for it than that.

(1) It is in fact a *strict commutative graded B-algebra*, i.e. $xy = (-1)^{|x||y|} yx$ and $x^2 = 0$ if $|x|$ =odd (this latter condition is what *strict* means. This is not immediate in char = 2).

(2) There's a map $\mathrm{d} : \Omega^j_{B/A} \to \Omega^{j+1}_{B/A}$ determined by

$$\mathrm{d}\,(b_0 \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_j) = \mathrm{d}b_0 \wedge \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_j.$$

This map is *not* B-linear (but is A-linear).

Thus, this object lives in strict commutative differential graded A-algebras (strict cdga over A).

Theorem 2.12. *Let $A \to B$ be a map of commutative rings. Then, $\Omega^\bullet_{B/A}$ is the initial strict cdga equipped with a map to its degree zero component, i.e. with $B \to \Omega^0_{B/A} = B$ (this is the identity).*

Proof. Recall that the exterior algebra $\bigwedge^\bullet_B(M)$ is the quotient of the tensor algebra

$$T^*_B(M) = B \oplus M \oplus (M \otimes M) \oplus \dots$$

by the two-sided ideal generated by $\{m \otimes m : m \in M\}$. In particular, this gives $\bigwedge^\bullet_B(M)$ the structure of a strict commutative graded algebra (strict cga) over A. We're out of time, so we'll continue next time... ■

3 Lecture 3 (9/8)

Recall we are wanting to prove the following result.

Theorem 3.1 (Deligne-Illusie)**.** *Let S be a scheme over a perfect field k of characteristic $p > 0$. Fix $\widetilde{S}$ a flat lift to $W_2(k)$. Let X be a smooth S-scheme. Then,*

(1) *If X lifts to $\widetilde{S}$ flatly, then the complex $\tau^{[0,p]} F_{X/S,*} \Omega^\bullet_{X/S}$ decomposes.*

(2) *The collection of decompositions are in bijection with lifts.*

Remark 3.2 (role of properness). Properness is not necessary for this statement, but it is needed to ensure that the Hodge numbers h^{ij} are finite. ◦

We ended last time in the middle of the proof of the following result

Theorem 3.3 (Theorem 2.12)**.** *Let $A \to B$ be a morphism of rings. Then, $\Omega^\bullet_{B/A}$ is the initial strict cdg A-algebra (A-cdga) equipped with an A-algebra map from B to its degree zero part.*

Recall 3.4. The natural map $B \to \Omega^1_{B/A}$ is not B-linear, only A-linear. ⊙

Proof. Continuing where we left off, last time we saw that the underlying strict cga of $\Omega^\bullet_{B/A}$ is simply $\bigwedge^\bullet_B(\Omega^1_{B/A})$. In general, to define an A-linear map ($C^\bullet$ a cdga)

$$\bigwedge\nolimits^\bullet_B(\Omega^1_{B/A}) \longrightarrow C^\bullet,$$

all we have to do is to define $B \to C^0$ (and then, e.g. the map $\Omega^1_{B/A} \to C^1$ is determined by the composition $B \to C^0 \xrightarrow{\mathrm{d}} C^1$). At this point we need to prove that in fact $\Omega^\bullet_{B/A}$ has an A-cdga structure as claimed. For this, we need to know that $\Omega^1_{B/A}$ as an A-algebra is generated by $\{b_0 \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_j\}$. ■

Remark 3.5 (First char p miracle). The failure of the de Rham differential to be B-linear comes from the fact that

$$\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f,$$

so '$g\mathrm{d}f$' acts as some sort of obstruction. In char $p > 0$, we have $\mathrm{d}f^p = pf^{p-1}\mathrm{d}f = 0$, so one has

$$\mathrm{d}\left(f^p g\right) = f^p \mathrm{d}g.$$

Thus, $\Omega^\bullet_{B/A}$ is linear over $B^{(1)} = B \otimes_{A,F_A} A \longrightarrow B$,[3] the Frobenius pullback of B. This makes sense of the object

$$F_{X/S,*}\Omega^\bullet_{X/S}.$$ ○

Here's a remarkable construction in char $p > 0$.

Lemma 3.6 (Cartier)**.** *Let S be an $\mathbb{F}_p$-scheme. Then, there exists $\mathscr{O}_{X^{(1)}}$-linear maps, called the **inverse Cartier maps**, of the form*[4]

$$C^{-1} : \Omega^j_{X^{(1)}/S} \longrightarrow \mathcal{H}^j\left(F_{X/S,*}\Omega^\bullet_{X/S}\right)$$

for all $j \geq 0$. These satisfy

(1) $C^{-1}(1) = 1$

(2) $C^{-1}(\omega \wedge \tau) = C^{-1}(\omega) \wedge C^{-1}(\tau)$

(3) $C^{-1}(\mathrm{d}f) = f^{p-1}\mathrm{d}f$.

If you want to be fancy, write $f^{p-1}\mathrm{d}f = f^p \frac{\mathrm{d}f}{f} = f^p \mathrm{d}\log f$.

Theorem 3.7 (Cartier)**.** *Let $X \to S$ be smooth. Then, the inverse Cartier maps are isomorphisms, so*

$$C^{-1} : \Omega^j_{X/S} \xrightarrow{\sim} \mathcal{H}^j\left(F_{X/S,*}\Omega^\bullet_{X/S}\right).$$

[3]Note $B^{(1)} \to B$ is dual to the relative Frobenius $\operatorname{Spec} B \to \operatorname{Spec} B^{(1)}$

[4]$\mathcal{H}^j(-)$ is the sheafification of the cohomology presheaf

Proof. To prove this result, let's consider first an easy case, say $\mathbb{A}^n_S = X \to S$. Let's observe that the cdga $F_{X/S,*}\Omega^\bullet_{\mathbb{A}^n_S/S}$ is the $\mathscr{O}_{(\mathbb{A}^n_S)^{(1)}}$-linear complex generated by

$$x_1^{w_1} \dots x_n^{w_n} \mathrm{d}x_{\alpha_i} \dots \mathrm{d}x_{\alpha_j},$$

where $w_i \in [0, p-1]$ and $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_j \le n$. Let $K(n)^\bullet$ be the $\mathbb{F}_p$-linear cdga generated as above. Then,

$$K(n)^\bullet \otimes_{\mathbb{F}_p} \mathscr{O}_{(\mathbb{A}^n_S)^{(1)}} \simeq F_{\mathbb{A}^n_S/S,*}\Omega^\bullet.$$

With this presentation, the cohomology of $\mathrm{H}^i(F_*\Omega^\bullet) \cong \mathrm{H}^i(K(n)^\bullet) \otimes_{\mathbb{F}_p} \mathscr{O}$. Let's observe further that

$$K(n)^\bullet \simeq K(1)^\bullet \otimes \cdots \otimes K(1)^\bullet$$

(normal tensor, not derived). Since all the objects in these complexes are free, Künneth holds. Therefore, it suffices to prove the result for $n = 1$, i.e. that

$$\mathrm{H}^j(K(1)^\bullet) = \begin{cases} \mathbb{F}_p & \text{if } j = 0 \\ \mathbb{F}_p \left\{x^{p-1}\mathrm{d}x\right\} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise. Prove the above. Key: consider $x^n \mathrm{d}x$ for $n < p-1$. Then, "$\int x^n \mathrm{d}x$"$= \frac{1}{n+1}x^{n+1}$ since $n+1$ is invertible. This will tell you that these classes die in cohomology.

Also, we saw earlier that $\mathrm{d}f^p = 0$. Note that

$$\underbrace{\int \cdots \int \int}_{n \text{ times}} x\mathrm{d}x = \frac{x^{n+1}}{(n+1)!}.$$

Let's look at the case of general X now. For fixed j, C^{-1} is a map

$$C^{-1} : \Omega^j_{(-)^{(1)}/S} \longrightarrow \mathcal{H}^j\left(F_{(-)^{(1)}/S,*}\Omega^\bullet_{(-)/S}\right).$$

It in fact gives a morphism of Zariski sheaves (over X). One of the characterizations of a smooth morphism is that $X \to S$ is smooth iff Zariski-locally on X, it is of the form

$$X \xrightarrow[\text{ét}]{g} \mathbb{A}^n_S \longrightarrow S.$$

We now conclude from the computation for $\mathbb{A}^n_S$ and the fact that if g is étale, then both $\Omega^j_{(-)^{(1)}/S}, \mathcal{H}^j(F_{(-)/S,*}\Omega)$ are stable under pullback.

Exercise. Unpack the above. Think: short exact sequence involving differential forms.

∎

Remark 3.8 (Popescu's approximation). Recall that a morphism of schemes $f : X \to Y$ is **regular** is every fiber X_y is locally noetherian, f is flat, and for any finite, purely inseparable extension $\kappa'/\kappa(y)$, $X_{\kappa'}$ is a regular scheme. ○

Theorem 3.9 (Popescu)**.** *Say $A \to B$ is a regular morphism of rings. Then, it can be written as a filtered colimit of smooth ring maps, i.e. $B \cong \operatorname{colim} A_\alpha$ with A_α smooth A-algebras.*

This is a big technical achievement (Elden was gushing about it). Being regular is a numerical condition (some equality of dimenions), while being smooth is a homological condition. This is relating the two in a surprising way.

Question 3.10 (Audience)**.** *How do you prove something like this?*

Answer. By being crazy. ⋆

Question 3.11 (Audience)**.** *What's the idea?*

Answer. It's Cauchy sequences. It's analysis. ⋆

From this, Cartier's theorem holds also for regular (noetherian) $\mathbb{F}_p$-schemes.

Definition 3.12 (Abstract Koszul complexes, Achinger-Suh)**.** Let $(\mathcal{X}, \mathscr{O})$ be a ring topos. A strict $\mathscr{O}$-cdga $K^\bullet$ which is **coconnective** (i.e. no negative grading) is an **abstract Koszul complex** if

(1) The map $\mathscr{O} \to \mathcal{H}^0(K^\bullet)$ is an isomorphism.

(2) For any $q \geq 1$, the below factorization (which exists since $K^\bullet$ is strict) is an isomorphism

$$\begin{tikzcd} \mathcal{H}^1(K^\bullet)^{\otimes q} \arrow[r] \arrow[rr, bend left] & \bigwedge^q_{\mathscr{O}} \mathcal{H}^1(K^\bullet) \arrow[r, "\sim"'] & \mathcal{H}^q(K^\bullet) \end{tikzcd}$$

This is meant to be (like) an axiomitization of the Cartier isomorphism. ⋄

This is not a weird definition.

Example. Say X is a torus $\simeq (S^1)^{2g}$. Then, $\mathrm{H}^0 \simeq \mathbb{Z}$ and $\bigwedge^q_{\mathbb{Z}} \mathrm{H}^1(X, \mathbb{Z}) \simeq \mathrm{H}^q(X, \mathbb{Z})$. △

We still need to go back and finish something up.

Proof Sketch of Cartier's Lemma 3.6. Say A is an $\mathbb{F}_p$-algebra. Say $\widetilde{A}$ is a flat lift to $\mathbb{Z}/p^2\mathbb{Z}$, and suppose there's some $\varphi : \widetilde{A} \to \widetilde{A}$ such that $\varphi/p = F_A$, i.e. that we have a lift of Frobenius. In this case, look at $\Omega^1_{\widetilde{A}/\mathbb{Z}/p^2\mathbb{Z}}$, and consider

$$\varphi^*(\mathrm{d}x) = \mathrm{d}\varphi(x) \equiv 0 \mod p.$$

Thus, we have $\varphi^* : \Omega^1_{\widetilde{A}/(\mathbb{Z}/p^2\mathbb{Z})} \to p\Omega^1_{\widetilde{A}/(\mathbb{Z}/p^2\mathbb{Z})}$. We can then form a 'divided Frobenius'

Question: Why is this well-defined? What about p-torsion?

$$\frac{\varphi^*}{p} : \Omega^1_{\widetilde{A}/(\mathbb{Z}/p^2\mathbb{Z})} \longrightarrow \Omega^1_{\widetilde{A}/(\mathbb{Z}/p^2\mathbb{Z})}$$

Note $(\varphi^*/p)(x)$ looks like "$x^{p-1}\mathrm{d}x$". In this case, we define $C^{-1} := \frac{\varphi^*}{p} \mod p$. Note this ensures $C^{-1}(\mathrm{d}g) = g^{p-1}\mathrm{d}g$. In this case, we even get a refinement

$$\begin{tikzcd} \Omega^1_{A/\mathbb{F}_p} \arrow[r, "C^{-1}"'] \arrow[rr, bend left, "C^{-1}"] & Z\Omega^1_{A/\mathbb{F}_p} \arrow[r] & \mathcal{H}^1(\Omega^1_{A/\mathbb{F}_p}). \end{tikzcd}$$

In general, you only get the map going to cohomology.

To make an official definition, key computation is

$$(x+y)^{p+1}\mathrm{d}(x+y) = x^{p-1}\mathrm{d}x + y^{p-1}\mathrm{d}y + \mathrm{d}\left(\sum \text{factorials}\right).$$

Can find a reference in the course notes. ■

This ends 'Lecture 1'.

3.1 'Lecture 2': In which we do some derived Linear algebra

"Before we learn AG, we learn LA, so before we learn derived AG, we better learn derived LA"

We now discuss the pure algebra part of the DI theorem 3.1. The theorem will tell us that $\tau^{[0,1]}F_{X/S,*}\Omega^\bullet$ will always split. This must be explained via geometry. We will talk about this later. For now, given this splitting, how do we split the entire complex?

Idea: we want to spread the splitting as much as possible. For example, it'd be nice if we could say something like

$$\mathrm{Sym}^p\left(\tau^{[0,1]}F_*\Omega\right) \simeq \tau^{[0,p]}F_*\Omega.$$

This is not quite correct as stated, but keep this in mind. The write functor (on the derived category) won't by Sym^p, but whatever it is, we'll get an iso like above as a consequence of the abstract Koszul duality.

Construction 3.13 (Divided powers). Let A be a ring. Let M be a f.g. projective A-module. Then, the **divided power algebra** on M is the commutative A-algebra generated by elements $\gamma_n(x)$, for $x \in M$, which behave like $x^n/(n!)$. We put $|x| = 1$ and $|\gamma_n(x)| = n$. One has a decomposition

$$\Gamma_A(M) = \bigoplus_{d\geq 0}\Gamma_A^d(M).$$

One also has

$$\left(M^{\otimes d}\right)^{\Sigma_d} \cong \Gamma_A^d(M).$$

Compare the above to the more familiar isomorphism

$$\mathrm{Sym}^d(M) = \left(M^{\otimes d}\right)_{\Sigma_d}.$$

4 Lecture 4 (9/13)

As always, keep in mind Figure 1.

4.1 Derived linear algebra

Today, we want to prove the following theorem.

Theorem 4.1. *Let X/S be smooth of relative dimension $< p$ (p characteristic of the base). Assume that the truncation $\tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S}$ decomposes. Then, there is a quasi-isomorphism*

$$\bigoplus_{j\geq 0} \Omega^j_{X^{(1)}/S}[-j] \simeq F_{X/S,*}\Omega^\bullet_{X/S}$$

inducing C^{-1} on $\mathcal{H}^j$.

Our usual lifting condition is replaced by the decomposition of the truncation $\tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S}$. The basic (but not literally correct) idea is to show something like "$\mathrm{Sym}^p(\tau^{\leq 1}) \simeq \tau^{\leq p}$."

Remark 4.2. The Cartier theorem/Kozsul condition essentially tells us that our de Rham complexes have cohomology generated in degree 1. ◦

Recall 4.3. If M is a f.g. projective A-module, one can form the divided power algebra

$$\Gamma^*_A(M) \cong \bigoplus_{d\geq 0} \Gamma^d_A(M) \text{ where } \Gamma^d_A(M) \cong \left(M^{\otimes d}\right)^{\Sigma_d}.$$

This algebra is generated by symbols $\gamma_n(x)$, for $x \in M$, which behave like $x^n/n!$ and live in degree n. ⊙

Remark 4.4. If M is a f.g. free A-module, then

$$\Gamma^d_A(M^\vee) \cong \left(\mathrm{Sym}^d_A(M)\right)^\vee.$$ ◦

Lemma 4.5. *If $d! \in A^\times$ and M is a f.g. projective A-module, then the "averaging map"*

$$\mathrm{Sym}^k_A(M) \longrightarrow \Gamma^k_A(M) \text{ for } k \leq d$$

is an isomorphism.

Note that many of the statements we've considered so far are "point-set" (e.g. looked at de Rham complex as a literal complex). However, for today's main theorem, we're really working in a derived setting.

Construction 4.6 (**animation**). Consider the category of f.g. projective A-modules, denoted $\mathrm{Mod}_A^{\mathrm{fgproj}}$. Say we're given a functor

$$F : \mathrm{Mod}_A^{\mathrm{fg}} \longrightarrow \mathrm{Mod}_A,$$

e.g. $F = \Gamma^d_A, \mathrm{Sym}^d_A, \bigwedge^d_A$. We want to extend F to derived categories. Note there are natural functors

$$\mathrm{Mod}_A^{\mathrm{fg}} \longrightarrow D(A)_{\geq 0} \text{ and } \mathrm{Mod}_A \longrightarrow D(A)_{\geq 0}.$$

We will produce some $LF : D(A)_{\geq 0} \to D(A)_{\geq 0}$, called the **nonabelian derived functor of** F (note F not assumed to be left or right exact).

Remark 4.7. In practice, one computes LF in the following way. Say M is an object in the derived category. Take (an equivalence[5]) $P_\bullet \to M$ where $P_\bullet$ is a simplicial object of free A-modules. Then,

$$LF(M) \simeq \mathrm{hocolim}_{\Delta^{\mathrm{op}}} F(P_n).$$ ◦

[5] colimit of $P_\bullet$ should be quasi-iso to M

I cannot tell if this is supposed to be the same as taking a projective resolution of M, and then simply applying F to that resolution (as one would for a left/right exact functor). I assume not, and this is why you need a homotopy colimit?

Definition 4.8 (**Higher Koszul complexes**)**.** Let $(X, \mathscr{O})$ be a ringed topos. Then, the **qth Koszul cohomology** of a map $f : M \to N$ between flat modules is defined to be

$$\mathrm{Kos}_q(f) := L\Gamma^q_A(\mathrm{fib}(f)).$$

Note $\mathrm{fib}(f)$ can be presented by the complex $[M \to N]$ where M is in degree 0 and N is in degree -1. $\diamond$

This is in homological grading

Remark 4.9. $\mathrm{fib}(f)[1] \simeq \mathrm{cof}(f)$. That is, fibers contain information about both fibers and cofibers. Have

$$M \xrightarrow{f} N \to \mathrm{cof}(f) \in D(A).$$

$\circ$

Fact.

$$L \bigwedge^q (\mathrm{cof})[-q] \simeq L\Gamma^q(\mathrm{fib}).$$

In general,

$$L \bigwedge^q (M[1])[-q] \simeq L\Gamma^q(M).$$

Note above that $\mathrm{fib}(f)[1] \in D(A)_{\geq 0}$ (which gets around the annoyance that $\mathrm{fib}(f) \in D(A)_{\geq -1}$ only in definition of Koszul cohomology). We write

$$\mathrm{Kos}^q(f) := L \bigwedge^q_A (\mathrm{fib}(f)).$$

Fact.

$$L\,\mathrm{Sym}^q_M(M[1])[-q] \simeq L \bigwedge^q_A (M) \text{ for } M \in D(A)_{\geq 0}.$$

Theorem 4.10 (Achinger-Suh)**.** *Let m be an integer such that $m!$ is invertible in $\mathscr{O}$. Let $q \geq m$. Assume*

(1) $q = m$; OR

(2) $m+1$ is a nonzero divisor in $\mathscr{O}$.

Let K be an abstract Koszul complex. Choose a representative

$$\tau^{\leq 1} K \simeq \left[K^0 \xrightarrow{\partial} Z^1 K\right]$$

such that $K^0, B^1K, Z^1K, \mathcal{H}^1(K)$ are flat. Then,

$$\tau^{\geq q-m}\,\mathrm{Kos}^q(\partial) \simeq \tau^{[q-m,q]} K.$$

In particular (taking $q = m$), if $\tau^{\leq 1}K$ is decomposable, then $\tau^{[0,q]}K$ is decomposable.

Remark 4.11. The statement above depends on the map δ, not just the complex $\tau^{\leq 1}K$. $\circ$

Construction 4.12. Say $f : M \to N$ is a morphism of flat $\mathscr{O}$-modules. We'll construct a complex $\mathrm{Kos}_q^\bullet(f)$. This will be of the form

$$0 \longrightarrow \Gamma^q_{\mathscr{O}}(M) \longrightarrow \bigwedge\nolimits^1_{\mathscr{O}} N \otimes \Gamma^{q-1}_{\mathscr{O}}(M) \longrightarrow \dots \longrightarrow \bigwedge\nolimits^q_{\mathscr{O}}(M).$$

The differential is given by

$$d\left(y \otimes \gamma_{e_1}(x_1) \otimes \dots \otimes \gamma_{e_r}(x_r)\right) = \sum_j \left(y \wedge f(x_j)\right) \otimes \left(\gamma_{e_1}(x_1) \otimes \dots \widehat{\gamma_{e_j}(x_j)} \dots \otimes \gamma_{e_r}(x_r)\right).$$

Lemma 4.13. *Say f as above. Then,*

$$\mathrm{Kos}_q^\bullet(f) \simeq_{qiso} \mathrm{Kos}_q(f).$$

The abstract definition of Kos_q let's one easily see that it's homotopy invariant. The computation given in the above lemma gives one an actual complex to work with.

Proof Sketch. We start with some general comments. Say we have an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of f.g. free modules. In this case, consider

$$\mathrm{Kos}^q_\bullet : \bigwedge\nolimits^q M' \longrightarrow \dots \longrightarrow \bigwedge\nolimits^2 M' \otimes \mathrm{Sym}^{q-2} M \longrightarrow M' \otimes \mathrm{Sym}^{q-1}(M) \longrightarrow \mathrm{Sym}^q_{\mathscr{O}}(M) \longrightarrow 0.$$

The above is the qth graded piece of the Koszul complex for $(f_1, \dots, f_n) : R^{\oplus n} \to R$, where R is the graded ring $R = \mathrm{Sym}^*_{\mathscr{O}}(M)$ and $f_1, \dots, f_n$ are generators for M' (thought of as elements of M). Hence, we have a quasi-isomorphism

$$\mathrm{Kos}^q_\bullet \simeq \mathrm{Sym}^q_{\mathscr{O}}(M'').$$

Since M'' is free, we see from this that

$$\mathrm{Sym}^*_{\mathscr{O}} M'' \simeq L\,\mathrm{Sym}^* M'' \simeq L\,\mathrm{Sym}^*_{\mathscr{O}}(\mathrm{cof}(M' \to M)) \simeq L\,\mathrm{Sym}^*_{\mathscr{O}}(\mathrm{fib}(M' \to M)[1]).$$

Now recall that $\Gamma^n_{\mathscr{O}}(M^\vee) \simeq \mathrm{Sym}^n_{\mathscr{O}}(M)^\vee$ and $\bigwedge_{\mathscr{O}}(M^\vee) \simeq (\bigwedge_{\mathscr{O}} M)^\vee$. If we dualize $\mathrm{Kos}^q_\bullet$ (note all objects are free), we get

$$0 \longrightarrow \Gamma^q_{\mathscr{O}}(M) \longrightarrow M'' \otimes \Gamma^{q-1}_{\mathscr{O}}(M) \longrightarrow \dots \longrightarrow \bigwedge\nolimits^q M'' \longrightarrow 0,$$

which is exactly the complex $\mathrm{Kos}_q^\bullet$. This complex computes

$$L\,\mathrm{Sym}^q\left((M')^\vee\right)^\vee \simeq L\Gamma^q(M') \simeq L\Gamma^q(\mathrm{fib}(M \to M'')).$$

This proves the Lemma when f is a surjection between f.g. free modules. ■

To finish, one wants to reduce to the above case using the fact that $L\Gamma^q$ is defined in general via left Kan extension from the case of free modules (something like this).

Proof of Theorem 4.10. We begin by constructing a map on stupid truncations; consider

$$0 \longrightarrow K^{q-m} \longrightarrow K^{q-m+1} \longrightarrow \dots \longrightarrow Z^q K \longrightarrow 0$$

We have a map

$$\begin{tikzcd}
\bigwedge_{\mathscr{O}}^{q-m} Z^1K \otimes \Gamma_{\mathscr{O}}^m(K^0) \arrow[d, "\wr"] & & \\
\bigwedge_{\mathscr{O}}^{q-m} Z^1K \otimes \mathrm{Sym}_{\mathscr{O}}^m(K^0) \arrow[r] \arrow[d] & \dots \arrow[r] & \bigwedge_{\mathscr{O}}^{q} Z^1K \arrow[r] \arrow[d] & 0 \\
K^{q-m} \arrow[r] & \dots \arrow[r] & Z^qK \arrow[r] & 0.
\end{tikzcd}$$

The point is that m is small enough to turn divided powers into symmetric powers, and those have natural maps into the K^{q-m}'s showing up. In this we, we construct

$$\sigma^{\geq q-m}\,\mathrm{Kos}_q^\bullet(\partial) \xrightarrow{\mu} \sigma^{\geq q-m}\tau^{\leq q}K.$$

To continue, we need to prove that we have a map on smart truncations, i.e. we need to know that

$$\mathrm{Im}(d) \subset \mathrm{Kos}^q(\partial)^{q-m} \longrightarrow K^{q-n}$$

factors through $\mathrm{Im}(d) \subset K^{q-n}$. This is not immediate (unless e.g. $q = m$).

Non-example. Consider the complex $\mathbb{F}_p[x] \to \mathbb{F}_p[x]\mathrm{d}x$ of $\mathbb{F}_p[x^p]$-modules. This complex is $\Omega^\bullet_{\mathbb{A}^1_{\mathbb{F}_p}/\mathbb{F}_p}$. Can look at

$$\Gamma^p_{\mathbb{F}_p[x^p]}(\mathbb{F}_p[x]) \longrightarrow \Gamma^{p-1}_{\mathbb{F}_p[x^p]}(\mathbb{F}_p[x]) \otimes \mathbb{F}_p[x]\mathrm{d}x \simeq \mathrm{Sym}^{p-1} \otimes \mathbb{F}_p[x]\mathrm{d}x \longrightarrow \mathbb{F}_p[x]\mathrm{d}x,$$

and ask whether the image lands in $\mathrm{d}\mathbb{F}_p[x] \subset \mathbb{F}_p[x]\mathrm{d}x$ (here, $m = p-1, q = p$). Note that

$$x^{[p]} \mapsto -x^{p-1} \otimes \mathrm{d}x \mapsto -x^{p-1}\mathrm{d}x$$

which is not a boundary ($\int x^{p-1}\mathrm{d}x$ DNE). $\triangledown$

This is why you need a hypothesis (this bit about $m+1$ being a non zero-divisor). Upon taking cohomology, we get a map

$$\bigwedge\nolimits_{\mathscr{O}}^{j} \left(\mathcal{H}^1(K)\right) \otimes_{\mathscr{O}} \Gamma^{q-j}\left(\mathcal{H}^0(K)\right) \longrightarrow \mathcal{H}^j(K).$$

One can show (see notes) that the LHS above is isomorphic to $\mathcal{H}^j(\mathrm{Kos}_q^\bullet(f))$. Also, the above map is an isomorphism by the Koszul condition. Thus, $\mathcal{H}^j(\mathrm{Kos}_q^\bullet(f)) \xrightarrow{\sim} \mathcal{H}^j(K)$, and this finishes the proof. ■

5 Lecture 5 (9/15)

Errata from last time

(1) Say we have $f : M \to N$. Then,

$$\mathrm{Kos}_q(f) := \left[0 \longrightarrow \Gamma^q_{\mathscr{O}}(M) \longrightarrow \dots \longrightarrow \bigwedge\nolimits^q_{\mathscr{O}} N\right].$$

We wrote the wrong formula for the differential last time. The correct formula is

$$\mathrm{d}\left(y \otimes \gamma_{e_1}(x_1) \otimes \cdots \otimes \gamma_{e_r}(x_r)\right) = \sum_j (y \wedge f(x_j)) \otimes \left(\gamma_{e_1}(x) \otimes \cdots \otimes \gamma_{e_j-1}(x) \otimes \cdots \otimes \gamma_{e_r}(x_r)\right).$$

(2) We also should have defined

$$L\Gamma^q(\mathrm{fib}) \simeq L\bigwedge\nolimits^q(\mathrm{fib}[1])[-q]$$

to get around fib not being connective. Recall $\mathrm{fib}[1] \simeq \mathrm{cof}$.

OH today at 5:30.

From the Achinger-Suh Theorem 4.10, we obtain the following corollary.

Corollary 5.1. *Let X/S have relative dimension $< p$. Assume that $\tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S}$ decomposes. Then, Achinger-Suh tells us that*

$$\bigoplus_j \Omega^j_{X^{(1)}/S}[-j] \simeq F_{X/S,*}\Omega^\bullet_{X/S}\left(\simeq \mathrm{Kos}^\bullet_p(\partial)\right).$$

More generally, we get that any truncation of the form $[a, a+p-2]$ for $p > 2$ or $[a, a+1]$ for $p = 2$ also splits in the same fashion.

We don't know that we can/cannot split beyond this range (sounds like Sasha is working on this/writing something up).

Remark 5.2. The statement of the result before 'More generally,' is from the 80's. The part after 'More generally' is from 2022. ◦

Recall the full Deligne-Illusie theorem had something to do with lifting to $W_2(k)$. Above, we instead have something about splitting of a truncation. In the next few lectures, we want to explain how to get this splitting from a lift. After that, onto Crystalline cohomology.

5.1 'Lecture 3': The one in which we speak some French

5.2 Deformation Theory

What's it all about?

Let's sketch the basic idea. Say we have $\widetilde{A} \twoheadrightarrow A$ map of rings ($\mathrm{Spec}\, A \hookrightarrow \mathrm{Spec}\, \widetilde{A}$ a closed immersion). Say we have some (flat) X/A. When we can lift this to $\widetilde{A}$?

$$\begin{tikzcd} \widetilde{X} \arrow[d, dashed] & X \arrow[l, dashed] \arrow[d] \\ \mathrm{Spec}\, \widetilde{A} & \mathrm{Spec}\, A \arrow[l, hook'] \end{tikzcd}$$

Question 5.3. *When does such a (Cartesian) diagram exist? When one does exist, how many lifts are there?*

(Also, how important is flatness?)

This is a hard thing to answer in general. To make things more tractable, we'll assume that $\widetilde{A} \xrightarrow{\pi} A$ is a **square-zero extension**, i.e. $\ker \pi = I$ with $I^2 = 0$. In this case, we can use (derived) linear algebra.

Remark 5.4. If you have something like $\mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z}$, break it up into a sequence of square-zero extensions, and lift one at a time. ∘

Notation 5.5. For X a scheme, we'll let $X_{\text{ét}}$ denote the small étale site, i.e. $X_{\text{ét}} = \{Y \to X \text{ étale}\}$ and covers are étale covers by finitely many schemes ("**coherent étale topology**").

For us, a (higher) stack is a functor $X_{\text{ét}}^{\text{op}} \to \text{An}$ which is an étale sheaf (An is the ∞-category of anima. If you want, replace it with Grpd).

Theorem 5.6 (Relèvements and Scindage, up to spelling)**.** *Let $X \to S$ be a morphism of $\mathbb{F}_p$-schemes. Fix $\widetilde{S}$ a flat lift of S to $\mathbb{Z}/p^2\mathbb{Z}$, so have*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \widetilde{S} \arrow[d] \\ \operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} \end{tikzcd}$$

(1) *There is an $X^{(1)}$-stack, called $\operatorname{Rel}(X^{(1)}, S)$, such that*

Note $\mathscr{O}$ below is $\mathscr{O}_{X^{(1)}}$, and these $\mathscr{O}$-linear mapping spaces

$$\begin{tikzcd} \operatorname{Rel}(X^{(1)}, S) \arrow[r] \arrow[d] & \operatorname{Map}(L_{X^{(1)}/\widetilde{S}}, \mathscr{O}[1]) \arrow[d] \\ \{\mathrm{id}\} \arrow[r] & \operatorname{Map}(\mathscr{O}, \mathscr{O}) \end{tikzcd}$$

is Cartesian.

(2) *There is also an object called $\operatorname{Sci}\left(\tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S}\right)$ sitting in a Cartesian square*

$$\begin{tikzcd} \operatorname{Sci}\left(\tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S}\right) \arrow[r] \arrow[d] & \operatorname{Map}\left(\tau^{\leq 1} F_{X/S,*}\Omega_{X/S}, \mathscr{O}\right) \arrow[d] \\ \{\mathrm{id}\} \arrow[r] & \operatorname{Map}(\mathscr{O}, \mathscr{O}) \end{tikzcd}$$

(3) *[DI] If $X \to S$ is smooth, then*

$$\operatorname{Rel}(X^{(1)}, S) \simeq \operatorname{Sci}\left(\tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S}\right)$$

as gerbes bounded by $\mathrm{H}^1\left(X^{(1)}, T_{X^{(1)}/S}\right)$.

Note that **(3)** above tells us that splittings of the truncated de Rham complex are exactly the same things as lifts.

5.3 Towards the cotangent complex

Fix k a base (discrete) base ring. We write $\mathrm{AniCAlg}_k$ for animated k-algebras. That is,

$$\mathrm{AniCAlg}_k := \mathrm{Fun}^{\times}\left(\mathrm{Poly}_k^{\mathrm{op}}, \mathrm{An}\right).$$

Note that contains the usual category Calg_k of discrete k-algebras.

Definition 5.7. A **k-linear derivation** of $A \in \mathrm{AniCAlg}_k$ valued in an A-module M is a k-linear morphism $A \xrightarrow{(\mathrm{id},\mathrm{d})} A \oplus M$ which is a k-algebra section of the projection map $A \oplus M \to A$. $\diamond$

(Compare with Remark 2.11).

Recall there's a universal derivation $\mathrm{d} : A \to \Omega^1_{A/k}$ in classical algebra. In higher algebra, there's something similar. We write $\mathrm{Der}_k(A, M)$ for the anima of k-linear derivations of A in M. The **cotangent complex** of A/k, denoted $L_{A/k} \in D(A) = \mathrm{Mod}_A$, is the universal k-linear derivation of A. That is, there's a map $\mathrm{d} : A \to L_{A/k}$ inducing

$$\mathrm{Map}_A(L_{A/k}, M) \simeq \mathrm{Der}_k(A, M).$$

Definition 5.8. A **square-zero extension** of A is a k-algebra map $\widetilde{A} \xrightarrow{s} A$ such that the following is Cartesian in $\mathrm{AniCAlg}_k$:

$M[1]$ is concentrated in homological degree 1 if M is discrete.

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d] & A \arrow[d, "(\mathrm{id},\mathrm{d}_s)"] \\ A \arrow[r, "(\mathrm{id},0)"] & A \oplus M[1] \end{tikzcd}$$

for some $M \in D(A)_{\geq 0}$ and some $\mathrm{d}_s : A \to M[1]$. $\diamond$

In the above diagram, the fiber of either row is M.

Remark 5.9. There's only a bit of derived AG showing up above, just the M shifted by 1. Alternatively, even in the classical world with $A, \widetilde{A}$ both discrete, there's a bit of the derived world hiding in the picture. $\circ$

Definition 5.10. If $\widetilde{A} \to A$ is a square zero extension of A, then a **deformation** of $B \in \mathrm{AniCAlg}_A$ to $\widetilde{A}$ is a pair $(\widetilde{B}, \widetilde{\alpha})$ such that $\widetilde{B} \in \mathrm{AniCAlg}_{\widetilde{A}}$ and

$$\widetilde{\alpha} : \widetilde{B} \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} A \xrightarrow{\sim} B.$$

Say $A, \widetilde{A}, B$ are all discrete. Then, a **flat deformation** is one where $\widetilde{B}$ is a flat $\widetilde{A}$-algebra so that $\widetilde{B} \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} A \simeq \widetilde{B} \otimes_{\widetilde{A}} A \simeq B$. $\diamond$

Sounds like in the flat case, $\widetilde{B}$ will always be discrete

Remark 5.11. Consider cofiber sequence

$$I \longrightarrow \widetilde{A} \longrightarrow A$$

in $\mathrm{Mod}_{\widetilde{A}}$. Apply $\overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B}$ to get

$$I \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B} \longrightarrow \widetilde{B} \longrightarrow B.$$

Let $J := \ker\left(\widetilde{B} \longrightarrow B\right)$. What can we say above

$$I \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B} \longrightarrow J?$$

This won't be an iso in general (image $\widetilde{B}, B$ both discrete. Then, J will be, but the derived tensor product certainly doesn't have to be). Sounds like this map is the sort of thing that would encode flatness. ○

Theorem 5.12 (cotangent complex). *Let $A \in \mathrm{AniCAlg}_k$. Then,*

(1) *For $A \to B$, can form relative cotangent complex $L_{B/A}$. This $L_{B/A}$ is in $D(B)_{\geq 0}$.*

(2) $\pi_0(L_{B/A}) \simeq \Omega^1_{\pi_0(B)/\pi_0(A)}$.
(Note π_0 is just H_0)

(3) *If $A \to B$ is a surjection of discrete rings with kernel I, then*

$$\pi_0(L_{B/A}) = 0 \text{ but } \pi_1(L_{B/A}) = I/I^2$$

is the conormal sheaf.

(4) *Say we have $A \to A'$. Then,*

$$L_{B/A} \overset{\mathrm{L}}{\otimes}_A A' \simeq L_{B \overset{\mathrm{L}}{\otimes}_A A'/A'}.$$

That is, the cotangent complex is compatible with (derived) base change. For this one, keep in mind the square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \overset{\mathrm{L}}{\otimes}_A A' \end{array}$$

(5) *Say we have $A \to B \to B'$. Then, we have a cofiber sequence*

$$L_{B/A} \overset{\mathrm{L}}{\otimes}_B B' \longrightarrow L_{B'/A} \longrightarrow L_{B'/B}$$

*(**transitivity sequence**).*

(6) *Say A, B are both discrete if you want.*

It sounds liked this might not be necessary, at least for the first bullet

- *If B is étale over A, then $L_{B/A} \simeq 0$.*
  (étale $\iff$ $L_{B/A} = 0$ + finite presentation).
- *Similarly, if B is smooth over A, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$.*
- *Lastly, if $A \to B$ is surjective with kernel generated by a regular sequence, then $L_{B/A} \simeq I/I^2[1]$.*

Remark 5.13 (Computing $L_{B/A}$). Take a simplicial object $P_\bullet$ whose space of n-simplices is a polynomial A-algebra. Choose an equivalence $P_\bullet \xrightarrow{\sim} B$. Then, $L_{B/A}$ can be computed as

$$L_{B/A} \simeq \left| B \overset{\mathrm{L}}{\otimes}_{P_\bullet} \Omega^1_{P_\bullet/A} \right|$$

($|\cdot|$ is geometric realization). $\circ$

5.4 Using all of this

Say we have $\widetilde{A} \to A$ square-zero, and we're given $A \xrightarrow{f} B$ (Assume $\widetilde{A}, A$ discrete if you want). We're interested in filling out

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d, dashed] & A \arrow[d, "f"] \\ \widetilde{B} \arrow[r, dashed] & B \end{tikzcd}$$

The first thing we want to do is reduce the number of dashed arrows. Since $\widetilde{A} \to A$ is square-zero, say with kernel I, it is equivalently given by a map $L_{A/k} \to I[1]$.[6] Using $f : A \to B$, we can form

$$\begin{tikzcd} L_{A/k} \arrow[r] \arrow[d] & I[1] \arrow[d] \\ L_{A/k} \overset{\mathrm{L}}{\otimes}_A B \arrow[r] & I[1] \overset{\mathrm{L}}{\otimes}_A B \end{tikzcd}$$

Defining the lift $\widetilde{B} \to B$ is the same as filling out

$$\begin{tikzcd} L_{A/k} \arrow[r] \arrow[d] & I[1] \arrow[d] \\ L_{A/k} \overset{\mathrm{L}}{\otimes}_A B \arrow[r] \arrow[d] & I[1] \overset{\mathrm{L}}{\otimes}_A B \arrow[d, equal] \\ L_{B/k} \arrow[r, dashed] & I[1] \overset{\mathrm{L}}{\otimes}_A B. \end{tikzcd}$$

The top row above isn't really necessary at this point. We just want to construct $L_{B/k} \dashrightarrow I[1] \overset{\mathrm{L}}{\otimes}_A B$ so that

$$\begin{tikzcd} L_{A/k} \overset{\mathrm{L}}{\otimes}_A B \arrow[rr] \arrow[dr] & & L_{B/k} \arrow[dl] \\ & I[1] \overset{\mathrm{L}}{\otimes}_A B & \end{tikzcd}$$

commutes (pretend the downward map on the right is dashed). That is, we only need to care about one arrow instead of two now. Note Theorem 5.12**(5)** applied to $k \to A \to B$ gives cofiber sequence

$$L_{A/k} \overset{\mathrm{L}}{\otimes}_A B \longrightarrow L_{B/k} \longrightarrow L_{B/A}.$$

[6] Since $\widetilde{A} \to A$ is square-zero, it comes equipped with a derivation $A \to I[1]$

Consider now

$$\begin{tikzcd}
L_{B/A}[-1] \arrow[d] \arrow[dr, "o(A,B,\widetilde{A})"] & \\
L_{A/k} \overset{\mathrm{L}}{\otimes}_A B \arrow[r] \arrow[d] & I[1] \overset{\mathrm{L}}{\otimes}_A B \\
L_{B/k} \arrow[ur, dashed] &
\end{tikzcd}$$

The column above is a cofiber sequence, so the dashed map exists iff $o(A, B, \widetilde{A})$ vanishes. That is, there is a **Kodaira-Spencer class**

$$o(A, B, \widetilde{B}) \in \left[L_{B/A}, I[2] \overset{\mathrm{L}}{\otimes}_A B\right] = \mathrm{Ext}^2_B \left(L_{B/A}, I \overset{\mathrm{L}}{\otimes}_A B\right)$$

which vanishes iff a lift exists. Note that 2 above is there since $1 - (-1) = 2$.

Now that we've figured out the obstruction to the existence of a lift, let's parametrize all possible lifts. Consider

$$\begin{tikzcd}
L_{A/k} \overset{\mathrm{L}}{\otimes}_A B \arrow[r, "e-s\simeq 0"] \arrow[d] & I[1] \overset{\mathrm{L}}{\otimes}_A B \arrow[d, equal] \\
L_{B/k} \arrow[r, "e-s"] \arrow[d] & I[1] \overset{\mathrm{L}}{\otimes}_A B \\
L_{B/A}. \arrow[ur] &
\end{tikzcd}$$

That is, we have two lifts e, s and we compare then by looking at $e - s$. They are both lifts, so $e - s|_{L_{A/k} \overset{\mathrm{L}}{\otimes}_A B} \simeq 0$ ('is homotopic to'). Therefore, since the left column is a cofiber sequence, we get an induced map $L_{B/A} \to I[1] \overset{\mathrm{L}}{\otimes}_A B$. Conversely, given such a map, you can add it to a lift to get another lift. Hence, the set of deformations is a torsor under

$$\left[L_{B/A}, I[1] \overset{\mathrm{L}}{\otimes}_A B\right] = \mathrm{Ext}^1_B \left(L_{B/A}, I \overset{\mathrm{L}}{\otimes}_A B\right)$$

6 Lecture 6 (9/20)

No OH this week.

Where are we? Say $\widetilde{S}$ is a fixed flat lift of S over $\mathbb{Z}/p^2\mathbb{Z}$. Then we have $X^{(1)}$-stacks

$$\begin{tikzcd}
\mathrm{Rel}(X^{(1)}, S) \arrow[r] \arrow[d] & \mathrm{Map}(L_{X^{(1)}/\widetilde{S}}, \mathscr{O}[1]) \arrow[d] \\
\{\mathrm{id}\} \arrow[r] & \mathrm{Map}(\mathscr{O}, \mathscr{O})
\end{tikzcd}$$

fitting into the above Cartesian diagram. We also have another Cartesian diagram

$$\begin{tikzcd} \mathrm{Sci}\left(\tau^{\le 1}F_{X/S,*}\Omega^{\bullet}_{X/S}\right) \arrow[r] \arrow[d] & \mathrm{Map}\left(\tau^{\le 1}F_{X/S,*}\Omega_{X/S}, \mathscr{O}\right) \arrow[d] \\ \{\mathrm{id}\} \arrow[r] & \mathrm{Map}(\mathscr{O}, \mathscr{O}) \end{tikzcd}$$

If X/S is smooth, there there's an equivalence

$$\mathrm{Rel}(X^{(1)}, S) \simeq \mathrm{Sci}(\tau^{\le 1}(blah)).$$

The above is the 'geometric part of the this Deligne-Illusie theorem' (and Achinger-Suh is its 'algebraic part'). Last time we were introduced to the cotangent complex. It parameterizes the following problem

$$\begin{tikzcd} \widetilde{A} \arrow[r] \arrow[d, dashed] & A \arrow[d] \\ \widetilde{B} \arrow[r, dashed] & B \end{tikzcd}$$

($\widetilde{A} \to A$ square zero extension; want (derived) cocartesian diagram). In particular, we saw that the obstruction to finding such a diagram is an element

$$o\left(\widetilde{A}, A, B\right) \in \mathrm{Ext}^2\left(L_{B/A}, I \overset{\mathrm{L}}{\otimes}_A B\right).$$

Furthermore, the set of such lifts forms a torsor under $\mathrm{Ext}^1(L_{B/A}, I \overset{\mathrm{L}}{\otimes}_A B)$. In fact, using the same sort of reasoning as last time, one can check that automorphisms of a lift are given by $\mathrm{Ext}^0(L_{B/A}, I \overset{\mathrm{L}}{\otimes}_A B)$ (note $\mathrm{Ext}^0 = \mathrm{Hom}$).

Now, we'd like conditions for $\widetilde{A} \to \widetilde{B}$ to be flat, so everything in sight will be classical.

Remark 6.1. Apparently often in derived AG, you do a bunch of derived stuff, and then at the end try to see if you can argue that the final answer is actually classical. ◦

6.1 Flat deformations

We want to do three things simultaneously

- ensure everything in sight is classical.
- ask for flat lifts.
- globalize from rings to schemes.

  (next week, joint OH held by Elden and Taeuk)

In other words, we want

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "f"', "\text{flat}"] & \widetilde{X} \arrow[d, dashed, "\text{flat}"] \\ S \arrow[r, hook] & \widetilde{S} \end{tikzcd} \tag{6.1}$$

Above, $S \hookrightarrow \widetilde{S}$ is a closed immersion with square-zero ideal sheaf. Also, all schemes are classical. Also, the square should be Cartesian.

Let's examine our constraints. From such a picture, we see that we have an $\widetilde{S}$-linear deformation of X given by $\widetilde{X}$ (ignore S). By general theory, such a thing is classified by a map

$$L_{X/\widetilde{S}} \longrightarrow \mathscr{J}[1],$$

for some $\mathscr{O}_X$-module $\mathscr{J}$ (think: ideal sheaf of $X \hookrightarrow \widetilde{X}$).

Warning 6.2. $X/\widetilde{S}$ is not necessarily smooth, so $L_{X/\widetilde{S}}$ is some actual non-trivial "complex". •

Since we insist on everything being discrete, then $\mathscr{J}$ better be a discrete $\mathscr{O}_X$-module. At this point, we can ask for two conditions which are natural.

(1) $Lf^*I \simeq \mathscr{J}$, where I is the ideal sheaf of S in $\widetilde{S}$.

Since I is square-zero, we can view it as a module on S or on $\widetilde{S}$

In particular, the derived pullback is actually discrete. Note that $Lf^* \simeq f^*$ since f is flat.

(2) Ask for (6.1) to be derived Cartesian ('no higher Tor information')

By some previous remark, **(2)** $\implies$ **(1)**. In the affine case ($S = \operatorname{Spec} A$ and $\widetilde{S} = \operatorname{Spec} \widetilde{A}$), tensoring $I \hookrightarrow \widetilde{A} \twoheadrightarrow A$ gives

$$I \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B} \longrightarrow \widetilde{B} \longrightarrow A \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B}, \tag{6.2}$$

and $A \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B} \simeq A \otimes_{\widetilde{A}} B$ if (6.1) is derived cartesian.

Lemma 6.3. *In the above situation, TFAE*

(1) *as above*

(2) *as above*

(3) *the morphism $\widetilde{f} : \widetilde{X} \to \widetilde{S}$ is flat*

Proof. Let's translate everything to algebra. We have

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & \widetilde{A} \arrow[r] \arrow[d, "\widetilde{f}"] & A \arrow[d] \\
J \arrow[r] & \widetilde{B} \arrow[r] & B
\end{tikzcd}$$

(**(2)** $\implies$ **(1)**) We have a map $\widetilde{B} \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} B \to B$ which we want to prove is an equivalence. By (6.2), if $I \overset{\mathrm{L}}{\otimes}_A B \simeq I \otimes_A B \simeq J$, then the map is an iso.

(**(1)**, **(2)** $\iff$ **(3)**) We want to show that $\widetilde{f}$ is flat. Let N be an $\widetilde{A}$-module. We want to show that $N \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} B \simeq N \otimes_{\widetilde{A}} B$. We have an exact sequence

$$0 \longrightarrow IN \longrightarrow N \longrightarrow N/IN \longrightarrow 0.$$

To check that $N \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} \widetilde{B}$ is discrete, we may check this for IN and N/IN, i.e. we may assume that N is killed by I. Hence, we may assume that N is naturally an A-module. In this case,

$$\widetilde{B} \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} N \simeq \widetilde{B} \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} A \overset{\mathrm{L}}{\otimes}_{A} N.$$

Now, we get discreteness iff $\widetilde{B} \overset{\mathrm{L}}{\otimes}_{\widetilde{A}} A \simeq \widetilde{B} \otimes_{\widetilde{A}} A$. ■

Hence, the flatness condition for the diagram (6.1) is governed by the condition

$$Lf^* I \simeq f^* I \simeq \mathscr{J}.$$

How do we use this?

Recall a diagram like (6.1) is classified by a map $L_{X/\widetilde{S}} \to \mathscr{J}[1]$. This fits into

$$\begin{tikzcd} f^* L_{S/\widetilde{S}} \arrow[r] \arrow[d] & L_{X/\widetilde{S}} \arrow[r] \arrow[d] & L_{X/S} \\ f^* I[1] \arrow[r] & \mathscr{J}[1] & \end{tikzcd}$$

If we assume that $S \hookrightarrow \widetilde{S}$ is l.c.i, then $f^* L_{S/\widetilde{S}} \xrightarrow{\sim} f^* I[1]$ is an equivalence above. Thus, the condition that $\widetilde{X} \to \widetilde{S}$ be flat is equivalent to the existence of a splitting map

There was some discussion about things not being quite right with this l.c.i condition (l.c.i should mean the kernel is generated by a regular sequence, but this is impossible if the kernel is square-zero). I'm confused by the resolution. Maybe things will be cleared up later?

$$L_{X/\widetilde{S}} \longrightarrow f^* L_{S/\widetilde{S}}.$$

Lemma 6.4. *Say we have a flat deformation*

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \widetilde{S} \arrow[d] \\ \operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} \end{tikzcd}$$

Then,

$$\begin{tikzcd} \mathrm{Rel}(X^{(1)}, S) \arrow[r] \arrow[d] & \mathrm{Map}(L_{X/\widetilde{S}}, f^* I[1] \simeq \mathscr{O}[1]) \arrow[d] \\ \{\mathrm{id}\} \arrow[r] & \mathrm{Map}(\mathscr{O}[1], \mathscr{O}[1]) \end{tikzcd}$$

(is Cartesian?)

Here's a sample application.

Proposition 6.5. *Let R be a perfect $\mathbb{F}_p$-algebra. Then, there is a unique flat $\mathbb{Z}_p$-algebra $W(R)$ such that $W(R)/p \simeq R$, i.e. we have*

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[d] & \operatorname{Spec} W(R) \arrow[d] \\ \operatorname{Spec} \mathbb{F}_p \arrow[r] & \operatorname{Spec} \mathbb{Z}_p. \end{tikzcd}$$

Furthermore, if S is a p-complete ring, then any map $R \to S/p$ lifts uniquely to a p-adically continuous map $W(R) \to S$.

Proof Idea. When R is perfect, $L_{R/\mathbb{F}_p} \simeq 0$ ('is acyclic'). ∎

(The idea is that $\mathrm{d}y = \mathrm{d}x^p = 0$ since R is perfect. Frobenius acts by 0 on the cotangent complex, but R is perfect, so it's also an automorphism).

Remark 6.6. In general, it's easy to map into the Witt vectors (think, easy to map into $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$). For perfect rings, you can map out of them easily too. ○

Let's see how $\mathrm{Res}(X^{(1)}, S)$ looks when X/S is smooth. Stare at (note $\Omega^1_{X^{(1)}/S} \simeq L_{X^{(1)}/S}$ by smoothness)

$$\begin{tikzcd}
\text{fiber of vertical map} \arrow[r, "\sim"] \arrow[d] & \mathrm{Map}(L_{X^{(1)}/S}, \mathscr{O}[1]) \arrow[r, "\sim"] \arrow[d] & \mathcal{H}^1\left(X^{(1)}, T_{X^{(1)}/S}\right) \\
\mathrm{Rel} \arrow[r] \arrow[d] & \mathrm{Map}(L_{X^{(1)}/S}, \mathscr{O}[1]) \arrow[d] & \\
\{\mathrm{id}\} \arrow[r] & \mathrm{Map}(\mathscr{O}[1], \mathscr{O}[1]). &
\end{tikzcd}$$

Somehow this is supposed to tell you that this Rel object is a gerbe whose band in this $\mathcal{H}^1$, whatever these words mean. We can examine the same picture for

$$\begin{tikzcd}
\mathcal{H}^0(F_{X/S,*})[0] \arrow[r] \arrow[d, "\wr"] & \tau^{\leq 1} F_{X/S,*}\Omega^\bullet_{X/S} \arrow[r] & \mathcal{H}^1(F_{X/S,*})[-1] \arrow[d, "\wr"] \\
\mathscr{O}[0] & & \Omega^1_{X^{(1)}/S}[-1]
\end{tikzcd}$$

Hom this into $\mathscr{O}$ to get

$$\begin{tikzcd}
\text{vert fiber} \arrow[r, "\sim"] \arrow[d] & \mathrm{Map}(\Omega^1_{X^{(1)}/S}, \mathscr{O}[1]) \arrow[d] \\
\mathrm{Sci}(\tau^{\leq 1}) \arrow[r] \arrow[d] & \mathrm{Map}(\tau^{\leq 1}, \mathscr{O}) \arrow[d] \\
\{\mathrm{id}\} \arrow[r] & \mathrm{Map}(\mathscr{O}, \mathscr{O})
\end{tikzcd}$$

This looks a lot like the picture we have before (recall $\Omega^1_{X^{(1)}/S} \simeq L_{X^{(1)}/S}$ by smoothness). We want to prove that, for X/S smooth, there's a natural map (equivalence?)

$$\mathrm{Rel}(X^{(1)}, S) \longrightarrow \mathrm{Sci}(\tau^{\leq 1} F_{X/S,*}).$$

Remark 6.7 (why should you believe this). Have fiber sequence/triangle

$$\mathscr{O}[1] \longrightarrow L_{X^{(1)}/\widetilde{S}} \longrightarrow L_{X^{(1)}/S}$$

as well as

$$\mathscr{O} \longrightarrow \tau^{\leq 1} \longrightarrow \Omega^1_{X^{(1)}/S}[-1]$$

Shifting the top sequence by $[-1]$ makes it look like the bottom sequence (at least on the outsides). Need a map realizing this sameness. $\circ$

Theorem 6.8. $L_{X^{(1)}/\widetilde{S}}[-1] \simeq \tau^{\leq 1}$

Instead of directly building a map between these two, it'll be more reasonable to build

$$\Phi : \mathrm{Rel}(X^{(1)}, S) \longrightarrow \mathrm{Sci}(\tau^{\leq 1} F_{X/S,*}).$$

We'll do so by "making the problem over-determined." We'll instead build a map

$$\mathrm{Rel}(\widetilde{X^{(1)}, S}) \longrightarrow \mathrm{Sci}(\widetilde{\tau^{\leq 1}})$$

and then show that it's nice enough to descend. The LHS parameterizes the datum of an open $U \subset X$ and an $\widetilde{S}$-lift of $U^{(1)}$, called $\widetilde{U^{(1)}}$ (so far this is data parameterized by $\mathrm{Rel}(X^{(1)}, S)$) + a map $\widetilde{U} \to \widetilde{U^{(1)}}$ lifting the relative Frobenius. That is, an entire lift of the our favorite Figure 1. The RHS parameterizes splittings in $K(\mathscr{O}_U)$ (in literal complexes). That is, choices $\tau^{\leq 1} \simeq [K^0 \longrightarrow Z^1K]$ along with a choice of $s : \mathcal{H}^1 \to Z^1K$ which is a splitting.

Warning 6.9. Both of these objects are secretly stacks. Above, we've suppressed what the automorphisms are. $\bullet$

To each such lift of Frob, can consider (recall proof of Theorem 3.7)

$$\begin{tikzcd}
\Omega^1_{U^{(1)}/S} \arrow[r] \arrow[d, equal] & Z^1 F_{X/S,*}\Omega_{X/S} \arrow[d] \\
\Omega^1_{U^{(1)}/S} \arrow[r, "C^{-1}", "\sim"'] & \mathcal{H}^1(-)
\end{tikzcd}$$

Thus, when we have a lift of Frobenius, we get a splitting of $\tau^{\leq 1}$.

7 Lectures 7,8 (9/22,27): Didn't go (see Elden's notes and also the reference [BLM21] therin)

Remark 7.1. One thing not mentioned in the notes, but that apparently was mentioned in class, is that it sounds like we'll use Crystalline cohomology to show that any **Fano variety**[7] X/k over a finite field has a k-point. $\circ$

8 Lecture 9 (9/29)

OH Today at 5pm (SC231)

Last time saw two processes on Dieudonné complexes

[7] A variety whose anticanonical bundle ω_X^{-1} is ample

- saturation

  Recall a Diedonné complex comes equipped with a map $M^* \xrightarrow{F} (\eta_p M)^*$. Saturation forces this to be an isomorphism.

- V-completion

  If you have a saturated complex, can produce operator $V : M^* \to M^*$. The completion is the limit of $W_r(M)^* = M^*/(V^r + dV^r)$

Remark 8.1. 'Saturation' includes torsion-free as a hypothesis. ◦

Theorem 8.2. *Let M^* be saturated. Then,*

$$M^* \longrightarrow W(M)^*$$

witnesses derived p-completion.

Have inclusions

$$\mathrm{DC}_{\mathrm{str}} \hookrightarrow \mathrm{DC}_{\mathrm{sat}} \hookrightarrow \mathrm{DC}$$

We saw last time that there's a left adjoint $\mathrm{Sat}(-) : \mathrm{DC} \to \mathrm{DC}_{\mathrm{sat}}$. This was given by

$$\mathrm{Sat}(M)^* = \eta_p^\infty (M/M_{tors})$$

(the '∞' is meant to indicate an infinite colimit).

Remark 8.3. Let's describe $\mathrm{Sat}(M)^*$ when F acts injectively, so $M^* \hookrightarrow M^*[F^{-1}] = \{F^{-r}x\}_{r\in\mathbb{Z}}$. Now, if M^* is saturated, then F gives an iso

$$F : M \xrightarrow{\sim} \left\{x : \mathrm{d}x \in pM^{*+1}\right\}.$$

In fact, (Prop 2.2.5 in [BLM21])

$$F^r : M^* \xrightarrow{\sim} \left\{x : \mathrm{d}x \in p^r M^{*+1}\right\}.$$

We'll use this observation to describe $\mathrm{Sat}(M)^* \hookrightarrow M^*[F^{-1}]$. It is the subgroup

$$\{x : \mathrm{d}(F^n x) \in p^n M^* \ \text{ for some } \ n \gg 0\}$$

(note: equivalent above to write for all $n \gg 0$). Notice we said sub*group*, not sub*complex* above. The differentials are given by

$$x \mapsto F^{-n} p^{-n} \mathrm{d}(F^n x).$$

n is not the degree of x, but the same sort of random really big n as in the description of $\mathrm{Sat}(M)^*$

We can recast the above in a more succient way.

First consider

$$\begin{array}{rccc} \mathrm{d} : & M^*[F^{-1}] & \longrightarrow & M^*[F^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[1/p] \\ & F^{-n}x & \longmapsto & p^{-n}F^{-n}\mathrm{d}x \end{array}$$

(map of graded groups. I *think* not a map of complexes; haven't checked). Then, $\mathrm{Sat}(M)^*$ is the subcomplex (of this two-object complex?) of those elements with *integral derivatives*, i.e. $y \in M^*[F^{-1}]$ s.t. $\mathrm{d}y \in M^*[F^{-1}]$. ○

We'll use this sort of more concrete description to later compute $W_k\Omega^*_{\mathbb{A}^r \times \mathbb{G}_m^s/\mathbb{F}_p}$.

We now want a left adjoint to the inclusion $\mathrm{DC}_{\mathrm{str}} \hookrightarrow \mathrm{DC}_{\mathrm{sat}}$. It's possible to show such a thing exists by abstract nonsense, but we really want a computable description.

Lemma 8.4. *If M^* is saturated, then*

$$W(M^*) := \varprojlim_r W_r(M)$$

is also saturated.

Remark 8.5. There are two sorts of 'localizations' (saturation and strictness). A priori, you might worry you need to do both infinitely many times to get something both saturated and V-complete, but this lemma tells us that that is not the case. ○

Proof. Let's begin by consolidating structures present in the tower

$$\ldots \longrightarrow W_r(M) \longrightarrow \ldots \longrightarrow W_2(M) \longrightarrow W_1(M).$$

(1) Restriction

$$\begin{tikzcd} W_{r+1}(M) \arrow[r, "R"] \arrow[d, equal] & W_r(M) \\ M/(V^{r+1} + \mathrm{d}V^{r+1}) & \end{tikzcd}$$

(2) Frobenius

$$W_{r+1}(M) \xrightarrow{F} W_r(M)$$

(3) Verschiebung

$$W_r(M) \xrightarrow{V} W_{r+1}(M)$$

(note this one increases r)

We'll prove the lemma in steps.

- Claim 1: $W(M)^*$ is p-torsion free

  Use the short exact sequence[8]

  $$0 \longrightarrow W_{r+1}(M)[p] \longrightarrow W_{r+1}(M) \xrightarrow{R} W_r(M) \longrightarrow 0$$

  (Exercise: prove above sequence is short exact). Given this, if we have p-torsion in stage $r+1$, then it must die in stage r under R. Hence, $W(M)^*$ must be p-torsion free.

- Claim 2: if $\mathrm{d}x$ is p-divisible, then x must be F-divisible (i.e. $x = Fy$ for some y).

[8] Think of $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z} \to 0$

Say $x \in W_r(M)^*$ at some finite level. Then, $\mathrm{d}x$ being p-divisible means that

$$\mathrm{d}x = py + V^r a + \mathrm{d}V^r b$$

for some y, a, b. Applying d tells us that $\mathrm{d}(V^r a) = -p\mathrm{d}y$ is p-divisible. Now we appeal to the following.

Lemma 8.6. *Say M^* is saturated. If x satisfies $\mathrm{d}(V^r x) \in pM^{*+1}$, then x is in the image of F.*

For the above lemma, we need only prove that $\mathrm{d}x$ is p-divisible (since M^* saturated). Recall that $F\mathrm{d}V = \mathrm{d}$, so $F^r \mathrm{d}V^r = \mathrm{d}$, so

$$\mathrm{d}x = F^r dV^r x \in F^r pM^{*+1} \subset pM^{*+1},$$

and this proves the lemma.

Back to claim, we conclude that $a = Fa'$, so that $\mathrm{d}(x - V^r b) = p(y + V^{r-1}a')$. As M^* is saturated, we then get that $x - V^r b = Fz$. Therefore, x is F-divisible modulo V^r. This is what we wanted. Going from finite stages to the limit is left as an exercise. ■

To really compute the left adjoint $L : \mathrm{DC} \to \mathrm{DC}_{\mathrm{str}}$ of the forgetful functor, we still need the following theorem.

Theorem 8.7. *If M^* is saturated, then $W(M)^*$ is in fact strict.*

(From this, one concludes that $L = W(\mathrm{Sat}(-)))$

By definition, to prove the theorem, we need to prove that

$$W(M)^* \longrightarrow W(W(M))^*$$

is an isomorphism (assuming M^* is saturated). We start with the following lemma (which maybe explains the choice of the word "strict").

Lemma 8.8. *Say $f : M^* \to N^*$ is a morphism of saturated Dieudonné complexes. Then, f/p is a quasi-isomorphism $\iff$ $W_1(f)$ is an isomorphism.*

The same statement holds with f/p^r and $W_r(f)$

Proof sketch. There is an induced map

$$W_1(M) \xrightarrow{F} H^*(M/pM).$$

It suffices to show that this map is an isomorphism.

(Injectivity) Say Fx is a boundary mod p, i.e. $Fx = py + \mathrm{d}z$. We want to show that $x \in \mathrm{Im}(V) + \mathrm{Im}(\mathrm{d}V)$. Note

$$px = VFx = V(py + \mathrm{d}z) = pVy + p\mathrm{d}Vz = p(Vy + \mathrm{d}Vz),$$

so we win since M^* is p-torsion free ($\Longleftarrow$ saturated).

(Surjectivity) A cocycle is simply an element s.t. $\mathrm{d}x$ is p-divisible. By saturation, this means x is F-divisible. ■

Proof of Theorem 8.7. We need to check that

$$W_r(M^*) \longrightarrow W_r(W(M))^*$$

is an iso. One can easily reduce this to the case $r = 1$. Lemma 8.8 then reduces this to checking an isomorphism on mod p cohomology. See [BLM21, Corollary 2.7.6]. ■

8.1 Lecture 5: The one where we get confused about indices

Let's see where we are at. We have W and Sat. The de Rham-Witt (dRW) complex is roughly given as follows: say $R \in \mathrm{Calg}^{\heartsuit}_{\mathbb{F}_p}$ is a discrete $\mathbb{F}_p$-algebra. Let $\widetilde{R}$ be a lift with a lift of Frobenius (e.g. $W(R)$ with its canonical lift). Then you form

$$W(\mathrm{Sat}(\Omega^*_{\widetilde{R}})) =: W\Omega^*_R.$$

The LHS is the whole construction, but we need to confirm that this is independent of choices. We'll do so by showing it satisfies some universal property.

Remark 8.9. If you want a cohomology theory for char p schemes, a goto example is étale ℓ-adic cohomology. But to define this, you first have to develop the formalism of sites. Arguably, this de Rham Witt complex is simpler by comparison. ○

Let's discuss algebra structures.

Definition 8.10. A **Dieudonné algebra** (D-algebra) is a commutative algebra object in DC such that

(1) $A^n = 0$ for $n < 0$

(2) $x \in A^0 \implies Fx \equiv x^p \mod p$

(3) It's a *strct* cdga, i.e. $x^2 = 0 \impliedby |x|$ odd. ◇

Assumption (for rest of class)**.** All cdga's we'll ever talk above are strict.

Remark 8.11. Should suspect that $W\Omega^0_R$ is the Witt vectors of R, so $W\Omega^*_R$ will be a module over $W(\mathbb{F}_p) = \mathbb{Z}_p$. ○

Remark 8.12. $W_1(A)^0 = A^0/VA^0$ is an $\mathbb{F}_p$-algebra (note $V(1) = V(F(1)) = p$ is killed), i.e. VA^0 is an ideal. ○

Lemma 8.13. *Let A^* be a Dieudonné algebra. Then, there is a unique ring structure on $W(\mathrm{Sat}(A))^*$ making the canonical map*

$$A^* \longrightarrow W(\mathrm{Sat}(A))^*$$

a map of cdga's. Furthermore, $W(\mathrm{Sat}(A))^*$ is a strict D-algebra.

(see Elden's notes for proof)

Definition 8.14. We say that a D-algebra is **strict** if it is saturated and $A^* \to W(A^*)$ is an isomorphism. ◇

Definition 8.15. Let R be an $\mathbb{F}_p$-algebra. Then, the **saturated de Rham-Witt complex** of R, denoted $W\Omega^*_R$, is the initial strict D-algebra equipped with a map

$$R \longrightarrow W\Omega^0_R/V \cdot W\omega^0_R.$$

$\diamond$

(compare with Theorem 3.3)

Warning 8.16. There's another notion of 'de Rham Witt complex' (due to Deligne?) and it differs from this one in general. They agree whenever R is a smooth $\mathbb{F}_p$-algebra though. •

Remark 8.17. You can take the hypercohomology of $W\Omega^*_R$ (this will be Crystalline cohomology), but you can also look at the graded pieces $W\Omega^j_R$, so you get more invariants than just a cohomology theory. ◦

Theorem 8.18. *The saturated dRW complex exists.*

We'll use two lemmas in the proof of this.

Lemma 8.19. *Let A^* be saturated. Then, A^0/VA^0 is reduced.*

Lemma 8.20 (Key)**.** *Let B^* be a strict D-algebra, and $R \in \mathrm{Calg}^{\heartsuit}_{\mathbb{F}_p}$. Then, TFAE*

(1) *A ring map $R \to B^0/VB^0$*

(2) *A ring map $W(R) \to B^0$ fitting into commutative diagrams*

$$\begin{array}{ccc} W(R) & \longrightarrow & B^0 \\ \downarrow & & \downarrow \\ R & \longrightarrow & B^0/VB^0 \end{array} \quad \textit{and} \quad \begin{array}{ccc} W(R) & \longrightarrow & B^0 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ W(R) & \longrightarrow & B^0 \end{array}$$

Remark 8.21. Mapping into the Witt vectors is easy. When R is perfect, mapping out of the Witt vectors is also easy. This key lemma tells us that, within the context of D-algebras, the Witt vectors have a left adjoint property. You can map out of them (even for non-perfect R) as long as the target is the degree 0 piece of a strict D-algebra. ◦

We will prove the lemmas later. For now, let's see how they imply the theorem.

Proof of Theorem 8.18. By Lemma 8.19, $R \to A^0/VA^0$ factors through its reduction, so we may (and do) assume R is reduced. In particular, this implies that $W(R)$ is p-torsion free. Hence, we have a D-algebra $\Omega^*_{W(R)}$ w/ F. Set

$$W\Omega^*_R := W\,\mathrm{Sat}(\Omega^*_{W(R)}).$$

Let B^* be a strict Dieudonné algebra. By various universal properties, the following data are equivalent

(1) A map $W\Omega^*_R \longrightarrow B^*$

(2) A map $\Omega^*_{W(R)} \longrightarrow B^*$
(equiv to **(1)** by adjunction)

(3) A map $W(R) \longrightarrow B^0$ intertwining Frobenius

(use Theorem 3.3 to get equiv to **(2)**)

(4) A map $R \longrightarrow B^0/VB^0$ of $\mathbb{F}_p$-algebras

(equiv to **(3)** by Lemma 8.20) ■

Remark 8.22. For simply showing $W\Omega_R^*$ exists, one could use the adjoint functor theorem. The above approach also gives a particular construction. ○

Since we have time, let's prove Lemma 8.19.

Proof of Lemma 8.19. Say $x \in A^0$ and assume $\overline{x}^p = 0$ ($\overline{x}$ is reduction mod V). We need to show that $x \in VA^0$. Recall $Fx \equiv x^p \mod p$. Thus, $Fx = Vy$ for some $y \in A^0$...

Ok, maybe we didn't have that much time. We'll finish up next time. ■

9 Lecture 10 (10/4)

Note 2. 3 minutes late

Today: 2 key lemmas. Next: all Fano/$\mathbb{F}_q$ have rational point.

Recall 9.1. **(1)** D-complex $M^* \xrightarrow{F} \eta_p M^*$. Saturated $\iff$ F an iso $\iff$ (dx p-divisible $\iff$ $x = Fy$)

(2) M saturated gives rise to $V : M \to M$ s.t. $FV = VF = p$. Can then define $W_k M = M/(V^m + \mathrm{d}V^m)$ and also $W(M) = \varprojlim W_k(M)$

Inclusions $\mathrm{DC}_{\mathrm{str}} \hookrightarrow \mathrm{DC}_{\mathrm{sat}} \hookrightarrow \mathrm{DC}$ have left adjoints.

⊙

Definition 9.2. If R is an $\mathbb{F}_p$-algebra, the **de Rham Witt complex** of R is the initial strict D-algebra $W\Omega_R^*$ equipped w/ a map $R \to W\omega_R^*/V$. ◇

Theorem 9.3 (Theorem 8.18)**.** *Such an object exists.*

Last time, we reduced this theorem to the following two lemmas.

Lemma 9.4 (Lemma 8.19)**.** *Let A^* be a saturated D-algebra. Then, A^0/VA^0 is a reduced $\mathbb{F}_p$-algebra.*

Proof. Note $V(1) = V(F(1)) = p$ so A^0/V is an $\mathbb{F}_p$-algebra. Say $x \in A^0$ such that $\overline{x}^p = 0$. We need to show that $x^p \in VA^0$. Since $Fx \equiv x^p \mod p$ (so also mod V) in degree 0, we see that $Fx = Vy$ for some y. Apply differential:

$$dVy = dFx = pFdx,$$

so dVy is p-divisible. Hence (Lemma 8.6), y is F-divisible, i.e. $y = Fz$. Now to prove that $x \in VA^0$, we need only check this after taking F (F injective). Observe that

$$Fx = Vy = VFz = FVz$$

(so $x = Vz$), and we win. ■

Lemma 9.5 (Lemma 8.20)**.** *Let B^* be a strict D-algebra, and $R \in \mathrm{Calg}_{\mathbb{F}_p}^{\heartsuit}$. Then, the follow are equivalent data*

(1) *A ring map $R \to B^0/VB^0$*

(2) *A ring map $W(R) \to B^0$ such that*

$$\begin{tikzcd} W(R) \arrow[r] \arrow[d] & B^0 \arrow[d] \\ R \arrow[r] & B^0/VB^0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} W(R) \arrow[r] \arrow[d, "F"'] & B^0 \arrow[d, "F"] \\ W(R) \arrow[r] & B^0 \end{tikzcd}$$

commute.

Remark 9.6. On $W(R)$, have Vershiebung $V : (a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$ given by shifting (even when R not perfect). ◦

Proof. Here's the Magic™:

$$B^0 \cong W(B^0/VB^0)$$

and the B^0-Frobenius is the Witt vector Frobenius. To prove the result from this, one needs a criterion for when a map $W(R) \to W(B^0/VB^0)$ is $W(f)$ for some $f : R \to B^0/VB^0$. We'll skip this later part (see Elden's notes or [BLM21]), and focus on showing the Magic™.

Let's write $S := B^0/VB^0$. By previous lemma, S is reduced, so $W(S)$ is p-torsion free. By the mapping in property of the Witt vectors construction, get a map

$$\begin{tikzcd} B^0 \arrow[r, "u"] \arrow[d] & W(S) \arrow[d] \\ B^0/VB^0 \arrow[r, "="] & S \end{tikzcd}$$

which is compatible with Frobenius (i.e. $uF = Fu$). We claim that u is an isomorphism.

- Step 1: $uV = Vu$

  Simply play F to both sides and observe that $FuV = uFV = up = pu = FVu$, so $uV = Vu$ as F is injective.

- Step 2

  We get maps $u_r : B^0/V^rB^0 \longrightarrow W_r(S)$. Since B^* is strict, it suffices to prove that u_r is an iso for any r (and then we take inverse limits). Consider now the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B^0/VB^0 \arrow[r, "V^{r-1}"] \arrow[d, "="] & B^0/V^rB^0 \arrow[r] \arrow[d, "u_r"] & B^0/V^{r-1}B^0 \arrow[r] \arrow[d, "u_{r-1}"] & 0 \\ 0 \arrow[r] & S \arrow[r] & W_r(S) \arrow[r] & W_{r-1}(S) \arrow[r] & 0 \end{tikzcd}$$

  By induction, we win. ■

Remark 9.7. In Step 2 above, only needed saturatedness in each finite step. Strictness comes in when you take a limit. ◦

Let's summarize. Say $R \in \mathrm{Calg}_{\mathbb{F}_p}^{\heartsuit}$. Consider its reduction R_{red}. We can set

$$W\Omega_R^* := W\,\mathrm{Sat}(\Omega^*_{W(R_{\mathrm{red}})}),$$

which works by universal property $W\Omega_R^* \longrightarrow A^* \iff R \longrightarrow A^0/VA^0$.

Remark 9.8. The functor $R \rightsquigarrow W\Omega_R^*$ is fully faithful on *reduced* $\mathbb{F}_p$-algebras. ◦

Remark 9.9. Note the above construction actually produces an element of $K(\mathbb{Z}_p)$, i.e. a literal complex and not just an element of the derived category. ◦

We extend de Rham Witt complexes via left Kan extension (LKE)

$$\begin{tikzcd} \mathrm{Calg}_{\mathbb{F}_p}^{\heartsuit} \arrow[r, "W\Omega^*/\mathbb{F}_p"] \arrow[d] & K(\mathbb{Z}_p) \arrow[d] \\ \mathrm{Ani}_{\mathbb{F}_p} \arrow[r, "LKE"] & D(\mathbb{Z}_p) \end{tikzcd}$$

We then right Kan extend (RKE) to animated $\mathbb{F}_p$-schemes

$$\begin{tikzcd} & \mathrm{Ani}_{\mathbb{F}_p} \arrow[r] \arrow[d] & D(\mathbb{Z}_p) \\ \mathrm{Sch}^{\mathrm{op}}_{\mathbb{F}_p} \arrow[r, phantom, "\subset"] & \mathrm{AniSch}^{\mathrm{op}}_{\mathbb{F}_p} \arrow[ur, dashed, "\mathrm{RKE}"'] & \end{tikzcd}$$

Definition 9.10. We define **Crystalline cohomology** to be this extension $LR\Gamma_{crys}(X/W)$ to (animated) schemes. ⋄

9.1 An example $\mathbb{G}_m^n$

Set $R := \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Let's compute $W\Omega^*_{R/p}$. First observe that

$$\Omega_R^* = \bigwedge\nolimits_R R\{\mathrm{d}\log x_i\} \text{ where } \mathrm{d}\log x_i = \mathrm{d}x_i/x_i.$$

The differential is given by $\mathrm{d}x_i^n = nx_i^n \mathrm{d}\log x_i$. To make this into a D-algebra, we've seen that

$$\begin{aligned} F(x_i) &= x_i^p \\ F(\mathrm{d}\log x_i) &= F(1/x_i)F(\mathrm{d}x_i) \end{aligned}$$

Note

$$F(1/x_i)F(\mathrm{d}x_i) = \frac{1}{x_i^p}\left(x_i^{p-1}\mathrm{d}x_i\right) = \mathrm{d}\log x_i \implies F(\mathrm{d}\log x_i) = \mathrm{d}\log x_i.$$

Now let's saturate (Keep in mind Remark 8.3). First define

$$R_\infty := \mathbb{Z}\left[x_1^{\pm 1/p^\infty}, \dots, x_n^{\pm 1/p^\infty}\right].$$

Observe that

$$\Omega^*_R[F^{-1}] = \bigwedge\nolimits_{R_\infty} R_\infty\{\mathrm{d}\log x_i\}$$

with differential

$$\begin{array}{rccc} \mathrm{d}: & \Omega^*_R[F^{-1}] & \longrightarrow & \Omega^*_R[F^{-1}, p^{-1}] \\ & F^{-n}x & \longmapsto & p^{-n}F^{-n}\mathrm{d}x. \end{array}$$

Hence (say $\gcd(a,p) = 1$),

$$\mathrm{d}\left(x_i^{a/p^n}\right) = \frac{a}{p^n} x_i^{a/p^n}\,\mathrm{d}\log x_i.$$

Now, recall that the saturation is smaller; we look only at the forms ω such that $\mathrm{d}\omega$ is also integral (i.e. no denominators).

Example. In degree 0, bx_i^{a/p^n} is in the saturation $\iff$ $p^n \mid b$. △

Example. In degree 1, have inclusion

In more than one variable, imagine e.g. $x^{1/p}\mathrm{d}\log y$

$$\mathrm{Sat}(\Omega^*_R)^1 \cong \bigoplus_{0\le i\le r} R_\infty \mathrm{d}\log x_i.$$

We'll later see what this is explicitly in one variable (spoiler: above is an equality if $r = 1$. $\mathrm{d}(blah) = 0$ always). △

Having said the above, let's compute $W_k\Omega^*_{\mathbb{F}_p[T^{\pm 1}]}$. To do this, we compute V (using $FV = p$):

$$V\left(bT^{a/p^k}\right) = pbT^{a/p^{k+1}}$$
$$V\left(\mathrm{d}\log T\right) = p\mathrm{d}\log T$$

Let $E^* :=$ integral forms $\subset \Omega^*_R[F^{-1}]$. Hence,

$$W_k\Omega^0_{\mathbb{F}_p[T^{\pm 1}]} = E^0/V^kE^0.$$

Note first that $p^k = V^k(1)$, so this object is a $\mathbb{Z}/p^k\mathbb{Z}$-module. Note there is a summand

$$\left(\frac{\mathbb{Z}}{p^k\mathbb{Z}}\right)[T^{\pm 1}] \simeq \bigoplus_{j\in\mathbb{Z}} \mathbb{Z}/p^k\mathbb{Z}\cdot T^j \subset W_k\Omega^0_R.$$

Can we understand the complement? The non-integral powers of T can be written as

$$\bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\} \\ (a,p)=1 \\ n\ge 1}} \mathbb{Z}\left\{p^nT^{a/p^n}\right\}.$$

We can express this using V's (since $V^n(T^a) = p^nT^{a/p^n}$):

$$\bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\} \\ (a,p)=1 \\ n\ge 1}} \mathbb{Z}\left\{p^nT^{a/p^n}\right\} = \bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\} \\ (a,p)=1 \\ n\ge 1}} \mathbb{Z}\left\{V^n(T^a)\right\}.$$

For fixed k, we also have

$$\bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\}\\(a,p)=1\\n\geq 1}} \frac{\mathbb{Z}}{p^{k-n}\mathbb{Z}}\left\{p^n T^{a/p^n}\right\}$$

(note if $n > k$, elements with T^{a/p^n} all die). This gives us a description of the "Witt vectors of $\mathbb{G}_m$" in degree 0.[9]

Recall that in this one variable case, $E^1 = \mathbb{Z}[T^{\pm 1/p^\infty}]\mathrm{d}\log T$. Note

$$\mathrm{d}\left(V^n(T^a)\right) = \mathrm{d}\left(p^n T^{a/p^n}\right) = p^n \frac{a}{p^n} T^{a/p^n}\mathrm{d}\log T = aT^{a/p^n}\mathrm{d}\log T.$$

One can use this to show

$$E^1 = \mathbb{Z}[T^{\pm 1}]\mathrm{d}\log T \oplus \bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\}\\(a,p)=1\\n\geq 1}} \mathbb{Z}\{dV^nT^a\}.$$

In particular, E^1 also breaks up into integral powers and fractional powers. Furthermore, the fractional power part is exactly the image of the differential on the fractional power part in degree 0. From above description, we see

$$W_k\Omega^1_{\mathbb{F}_p[T^{\pm1}]} \cong \frac{\mathbb{Z}}{p^k\mathbb{Z}}\left[T^{\pm 1}\right]\mathrm{d}\log T \oplus \bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\}\\(a,p)=1\\n\geq 1}} \frac{\mathbb{Z}}{p^{k-n}\mathbb{Z}}\mathrm{d}V^nT^a.$$

Lemma 9.11 (Deligne)**.** *Let $R := \mathbb{F}_p[T^{\pm1},\dots,T_r^{\pm1}]$. Then, $W_k\Omega^*_{R/\mathbb{F}_p}$ is isomorphic to*

$$\Omega^*_{\mathbb{Z}/p^k\mathbb{Z}[T_1^{\pm1},\dots,T_r^{\pm1}]/\mathbb{Z}/p^k\mathbb{Z}} \oplus (\textit{something acyclic}).$$

(Our computation essentially shows that when $r = 1$. Can get it for larger r with more work along the same lines).

9.2 Smooth and de Rham

Theorem 9.12. *Let R be a commutative ring which is p-torsion free. Say R/p is smooth over a perfect $\mathbb{F}_p$-algebra k. Assume that $\varphi : R \to R$ is a lift of Frobenius. Then, there is a quasi-isomorphism*

$$\mu : \widehat{\Omega}^*_R \xrightarrow{\sim} W\Omega^*_{R/p}$$

of D-algebras such that

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \widehat{\Omega}^0_R \arrow[r, "\mu"] & W\Omega^0_{R/p} \arrow[d] \\ R/p \arrow[rr] & & W_1\Omega^0_{R/p} \end{tikzcd}$$

[9]

$$W_k\Omega^0_{\mathbb{F}_p[T^{\pm1}]} = \frac{\mathbb{Z}}{p^k\mathbb{Z}}[T^{\pm1}] \oplus \bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\}\\(a,p)=1\\n\geq 1}} \mathbb{Z}\left\{p^nT^{a/p^n}\right\} = \bigoplus_{\substack{a\in\mathbb{Z}\setminus\{0\}\\(a,p)=1\\n\geq 1}} \mathbb{Z}\left\{V^n(T^a)\right\}.$$

commutes.

10 Lecture 11 (10/6): Didn't go (see Lectures 5 and 6 of Elden's notes)

11 Lecture 12 (10/11)

OH Thursday at 6pm (led by Natalie): "Completions after dark"

At this point, we have a **slope spectral sequence** coming from the following filtration:

$$\mathrm{Fil}^{\geq *}_{\text{slope}} R\Gamma_{crys}(X/W) := R\Gamma_{Zar}(X, W\Omega^{\geq *}_X) \longrightarrow R\Gamma_{crys}(X/W).$$

Assumption. For today, we want X smooth over k (or maybe smooth and projective).

From this filtration, we get a spectral sequence

$$E_1^{i,j} = \mathrm{H}^j(X, W\Omega^i) \implies \mathrm{H}^{i+j}_{\text{crys}}(X/W)$$

(rth page has differentials of bidegree $(1-r, r)$). Elden drew this with horizontal axis the j-axis.

Theorem 11.1 (Illusie, after Bloch)**.** *Say X is proper (in addition to be smooth/k). Then, $d_r \otimes K = 0$ for all r. Hence, SS collapses at E_1 (after tensoring with $K = \operatorname{Frac} W$).*

Remark 11.2. We got this spectral sequence easily since crystalline cohomology was defined in terms of an actual complex. If we got it without a complex, it'd be harder to obtain this sequence. Also, there's such a thing as relative crystalline cohomology, but the relative de Rham Witt complex is harder to obtain. ○

Assumption. For rest of class, X is smooth and proper over k (= perfect field).

11.1 Some analysis of this s.s.

Notation 11.3.

$$\mathrm{H}^i(W_{(r)}\Omega^j_X) := \mathrm{H}^i_{\text{Zar}}(X, W\Omega^j_X) \text{ and } \mathrm{H}^i(X) := \mathrm{H}^i_{\text{crys}}(X/W).$$

For a surface X, only nonzero objects in (E_1 page of the) spectral sequence are

$$\begin{array}{ccc} \mathrm{H}^2(W\mathscr{O}_X) & \mathrm{H}^2(W\Omega^1_X) & \mathrm{H}^2(W\Omega^2_X) \\ \mathrm{H}^1(W\mathscr{O}_X) & \mathrm{H}^1(W\Omega^1_X) & \mathrm{H}^1(W\Omega^2_X) \\ \mathrm{H}^0(W\mathscr{O}_X) & \mathrm{H}^0(W\Omega^1_X) & \mathrm{H}^0(W\Omega^2_X) \end{array}$$

Remark 11.4. Serre studied Witt vector cohomology $\mathrm{H}^*(W\mathscr{O}_X)$ as a first approximation to crystalline cohomology. ○

Lemma 11.5. *The maps $d_1 : \mathrm{H}^0(W\Omega^j_X) \to \mathrm{H}^0(W\Omega^{j+1}_X)$ are all zero. That is, the bottom row above has trivial differentials.*

Proof. By previous theorem, $d_1[1/p] = 0$ (adjoining $1/p$ is tensoring with K). At the same time, $\mathrm{H}^0(W\Omega^j_X)$ is p-torsion free, so $d_1 = 0$. For this p-torsion free claim, it suffices to know that multiplication by p is injective on $W\Omega^*$ (since saturated D-complex). ■

Corollary 11.6. *$(W(\Gamma(X, \mathscr{O})) =)\mathrm{H}^0_{Zar}(X, W\mathscr{O}_X) \simeq \mathrm{H}^0_{crys}(X/W)$*

Furthermore, one can compute $\mathrm{H}^0_{\mathrm{Zar}}(X, W\mathscr{O}_X) \simeq W^{\pi_0^{geom}(X)}$ is W-free of rank equal to the number of geometric connected components of X.

Theorem 11.7 (Serre)**.** *$\mathrm{H}^1(X, W\mathscr{O}_X)$ is also p-torsion free.*

Remark 11.8. $\mathrm{H}^1(X, \mathbb{Z})$ is always torsion free. Here's one proof: $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$ gives rise to

$$\mathrm{H}^0(X, \mathbb{Z}) \twoheadrightarrow \mathrm{H}^0(X, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \mathrm{H}^1(X, \mathbb{Z}) \longrightarrow \mathrm{H}^1(X, \mathbb{Z}),$$

from which we see that $\mathrm{H}^1(X, \mathbb{Z})[m] = 0$ (note map on H^0's above surjective since both groups count connected components).

A similar argument can be made to work to show Serre's theorem. ○

Lemma 11.9. *$d_1 : \mathrm{H}^1(W\mathscr{O}_X) \to \mathrm{H}^1(W\Omega^1_X)$ is always zero.*

(Use that source is p-torsion free)

Corollary 11.10. *$\mathrm{H}^1_{crys}(X/W)$ is also p-torsion free.*

It sits in an exact sequence $0 \to \mathrm{H}^0(W\Omega^1_X) \to \mathrm{H}^1(X) \to \mathrm{H}^1(W\mathscr{O}_X) \to 0$ with kernel and cokernel both p-torsion free.

Remark 11.11. If X is a curve, the spectral sequence collapses on the E_1-page. The extension in degree 1 may or may not split. There is a Serre duality relating $\mathrm{H}^0(W\mathscr{O}_X)$ and $\mathrm{H}^1(W\Omega^1_X)$, so both count connected components. ○

11.2 Slopes

Notation 11.12. Let κ be a perfect field of characteristic $p > 0$.

- W is the Witt vectors, with Frobenius denoted by $\varphi : W \to W$. We also set $K = W[1/p]$.

Definition 11.13. An **F-isocrystal** is a pair (M, F), where M is a f.g. free W-module and $F : M \to M$ is a φ-semilinear endomorphism[10] s.t. $F[1/p]$ is bijective. We write $\mathrm{Isoc}_F(\kappa)$ for the category of F-isocrystals. We write $\mathrm{Isoc}_F(\kappa)_{\mathbb{Q}}$ to denote the isogeny category (replace Hom's with $\mathrm{Hom}(-,-)_{\mathbb{Q}_p}$) ◇

Remark 11.14.

- This is the definitiion in Katz's paper

Question: Which one?

- What people call isocrystals: just $M[1/p]$

○

Example. Let's make one up. Choose $\lambda = \frac{n}{m} \in \mathbb{Q}$ written in lowest terms. Take

Frobenius here has eigenvalue p^λ

$$M(\lambda) = \frac{\mathbb{Z}_p[T]}{(T^m - p^n)} \otimes_{\mathbb{Z}_p} W \text{ with } F = T \otimes \varphi$$

(F multiplies by T in first fact, and applies φ in second factor). If we pick a basis given by $\{1, \dots, T^{m-1}\}$, then

$$F(X_1, \dots, X_m) = (p^n \varphi(X_m), \varphi(X_1), \dots, \varphi(X_{m-1}))\,. \qquad \triangle$$

Example. X/κ smooth, projective. Let $F_X : X \to X$ be absolute Frobenius. This induces a map $F_X^* : \mathrm{H}^i_{\mathrm{crys}}(X/B) \to \mathrm{H}^i_{\mathrm{crys}}(X/W)$ which is φ-semilinear. We claim tha $F_X^*[1/p]$ is an isomorphism. In fact

Claim 11.15. *F_X^* on $\mathrm{H}^i(X)/tors$ is injective.*

This is a consequence of **Poincaré duality**. If $\dim X = d$, then there is a pairing

$$\mathrm{H}^i(X) \otimes \mathrm{H}^{2d-i}(X) \longrightarrow W$$

which is nondegenerated modulo torsion.

Exercise. Prove claim, assuming Poincaré duality. (Hint: the action of F on W is φ) △

Remark 11.16. Poincaré duality can be proved using de Rham comparison. ○

Notation 11.17. $\Phi_X := F_X^*$ from previous example.

Warning 11.18. Φ_X differs on the chain complex level from the internal ('Dieudonné theoretic') Frobenius by $\Phi_X = p^i F$. •

Theorem 11.19 (Dieudonné-Manin)**.** *Let κ be an algebraically closed field. Then, the isogeny category $\mathrm{Isoc}_F(\kappa)_{\mathbb{Q}}$ is **semisimple**[11], and the simple objects are exactly the $M(\lambda)$'s from our first example.*

Remark 11.20. If κ is not algebraically closed and (M, F) is an F-isocrystal, then we can look at $W(\overline{\kappa})$ with fraction field $\overline{K}$ (not algebraically closed). Then, $(M \otimes W(\overline{\kappa}), F \otimes \varphi_{W(\overline{\kappa})})$ is an F-isocrystal over $\overline{\kappa}$, so we can write

$$M \cong \bigoplus M_\lambda,$$

where M_λ is the largest sub-F-isocrystal (up to isogeny) s.t. $M_\lambda \otimes \overline{K}$ is isomorphic to a sum of $M(\lambda)$'s. ○

Definition 11.21. The **(Newton) slopes** of an F-isocrystal form the sequence of rational number's (up to isogeny) $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_r$ given as

$$\left(\frac{n_1}{m_1}, \dots, \frac{n_1}{m_1}, \frac{n_2}{m_2}, \dots, \frac{n_2}{m_2}, \dots\right).$$

◇

Example. A **Dieudonné module** is an F-isocrystal with slopes $\in [0, 1]$. △

[10] A linear map $\varphi_* M \to M$

[11] i.e. an abelian category where every object is a direct sum of **simple objects** (objects X whose only quotients are $0, X$)

Let's elaborate on this. Say M is a Dieudonn'e module with $F : M \to M$. This slope condition tells us that $pM \subset F(M)$.

Exercise. Prove this.

Grothendieck gave an equivalence of categories

$$\left\{\begin{matrix} p\text{-divisible} \\ \text{groups}/\kappa \end{matrix}\right\} \xrightarrow[M]{\sim} \left\{\begin{matrix} \text{Dieudonné} \\ \text{modules} \end{matrix}\right\}.$$

Example.

(1) If G is the p-divisible group associated to $A[p^\infty]$ for A/κ an abelian variety, then

$$M(G) \cong \mathrm{H}^1_{\text{crys}}(A/W).$$

(2) If $G = \mathbb{Q}_p/\mathbb{Z}_p$, then $M(G) = W$ with $F = p\varphi$

(3) If $G = \mu_{p^\infty}$, then $M(G) = W$ with $F = \varphi$ △

Remark 11.22. To collapse a spectral sequence, one uses slopes. ○

Theorem 11.23. *Let X be a smooth, proper κ-variety. Then, the canonical map*

$$\mathrm{H}^*(X) \longrightarrow \mathrm{H}^*(W\Omega_X^{\leq i})$$

induces an isomorphism

$$\mathrm{H}^*(X)_{[0,i[} \longrightarrow \mathrm{H}^*(W\Omega_X^{\leq i})$$

This subscript is the submodule of slopes $< i$

as isocrystals.

11.3 Esnault's theorem

(Will probably finish next time)

Setup 11.24. Let $\kappa = \mathbb{F}_q$ with $q = p^a$.

Lemma 11.25. *Let X be geometrically connected over κ. Assume $\mathrm{H}^i(X, W\mathscr{O}) = 0$ for all $i > 0$ ('**$W\mathscr{O}$-acyclic**'). Then,*

$$\#X(\kappa) \equiv 1 \mod p.$$

In particular, a κ-rational point must exist.

(Recall: everything smooth + proper)

Proof. We have the **Lefschetz trace formula for crystalline cohomology** (Étesse)

$$\#X(\kappa) = \sum_{i=0}^{2\dim X} (-1)^i \operatorname{Tr}\left(\Phi_X^a \mid \mathrm{H}^1(X) \otimes K\right).$$

Hence, if $\mathrm{H}^i(X, W\mathscr{O}) = 0$, slope considerations (i.e. Theorem 11.23) tell us that all the Frobenius eigenvalues for H^i have strictly positive[12] p-adiv valuation. For $i = 0$, $\Phi_X \curvearrowright \mathrm{H}^0(X/W) = W$ by the identity (so has trace 1). This gives the claim. ■

Example (Audience, assuming I heard correctly)**.** Can take $X \hookrightarrow \mathbb{P}^n$ hypersurface of degree $d \le n$. △

Remark 11.26. Can apparently get stronger congruences by using more of the full force of the Weil conjectures. ○

Theorem 11.27 (Esnault, Lang's conjecture)**.** *Let X be a Fano variety over a finite field. Then, X has a rational point.*

This requires two ingredients. We'll prove one Thursday, but blackbox the other.

Theorem 11.28 (Kollar, Miyaoka (spelling?), Mori)**.** *Any Fano is rationally chain connected.*

Theorem 11.29. *Any rationally connected variety has a 'decomposition of the diagonal'*

This last thing says (something like) $\Delta \in \mathrm{Ch}^{\dim X}(X \times X)$ breaks up as a sum of something horizontal and something vertical.

12 Lecture 13 (10/13)

OH 6pm: 'completions after dark'

Errata from last time:

$$\mathrm{H}_{\mathrm{crys}}(X/W)_{[0,i[} \cong \mathrm{H}^*(X, W\Omega^{\le i-1})$$

(isogenous, not necessarily isomorphic). We forgot the -1 last time.

Speaking of last time, we saw this lemma (recall $\kappa = \mathbb{F}_q$ and all schemes assumed smooth and projective):

Lemma 12.1. *Say X/κ is geometrically connected. If $\mathrm{H}^i(X, W\mathscr{O}) = 0$ for $i > 0$ (so all slopes are ≥ 1), then*

$$|X(K)| \equiv 1 \mod p.$$

Theorem 12.2. *Let X/κ be Fano (i.e. ω_X^{-1} is ample). Then, X has a rational point.*

Exercise. If X has a lift to $W_2(\kappa)$, the result is easier (use Kodaira vanishing).

Fact. A result of Kollar, Miyaoka, Mori, Campana says that any Fano variety is rationally chain connected.

If you have a (geometrically connected) variety over a field, then any two points can be connected by some curve. Being **rationally chain connected** means you can connect any two points by a chain of $\mathbb{P}^1$'s (after suitable field extension). Question: Is this obvious?

Non-example. Elliptic curves are not rationally chain connected (any map $\mathbb{P}^1 \to E$ is constant for genus reasons). ▽

[12]not necessarily integral

A consequence of this fact is that X (our Fano) will have zero cycle of degree 1, and also

$$\mathrm{CH}_0\left(X \times \overline{k(X)}\right) \xrightarrow{\deg} \mathbb{Z}$$

is an isomorphism.

12.1 Digression: decomposition of the diagonal

Let k be a field. We say that X is **universally** CH_0**-trivial** (or **motivically connected**) if the map

$$\deg : \mathrm{CH}_0(X_F) \longrightarrow \mathbb{Z}$$

is an isomorphism for all field extensions F/k. An element of $\mathrm{CH}_0(X)$ is a formal linear combination of 0-dimensional (so closed) points. The degree map is

$$\deg : \sum n_i[p_i] \mapsto \sum n_i \deg(\kappa(p_i)/\kappa).$$

In CH_0, these formal combinations are only considered up to rational equivalence. We'll give a different definition later.

Non-example. If $X = \operatorname{Spec} L \to \operatorname{Spec} \kappa$, then $\mathrm{CH}_0(\operatorname{Spec} L) \cong \mathbb{Z}$, but $\mathrm{CH}_0(X_L) \cong \bigoplus_{|G|} \mathbb{Z}$, where $G = \mathrm{Gal}(L/\kappa)$. $\triangledown$

Example. $\mathrm{CH}_0(\mathbb{P}^n_F) \cong \mathbb{Z}$ always. $\triangle$

This universally CH_0-trivial thing is a definition which is (somewhat) checkable. We'd like to relate it to the 'definition that we want'. First, here's an even more checkable definition.

Lemma 12.3. *Assume that X is geometrically connected. Then, TFAE*

(1) *X is universally CH_0-trivial*

(2) *X has a zero-cycle of degree 1 +*

$$\deg : \mathrm{CH}_0(X \times k(X)) \xrightarrow{\sim} \mathbb{Z}.$$

(note that is a little different from what he had for Fano X, where we base changed to $\overline{k(X)}$ instead). Bloch-Srinivas (spelling?) saw the importance of this definition.

Definition 12.4. Fix a smooth, proper X of dimension d. A cycle $Z \in \mathrm{CH}_k(X \times X)$ is called **narrow** if it is supported on $X \times V$ for[13] $V \hookrightarrow X$ of codimension ≥ 1. $\diamond$

In this case, if $j : X \setminus V \hookrightarrow X$ is the open embedding of the complement, we have $j^*(Z) = 0 \in \mathrm{CH}_k(X \times (X \setminus V))$.

Definition 12.5. X has a **decomposition of the diagonal** if the following equality holds in $\mathrm{CH}_d(X \times X)$:

$$[\Delta_X] = \alpha \times X + Z,$$

where Z is narrow and $\alpha \in \mathrm{CH}_0(X)$ is zero cycle of degree 1. $\diamond$ TODO: Add picture?

Example. Take $X = \mathbb{P}^1$. Note that $\mathrm{CH}_1(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. In here, we have $(0,1) + (1,0) = (1,1)$, i.e. the diagonal is a horizontal $\mathbb{P}^1$ + a vertical $\mathbb{P}^1$. Thus, $\mathbb{P}^1$ has a decomposition of the diagonal. $\triangle$

Theorem 12.6 (Bloch-Srinivas)**.** *Say X/κ is geometrically connected. Then, TFAE*

(1) *X is universally CH_0-trivial.*

(2) *X has a zero-cycle of degree 1 + $\mathrm{CH}_0(X \times \kappa(X)) \cong \mathbb{Z}$.*

(3) *X has a decomposition of the diagonal.*

Proof. **(1)** $\implies$ **(2)** is clear.

(**(2)** $\implies$ **(3)**). Let $K = \kappa(X)$ be the fraction field. Consider $j : X_K \to X \times X$ (inclusion of generic point in second factor). Let $\alpha \in \mathrm{CH}_0(X)$ be of degree 1. Then, note that both

$$j^*([\Delta_X]) \text{ and } j^*(\alpha \times [X])$$

are cycles of degree one. By **(2)**, this means that they are equal in $\mathrm{CH}_0(X_K)$. At the same time,

$$\mathrm{CH}_0(X \times \kappa(X)) \cong \operatorname{colim} \mathrm{CH}^d(X \times U)$$

(colimit over U and $d = \dim U$?) But now

$$j_U^*\left([\Delta_X] - [\alpha \times X]\right) = 0$$

for some open $U \hookrightarrow X$. Consider the exact sequence

$$\mathrm{CH}^{d-s}(Z) \longrightarrow \mathrm{CH}^d(X \times X) \longrightarrow \mathrm{CH}^d(X \times U) \longrightarrow 0$$

where $Z = X \times X \setminus X \times U$. This tells us that the difference $[\Delta_X] - [\alpha \times X]$ must be narrow.

(**(3)** $\implies$ **(1)**) Consider the map

$$\mathrm{CH}_d(X \times X) \longrightarrow \mathrm{End}(\mathrm{CH}_0(X))$$

via "pushing and pulling": take a cycle, pull to $X \times X$, multiply by element in CH_d, and then push back down, i.e. $\beta_*(\alpha) = p_{2,*}\left(p_1^*(\alpha) \cup \beta\right)$ (here $\beta \in \mathrm{CH}_d(X \times X)$ and $\alpha \in \mathrm{CH}_0(X)$). Note that if $f : X \to X$, then $(\Gamma_f)_* = f^*$. In particular, $(\Delta_X)_* = \mathrm{id}$.

Slogan. Having a decomposition of the diagonal tells us what the identity looks like.

The upshot is that for any 0-cycle β, we have

$$\beta = \deg(\beta)\alpha$$

[13] V closed subscheme with reduced induced structure

(where α s.t. $[\Delta_X] = \alpha \times X + Z$ with Z narrow), and so we win. Why does the above hold? Projection formula gives

$$\begin{aligned}(\alpha \times [X])_* (\beta) &= p_{1,*} (\alpha \cup p_2^* \beta) \\ &= \alpha \cup p_{1,*} p_2^* \beta && \text{projection formula} \\ &= \alpha \deg(\beta).\end{aligned}$$

At the same time, for any $p \in X$,

$$Z_*([p]) = (\iota_* Z)_*([p]) = p_{2,*}((\{p\} \times X) \cup \iota_* Z),$$

where $\iota : X \times V \hookrightarrow X \times X$. This vanishes as soon as we can assume that

$$(\{p\} \times X) \cap ((X \setminus V) \times X) = \emptyset.$$

Lemma 12.7. *As above, for any zero cycle $z \in \mathrm{CH}_0(X)$, for any open $U \subset X$, there exists a cycle z' supported away from U s.t. $z' = z$ in $\mathrm{CH}_0(X)$.*

(Easy for curves. In general, use that any two points on a variety can be connected by a curve. Details left as exercise). This completes the proof. ■

What we need is a variant.

Definition 12.8. A **rational decomposition of the diagonal** means there exists some N such that

$$N[\Delta_X] = \alpha \times X + Z$$

of the form from before. Equivalently, $[\Delta_X]$ is of the desired form in $\mathrm{CH} \otimes \mathbb{Q}$. ◇

Lemma 12.9. *As soon as*

($\star$) *There exists a degree one zero cycle α, and*

$$\mathrm{CH}_0(X \times \overline{\kappa(X)}) = \mathbb{Z},$$

then X has a rational decomposition of the diagonal.

The main point is that if F'/F is a finite extension of fields, then you get a pushfoward map $\mathrm{CH}_0(X_{F'}) \to \mathrm{CH}_0(X_F)$ (and pull-push is multiplication by degree).

12.2 Back to Esnault's theorem

Theorem 12.10. *Let X/κ be Fano (i.e. ω_X^{-1} is ample). Then, X has a rational point.*

Proof. By previous discussion, we have decomposition $[\Delta_X] = \alpha \times X + Z$. Note, Chow acts also on crystalline cohomology

Question: Do we have this in integral Chow or only rational Chow?

$$\mathrm{CH}_d(X \times X) \longrightarrow \mathrm{End}(\mathrm{H}^*(X)).$$

Now, Z applied to $\mathrm{H}^i(X) \otimes K$ lands in

$$Z_* \left(\mathrm{H}^*(X) \otimes K\right) \subset \ker\left(\mathrm{H}^*(X) \otimes K \longrightarrow \mathrm{H}^*(X \setminus V) \otimes K\right) = \mathrm{Im}\left(\mathrm{H}^*_V(X) \otimes K \longrightarrow \mathrm{H}^*(X) \otimes K\right).$$

Claim 12.11. *Frobenius acts on $\mathrm{H}^i_V(X) \otimes K$ by slopes ≥ 1.*

Given this, note that $(\alpha \times X)_*$ factors through $\mathrm{H}^*(\operatorname{supp} \alpha) \otimes K$ which is concentrated in degree 0.

In fact, if $Z \hookrightarrow X$ of codim $c \geq 1$, then $\mathrm{H}^i_Z(X) \otimes K$ does have slopes ≥ 1. Indeed, if Z/κ is smooth, then

$$\mathrm{H}^i_Z(X) \otimes K \cong \mathrm{H}^{i-2c}(Z) \otimes K$$

(**purity isomorphism**[14]). Under this isomorphism Frobenius $\Phi_X \mapsto p^c \Phi_X$, so get slopes $\geq c$. For general Z, need to stratify $\cdots \subset Z^2 \subset Z^1 \subset Z^0$ with smooth differences. In stratification, c never decreases, so still get slopes ≥ 1. ■

12.3 Lecture 7: K is for K-theory

What do we want from K-theory/how should we think about it?

Thomason 1984: K-theory is black magic

Why did he say this? He was trying to prove something like this: say X is a regular noetherian scheme with $Z \hookrightarrow X$ a regular, closed subscheme. Thomason proved that

$$\mathrm{H}^i_{\text{ét},Z}(X, \mathbb{Z}_\ell(j)) \cong \mathrm{H}^{i-2c}_{\text{ét}}(Z, \mathbb{Z}_\ell(j-c))$$

(**purity**). His proof made use of K-theory, despite it not showing up in the statement.

For Thomason, **algebraic K-theory** is some collection of functors $\{K_j : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Ab} : j \in \mathbb{Z}\}$ satisfying

(1) rank map

$$\operatorname{rank} : K_0 \twoheadrightarrow \mathrm{H}^0_{\mathrm{Zar}}(-, \mathbb{Z})$$

(2) If X qcqs and $X = U \cup V$, get LES

$$\ldots \longrightarrow K_j(X) \longrightarrow K_j(U) \oplus K_j(V) \longrightarrow K_j(U \cap V) \longrightarrow K_{j-1}(X) \longrightarrow \ldots$$

(Mayer-Vietoris)

(3) As in the statement of purity, get LES

$$\ldots \longrightarrow K_j(Z) \xrightarrow{i_*} K_j(X) \longrightarrow K_j(X \setminus Z) \longrightarrow K_{j-1}(Z) \longrightarrow \ldots$$

(4) If X is regular, then

$$K_j(X) \cong K_j(X \times \mathbb{A}^1) \text{ and } K_j(X) = 0 \text{ for } j < 0.$$

[14] proof uses rigid cohomology which apparently means it uses some analysis

(5) For any qcqs X,

$$K(\mathbb{P}^1_X) \cong K_j(X) \oplus K_j(X)$$

with generators $\{\mathscr{O}\}$ and $\{\mathscr{O} - \mathscr{O}(1)\}$.

Goal. Use K-theory to control cycles.

Remark 12.12 (how did this help Thomason). There's a modification of K-theory which supports a spectral sequence with étale cohomology on its E_2-page. Thomasson was able to degenerate the sequence enough to deduce purity from **(3)** above. ◦

Let's now give an 'official definition' of K-theory, for the sake of completeness.

Definition 12.13. Let R be a commutative ring. Let $\mathbf{Proj}^{\mathrm{fg}}(R)$ be the 1-category of f.g. projective modules over R. Let $\mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq} \subset \mathbf{Proj}^{\mathrm{fg}}(R)$ be the maximal subgroupoid. ⋄

Note that $\left(\mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq}, \oplus, 0\right)$ is a unital H-space, but 'H-space' is a "terrible notion". It's better to say this is an $\mathbb{E}_\infty$-space. Note that

'space' because it's a groupoid

$$\pi_0\left(\mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq}\right) = \left\{\begin{array}{c}\text{iso classes of}\\ \text{f.g. projective}\end{array}\right\}.$$

Furthermore,

Question: Is this obvious

$$\pi_1(blah, [P]) = \mathrm{Aut}_R(P) \text{ and } \pi_2(blah, [P]) = 0 \text{ for } j \geq 2.$$

To get something interesting from this, we'll need to modify. Note that π_0 above is a monoid (because $\oplus$). To get something interesting, we group complete. **Connective K-theory** is

$$K_{\geq 0}(R) := \left(\mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq}, \oplus, 0\right)^{\mathrm{gp}}.$$

13 Lecture 14 (10/18)

Note 3. About 3 minutes late

Guest speaker on Thursday: Hyungseop Kim from U. Toronoto talking about "Adelic descent for K-theory"

Something about K-theory have two "directions," the "Picard direction" and the "Milnor direction".

Remark 13.1. We defined connective K-theory last time as the group completion of the $\mathbb{E}_\infty$-space given by $\left(\mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq}, \oplus, 0\right)$. In particular, $\pi_0(K_{\geq 0}(R))$ is the Grothendieck group of (iso classes) of f.g. projective modules (note: $[M] = [M'] + [M'']$ is $M \cong M' \oplus M''$). ◦

I think we'd like to say something about accessing the first bit of higher homotopical information.

13.1 K_1 and units

Let $\mathbf{Pic}(R)$ be the groupoid of $\otimes$-invertible R-modules (i.e. of line bundles on $\operatorname{Spec} R$). Then,

Question: How hard is it to compute π_1? Presumably this is just automorphisms of line bundles?

$$\pi_j\left(\mathbf{Pic}(R)\right) = \begin{cases} \mathrm{Pic}(R) & \text{if } j = 0 \\ R^\times & \text{if } j = 1 \\ 0 & \text{otherwise.}\end{cases}$$

Note that there is a map

$$\begin{array}{rccc} \det : & \mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq} & \longrightarrow & \mathbf{Pic}(R) \\ & M & \longmapsto & \bigwedge^{\mathrm{top}} M. \end{array}$$

Note that $\mathscr{O} \mapsto \mathscr{O}$ (so preserving natural basepoints) and also $\det(M \oplus N) \cong \det(M) \otimes \det(N)$ so "$\oplus \mapsto \otimes$". We'd like to use this to say that there is an extension

$$\begin{tikzcd} \mathbf{Proj}^{\mathrm{fg}}(R)^{\simeq} \arrow[r] \arrow[d] & \mathbf{Pic}(R) \\ K_{\geq 0}(R) \arrow[ur, dashed] & \end{tikzcd}$$

since $\mathbf{Pic}(R)$ is grouplike (has π_0 a group). However, this isn't quite correct. The natural isomorphisms below do not commute (they commute up to $(-1)^{\mathrm{rank}(M)\,\mathrm{rank}(N)}$)

$$\begin{tikzcd} \det(M \oplus N) \arrow[r] \arrow[d] & \det(M) \otimes \det(N) \arrow[d] \\ \det(N \oplus M) \arrow[r] & \det(N) \otimes \det(M). \end{tikzcd}$$

To take care of this, we introduce notion of graded determinant.

Define

$$\mathbf{Pic}^{\mathbb{Z}}(R) := \mathbf{Pic}(R) \times \mathrm{Hom}(\mathrm{Spec}\, R, \mathbb{Z})$$

as well as the **graded determinant** $\det_*(M) = (\det(M), \mathrm{rank}(M))$. This gives rise to a map

$$\det_* : K_{\geq 0}(R) \longrightarrow \mathrm{Pic}^{\mathbb{Z}}(R).$$

Remark 13.2. Above modification only adjusts what happens in π_0. If R is a local ring (or just $\mathrm{Spec}\, R$ connected), then

$$\pi_j\left(\mathbf{Pic}^{\mathbb{Z}}(R)\right) \cong \begin{cases} \mathrm{Pic}(R) \times \mathbb{Z} & \text{if } j = 0 \\ R^{\times} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

○

Question 13.3 (Audience)**.** *Why does* $\boldsymbol{Pic}(R)$ *not have homotopy groups in degrees* ≥ 2*?*

Answer. Groupoids only have two homotopy groups, objects and isomorphisms. ⋆

Lemma 13.4. *Let* R *be a local ring. Then,*

$$\det_* : K_1(R) \longrightarrow R^{\times}$$

is an isomorphism.

How to think about this? π_0 is a group, so can choose any basepoint, so can choose trivial module R as the basepoint. Note that K-theory doesn't see rank. So should think of $K_1(R)$ as a generic automorphism of a module which doesn't see rank. These are just multiplication by units. Sounds like one can turn this intuition into an actual proof?

Lemma 13.5.

$$K_0(R) \longrightarrow \mathrm{Pic}(R) \times \mathrm{H}^0_{\mathrm{Zar}}(\operatorname{Spec} R; \mathbb{Z})$$

is surjective.

Proof. Note that if $\mathscr{L}$ is a line bundle, then $\det([\mathscr{L}] - [\mathscr{O}]) = \det(\mathscr{L})\det(\mathscr{O})^{-1} = \mathscr{L}$. ∎

Example (globalizing K-theory)**.** Say we want to define $K_{\geq 0}(X)$ for X a scheme. Can use Kan extension and so define

$$K_{\geq 0}(X) = \varprojlim_{\operatorname{Spec} R \to X} K_{\geq 0}(R).$$

However, this is *not a good idea*. Suppose that $X = \operatorname{Spec} R$ is itself affine, and write $X = U \cup V$ s.t. $U, V, U \cap V$ are all affine (opens). Do you get a Mayer-Vietoris sequence

$$K_j(X) \longrightarrow K_j(U) \oplus K_j(V) \longrightarrow K_j(U \cap V) \longrightarrow K_{j-1}(X)?$$

(i.e. is K-theory a Zariski sheaf?) If this is the case, then this would be a good definition. Anytime you left Kan extend something which satisfies Zariski descent, the result satisfies Zariski descent. So there'd be no weird pathologies.

However, we should not exact this to be the case. In degree 0, we have

$$K_0(X) \longrightarrow K_0(U) \oplus K_0(V) \longrightarrow K_0(U \cap V) \longrightarrow 0.$$

Is this last map always surjective? No. Imagine placing K_0 with Pic. If your scheme is locally factorial, can extend line bundles by taking closures of Weil divisors, but in general, no such luck.[15]

Repairing this failure is one reason for the existence of negative K-groups

On the other hand, things are ok if X is regular noetherian. That is, we can safely define K-theory of a regular, noetherian scheme X via

$$K(X) = \varprojlim_{\operatorname{Spec} R \to X} K_{\geq 0}(\operatorname{Spec} R). \qquad \triangle$$

Let's try and unpack the meaning of this definition of $K(X)$ for regular, noetherian X. Among other things, it gives us a **descent spectral sequence**

p is cohomological, so it pushes homotopy groups down

$$\mathrm{H}^p_{\mathrm{Zar}}(X, \pi_q^{\mathrm{Zar}}(K)) \implies K_{q-p}(X),$$

where $\pi_q^{\mathrm{Zar}}(K)$ is the sheafification of $U \mapsto K_q(U)$.

Lemma 13.6. *Say X is a regular noetherian curve (in particular, $\dim X = 1$). Then,*

$$K_0(X) \cong \mathrm{Pic}(X) \oplus \mathrm{H}^0_{\mathrm{Zar}}(X, \mathbb{Z}).$$

Proof. Use the spectral sequence:

Sheaves below determined by values on a local ring

[15] Apparently, one might think to try $(j_*\mathscr{L})^{\vee\vee}$, but even this doesn't work for extending line bundles

$$\begin{array}{lll} \mathrm{H}^0(X, \pi_2^{\mathrm{Zar}}(K)) & \mathrm{H}^1(X, \pi_2^{\mathrm{Zar}}(K)) & 0 \\ \mathrm{H}^0(X, \pi_1^{\mathrm{Zar}}(K) = \mathbb{G}_m) & \mathrm{H}^1(X, \mathbb{G}_m) & 0 \\ \mathrm{H}^0_{\mathrm{Zar}}(X, \pi_0^{\mathrm{Zar}}(K) \cong \mathbb{Z}) & \mathrm{H}^1_{\mathrm{Zar}}(X, \mathbb{Z}) & 0 \end{array}$$

Thus, the only contributions to K_0 are $\mathrm{H}^0(X, \mathbb{Z})$ and $\mathrm{H}^1(X, \mathbb{G}_m)$, so get exact sequence

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{Zar}}(X, \mathbb{G}_m) \longrightarrow K_0(X) \longrightarrow \mathrm{H}^0_{\mathrm{Zar}}(X, \mathbb{Z}) \longrightarrow 0.$$

Split exact sequence using $\det_*$. ■

Furthermore, the spectral sequence in the proof has no nontrivial maps because it's so narrow (a map would go to the right 2 and down 1).

Example. Let $F/\mathbb{Q}$ be a number field, and let $X = \operatorname{Spec} \mathscr{O}_F$.

When $F = \mathbb{Q}$, have Riemann zeta $\zeta(s)$. We'll write $\zeta^*(s)$ for the coefficient of the leading term of its Taylor expansion around s. It's a fact that $-\frac{1}{2} = \zeta^*(0)$. Note that this is

$$\zeta^*(0) = -\frac{\#\,\mathrm{Cl}(\mathbb{Q})}{\#\mathscr{O}_{\mathbb{Q}}^\times}.$$

In fact, the above holds for any number field F (up to regulator nonsense?) By previous computation, note that

$$\mathrm{Cl}(F) \cong \mathrm{Pic}(\mathscr{O}_F) \cong K_0(\mathscr{O}_F)_{\mathrm{tors}}.$$

Furthermore, Bass-Milnor-Tate tell us that $K_1(\mathscr{O}_F) \cong \mathscr{O}_F^\times$. In general, one has

$$\zeta_F^*(s) \sim \text{ratio of (even-torsion)/odd } K\text{-groups}$$

(this is a theorem of Rost-Voevodsky). Above, $\sim$ means up to simple factors. △

Question 13.7 (Audience)**.** *Is there a categorification of the LHS?*

Answer (paraphrase)**.** It's related to motivic cohomology with (fractional) Tate twists. ⋆

13.2 Milnor K-theory

Goal of today and next Tuesday is to make serious computations in Milnor K-theory, and to try and understand what it means.

Recall 13.8. Fields are local rings, so $K_1(F) = F^\times$ for F a field. ⊙

What is $K_2(F)$?

Milnor proposed the following definition, which takes serious the idea that K-theory is some kind of multiplicative extension of units.

Definition 13.9 (Milnor)**.** Let F be a field. Define the graded ring

$$K_*^M(F) := \frac{T_{\mathbb{Z}}(F^\times)}{\langle a \otimes (1-a) = 0 : a \neq 0, 1\rangle}$$

($T =$ tensor algebra), and call it **Milnor K-theory**. $\diamond$

Remark 13.10. The relation above is called the **Steinberg relation**. Also $T_{\mathbb{Z}}(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes_{\mathbb{Z}} \mathbb{F}^\times) \oplus \dots$. $\circ$

Remark 13.11. $K_0^M(F) = \mathbb{Z} = K_0(F)$ and $K_1^M(F) = F^\times = K_1(F)$. $\circ$

Notation 13.12. An element of Milnor K-theory is written using the symbol

$$\{a_1, \dots, a_j\} := a_1 \otimes \dots \otimes a_j.$$

With this notation, products are

$$\{a_1, \dots, a_j\}\{b_1, \dots, b_k\} = \{a_1, \dots, a_j, b_1, \dots, b_k\},$$

and sums are given by bi-linearity, e.g.

$$\{a_1, \dots, a_i, \dots, a_j\} + \{a_1, \dots, a_i', \dots, a_j\} = \{a_1, \dots, a_i a_i', \dots, a_j\}.$$

More specifically, $\{a\} + \{b\} = \{ab\}$ ("logarithmic nature" of bracket)

Claim 13.13. *The product is graded commutative, i.e.* $\{\alpha, \beta\} = (-1)^{|\alpha||\beta|}\{\beta, \alpha\}$. *Furthermore,*

Here, α, β are sequences of elements of $F^\times$

(1) $\{x, -x\} = 0$

(2) $\{x, x\} = \{x, -1\}$

Proof. Once we have "Furthermore," we can prove graded commutativity as follows:

$$\begin{aligned}\{x, y\} + \{y, x\} &= \{x, y\} + \{x, -x\} + \{y, -y\} + \{y, x\} \\ &= \{x, -yx\} + \{y, -yx\} \\ &= \{xy, -yx\} \\ &= 0.\end{aligned}$$

To prove **(1)**,

$$\{x, -x\} + \{x, -x^{-1}(1-x)\} = \{x, -x(-x^{-1}(1-x))\} = \{x, 1-x\} = 0,$$

so we want to show that $-\{x, -x^{-1}(1-x)\} = 0$. Note that $-\{a, b\} = \{a^{-1}, b\} = \{a, b^{-1}\}$ by bilinearity. Hence,

$$-\{x, -x^{-1}(1-x)\} = -\{x, 1-x^{-1}\} = \{x^{-1}, 1-x^{-1}\} = 0.$$

For **(2)**,

$$\{x, x\} - \{x, -1\} = \{x, -x\} = 0.$$

■

Example. Let $F = \mathbb{F}_q$ be a finite field. Then,

$$K_j^M(F) = \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{F}_q^\times \cong \dfrac{\mathbb{Z}}{(q-1)\mathbb{Z}} \cdot \tau & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let τ be a generator in degree 1. Then, we only need to show that $\{\tau, \tau\} = 0$ since K_*^M is generated in degree 1. Observe $\{\tau, \tau\} = \{\tau, -1\}$ and also

$$2\{\tau, -1\} = \{\tau, -1\} + \{\tau, -1\} = \{\tau, 1\} = 0,$$

so $\{\tau, \tau\}$ is 2-torsion (this also would have followed from graded commutativity). To finish proof, suffices to show that it is also odd-order-torsion.

- If $\operatorname{char} F = 2$, then $0 = \{1, \tau\} = \{\tau^{\text{odd}}, \tau\} = \text{odd}\{\tau, \tau\}$ since $q - 1$ is odd.
- If $\operatorname{char} F \neq 2$, we can find elements $a, b \in F^\times$ which are non-squares but which satisfy $a + b = 1$. Note $a, b = \tau^{\text{odd}}$, so
$$0 = \{a, b\} = \text{odd} \cdot \text{odd}\{\tau, \tau\}. \qquad \blacksquare$$

$\triangle$

Remark 13.14. Even for higher K-theory, all higher groups have order coprime to q, so mod p only get something in degree 0. ◦

Remark 13.15. There are maps $K_*^M(F) \longrightarrow K_*(F)$ which are isos in degrees $* \leq 2$. This map is generally not nice for $* \geq 3$. For example, when $F = \mathbb{Q}$, get

$$\mathbb{Z}/2\mathbb{Z} = K_3^M(\mathbb{Q}) \longrightarrow K_3(\mathbb{Q}) = \mathbb{Z}/48\mathbb{Z}. \quad \circ$$

Warning 13.16. K_*^M is not really a cohomology theory in any reasonable sense. •

14 Lecture 15 (10/20): Didn't go (guest lecture, so no notes to link to)

15 Lecture 16 (10/25)

OH on Thursday: "Geometry after dark" w/ Dori, 4:30Pm

15.1 Bloch's higher Chow groups

Definition 15.1. For $q \geq 0$, we set

$$\Delta^q := \frac{\operatorname{Spec} \mathbb{Z}[T_0, \ldots, T_q]}{\sum_{i=0}^q T_i = 1} \cong \mathbb{A}^q.$$

◇

Remark 15.2. The vanishing locus $V(T_i) \hookrightarrow \Delta^q$ gives $q+1$ divisors in Δ^q. ○

Example. $q = 0$ is a point.

$q = 1$ is a line with divisors $0, 1$

$q = 2$ is a plane with 3 lines forming an equilateral triangle △

TODO: Draw pictures

Definition 15.3. A **face** is an arbitrary intersection of such divisors. ◇

Example. For $q = 1$, there are only two (non empty) faces.

For $q = 2$, some faces are lines and some are points. △

From this data, we obtain a cosimplicial scheme

Really, semi-cosimplicial since we haven't given co-face maps

$$\Delta^0 \underset{V(T_1)}{\overset{V(T_0)}{\rightrightarrows}} \Delta^1 \overset{\to}{\rightrightarrows} \Delta^2 \underset{\to}{\overset{\to}{\rightrightarrows}} \dots$$

TODO: Come back and tikz this

Definition 15.4. Let k be a field, and X a k**-variety** (i.e. integral, separated, f.type$/k$). Suppose $Z, W \hookrightarrow X$ are closed subvarieties. Then, we say that Z **intersects** W **properly** if any irreducible component P of $Z \cap W$ has the expected dimension, i.e. $\operatorname{codim}_X(Z) + \operatorname{codim}_X(W) = \operatorname{codim}_X(P)$. We denote this by writing $Z \pitchfork W$. ◇

(Note this is weaker than intersecting transversally).

Remark 15.5. One always has the inequality $\operatorname{codim}_X(P) \le \operatorname{codim}_X(Z) + \operatorname{codim}_X(W)$. ○

Convention. We'll say the empty set has arbitrary codimension.

Example. $X = \mathbb{A}^2$, Z a point, and W a curve. If $Z \in W$, then $\operatorname{codim}_X(P) = 2 < 3 = \operatorname{codim}_X(Z) + \operatorname{codim}_X(W)$, so this intersection is *not* proper. Note we have $\operatorname{codim}_X(P) = 3 \iff P = \emptyset$, i.e. $Z \pitchfork W \iff Z \cap W = \emptyset$. △

Example. Say $Z = C, W = C'$ are both curves. If $C = C'$, then $\operatorname{codim}_X(P) = 1 < 2$. Here, $C \pitchfork C' \iff C \cap C'$ is a finite collection of points. △

Construction 15.6 (Bloch). Let X be a k-variety. Then, $Z^j(X, \bullet)$ (here, $\bullet \in \mathbb{N}$) is the free abelian group

$$Z^j(X, \bullet) := \mathbb{Z}\left\{Z \hookrightarrow X \times \Delta^\bullet \text{ codim } j : Z \pitchfork (X \times F) \text{ for all } F \text{ a face}\right\}.$$

These together form a simplicial abelian group (for fixed $j \ge 0$)

TODO: tikzcd this

$$Z^j(X,0) \leftleftarrows Z^j(X,1) \leftleftarrows\!\!\leftarrow Z^j(X,2) \dots .$$

The two maps at the end are intersection with $V(T_0)$, $V(T_1)$.

Remark 15.7. Note $Z^j(X, 0)$ is codim j subvarieties of X w/ no other conditions. ○

Definition 15.8. Bloch's higher Chow groups are

$$\mathrm{CH}^j(X, n) := \pi_n Z^j(X, \bullet) \cong \mathrm{H}_n\left(Z^j(X, \bullet), \sum (-1)^i d_i\right).$$

◇

Remark 15.9. If X is smooth, these give a good definition of "**motivic cohomology of** X". ◦

Remark 15.10. $\mathrm{CH}^j(X,0) \cong \mathrm{CH}^j(X)$ gives the usual Chow groups. ◦

Remark 15.11. Suppose $X \xrightarrow{f} Y$ is flat. Then, the map $\Delta^n_X \to \Delta^n_Y$ induces a pullback

$$f^* : Z^j(Y,\bullet) \longrightarrow Z^j(X,\bullet).$$

Flatness let's you preserve codimension of cycles ($\mathrm{Bl}_0\,\mathbb{P}^2 \to \mathbb{P}^2$ for example, does *not* preserve codimension under pullback).

If f is proper of relative dimension d, then you get a pushforward

$$f_* : Z^{j+d}(X,\bullet) \longrightarrow Z^j(Y,\bullet).$$ ◦

Theorem 15.12 (**Chow's Moving Lemma**, Bloch-Levine)**.** *There is a functor* $\mathrm{CH}^j(-,n) : \left(\mathrm{Sm}_K^{\mathit{flat}}\right)^{op} \longrightarrow \mathrm{Ab}$ *by the previous remark. This in fact extends to a functor* $(\mathrm{Sm}_K)^{op} \longrightarrow \mathrm{Ab}$*, still denoted by* $\mathrm{CH}^j(-,n)$*.*

(hard; we won't prove)

15.2 Nesterenko-Suslin, Totaro

Example. For F a field,

$$\mathrm{CH}^j(\operatorname{Spec} F,0) \cong \mathrm{CH}^j(\operatorname{Spec} F) = \begin{cases} \mathbb{Z} & \text{if } j=0 \\ 0 & \text{otherwise.} \end{cases}$$ △

Theorem 15.13 (NS,T)**.** *Let F be a field. Then, there is a natural isomorphism (functorial in maps of fields)*

$$\mathrm{CH}^j(F,j) \cong K^M_j(F).$$

The above theorem gives a generators and relations presentation of the LHS.

Corollary 15.14. $\mathrm{CH}^1(F,1) \cong F^\times$*.*

Corollary 15.15.

$$\bigoplus_{j\in\mathbb{N}} \mathrm{CH}^j(F,j)$$

is a graded ring (which is graded commutative) generated by elements in degree 1.

Remark 15.16. Milnor K theory is interesting because of relation to class field theory. This theorem gives one geometric realization of CFT, i.e. connected Milnor K-theory to something geometric. Usually one uses these higher Chow groups to prove things about Milnor K-theory instead of vice versa. ◦

Let's work towards the proof of theorem 15.13.

(Step 0) Reparameterize higher Chow groups into something cubical instead of something simplicial. This is kinda of technical, so feel free to kinda ignore.

First define

$$\square^n = \mathbb{A}^n \cong \left(\mathbb{P}^1 \setminus \{1\}\right)^n$$

with coordinates $T_1, \ldots, T_n$. Now, instead of having $n+1$ divisors, we'll have $2n$ divisors.

Example. When $n = 2$, get $V(T_1), V(T_1 - \infty)$, $V(T_2)$, and $V(T_2 - \infty)$. Think of $\square^2$ as a plane with x-axis $V(T_2)$ and y-axis $V(T_1)$. $\triangle$

Remark 15.17. Removing 1 from $\mathbb{P}^1$ and keeping $0, \infty$ let's you think of zeros/poles (of rational functions). For example, if you have a curve $C \to \mathbb{A}^2$, this extends to $(f, g) : \widetilde{C} \to (\mathbb{P}^1)^2$ and you can see the zeros/poles of f, g in $\mathbb{A}^2 = (\mathbb{P}^1 \setminus \{1\})^2$. $\circ$

Back to the general case. We consider the $2n$ divisors

$$\square^{n-1} \cong D^{\varepsilon}_{T_i} := V(T_i - \varepsilon) \text{ for } \varepsilon \in \{0, \infty\} \text{ and } 1 \le i \le n.$$

Definition 15.18. We define **faces** to be intersections of these divisors. We also define

$$Z^j_{\square}(X, n) := \frac{\mathbb{Z}\{Z \hookrightarrow X \times \square^n \text{ codim } j : Z \pitchfork F\}}{d^j(X, n)}.$$

Above, we mod out by the subgroup of **degenerate cycles**, where Z is *degenerate* if $\cong \pi_k^{-1}$ where

$$\pi_k : X \times \square^n \longrightarrow X \times \square^{n-1} \text{ for } k = 1, \ldots, n$$

are the natural projection maps. $\diamond$

We consider the differentials ($\cdot$ below is intersection product)

$$\partial = \sum_{i=1}^{n} (-1)^i \left(D^{\infty}_{T_i} \cdot (-) - D^0_{T_i} \cdot (-) \right) : Z^j_{\square}(X, n) \longrightarrow Z^j_{\square}(X, n-1).$$

Lemma 15.19. *These differentials form a chain complex such that*

$$\mathrm{H}_n(Z^j_{\square}(X, \bullet)) \cong \mathrm{CH}^j(X, \bullet).$$

One of the benefits of this is that it makes it easier to construct products in higher Chow.

Construction 15.20. Say X/k is smooth (so $X \times_k X$ is as well). By previous remark, get pullback

$$\Delta^* : \mathrm{CH}^j(X \times X, n) \longrightarrow \mathrm{CH}^j(X, n).$$

We want to use this to construct a product. We start with the **external product**

$$\boxtimes : \mathrm{CH}^j(X, n) \times \mathrm{CH}^i(Y, m) \longrightarrow \mathrm{CH}^{i+j}(X \times Y, m+n).$$

This ultimately comes from the isomorphism (i.e. $(Z, W) \mapsto Z \boxtimes W$, its image under below iso)

$$(X \times \square^n) \times (Y \times \square^m) \cong X \times Y \times \square^{n+m}.$$

Warning 15.21. The above does not hold with Δ in place of $\square$. $\bullet$

We use this now to give

$$\cap : \mathrm{CH}^j(X,n) \otimes \mathrm{CH}^i(X,m) \xrightarrow{\boxtimes} \mathrm{CH}^{i+j}(X \times X, n+m) \xrightarrow{\Delta^*} \mathrm{CH}^{i+j}(X, n+m).$$

We wanted a product because we want a map out of Milnor K-theory which is defined by generators (in degree 1) and relations.

(Step 1) Construct map

$$K_*^M(F) \longrightarrow \mathrm{CH}^*(F,*)$$

of graded rings.

Remark 15.22. A generator of $Z^1_\square(F,1)$ is a closed point of $\mathbb{P}^1_F \setminus \{1\}$ which intersects faces properly (meets them in codimension 2). That is, the generators are closed points of $\mathbb{P}^1_F \setminus \{0,1,\infty\}$. We'll denote the generator corresponding to the closed point $a \in \mathbb{P}^1$ via $[a] \in Z^1_\square(F,1)$. ○

Remark 15.23. What about $Z^2(F,2)$. Generators look like

- curves in $(\mathbb{P}^1 \setminus \{1\})^2$
- meeting the four codim 1 faces at points (e.g. the curve can't contain one of these faces as a component)
- meeting the four points $(0,0)$, $(\infty,0)$, $(0,\infty)$, and (∞,∞) at nothing (i.e. must avoid these 4 points) ○

To construct our map, we start with

$$\begin{array}{cccc} \mathrm{cyc}^1 : & F^\times & \longrightarrow & Z^1_\square(F,1) \\ & \{a\} & \longmapsto & \begin{cases} 0 & \text{if } a = 1 \\ [a] & \text{otherwise.} \end{cases} \end{array}$$

It's not clear that this is a group homomorphism, and it's not before modding.

Lemma 15.24. *$[a]+[b] = [ab] \in \mathrm{CH}^1(F,1)$ if $a,b,ab \neq 0,1$. Also, $[a] + [1/a] = 0 \in \mathrm{CH}^1(F,1)$ if $a \neq 0,1$.*

Proof. We need a curve in $\mathbb{A}^2$ whose boundary is $[a]+[b]-[ab]$. For this, we consider the curve $C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^2$ given by

$$\left(x, f(x) = \frac{ax-ab}{x-ab}\right).$$

If $ab \neq 1$, then $f(0) = 1$. Check $C(a,b)$ meets the four divisors at $(\infty,a),(b,0),(ab,\infty)$, so $[a]+[b]-[ab]=0$.

Otherwise (if $ab=1$), $C(a,b)$ meets them at (∞,a) and $(1/a,0)$, so $[a]+[1/a]=0$. ■

Exercise. Convince yourself the signs above are all correct.

Lemma 15.25 (Steinberg)**.** *$\mathrm{cyc}^1(a)\,\mathrm{cyc}^1(1-a) = 0 \in Z^2(F,2)$ if $a \neq 0,1$.*

For this, we'll need an element of $Z^2(F,3)$ witnessing this relation, so let's first take a moment to describe this group.

Remark 15.26. $Z^2(F,3)$ consists of

- curves in $(\mathbb{P}^1 \setminus \{1\})^3$
- meeting the six planes at points
- meeting the lines nowhere
- meeting the points nowhere $\circ$

Proof of Lemma 15.25. Take $C(a) = \left(x, 1-x, \frac{a-x}{1-x}\right)$. Can check that this only meets $V(T_3) = D^0_{T_3}$ at $(a, 1-a, 0)$. This implies that $\operatorname{cyc}^1(a)\operatorname{cyc}^1(1-a) = 0 \in \mathrm{CH}^2(F,2)$. ■

Since we've verified the relations, we conclude that we get a map

$$K_*^M(F) \longrightarrow \mathrm{CH}^*(F,*).$$

(Step 2) We want an inverse map

$$\mathrm{CH}^*(F,*) \longrightarrow K_*^M(F).$$

Note we don't (yet) know that the LHS is generated in degree 1 (with relations in degree 2), so this is harder to define.

Remark 15.27. Secretly, the secret to this construction is 'Weil reciprocity in action' (apparently, Weil reciprocity is the function field analogue of quadratic reciprocity). $\circ$

Say $F = \overline{F}$ is algebraically closed. Then, any closed point of $(\mathbb{P}^1 \setminus \{1\})^{\times j}$ is simply given by a tuple $(a_1, \ldots, a_j)$ with $a_i \in (F \cup \{\infty\}) \setminus \{1\}$. In this situation, we get a map

Higher Chow does not satisfy Galois descent, so general case cannot be reduced to this case

$$\begin{array}{ccc} Z^j_\square(F,j) & \longrightarrow & K_j^M(F) \\ [(a_1,\ldots,a_j)] & \longmapsto & \{a_1,\ldots,a_j\}. \end{array}$$

Question 15.28. *How do you know that boundaries map to* 0*?*

Example $(j=2)$**.** Have curve $C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^2$. To get access to nice theory of divisors and rational functions, consider the normalization $\nu : \widetilde{C} \to C$, so have finite map $\widetilde{C} \to (\mathbb{P}^1 \setminus \{1\})^2$ from something smooth. This will be given by a pair (f,g) of rational functions on $\widetilde{C}$, i.e. $f, g \in F(C)^\times$. Note C meets $D^\varepsilon_{T_i}$ (for $i = 1,2$ and $\varepsilon = 0, \infty$) at points. This means that f, g are not uniformly 0 or ∞. Also, C does not meet $(0,0), (0,\infty), (\infty,0), (\infty,\infty)$. This means that if $w \in C$ such that $f(w) \in \{0,\infty\}$, then $g(w) \notin \{0,\infty\}$ (and vice versa). Then,

$$C \cdot D^0_{T_2} = C \cdot \{x\text{-axis}\} = [(f(w),0)] \cdot \operatorname{ord}_w(g) \mapsto \{\overline{f}^{\operatorname{ord}_w(g)}\} \in K_2^M(F),$$

where $\overline{f}$ is the image of f is the residue field of $F(C)$ at w (note this residue field is $F = \overline{F}$ and that $\operatorname{ord}_w(f) = 0$, so $\overline{f}$ is well-defined and $\neq 0$. In other words, $\overline{f}$ is the number $f(w) \in F$). $\triangle$

Out of time, so pick things up later...

16 Lecture 17 (10/27)

Recall 16.1. Last time we were in the middle of constructing a map

$$\mathrm{CH}^j(F,j) \longrightarrow K_j^M(F).$$

Recall that this is more more difficult to construct than its inverse since $\mathrm{CH}^j(F,j)$ is not a priori generated in degree 1. To make life easier we had assumed $F = \overline{F}$. ⊙

Warning 16.2. Higher Chow does *not* satisfy Galois descent. A separate argument from the below is needed to handle the non algebraically closed case (even over perfect fields).[16] •

In this case, we had written

$$\begin{array}{ccc} \mathrm{CH}^j(F,j) & \longrightarrow & K_j^M(F) \\ (a_1,\dots,a_j) & \longmapsto & \{a_1,\dots,a_j\} \end{array}$$

We need to show

Claim 16.3. *If $C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^2 \in Z^1(F,2)$, then $\mathrm{d}(C) \mapsto 0$ under the above map.*

Proof. Assume that C is smooth (see last time for why this is ok), so $(f,g) : C \to (\mathbb{P}^1 \setminus \{1\})^2$. Recall

- C meets $D_{T_i}^{\varepsilon}$ ($i = 1,2$ and $\varepsilon = 0,\infty$) at points $\iff$ f,g not uniromly 0 or ∞
- C does not meet $(0,0),(0,\infty),(\infty,0),(\infty,\infty)$ $\iff$ $f(w) \in \{0,\infty\}$ implies $g(w) \notin \{0,\infty\}$ (and vice versa).

Note $C \cdot D_{T_2}^0 = C \cdot \{x\text{-axis}\} = [(f(w),0)]\nu_w(g)$ (where ν_w is order of vanishing at w, i.e. $\nu_w : F(C)^\times \to \mathbb{Z}$ the valuation at w). Note that $\nu_w(f) = 0$ since $f(w) \notin \{0,\infty\}$ (since $g(w) = 0$). That is, we can write

$$\begin{aligned} f(T) &= a_0 + a_1 T + \dots & \qquad a_0 &\neq 0 \\ g(T) &= a_{\nu_w(g)} T^{\nu_w(g)} + \dots & \qquad a_{\nu_w(g)} &\neq 0 \end{aligned}$$

near w. The class $[(f(w),0)]\nu(g) \mapsto \{a_0^{\nu(g)}\} = \nu(g)\{a_0\} = \nu(g)\{\overline{f}\}$, where $\overline{f}$ is the reduction of f under $\mathscr{O}_{\nu_w} \to \mathscr{O}_{\nu_w}/\mathfrak{m}_w$. Now, we want to show that

$$(-1)\left([\infty, g(w_1)]\nu_{w_1}(f) - [(0,g(w_2))]\nu_{w_2}(f)\right) + \left([f(w_3),\infty]\nu_{w_3}(g) - [(f(w_4),\infty)]\nu_{w_4}(g)\right) \longmapsto 0 \in K_1^M(F).$$

How do we prove this?

Definition 16.4. $F = \overline{F}$ and $w \in C$ (a smooth, projective F-curve). For $f,g \in F(C)^\times$, set

$$\partial_w(\{f,g\}) := (-1)^{\nu_w(f)\nu_w(g)} \overline{\left(\frac{g^{\nu_w(f)}}{f^{\nu_w(g)}}\right)}$$

(**local factor**). ◇

[16] If I heard/recall correctly, the étale sheafification of higher Chow should essentially be singular cohomology

Remark 16.5. If π is a unit of $\mathscr{O}_{\nu_w}$ and $g \in F(C)^\times$, then we can write $g = u\pi^j$ for some $j = \nu_w(g) \in \mathbb{Z}$ and $u \in \mathscr{O}_{V_w}^\times$. Now,

$$\overline{u\pi^j} = \begin{cases} \overline{u} & \text{if } j \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

○

Theorem 16.6 (Weil Reciprocity).

$$\prod_{w \in C} \partial_w(\{f, g\}) = 1$$

(with product over closed points of C)

Example. For $f, g \in F(C)^\times$, Weil Reciprocity tells us that[17]

$$g(\operatorname{div}(f)) = f(\operatorname{div}(g))$$

if $\operatorname{supp}\operatorname{div}(f) \cap \operatorname{supp}\operatorname{div}(g) = \emptyset$. △

Corollary 16.7. *If $F = \overline{F}$, then $Z^1(F, 1) \ni \partial(C) \longmapsto 0 = \{1\} \in K_1^M(F)$.*

In other words, Weil reciprocity proves the claim. ■

Question 16.8. *How do you deal with $\mathrm{CH}^j(F, j) \longrightarrow K_j^M(F)$, i.e. how do you show the obvious such map is well-defined? Also, how do you deal with F not algebraically closed?*

Answer. Suslin's reciprocity. ⋆

To set this up, we'll need to talk more about Milnor K-theory.

16.1 Symbols

Definition 16.9. A **Steinberg symbol** is an abelian group A and a map $c : F^\times \otimes_{\mathbb{Z}} F^\times \to A$ which is bilinear and which satisfies $c(r, r-1) = 0$ (whenever $r \neq 0, 1$), i.e. it's simply a homomorphism

$$K_2^M(F) \longrightarrow A.$$

◇

(This definition predated that of Milnor K-theory)

Example (Galois symbol). Say $\ell \in F^\times$, and consider the Kummer sequence

$$1 \longrightarrow \mu_\ell \longrightarrow \mathbb{G}_m \xrightarrow{(-)^\ell} \mathbb{G}_m \longrightarrow 1$$

(exact sequence of sheaves on $\operatorname{Spec} F_{\text{ét}}$). This gives rise to

$$F^\times \xrightarrow{\partial} \mathrm{H}^1_{\text{ét}}(F, \mu_\ell),$$

[17] If $\operatorname{div}(f) = \sum n_i[p_i]$, then $g(\operatorname{div}(f)) = \prod g(p_i)^{n_i}$

so we can now form

$$F^\times \otimes_{\mathbb{Z}} F^\times \xrightarrow{\partial\otimes\partial} \mathrm{H}^1_{\text{ét}}(F,\mu_\ell)\otimes \mathrm{H}^1_{\text{ét}}(F,\mu_\ell) \xrightarrow{\cup} \mathrm{H}^2_{\text{ét}}(F,\mu_\ell^{\otimes 2}).$$

This turns out to factor through $K_2^M(F) \to \mathrm{H}^2_{\text{ét}}(F,\mu_\ell^{\otimes 2})$ and is called the **Galois symbol**. △

Fact. Suslin (and someone else, I think? I missed the names) showed this gives an iso after modding by ℓ. Voevodsky proves something analogous with all the 2's replaced by any $j \geq 0$, i.e.

$$K_*^M(F)/\ell \simeq \mathrm{H}^*_{\text{ét}}(F,\mu_\ell^{\otimes *}).$$

Example (Hilbert symbol). Say $r \neq 0 \in \mathbb{Q}$. Write $r = (-1)^i 2^j 5^k u$ with $u \equiv 1 \pmod 8$. Get a symbol

$$\begin{array}{ccc} \mathbb{Q}^\times \otimes_{\mathbb{C}} \mathbb{Q}^\times & \longrightarrow & \{\pm 1\} \\ (r,s)_2 & \longmapsto & (-1)^{ii'+jj'+kk'}. \end{array}$$

△

Definition 16.10. A **higher symbol** is a map of abelian groups $K_j^M(F) \to A$ for some j. ⋄

Construction 16.11 (**tame symbol**). Let K be a field with discrete valuation $\nu : K^\times \to \mathbb{Z}$. In particular, Should not have put this all under one 'construction' block

$$\nu(ab) = \nu(a)+\nu(b) \;\text{ and }\; \nu(a,b) \geq \min\{\nu(a),\nu(b)\}.$$

Let $\mathscr{O} := \nu^{-1}(\mathbb{Z}_{\geq 0})$ which is a local ring with maximal ideal $(\pi) = \mathfrak{m}_{\mathscr{O}}$ (here, π element such that $\nu(\pi)=1$). Also,

$$x \in \mathscr{O}^\times \iff \nu(x) = 0$$

(think: power series with nonzero constant term is invertible). We call such a field K **discretely valued**.

Lemma 16.12. *If F is a discretely valued field, then its jth Milnor K group has generators*

$$\{\{u_1,\dots,u_{j-1},\pi\},\{u_1,\dots,u_j\}\} \twoheadrightarrow K_j^M(F)$$

with all $u_i \in \mathscr{O}^\times$. That is, it's generated by symbols given by all units or all units with one π at the end.

Proof for $j \leq 2$. For $j=1$, simply write $x = u\pi^j \in F^\times$, so $\{x\} = \{u\} + j\{\pi\}$. For $j=2$, compute

$$\begin{aligned} \{u\pi^r, v\pi^s\} &= \{u,v\} + \{u,\pi^s\} + \{\pi^r, v\} + \{\pi^r,\pi^s\} \\ &= \{u,v\} + s\{u,\pi\} - r\{v,\pi\} + rs\{\pi,\pi\}. \end{aligned}$$

To finish, recall $\{\pi,\pi\} = \{-1,\pi\}$. ■

The proof for higher j is similar.

Theorem 16.13 (Milnor, Serre). *For $j \geq 1$, $\exists!$ map of abelian groups*

$$\partial^M : K_j^M(F) \longrightarrow K_{j-1}^M(\kappa),$$

*where $\kappa = \mathscr{O}/\pi$ is F's residue field. This is the **tame symbol**. There's also a map*

$$S_\pi^M : K_j^M(F) \longrightarrow K_j^M(\kappa)$$

depending on π such that

$$\begin{aligned} \partial^M\left(\{u_1, \dots, u_j, \pi\}\right) &= \{\overline{u}_1, \dots, \overline{u}_j\} \\ \partial^M(\{u_1, \dots, u_j\}) &= 0 \\ S_\pi^M(\{\pi^{i_1}u_1, \dots, \pi^{i_j}u_j\}) &= \{\overline{u}_1, \dots, \overline{u}_j\} \end{aligned}$$

Remark 16.14 (Morel's heuristic). Picture $\operatorname{Spec} F \hookrightarrow \operatorname{Spec} \mathscr{O} \overset{i}{\hookleftarrow} \operatorname{Spec} \kappa$. If there were such a thing as a 'tubular neighborhood' in this context, then there would be a map

Think of F as an open, $\mathscr{O}$ as the whole space, and κ as a closed point

$$\mathbb{V}(N_i) \setminus \{0\} \longrightarrow \operatorname{Spec} F$$

($\mathbb{V}(N_i)$ the vector bundle associated to the normal sheaf of i). Suppose we can evaluate Milnor K-theory on these objects. We would get

$$K_*^M(F) \longrightarrow \text{“}K_*^M\left(\mathbb{V}(N_i \setminus \{0\})\right)\text{”}.$$

Playing with this geometric imagination, one would expect that $\mathbb{V}(N_i \setminus \{0\}) \simeq \mathbb{G}_{m,\kappa}$ (punctured line bundle on a point), with identification dependent on some choice of uniformizer π. This thing in quotations, we'll define to be

$$\frac{K_*^M(\kappa)[\xi]}{(\xi^2 = \xi\{-1\})} \quad \text{where} \quad |\xi| = 1.$$

(Think of this as pretending to be the Milnor K-theory of a circle by comparison with $\mathrm{H}^*(S^1;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^2)$ with x in degree 1). Now, we can literally define a map (S for Serre)

I don't know how to explain the $\{-1\}$

$$S : K_*^M(F) \longrightarrow \frac{K_*^M(\kappa)[\xi]}{(\xi^2 = \xi\{-1\})}$$

which is given in degree 1 by

$$\begin{array}{ccc} F^\times & \longrightarrow & \kappa^\times \oplus \mathbb{Z}\xi \\ u\pi^j & \longmapsto & (\overline{u}, j\xi). \end{array}$$

This extends to the alleged map S once we check the Steinberg relation. The graded pieces of S give

$$S_\pi : K_j^M(F) \longrightarrow K_j^M(\kappa) \text{ and } \partial : K_j^M(F) \longrightarrow K_{j-1}^M(\kappa).$$ ◦

Example ($j = 1$)**.** Get $K_1^M(F) = F^\times \xrightarrow{\nu} \mathbb{Z} = K_0^M(\kappa)$, just the valuation map. △

Example ($j = 2$)**.** Get $K_2^M(F) \longrightarrow K_1^M(\kappa) = \kappa^\times$ given by

$$\partial\left(\{u_1, u_2\}\right) = (-1)^{\nu(u_1)\nu(v_2)}\overline{\left(\frac{u_2^{\nu(u_1)}}{u_1^{\nu(u_2)}}\right)},$$

i.e. we've rediscovered the local factors. △

Let's sketch a proof of (a generalization of?) Weil's theorem. To do so, we'll need

Theorem 16.15 (Milnor)**.** *We have a short exact sequence*

$$0 \longrightarrow K_j^M(K) \longrightarrow K_j^M(K(t)) \longrightarrow \bigoplus_{p\in\mathbb{P}^1\setminus\{\infty\}} K_{j-1}^M(\kappa(p)) \longrightarrow 0,$$

and there is a splitting on the left given by $S_\infty : K_j^M(K(t)) \longrightarrow K_j^M(K)$.

I this point I got by K's and my κ's mixed up. Probably all κ's should be K's (except the ones that shouldn't). Sorting this out left to the reader

Remark 16.16. K any field. For any point $p \in \mathbb{P}^1$, one gets a valuation $v_p : K(t)^\times \to \mathbb{Z}$ and so one gets $\partial_p : K_j^M(K(t)) \longrightarrow K_{j-1}^M(\kappa(p))$. These are the maps appearing above. The splitting map comes from picking a uniformizer since $\mathscr{O}_{v_\infty}/\pi \simeq K$ (i.e. ∞ is a K-rational point). ○

This looks like the residue exact sequence for Brauer groups

Corollary 16.17. *There are maps*

$$\psi_p^M : K_{j-1}^M(\kappa(p)) \longrightarrow K_j^M(K(t))$$

such that

$$\partial_p^M \circ \psi_p^M = \begin{cases} \text{id} & \textit{if } p = q \\ 0 & \textit{otherwise.} \end{cases}$$

This is for $p, q \in \mathbb{P}^1 \setminus \{\infty\}$.

We will use this to construct norms to deal w/ the fact that $\kappa \neq \overline{\kappa}$.

Construction 16.18. Say $p \in \mathbb{P}^1$ a closed point (and $p \neq \infty$). We define a **norm map**

$$N_p : \quad \begin{tikzcd} K_j^M(\kappa(p)) \arrow[r, "\psi_p"] \arrow[dr, "N_p"'] & K_{j+1}^M(K(t)) \arrow[d, "\partial_\infty"] \\ & K_j^M(\kappa) \end{tikzcd}$$

We also set $N_\infty = \text{id}$.

Lemma 16.19. $N_p : K_0^M(\kappa(p)) \longrightarrow K_0^M(\kappa)$ *is multiplication by* $[\kappa(p) : \kappa] : \mathbb{Z} \to \mathbb{Z}$

Lemma 16.20. *In degree 1,* $N_p : K_1^M(\kappa(p)) \longrightarrow K_1^M(\kappa)$ *is the field norm* $\kappa(p)^\times \to \kappa^\times$.

Theorem 16.21 (**Higher Weil reciprocity for** $\mathbb{P}^1$)**.** *Say* $\alpha \in K_j^M(k(t))$. *Then,*

$$\sum_{p\in\mathbb{P}^1} (N_p \circ \partial_p^M)(\alpha) = 0.$$

(We'll prove next time)

Remark 16.22. If $j = 2$ and $k = \overline{k}$, then ∂_p^M are the local factors, and N_p is nothing, so recover normal Weil reciprocity. ○

Remark 16.23. Still true if $\mathbb{P}^1$ is replaced by C a smooth projective curve over k (due to Suslin). One has

$$\sum_{p\in C} N_p \circ \partial_p^M(\alpha) = 0. \qquad ○$$

17 OH: Geometry after dark

Note 4. For everything today, see Dori's notes from a class he taught.

Goal. Fano varieties are rationally chain connected.

This is due to Kollar-Miyaoka-Mori and also independently to Campana.

Remark 17.1. In char 0, can remove the word chain. ○

Remark 17.2. In any version, need $k = \overline{k}$. ○

Definition 17.3. A smooth projective variety X/k ($k = \overline{k}$) is **rationally chain connected** (RCC) if any two points on X are connected by a chain of $\mathbb{P}^1$'s, i.e. for general $x, y \in X$, there exists a 2-pointed map $f : (C, 0, \infty) \to (X, x, y)$ with C a 2-pointed tree of $\mathbb{P}^1$'s. It is **rationally connected** (RC) if you can always take $C = \mathbb{P}^1$. ◇

In particular, $p_a(C) = 0$

Remark 17.4.

(1) In char 0, RCC $\iff$ RC.

(2) In char $p > 0$, they are not equivalent. This has to do with the existence of inseparable curves.

In the sort of smoothening/deforming arguments on makes, one often uses that maps of schemes in char 0 are generically smooth. In positive characteristic, this is only true if the map is (generically?) separable.

(3) Can remove the word 'general' in the definition.

The main point is that limits of rational curves will be rational curves (can't acquire genus). If you want to connect x, y, approximate them by x_t, y_t which you can connected, and then take a family $C_t \to X$ of chains through these. The limit/closure will still be a rational curve. ○

Definition 17.5. A smooth projective variety X is **Fano** if $-K_X = \det T_X$ is ample. ◇

Example.

(1) A hypersurface $X_d \subset \mathbb{P}^N$ is Fano if $d \leq N$

(2) A complete intersection $X_{d_1,\dots,d_k} \subset \mathbb{P}^N$ is fano if $\sum d_i \leq n$. △

Slogan. Low degree hypersurfaces are very close to projective space

For example, any quadric with a point is rational, e.g. project away from that point.

Fact. cubic hypersurfaces (if $n \geq 3$) are **unirational**, i.e. there's a dominant rational map $\mathbb{P}^n \dashrightarrow X$ for some n.

Remark 17.6. The number of points of $\mathbb{P}^n$ over $\mathbb{F}_q$ is $1 \mod p$. This holds also for Fano varieties. ○

You should think that Fano hypersurfaces are close to being unirational.

Conjecture 17.7. *"Most" Fano hypersurfaces are not unirational.*

(Apparently there's no known example of a hypersurface of degree $\leq n$ which is not unirational)

Remark 17.8. Can think of being rationally chain connected as saying you have a lot of rational curves. ○

Theorem 17.9. *If $X/k = \overline{k}$ is Fano, then X is RCC.*

(Note that unirational $\implies RC \implies RCC$)

Theorem 17.10 (Mori)**.** *Projective space is the unique smooth, projective variety whose tangent bundle is ample.*

(This, or rather the techniques used to prove it, won him a fields medal)

This theorem is maybe one motivation for the definition of Fano, asking $\det T_X$ to be ample is weaker than asking T_X to be ample.

Conjecture 17.11. *In characteristic 0, there exists rationally connected but not unirational varieties.*

Apparently every invariant we know of that vanishes for unirational varieties also vanishes for RC ones.

Proposition 17.12. *In char 0, if X is RC, then*

$$\mathrm{H}^0(X, (\Omega^m)^{\otimes n}) = 0$$

for $m, n \geq 1$.

Conjecture 17.13 (Mumford's conjecture). *The above is an iff*

History. Apparently there's no record of Mumford writing it down, but at least Joe Harris says it's Mumford's conjecture. ⊖

To prove the proposition, the idea is to show that if you have something nonzero, then there are so many $\mathbb{P}^1$'s that it must pullback to something nonzero on one of them, but this is impossible ($\mathbb{P}^1$ has no global differential forms).

Remark 17.14. Apparently there's work to classify Fano varieties, and it was recently proved that smooth Fano's of a fixed dimension form a f.type moduli space (so only f.many components, so hope to actually classify/enumerate them all). ○

17.1 Deformation Theory of Curves

Setup 17.15. Have $X, Y \to S$ flat, q.proj maps to a locally noetherian base. Say we have $B \subset X$ which is also flat over S, and fix a morphism $g : B \to Y$. Visualize this as

$$\begin{tikzcd} X & Y \arrow[r] & S \\ \cup\!| & & \\ B \arrow[uur, "g"] & & \end{tikzcd}$$

Theorem 17.16. *There exists a quasi-projective S-scheme $\mathrm{Hom}_S(X, Y, g)$ whose T points are*

Note formation of the Hom-scheme commutes with arbitrary base change $T \to S$

$$\mathrm{Hom}_S(X, Y, g)(T) = \{f \in \mathrm{Hom}(X_T, Y_T) : f|_{B_T} = g_T\} \subset \mathrm{Hom}(X_T, Y_T).$$

A situation to keep in mind is

- Y some variety we care about, e.g. our Fano
- X is some curve
- $B \xrightarrow{g} Y$ is picking out points of Y, e.g. $B = \{0, \infty\}$ and we choose $x, y \in Y$ s.t. $g(0) = x$ and $g(\infty) = y$.

In the above situation, we are looking at deformations of a curve fixing two points.

The idea is that projective varieties always have curves (of some high genus). If this moduli space has high dimension, then we'll show that we can start with a curve and then deform it so much (i.e. bend it) that it eventually breaks into multiple components (we call this **bend and break**, and it's due to Mori). If you run induction with this, you should eventually break things apart into so many pieces that each has genus 0, and then you're thing is RCC.

Theorem 17.17. *Fix a point $s : \operatorname{Spec} k \to S$ and suppose that S is equidimensional at s. Suppose also that the fiber X_s is reduced (by flatness, get reduced fibers in a neighborhood), and that we fix a map*

$$f_s : X_s \to Y_s$$

with image in the smooth locus of Y_s, i.e. f_s factors through Y_s^{sm}.

(1) $T_{[f_s]} \operatorname{Hom}_k(X_s, Y_s) = \mathrm{H}^0(X, f^* T_{Y_s})$

  Slogan. Vector fields along the curve (X) are the tangent space.

  If you want to deform X in Y, choose a vector field, and then push it in that direction.

(2) every component $Z \subset \operatorname{Hom}_S(X, Y)$ containing $[f_s]$ satisfies

$$\dim Z = \dim_{[f_s]} Z \geq h^0(f_s^* T_{Y_s}) - h^1(f_s^* T_{Y_s}) + \dim_s S.$$

  Note the there's an obvious bound of $\dim Z \leq \dim_s S + \mathrm{H}^0(f_s^* T_{Y_s})$ since scheme can't have bigger dim than tangent space.

  Remark 17.18. On a curve, $h^0 - h^1$ is the Euler characteristic is easy to compute, so deformation theory on curves is easier than on higher dimensional varieties. ∘

(3) If above is an equality, then $\operatorname{Hom}_S(X, Y) \to S$ is lci (+ flat?) at $s \in S$

(4) If $h^1(X, f_s^* T_{Y_s}) = 0$, then $\operatorname{Hom}_S \to S$ is smooth at $[f_s]$.

What if $B \neq \emptyset$? Everything is the same, but everywhere you see a tangent bundle (a $f_s^* T_{Y_s}$), you should tensor with I_{B_s/X_s} (think of this as restricting to deformations which vanish along B). This modification is independent of the choice of g.

Proposition 17.19. *Say C/k is a smooth, projective curve. Let $B \subset C$ be a collection of distinct points. Let $f : C \to Y$ be a morphism to some smooth quasi-projective Y, and choose some $g : B \to Y$. Then,*

$$\dim_{[f]} \operatorname{Hom}_k(C, Y, g) \geq -K_Y \cdot f_* C + d(1 - g - n),$$

where $n = \#B$, $d = \dim Y$, and the $\cdot$ is intersection product.

Remark 17.20. If Y is Fano, then $-K_Y \cdot f_*C$ is positive as long as f is not constant (it's $\deg_C(f^*(-K_Y))$ if f is not constant, and this is positive since $-K_Y$ is ample). ∘

Proof. $\dim \geq \chi(f^*T_Y(-B)) = \deg(f^*T_Y(-B)) + \operatorname{rank}(f^*T_Y(-B))(1-g)$. Note rank $= \dim Y$. Also, $\deg(f^*T_Y) = -K_Y \cdot f_*C$. Twisting by $-B$ changes degree by $-nd$. All together, we get the statement. ∎

Remark 17.21. To get a lot of deformations, we want to make the curve have large degree (w.r.t $-K_Y$) w/o having large genus. In positive characteristic, we can use Frobenius to achieve this. ∘

Proposition 17.22 (**bend and break I**). *Work over $k = \overline{k}$. Take $f : C \to X$ with X smooth projective variety (and C a nice curve still), with one special point $c \in C$ whose image is fixed (i.e. $n = 1$). Suppose that*

So no difference between rational curves and genus 0 curves

$$-K_X \cdot f_*C - g \dim X > 0$$

(so also $\dim_[f] \operatorname{Hom}(C, X, g = f|_c) > 0$). Then, there exists a rational curve through $f(c) \in X$.

Proof. $\dim \operatorname{Hom} \geq 1$, so pick a curve $T \subset \operatorname{Hom}$ (any curve). Then, we get a family of maps

$$\begin{array}{ccccc} C \times \{0\} & \subset & C \times T & \xrightarrow{F} & X \\ \downarrow & & \downarrow & & \\ 0 & \in & T & & \end{array}$$

such that $F|_{C\times 0} = f$. We have more than this. We also know, by construction, that $F(\{c\} \times T) = f(c)$ is constant. Can assume that T is smooth (normalize if not). Let $\overline{T} \supset T$ be its smooth, projective completion. Can complete $C \times T \subset C \times \overline{T}$, so F gives a rational map

$$F : C \times \overline{T} \dashrightarrow X.$$

Claim 17.23. *F has an indeterminancy point along $\{c\} \times \overline{T}$, i.e. there is at least one point of indeterminancy there.*

This is a consequence of the rigidity lemma,[18] and the fact that we chose a nontrivial deformation to start with.

CTRL + F this in Mumford's abelian varieties book, for example

Remark 17.24. One needs to be a little careful. Really, rigidity would show that if no indeterminancy, then the deformation would be trivial up to $\operatorname{Aut}(C, c)$. However, $\operatorname{Aut}(C, c)$ if finite if $g > 0$. ∘

To finish, replace $C \times \overline{T}$ with blowup $\operatorname{Bl}_{(c,p)}(C \times \overline{T})$ with (c, p) a point of indeterminancy. F extends to a morphism on this, will no nontrivial on the exceptional divisor which is a $\mathbb{P}^1$. This gives our rational curve. ∎

Proposition 17.25 (**bend and break II**). *Work over $k = \overline{k}$. Take $f : \mathbb{P}^1 \to X$ with X smooth projective variety and with two special points $0, \infty \in \mathbb{P}^1$ whose images are fixed (i.e. $n = 2$). Suppose that*

$$-K_X \cdot f_*C - g \dim X \geq 2.$$

[18] If $\mathscr{L}$ is ample on X, it'll intersect $c \times \{\overline{T}\}$ in degree 0, so it'll intersect all such fibers in degree 0. This is only possible if it contracts all horizontal lines.

Then, there is a rational curve of smaller degree.

Remark 17.26. This one is somehow useful for lifting to char 0. Something like it gives a uniform bound on degrees which let's you study things by looking at finitely many (one?) Hilbert scheme instead of needing some non-finite type moduli. ∘

Let's not look at a relative situation.

Proposition 17.27 (bend and break III). *Work over $k = \overline{k}$. Say we have*

$$\begin{tikzcd} B & \subset & C \arrow[r, "f"] \arrow[dr, "g"'] & X \arrow[d, "\pi"] \\ & & & Y \end{tikzcd}$$

(Above, π projective). Consider

$$\begin{tikzcd} \rho: & \mathrm{Hom}(C, X, f|_B) \arrow[r, "\pi\circ -"] \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & \mathrm{Hom}(C, Y, \pi \circ f|_B) \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\in$}"] \\ & \mathrm{Hom}(C, X, g, f|_B) \arrow[r] & {[g]} \end{tikzcd}$$

Say $g(C) > 0$ and we have $c \in C$ such that

$$\dim_{[F]} \mathrm{Hom}(C, X, g, f|_c) \geq 1.$$

Then, $f_* C \sim_{rat} f'_* C + Z$ where Z is an effective rational cycle, and $f' : C \to X$ is such that $\pi \circ f' = g$ and $Z \subset$ fiber of π.

(I missed exactly what hypotheses we need on X, Y above)

Proposition 17.28 (bend and break IV). *Same sort of setup as last time. If ρ is non-constant on some component of the Hom-scheme $\mathrm{Hom}(C, X, f|_B)$ containing $[f]$, then we can find an effective rational cycle Z such that $\pi_*(Z) \neq 0$.*

Some deformation not contracted by ρ

Theorem 17.29. *Suppose $X/k = \overline{k}$ is Fano. Say we have*

$$\begin{tikzcd} X_0 \arrow[r, phantom, "\overset{dense}{\subset}"] \arrow[d, "proj+surj"', "\pi"] & X \arrow[d, dashed] \\ Y_0 \arrow[r, phantom, "\subset"] & Y \end{tikzcd}$$

Then, for any point $y \in Y_0$, there exists a rational curve $f : \mathbb{P}^1 \to X$ such that $f_*(\mathbb{P}^1) \not\subset X_y$, but $f_* \mathbb{P}^1 \cap X_y \neq \emptyset$.

Proof Sketch. Can reduce to the case where π is a morphism.[19] Suppose char $k = p > 0$. Now, find any $f : (C, c) \to X$ with $f(c) \in X_y$, but $f(C) \notin X_y$. The existence of such a thing comes from π being

[19] Blowup of Fano is not Fano, but blowup has lots of horizontal curves (the exceptional divisors), so things work out. Sounds like it's not super immediate to work out, but can not too bad?

projective. Suppose $g(C) > 0$ (if $g(C) = 0$, we're done). Say H ample on Y. Choose $0 < \alpha \ll 1$ such that

$$-K_X - \alpha\pi^* H$$

is ample (ampleness is open, so preserved under sufficiently small perturbation). Let $F_m : C_m \to C$ be the mth power Frobenius. Consider the intersection number

$$\alpha H \cdot (\pi \circ f \circ F_m)_* C_m - g(C) \dim X = \alpha p^m H \cdot (\pi \circ f)_* C - g(C) \dim X \gg 0 \text{ for } m \gg 0$$

(Note $H \cdot (\pi \circ f)_*(C) > 0$ since H ample). So, without loss of generality, we can assume

$$\alpha H \cdot (\pi \circ f)_* C \geq g \dim X.$$

By our choice of α, we also have $-K_X \cdot f_* C \geq \alpha H \cdot (H \circ f)_* C$.

Remark 17.30. This suffices for bend and break I. If we only wanted to know that there was a single rational curve, then this would do it. ○

Let's look at

$$\rho : \operatorname{Hom}(C, X, f|_C) \longrightarrow \operatorname{Hom}(C, Y, (f \circ \pi)|_C).$$

If ρ is non-constant on components containing $[f]$, then bend and break IV implies the theorem (gives a horizontal rational curve), so suppose that ρ is constant on every component containing $[f]$. At this point, one derives a contradiction using bend and break III. Note this assumption gives

$$\rho^{-1}(f \circ \pi) = \operatorname{Hom}(C, X, f|_c),$$

so we get the same dimension bound for the relative Hom in BnB III. In particular, we can make it big with the same Frobenius trick. Now apply BnB III. Each time we do this we get some new curve (the $f'_* C$) with strictly smaller degree (since $-K_X \cdot Z > 0$ since Z effective and $-K_X$ ample). This must end eventually, but that contradicts our frobenius trick. ■

Theorem 17.31 (**MRC fibrations** (Maximally rationally connected))**.** *Say X is smooth projective. Then, $\exists X_0 \subset X$ and $Y \supset Y_0$ along with a projective morphism $\pi : X_0 \to Y_0$ such that*

(1) *The fibers of π are RCC*

(2) *For very general $y \in Y$, $x \in \pi^{-1}(y) \iff x$ can be connected to $\pi^{-1}(y)$ by a chain of rational curves.*

(Get this morally by forming the quotient of X by the equivalence relation, any two (very general) points which are rationally chain connected get identified)

Proof. For any X, X if RCC $\iff$ MRC$(X) = pt$ (MRC(X) is the Y in previous theorem). Suppose X is Fano, and let $\pi : X \dashrightarrow Y$ be the MRC fibration. If Y is not a point, then we can find some very general $y \in Y$ satisfying **(2)** above. This contradicts the previous theorem we just proved. ■

18 Lecture 18 (11/1)

Recall 18.1. Say $\mathscr{O}_v$ is a dvr with fraction field F and residue field κ. Let $\pi \in \mathscr{O}_v$ be a uniformizer. Then we get the tame symbol

$$\partial_v : K_j^M(F) \longrightarrow^M_{j-1} (\kappa).$$

This is determined by

$$\partial_v\{\pi, a_2, \ldots, a_j\} = \{\overline{a}_2, \ldots, \overline{a}_j\} \text{ and } \partial_v\{a_1, \ldots, a_j\} = 0,$$

where $a_j \in \mathscr{O}_v^\times$. In fact, with π chosen, one also gets a **specialization**

$$\begin{array}{rccc} S_\pi : & K_j^M(F) & \longrightarrow & K_j^M(\kappa) \\ & \{\pi^{i_1}a_1, \ldots, \pi^{i_j}a_j\} & \longmapsto & \{\overline{a}_1, \ldots, \overline{a}_j\}. \end{array}$$

$\odot$

Theorem 18.2 (Milnor)**.** *We have a short exact sequence*

$$0 \longrightarrow K_j^M(\kappa) \longrightarrow K_j^M(\kappa(t)) \longrightarrow \bigoplus_{p \in \mathbb{P}^1 \setminus \{\infty\}} K_{j-1}^M(\kappa(p)) \longrightarrow 0$$

which is split on the left by $S_{t^{-1}}$.

Corollary 18.3. *There is a splitting on the right, called **cospecialization**, of the form*

$$\psi_{p,j} : K_j^M(\kappa(p)) \longrightarrow K_{j+1}^M(\kappa(t)).$$

Corollary 18.4. *For any $j > 0$, any $\alpha \in K_j^M(\kappa(t))$ can be written in the form*

$$\alpha = S_{t^{-1}}(\alpha) + \sum_{p \in \mathbb{P}^1 \setminus \{\infty\}} \psi_p^M \circ \partial_p^M(\alpha).$$

Recall we were previously interested in constructing a map

$$\mathrm{CH}^j(F, j) \longrightarrow K_j^M(F)$$

when F is a field. To do such a thing, we need(ed) to know that our proposed construction kills boundaries, i.e. if $C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^{\times(j+1)} \in Z^j_\square(F, j+1)$, then $\partial(C) \mapsto 0$.

Recall 18.5. We saw that when $F = \overline{F} = \mathbb{C}$, this is a consequence of a generalization of Weil reciprocity.

$\odot$

We want to discuss the general case.

Recall 18.6. We have **norm maps**

$$N_p : K_j^M(\kappa(p)) \xrightarrow{\psi_{p,j}^M} K_{j+1}^M(\kappa(t)) \xrightarrow{-\partial_\infty} K_j^M(\kappa)$$

for $p \in \mathbb{P}^1 \setminus \{\infty\}$. We also set $N_\infty := \mathrm{id}$. $\odot$

(Think of this as some sort of 'vanishing cycles' construction).

Theorem 18.7 (General Weil Reciprocity). *For any $\alpha \in K_j^M(\kappa(t))$,*

$$\sum_{p \in \mathbb{P}^1} (N_p \circ \partial_p^M)(\alpha) = 0.$$

Proof. We use Milnor's exact sequence, Theorem 18.2. Consider

$$\partial_p^M \left(\underbrace{\alpha - \sum_{q \in \mathbb{P}^1 \setminus \{\infty\}} \psi_q^M \circ \partial_q^M(\alpha)}_{\beta \in K_j^M(\kappa(t))} \right) = \partial_p^M(\alpha) - \partial_p^M(\alpha) = 0.$$

Thus, we have $\beta \in K_j^M(\kappa)$. Applying ∂_∞^M, we get

$$0 = \partial_\infty^M(\beta) = \partial_\infty^M(\alpha) - \sum_p \partial_\infty^M \circ \psi_p^M(\partial_p^M(\alpha)).$$

Rearrange and recall the definition of N_p to finish. ∎

For concreteness (to make the norm less mysterious), we remark

Remark 18.8. For E/F a finite extension, we can always construct

$$N_{E/F} : K_j^M(E) \longrightarrow K_j^M(F).$$

(To do this, the rough idea is to factor $E \to F$ as a sequence of monogenic extensions and construct this norm map inductive + prove that it's independent of choices) ○

Proposition 18.9. *When $j \leq 1$, we have*

$$\begin{array}{ccc} K_0^M(E) & \xrightarrow{N_{E/F}} & K_0^M(F) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{Z} & \xrightarrow{[F:E]} & \mathbb{Z} \end{array} \quad \text{and} \quad \begin{array}{ccc} K_1^M(E) & \xrightarrow{N_{E/F}} & K_1^M(F) \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ E^\times & \xrightarrow{\mathrm{Nm}_{E/F}} & F^\times. \end{array}$$

Construction 18.10. Consider a point $\operatorname{Spec} \kappa(p) \xrightarrow{(f_1,\ldots,f_j)} (\mathbb{P}^1 \setminus \{1\})^{\times j}$, viewed as an element of $Z^j_\square(F,j)$. Note that this map includes an isomorphism $\kappa(p) \simeq k(f_1, \ldots, f_j)$. This cocycle gets mapped to

Think of this as a 'framing' of the point $\operatorname{Spec} \kappa(p)$

$$N_{\kappa(p)/k}\{f_1, \ldots, f_j\} \in K_j^M(\kappa).$$

(Note $\{f_1, \ldots, f_j\} \in K_j^M(\kappa(p))$).

For the boundary map

$$Z^j_\square(F, j+1) \xrightarrow{\mathrm{d}} Z^j_\square(F, j),$$

we want to show that $d(C) \mapsto 0 \in K_j^M(F)$ ($\implies$ map descends to $\mathrm{CH}^j(F,j) \to K_j^M(F)$).

Proof Sketch. Consider a curve $C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^{j+1}$ and first replace it with normalization $\widetilde{C} \xrightarrow{\nu} C$. We then get the composition $(f_1, \ldots, f_j) : \widetilde{C} \xrightarrow{\nu} C \hookrightarrow (\mathbb{P}^1 \setminus \{1\})^{j+1}$.

Exercise. Show the boundary only depends on the normalization, so we're justified in passing to the smooth case.

The conditions for being in Z^j translate to the following:

- f_j is neither constantly 0 or constantly ∞ for any j.
- If $w \in \widetilde{C}$ such that $f_j(w) \in \{0, \infty\}$, then $f_i(w) \notin \{0, \infty\}$ for all $i \neq j$.

Now, C may not be proper, so let $\widetilde{C} \hookrightarrow \overline{\widetilde{C}}$ be its unique smooth compactification. We now appeal to

Theorem 18.11 (**Suslin's reciprocity**). *Let X/F be a smooth, proper curve. Then,*

$$\sum_{w \in X} N_{\kappa(w)/F} \circ \partial_w(\alpha) = 0.$$

(Sounds like you can reduce to the case of $\mathbb{P}^1$ by playing around with a branched cover $X \to \mathbb{P}^1$)

Now, we have $f_1, \ldots, f_{j+1}$ which are rational functions on $\overline{\widetilde{C}}$. Suslin reciprocity tells us that

$$\sum_{w \in \overline{\widetilde{C}}} \mathrm{Nm}_{k(w)/F} \circ \partial_w\{f_1, \ldots, f_{j+1}\} = 0.$$

We want to prove that

$$\sum_{w \in \widetilde{C}} \mathrm{Nm}_{k(w)/F}\, \partial_w\{f_1, \ldots, f_{j+1}\} = 0.$$

If we have this, we're done. Now, suppose that $w \in \overline{\widetilde{C}} \setminus \widetilde{C}$ is in the boundary. Then, there must exist an f_i such that $f_i(w) = 1$ (otherwise, $\widetilde{C} \to (\mathbb{P}^1 \setminus \{1\})^{\times j}$ would not be proper). Observe that

Alternatively, use that $\overline{\widetilde{C}} \to \mathbb{P}^1$ is either surjective or constant... it's not clear to me this implies the claim. I only see this implying that for all i there's some w in the boundary s.t. $f_i(w) = 1$, but this is not the same?

$$\partial_w\{f_1, \ldots, f_{j+1}\} = 0$$

if one of the f_i's has $f_i(w) = 1$ (in Milnor K-theory, "$1 = 0$"). Thus, we can omit those w's in the sum in Suslin reciprocity.

This is all enough because the map $Z^j_\square(F, j) \longrightarrow K^M_j(F)$ is given by

$$\left(\operatorname{Spec} k \xrightarrow{(f_1, \ldots, f_j)} (\mathbb{P}^1 \setminus \{1\})^{\times j}\right) \longmapsto N\{f_1, \ldots, f_j\}.$$

Furthermore,

$$\partial_w\{f_1, \ldots, f_{j+1}\} = \nu_w(f_i)\left\{f_1(w), \ldots, \widehat{f_i(w)}, \ldots, f_{j+1}(w)\right\}$$

(this is because the zeros/poles only occur once, so there's at most one i with $f_i(w) \in \{0, \infty\}$, i.e. with $\nu_w(f_i) \neq 0$). ∎

The upshot is that we have now constructed maps

$$K^M_j(F) \longrightarrow \mathrm{CH}^j(F, j) \longrightarrow K^M_j(F)$$

(first map easy, second map hard). One can check that the composite is the identity, so we have a split injection

$$K_j^M(F) \hookrightarrow \mathrm{CH}^j(F, j).$$

Why is this surjective?

Lemma 18.12. *Any $\alpha \in \mathrm{CH}^j(F, j)$ is equivalent to a sum of F-rational points.*

Proof when $j = 1$. Say we have some point $p \in \mathbb{P}^1_F \setminus \{0, 1, \infty\}$. Let $x \in \kappa(p) \setminus \{0, 1\}$ be a generator over F. Let $E(T) = T^d - a_{d-1}T^{d-1} + \cdots + (-1)^d a_0 \in F[T]$ be the minimal polynomial of x, and note that $a_0 = N(x)$ is the norm of x. Consider the following curve:

$$V(P(T, S)) \subset (\mathbb{P}^1 \setminus \{1\})^2 \text{ where } P(T, S) = E(T) - (T - 1)^{d-1}(T - a_0)S.$$

This intersects

- $S = 0$ at the point p
- $S = \infty$ at the point a_0 (an F-rational point)
- $T = 0$ nowhere
- $T = \infty$ nowhere

Thus, $[p] = [a_0] = [N_{k(p)/F}(x)]$. ■

Theorem 18.13 (Nesterenko-Suslin, Totaro)**.** *Let F be a field. Then,* $\mathrm{CH}^j(F, j) \cong K_j^M(F)$.

18.1 Lecture 9: Boston in the 90's

18.1.1 Geisser-Levine, a retrospective

The story starts with.

Proposition 18.14. *Say R is a perfect $\mathbb{F}_p$-algebra. Then,*

$$K_j(R, \mathbb{Z}/p\mathbb{Z}) = 0 \text{ for all } j > 0.$$

Proof. $\mathrm{Frob}_R^* \curvearrowright K_j(-, \mathbb{Z}/p\mathbb{Z})$. One can show that, in degree j, this induces multiplication by p^j. Thus, it's the zero map for $j \geq 1$. However, R is perfect, so it's an isomorphism for $j \geq 0$. ■

Here's a corollary of the main result of Geisser-Levine.

Theorem 18.15 (G-L)**.** *Let X be ess (?) smooth over k with k perfect of characteristic $p > 0$. Then,*

$$K_{>\dim X}(X; \mathbb{Z}/p\mathbb{Z}) = 0$$

Remark 18.16. K-theory is almost never bounded above, so the above should be v. surprising. ○

Other consequences of their main theorem

Theorem 18.17. *Let F be a char $p > 0$ field. Then, $K_j^M(F)$ and $K_j(F)$ are both p-torsion free (This generalizes a result of Izhboldin). Furthermore,*

$$K_j^M(F)/p^r \longrightarrow K_j(F)/p^r$$

is an isomorphism.

Theorem 18.18. *Let $K^{ét} = L_{ét}K_{\geq 0}$ (étale sheafification of K-theory). Then,*

$$K^{ét}(-, \mathbb{Z}/p\mathbb{Z})$$

is a flat sheaf on $\mathrm{Sch}_{\mathbb{F}_p}$.

We'll spend the next 15 minutes (and up to half an hour next time) telling the story of how this result came about.

History (Prehistory)**.** Let V be a dvr of mixed characteristic $(0, p)$. Let κ be its residue field (assumed perfect), and let F be its fraction field.

Theorem 18.19 (Faltings '88, "p-adic Hodge theory")**.** *Let X/F be smooth and proper. Then, there exists a Galois-equivariant natural isomorphism*

$$\mathrm{H}^n_{ét}(X_{\overline{F}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_p \simeq \bigoplus_{i+j=n} \mathrm{H}^i(X, \Omega^j) \otimes_F \mathbb{C}_p(-j),$$

where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ is the completion of the algebraic closure of $\mathbb{Q}_p$. Above, we've implicitly chosen an embedding $F \hookrightarrow \mathbb{C}_p$.

Above,

$$\mathbb{Z}_p(q) := \varprojlim \mu_{p^n} \text{ and } \mathbb{Z}_p(-1) = \mathbb{Z}_p(1)^\vee$$

as usual.

Example. Say $X = \mathbb{P}^1$ and $n = 2$. Then,

$$\mathrm{H}_{\text{ét}}(\mathbb{P}^1, \mathbb{Z}_p) \cong \mathbb{Z}_p(-1).$$

This implies that $\mathrm{H}^0(\Omega^2) = 0$, $\mathrm{H}^2(\mathscr{O}) = 0$, and $\mathrm{H}^1(\Omega^1)$ is 1-dimensional. In fact, could replace $\mathbb{P}^1$ with $\mathbb{P}^k$ and still have $\mathrm{H}^2_{\text{ét}}(\mathbb{P}^k, \mathbb{Z}_p) \cong \mathbb{Z}_p(-1)$. △

Example. Say $X = E$ is an elliptic curve. Consider the p-adic Tate module $T_p(E) := \varprojlim E[p^n]$. It's well-known that $\mathrm{H}^1_{\text{ét}}(E, \mathbb{Z}_p) \cong T_p(E)^\vee$. Thus,

$$T_p(E) \otimes \mathbb{C}_p \cong \left(\mathrm{H}^1(E, \mathscr{O})^\vee \otimes_K \mathbb{C}_p\right) \oplus \left(\mathrm{H}^0(E, \Omega^1)^\vee \otimes \mathbb{C}_p(1)\right).$$ △

A more serious consequence is the following

Theorem 18.20 (Kontsevich, Deref-Loeser (spelling?), Ito)**.** *Say X, Y are smooth projective over $\mathbb{C}$ are **Calabi-Yau**'s ($K_X, K_Y \sim 0$) which are birational. Then, $h^{ij}_X = h^{ij}_Y$.*

(Faltings let's you reduce to comparing étale cohomology, and this is something accessible via point counts. Matching up of point counts comes from the birational assumption) "Faltings is a serious guy, but these guys are much less serious. So, Bloch and Kato... " (paraphrased, referring to their demeanor)

Bloch and Kato proved special cases of Faltings in '84 using K-theory. The setup is

$$\begin{tikzcd} X_k \arrow[r, hook] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook'] \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spec} V & \operatorname{Spec} F. \arrow[l, hook'] \end{tikzcd}$$

Then there are two spectral sequences

$$\mathrm{H}^p_{\text{ét}}(X_{\overline{k}}, \overline{i}^* R^q j_* \mathbb{Z}/p^r\mathbb{Z}) \implies \mathrm{H}^{p+q}_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Z}/p\mathbb{Z})$$

(**vanishing cycle spectral sequence**) and the slop spectral sequence. By comparing the two (using K-theory, more on this later), they proved Faltings' (later) result in special cases. $\ominus$

19 Lecture 19 (11/3)

19.1 Retrospective, continued

History (continuation from last time)**.** Recall the setup

$$\begin{tikzcd} X_k \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X_\eta \arrow[l, hook', "j"'] \arrow[d] \\ \operatorname{Spec} \kappa \arrow[r, hook] & \operatorname{Spec} V & \operatorname{Spec} F. \arrow[l, hook'] \end{tikzcd}$$

Above, V is a dvr of mixed characteristic $(0, p)$ with perfect residue field κ. Assume throughout that X/V is smootha and proper. Bloch and Kato wanted to understand $\mathrm{H}^q_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/p^n\mathbb{Z})$ in terms of $\mathrm{H}^i(X_F, \Omega^j)$. One idea was to try and construct a spectral sequence going from Hodge cohomology to étale cohomology. They looked at the *vanishing cycles spectral sequence*. First consider

$$M_n^q := i^* R^q j_* \mu_{p^n}^{\otimes q}$$

(note M_n^q is a sheaf on $X_{K,\text{ét}}$). To get a sense of this, we note that its stalks are

$$(M_n^q)_{\overline{y}} \cong \mathrm{H}^q_{\text{ét}}(\mathscr{O}^{\text{sh}}_{X,\overline{y}}[1/p], \mu_{p^n}^{\otimes q})$$

(for $\overline{y} \in X_\kappa$)

Remark 19.1. Above, we are *not* doing weird things like looking at étale coh w/ mod p coeffs in char p. $\circ$

From these objects, one gets the **vanishing cycles spectral sequence**

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(X_{\overline{\kappa}}, \overline{M}_n^q) \implies \mathrm{H}^{p+q}_{\text{ét}}(X_{\overline{F}}, \mathbb{Z}/p^n\mathbb{Z}).$$

BK constructed their filtration ("one of the most underused in algebraic K-theory")

$$U^r \subset \cdots \subset U^1 \subset U^0 M_n^q = M_n^q$$

('U' units, the **unit filtration**).

Example ($n = 1$)**.** When $n = 1$, we have

(i) $\mathrm{gr}_U^0(M_1^q) \cong \Omega^q_{X_\kappa,\log} \oplus \Omega^{q-1}_{X_\kappa,\log}$. We'll define this properly in a moment, but for now know they are sheaves on the Zariski site consisting of logarithmic differential forms.

(ii) Let $e' = \frac{ep}{p-1}$ with e the ramification degree of V (over $W(\kappa)$). For $1 \leq m < e'$ with $(m, p) = 1$, one has

$$\mathrm{gr}^m(M_1^q) \cong \Omega^{q-1}_{X_\kappa},$$

a coherent sheaf.

(iii) If $p \mid m$, then

$$\mathrm{gr}^m \cong \frac{\Omega^{q-1}_{X_\kappa}}{Z^{q-1}} \oplus \frac{\Omega^{q-2}_{X_\kappa}}{Z^{q-2}},$$

where we'll say what these cycles Z^* are later.

(iv) If $m \geq e'$, then $U^m = 0$.

We have a filtration on this étale object M_n^q whose graded pieces are of logarithmic and coherent natures. △

Theorem 19.2 (Bloch-Kato '84)**.** *If $e = 1$ and $n = 1$, then we get a short exact sequence*

$$0 \longrightarrow \Omega^{q-1}_{X_\kappa} \longrightarrow i^* R^q j_* \mu_p^{\otimes q} \longrightarrow \Omega^{q-1}_{X_\kappa,\log} \oplus \Omega^q_{X_\kappa,\log} \longrightarrow 0.$$

More generally, you have

$$0 \longrightarrow U^1 \longrightarrow i^* R^q j_* \mu_{p^n}^{\otimes q} \longrightarrow W_n\Omega^{q-1}_{X_\kappa,\log} \oplus W_n\Omega^q_{X_\kappa,\log} \longrightarrow 0.$$

Definition 19.3. Say X is a regular $\mathbb{F}_p$-scheme. Consider the following map in X_{Zar}:

$$\begin{array}{ccc} \mathbb{G}_m^{\otimes j} & \longrightarrow & W_n\Omega^j_X \\ f_1 \otimes \cdots \otimes f_j & \longmapsto & \mathrm{d}\log[f_1] \wedge \cdots \wedge \mathrm{d}\log[f_j]. \end{array}$$

Let $W_n\Omega^j_{X,\log} \in \mathrm{Ab}(X_{\mathrm{Zar}})$ denote the sheafification of the image of this map. ⋄

Remark 19.4. There is a multiplicative lift $[-] : R \to W(R)$. Got distracted, but Elden wrote

$$\frac{\mathrm{d}[f_1]}{[f_1]} \wedge \cdots \wedge \frac{\mathrm{d}[f_j]}{[f_j]} \in W\widehat{\Omega}^j_{W(R)} \longrightarrow W\,\mathrm{Sat}(W\widehat{\Omega}^j_R) = W\Omega^j_R.$$

∘

Exercise. Show that $\mathrm{d}\log[f] \wedge \mathrm{d}\log[1-f] = 0$ and then think of Milnor K-theory.

If I heard correctly, sounds like you can sometimes define Milnor K-theory for rings (as opposed to just fields) and then get a factorization

$$\begin{tikzcd} \mathbb{G}_m^{\otimes j} \arrow[r] \arrow[rr, bend left] & \text{``}\mathcal{K}_j^M\text{''} \arrow[r] & W_n\Omega^j_{\log,X_\kappa} \end{tikzcd}$$

Warning 19.5. Constructing the map $i^*R^qj_*\mu_p^{\otimes q} \to \Omega^{q-1}_{X_\kappa,\log} \oplus \Omega^q_{X_\kappa,\log}$ is hard™. •

What BK ended up proving is the following.

Theorem 19.6. *Assume X_κ is **ordinary**, by which we mean*

$$\mathrm{H}^q\left(X_{\overline{\kappa}}, \Omega^r_{\log,X_{\overline{\kappa}}}\right) \otimes_{\mathbb{Z}/p\mathbb{Z}} \overline{\kappa} \cong \mathrm{H}^q(X_{\overline{\kappa}}, \Omega^r_{X_{\overline{\kappa}}}).$$

Then, there exists a $\mathrm{Gal}(\overline{F}/F)$*-equivariant isomorphism*

$$\mathrm{gr}^{q-i}_{\text{vanishing cycles}}\mathrm{H}^q_{\acute{e}t}(X_{\overline{F}}, \mathbb{Q}_p) \cong \check{\mathrm{H}}^q(X_{\overline{\kappa}/W(\overline{\kappa})})^{\Phi_X=p^i}\left[\frac{1}{p}\right](-i)$$

(LHS E_2 page of vanishing spectral sequence). Also

$$blah, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\overline{\kappa}) \cong \mathrm{H}^{q-1}(X_{\overline{\kappa}}, W\Omega^i)(-i)$$

(RHS E_2-page of slop spectral sequence), and

$$blah, \mathbb{C}_p) \cong \mathrm{H}^{q-i}(X_F, \Omega^i_F) \otimes \mathbb{C}_p(-i).$$

Exercise. Stare at everything above (in this history block) until you get a sense of how it all fits together.

⊖

(I've moved out of the history block at this point. I don't know if this was the right place to do it, but it doesn't really matter)

We want to understand the short exact sequence constructed in the Bloch-Kato result. Let R be a strictly henselian local ring with residue characteristic $p > 0$. We'd like to construct a map

$$\mathrm{H}^q_{\text{ét}}(R[1/p], \mu_p^{\otimes q}) \dashrightarrow \Omega^q_{\log,R/p} \oplus \Omega^{q-1}_{\log,R/p}.$$

We'd even like it to factor through

$$\text{``}K_q^M(R/p)\text{''} \oplus \text{``}K_{q-1}^M(R/p)'' \xrightarrow{\mathrm{d}\log\oplus\mathrm{d}\log} \Omega^q_{\log,R/p} \oplus \Omega^{q-1}_{\log,R/p}$$

(for objects in quotes, defined them using same generators and relations as you would for fields). Recall we have the Galois symbol

$$\text{``}K_q^M(R[1/p])\text{''} \longrightarrow \mathrm{H}^q_{\text{ét}}(R[1/p], \mu_p^{\otimes q}).$$

Is this map an isomorphism, or even surjective? This was posed in '84 and solved in 2010.

Recall 19.7. We care about this because we have $K_q^M(R[1/p]) \to K_q^M(R/p) \oplus K_{q-1}^M(R/p)$. Recall the tame symbols (when R is a dvr). ⊙

Remark 19.8. The situation above is summarized in the diagram

$$\begin{tikzcd} \mathrm{H}^q_{\text{ét}}(R[1/p], \mu_p^{\otimes q}) \arrow[r, dashed, "?"] \arrow[dr, dashed, "?"] & K_q^M(R/p) \oplus K_{q-1}^M(R/p) \arrow[d, "\mathrm{d}\log \oplus \mathrm{d}\log"] \\ K_j^M(R[q/p]) \arrow[u, "\cong ?"] \arrow[ur] & \Omega^q_{\log, R/p} \oplus \Omega^{q-1}_{\log, R/p} \end{tikzcd}$$

○

Conjecture 19.9 (Bloch-Kato)**.** *Let F be a field in which p is invertible. Then, the Galois symbol*

$$K_q^M(F)/p^r \longrightarrow \mathrm{H}^q_{\acute{e}t}(F, \mu_{p^r}^{\otimes q})$$

is an isomorphism for all q, r.

History (History of Bloch-Kato)**.** When $q = 0, 1$, you can do things by hand. When $q = 2$, it was proven by Merkurjev-Suslin in '83 (For F a discretely valued field with residue field perfect of characteristic p?).

Question 19.10. *Is there a uniform statement of Bloch-Kato, independent of characteristic (e.g. no $1/p \in F$ assumption)? Also, what if instead of F, we have a scheme X?*

Lichtenbam '84, Beilinson '87, and Milne '88 formulated and conjectured the following picture. Say X is smooth over perfect F. There should be some objects

$$\mathbb{Z}(j)_X^{\text{mot}} \in D(X_{\text{Zar}}) \text{ and } \mathbb{Z}(j)_X^{\text{ét-mot}} \in D(X_{\text{ét}})$$

satisfying

- $\mathbb{Z}(j)_X^{\text{ét}}/\ell^r \cong \mu_{\ell^r}^{\otimes j}$ if $\ell \in \mathscr{O}_X^\times$
- $\mathbb{Z}(j)_X^{\text{ét}}/p^r \cong W_r\Omega^j_{\log}[-j]$ if $p = 0 \in \mathscr{O}_X$

  This comes from zeta value considerations.[20]

The mot and ét-mot complex are related via

$$D(X_{\text{ét}}) \xrightarrow{R\varepsilon_*} D(X_{\text{Zar}})$$

(forgetful functor, every étale sheaf is a Zariski sheaf).

Conjecture 19.11 (Beilinson-Lichtenbaum)**.** *For all p*

$$\mathbb{Z}(j)_X^{mot}/p^r \longrightarrow R\varepsilon_*\mathbb{Z}(j)_X^{\acute{e}t\text{-}mot}/p^r$$

factors through an isomorphism

$$\mathbb{Z}(j)_X^{mot}/p^r \xrightarrow{\sim} \tau^{\leq j}_{Zar} R\varepsilon_*\mathbb{Z}(j)^{\acute{e}t\text{-}mot}/p^r.$$

[20] Sounds like Milne gave a proof of the Weil conjectures in '88 using char p methods, and in particular, using the sheaves $W_r\Omega^j_{\log}$

For now, let's simply define (think of this as a sheaf of complexes)

$$\mathbb{Z}(j)_X^{\mathrm{mot}} := \left(U \mapsto Z^j(U, \bullet)[-2j]\right)$$

(or the sheafifcation of this). Now, in this notation, one has

$$\mathrm{H}^j(\mathbb{Z}(j)_X^{\mathrm{mot}}(L)) \cong \mathrm{CH}^j(L, j) \cong K_j^M(L)$$

for L a field. If $1/\ell \in L$, you should get a map

TODO: Convince yourself BL says you should have an isomorphism like below

$$K_j^M(L) \longrightarrow \mathrm{H}^j_{\text{ét}}(L, \mu_\ell^{\otimes j})$$

that should be an iso, and so Beilinson-Lichtenbaum recovers Bloch-Kato.

Notation 19.12. Define $\mathrm{H}^i_{\mathrm{mot}}(X; \mathbb{Z}(j)) := \mathrm{H}^i(\mathbb{Z}(j)^{\mathrm{mot}}(X))$.

Concretely, say $\mathscr{O}$ is an equicharacteristic regular local ring. Then, BL conjecture says that

$$\mathrm{H}^i_{\mathrm{mot}}(\operatorname{Spec}\mathscr{O}; \mathbb{Z}/p^r\mathbb{Z}(j)) = \begin{cases} \mathrm{H}^i_{\text{ét-mot}}(\mathscr{O}; \mathbb{Z}/p^r\mathbb{Z}(j) & \text{if } i \le j \\ 0 & \text{otherwise.} \end{cases}$$

We can break this up further.

- If $1/p \in \mathscr{O}_X$, this is
$$\begin{cases} \mathrm{H}^i_{\text{ét}}(\mathscr{O}, \mu_{p^r}^{\otimes j}) & \text{if } i \le j \\ 0 & \text{otherwise.} \end{cases}$$
- If $p = 0 \in \mathscr{O}_X$, this is
$$\begin{cases} \mathrm{H}^0_{\text{ét}}(\mathscr{O}, W_r\Omega^j_{\log} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Using hypercohomology spectral sequence, we get all other values of $\mathrm{H}^i_{\mathrm{mot}}(X, \mathbb{Z}/p^r\mathbb{Z}(j))$ (for general X).

19.2 Where are we going?

For the rest of the semester, our goal is to prove

$$\mathrm{H}^i_{\mathrm{mot}}(\operatorname{Spec}\mathscr{O}; \mathbb{Z}/p^r\mathbb{Z}(j)) = \begin{cases} \mathrm{H}^0_{\text{ét}}(\mathscr{O}, W_r\Omega^j_{\log} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

when $p = 0$. This it the **Geisser-Levine theorem** from '99.

Remark 19.13 ("Motives away from p"). Some remarks about $1/p \in \mathscr{O}_X$.

Theorem 19.14 (Rost-Voevodsky, 1999-2010). *BL w/ $1/\ell \in \mathscr{O}_X$ is true.*

This uses motivic homotopy theory, with two crucial ingredients being the construct of Steenrod operations and Rost motives. $\circ$

Next week, we want to reformulate Gisser-Levine in the language of motivic homotopy theory. In the intervening years, the subjects of K-theory and p-adic Hodge theory have separated. Often, you want to prove something like

$$\mathrm{H}^*(X_{\overline{F}}, \mathbb{Z}/p\mathbb{Z}) \otimes B \cong \mathrm{H}^*(X_{dR}) \otimes B$$

for B some "huge ring".

Example.

- If $B = B_{dR}$, a specific ring with a filtration whose graded pieces are $\bigoplus \mathbb{C}(-j)$, then a theorem of this form is due to Faltings.
- If $B = B_{crys}$, a specific ring with a filtration and frobenius action, then a theorem of this form is also due to Faltings.

  Sounds like both this these results mostly ignore the methods/contributions of Bloch-Kato. However, later on, Niziol (one of Faltings' students) reproved this using K-theory. For log schemes, Tsuji proved this using the BK method.
- BMS 2 '18 (Bhatt-Morrow-Scholze, "THH and integral p-adic Hodge") has basically reunified the two subjects. $\triangle$

20 Lecture 20 (11/8)

OH Thursday 5:30pm by Sanath (Witt vectors after dark)

20.1 Motivic homotopy theory

We'll need to setup some motivic homotopy theory in order to discuss (and prove?) the following theorem.

Theorem 20.1 (Gisser-Levine)**.** *Let $\mathscr{O}$ be a regular local $\mathbb{F}_p$-algebra. Then, for $r \geq 1$ and $i, j \geq 0$, one has*

$$\mathrm{H}^i_{mot}(\mathscr{O}, (\mathbb{Z}/p^r\mathbb{Z})(j)) \cong \begin{cases} 0 & \text{if } i \neq j \\ W_r\Omega^j_{\log,R} & \text{otherwise.} \end{cases}$$

(second object above is subgroup of $W_r\Omega^j_R$ generated by log forms, see previous lecture)

A rapid primer to motivic homotopy theory Throughout, all schemes are qcqs.

Definition 20.2. Let B be a (base) scheme. Then,

(1) A **Nisnevich square** is a pullback square of schemes of the form

$$\begin{tikzcd} Y \times_X U \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ U \arrow[r, "j"] & X \end{tikzcd}$$

with p étale, j an open immersion, and p inducing an isomorphism

$$p^{-1}\left((X \setminus U)_{\text{red}}\right) \xrightarrow{\sim} (X \setminus U)_{\text{red}}.$$

(2) A functor $E : \mathrm{Sm}_B^{\mathrm{op}} \to \mathrm{Sp}$ (spectra) is said to be an **$\mathbb{A}^1$-invariant Nisnevich sheaf** if both

- **(i)** E takes Nisnevich squares to (homotopy) Cartesian squares
- **(ii)** The natural map $E(X) \to E(X \times \mathbb{A}^1)$ is always an equivalence

Apparently not obvious this is enough to really be a Nisnevich sheaf, but it's true when X is qcqs. For us, we'll take what's written as a definition.

(3) Set $\Omega_{\mathbb{P}^1} E(X)$ (**$\mathbb{P}^1$-loops on** E) to be the fiber

$$\Omega_{\mathbb{P}^1} E(X) := \mathrm{fib}\left(E(\mathbb{P}^1 \times X) \xrightarrow{\infty^*} E(X)\right).$$

Note that this map is split, so there's a functorial splitting $E(\mathbb{P}^1 \times X) \simeq \Omega_{\mathbb{P}^1} E(X) \oplus E(X)$.

Think of this as giving reduced cohomology, e.g. image $X = pt$

(4) Suppose that $\{E(\bullet)\}_{\bullet \in \mathbb{Z}}$ is a collection of functors $E(j) : \mathrm{Sm}_B^{\mathrm{op}} \to \mathrm{Sp}$. A **$\mathbb{P}^1$-bundle datum at level** j is a map

$$E(j) \longrightarrow \Omega_{\mathbb{P}^1} E(j+1).$$

Think of the adjoint maps in a suspension spectrum?

If $\{E(\bullet)\}$ has $\mathbb{P}^1$-bundle datum at all levels j, we say that it is a **$\mathbb{P}^1$-bundle datum**.

(5) A **homotopy $\mathbb{A}^1$-invariant motivic cohomology theory** is the data of

- **(i)** $\{E(\bullet)\}_{\bullet \in \mathbb{Z}}$
- **(ii)** $\mathbb{P}^1$-bundle datum

such that

- **(a)** The $E(j)$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf for all j
- **(b)** The $\mathbb{P}^1$-bundle data are all equivalences $\diamond$

Example. Assume $1/p \in \mathscr{O}_B$, and set $E(j) = R\Gamma_{\text{ét}}(-, \mu_{p^r}^{\otimes j})[2j]$. Recall that for each line bundle $\mathscr{L}$ on X, it has a **first Chern class** $c_1(\mathscr{L}) \in \mathrm{H}^2_{\text{ét}}(X, \mu_{p^r}^{\otimes 1})$. We'll use this to construct boundary maps between our $E(j)$'s.

Fact.

$$\begin{tikzcd} R\Gamma_{\text{ét}}(X, \mu_{p^r}^{\otimes j}) \arrow[dr, "\pi^*"] & \oplus & R\Gamma_{\text{ét}}(X, \mu_p^{\otimes j-1})[-2] \arrow[dl, "\pi^*(-)\cup c_1(\mathscr{O}(-1))"] \\ & R\Gamma_{\text{ét}}(X \times \mathbb{P}^1, \mu_{p^r}^{\otimes j}) & \end{tikzcd}$$

is an isomorphism (**projective bundle formula**).

Furthermore,

- $R\Gamma_{\text{ét}}(-, \mu_{p^r}^{\otimes j})$ is an $\mathbb{A}^1$-invariant Nisnevich sheaf.

  $\mathbb{A}^1$-invariance requires p invertible on the base. It's always a Nis sheaf.

Thus, $\{E(j)\} =: \mathrm{H}_{\text{ét}}\, \mu_{p^r}$ is an $\mathbb{A}^1$-invariant motivic cohomology theory. $\triangle$

Example. Say B is a regular (noetherian) base. Then,

$$K(X \times \mathbb{A}^1) \simeq K(X)$$

if X/B is smooth ($\implies X$ regular). Thomosan-Trobough proved that this is in fact a Nisnevich sheaf (recall we're working over Sm_B).

Recall 20.3.

$$K(\mathbb{P}^1_X) \simeq K(X) \oplus K(X)$$

with first summand generated by $\{\mathscr{O}\}$ and second generated by $\{\mathscr{O} - \mathscr{O}(-1)\}$. Thus,

$$\Omega_{\mathbb{P}^1} K \simeq K. \qquad \odot.$$

So one can set $E(j) := K$ for all j which gives a (weightless) cohomology theory called KGL. $\triangle$

Example. Let k be a field and $X \in \mathrm{Sm}_k$.

Fact.

$$\mathrm{H}^i_{\mathrm{mot}}(X, \mathbb{Z}(1)) = \begin{cases} \mathscr{O}(X)^\times & \text{if } i = 1 \\ \mathrm{Pic}(X) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we can make sense of first Chern classes $c_1(\mathscr{L}) \in \mathrm{H}^2_{\mathrm{mot}}(X, \mathbb{Z}(1))$. In particular, c_1 for motivic cohomology is an isomorphism (this is not the case e.g. in étale cohomology). Just as in the étale story, can set

$$\{E(j) := \mathbb{Z}(j)^{\mathrm{mot}}[2j]\} =: \mathrm{H}\,\mathbb{Z}$$

(note $\mathbb{Z}(j)^{\mathrm{mot}} = 0$ if $j < 0$). $\triangle$

Exercise. Try to realize de Rham cohomology in characteristic 0 as one of these objects.

Remark 20.4. We can package everything into an ∞-category, but it's only useful once we need symmetrical monoidal structures $\otimes$. If you only need to talk about individual motivic spectra, then you can just do that. Let's give a summary of the general context though.

- There is a symmetric monoidal stable ∞-category called the category $\mathrm{SH}(B)$ of **motivic spectra**, and any $\mathbb{A}^1$-invaariant motivic cohomology theory defines an object in $\mathrm{SH}(B)$.

- There is a functor

$$M_B(-) : \mathrm{Sm}_B \longrightarrow \mathrm{SH}(B)$$

  associating to X the "*relative B-motive*" $M_B(X)$. This functor is symmetric monoidal, with product structure on Sm_B the usual product of schemes (over B).

- Sometimes we want *pointed* motives

$$\begin{tikzcd} X \arrow[d, shift right] \\ B \arrow[u, bend right, "x"'] \end{tikzcd}$$

In this case,

$$M_B(X,x) :=\rightarrowtail \Big(M_B(B) \xrightarrow{x_*} M_B(X)\Big).$$

Notation 20.5. We write $\mathbb{S}_B := M_B(B)$ for the "**sphere spectrum**".

- There are new relations in the essential image of M_B

  Example. $M(\mathbb{P}^1, 1) \simeq M(\mathbb{G}_m, 1)[1]$.

  To see this, consider the Nisnevich square

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1 \end{tikzcd}.$$

  After applying $M_B(-,1)$, must have a homotopy (bi)cartesian square

$$\begin{tikzcd} M(\mathbb{G}_m) \arrow[r] \arrow[d] & M(\mathbb{A}^1) \arrow[d] \\ M(\mathbb{A}^1) \arrow[r] & M(\mathbb{P}^1). \end{tikzcd}$$

  At the same time $M(\mathbb{A}^1) = M(*)$ by $\mathbb{A}^1$-invariance, so this says that $M(\mathbb{P}^1) \simeq M(\mathbb{G}_m)[1]$. △

- The object $M(\mathbb{P}^1, 1)$ is $\otimes$-invertible in $\mathrm{SH}(B)$

  (This is the consequence of the equivalences in $\mathbb{P}^1$-bundle data). Furthermore, this tells us that $M(\mathbb{G}_m, 1)$ is also $\otimes$-invertible.

  Notation 20.6. We have bigraded-spheres

  Below, one thinks of q as *weight* and of $p-q$ as *cohomological grading*

$$\mathbb{S}^{p,q} := M(\mathbb{G}_m, 1)^{\otimes q}[p-q] \ \text{ for any } \ p, q \in \mathbb{Z}.$$

  In particular $\mathbb{S}^{1,1} \simeq M(\mathbb{G}_m, 1)$ and $\mathbb{S}^{2,1} \simeq M(\mathbb{P}^1, 1)$.

- We can compute the values of $E = \{E(\bullet)\}$ any $\mathbb{A}^1$-invariant motivic cohomology theory using Hom's in $\mathrm{SH}(B)$ via

$$\pi_{q-p}(E(q)(X)) \simeq [M_B(X), \mathbb{S}^{p,q} \otimes E]$$

  $([-,-] := \pi_0 \operatorname{Map}_{\mathrm{SH}(B)}(-,-))$ ○

Definition 20.7. Given E, we study it using **homotopy sheaves**, which will be certain sheaves of abelian groups on Sm_B (in the Nisnevich topology). One first defines

$$\underline{\pi}_i(E)_{-j}$$

to be the Nisnevich sheafification (in the conventional abelian sense) of the presheaf

$$U \longmapsto \big[M(U)[1], \mathbb{S}^{j,j} \otimes E\big] \simeq [M(U)[1], E(j)[-j]].$$ ◇

Example. Say $B = \operatorname{Spec} k$ with k a field. Let L/k be a field extension. Then,

$$\underline{\pi}_0(H\mathbb{Z})_{-j}(L) \cong K_j^M(L)$$

by Totaro and Nesterenko-Suslin (Theorem 18.13). △

Question: Is the claim that $\underline{\pi}_0(H\mathbb{Z})_{-j}(L) \cong \mathrm{H}^j_{\mathrm{mot}}(L, \mathbb{Z}(j))$ when you unpack definitions?

In general, it's useful to think of the full graded object $\{(\underline{\pi}_0 E)_*\}$. Also, there is a hypercohomology style spectral sequence relating these homotopy sheaves to global values (we won't explain this until we need it).

20.1.1 Input I: Homotopy t-structure and Gersen (spelling?) principle

This is a way to reduce questions about rings/schemes to ones about fields.

Remark 20.8. Say R is a noetherian domain. Then, $R \hookrightarrow \operatorname{Frac}(R)$. Setting $X = \operatorname{Spec} R$, one also has[21]

$$\bigcap_{x \in X^{(1)}} \mathscr{O}_{X,x} \hookrightarrow \operatorname{Frac}(R).$$

If R is normal (integrally closed in fraction field), then

$$R \cong \bigcap_{x \in X^{(1)}} \mathscr{O}_{X,x}$$

(compare this to Hartog's theorem). ○

To study objects in $\mathrm{SH}(B)$, we use something called a '***t*-structure**'. "Recall" that this is a datum on a stable ∞-category $\mathcal{C}$ of two subcategories $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ such that

- $\mathcal{C}_{\geq 0}[1] \subset \mathcal{C}_{\geq 0}$
- $\mathcal{C}_{\leq 0}[-1] \subset \mathcal{C}_{\leq 0}$
- There are no nonzero maps $\mathcal{C}_{\geq 0} \to \mathcal{C}_{\leq 0}[-1]$ (from objects in LHS to objects in RHS)
- Any X can be functorially decomposed into

$$X_{\geq 0} \longrightarrow X \longrightarrow X_{\leq -1}$$

  with $X_{\geq 0} \in \mathcal{C}_{\geq 0}$ and $X_{\leq -1} \in \mathcal{C}_{\leq -1}$.

Example. Say Λ is a ring. Can set (homological grading on subscript)

$$D(\Lambda)_{\geq 0} := \left\{ M : \mathrm{H}^i(M) = 0 \text{ for } i > 0 \right\} \text{ and } D(\Lambda)_{\leq 0} := \left\{ M : \mathrm{H}^i(M) = 0 \text{ for } i < 0 \right\}.$$ △

Notation 20.9. If $\mathcal{C}$ has a t-structure, one sets

$$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}.$$

Fact (Beilinson-Bernstein-Deligne-Gabber)**.** $\mathcal{C}^{\heartsuit}$ is an abelian category.

[21] $X^{(1)}$ is the set of codim points of X

This gives a way of passing from (stable) ∞-categories back to a more concrete (abelian) 1-category.

Example. Have $\mathrm{Sp}, \mathrm{Sp}_{\geq 0}, \mathrm{Sp}_{\leq 0}$. Then,

$$\mathrm{Sp}^{\heartsuit} \simeq \mathrm{Ab}\,.$$

Also, $D(\mathbb{Z})^{\heartsuit} \simeq \mathrm{Ab}$. That is, two different categories can have the same heart. △

Note $\mathrm{Sp}^{\heartsuit} \hookrightarrow \mathrm{Sp}$ is fully faithful, and this is the Eilenberg-Maclane functor $H(-)$.

Theorem 20.10 (Morel). *Let k be a perfect field. Then, there exists a t-structure on $\mathrm{SH}(k)$ such that $E \in \mathrm{SH}(k)_{\geq 0}$ iff*

$$\underline{\pi}_i(E)_*(L) = 0 \ \textit{for}\ i < 0$$

when L is a f.g. field extension of k. Furthermore, $E \in \mathrm{SH}(k)_{\leq 0}$ iff

$$\underline{\pi}_i(E)_*(L) = 0 \ \textit{for}\ i > 0$$

when L/k f.g. Furthermore, $\pi_0(E)_*$ defines canonical objects in $\mathrm{Sh}(k)$ such that

$$[M(X), \mathbb{S}^{p,q} \otimes \pi_0(E)_*] \cong \mathrm{H}^{p,q}_{\mathrm{Zar}}(X, \underline{\pi}_0(E)_{-q}).$$

In particular, we can check membership of $\mathrm{SH}(k)_{\geq 0}$ by evaluating at fields instead of needing to look at stalks (henselian rings?).

Remark 20.11. We make sense of values on L/k (which may not be smooth) via colimits. That is, we start with some $\mathscr{F} : \mathrm{Sm}_B^{\mathrm{op}} \to \mathrm{Sp}$ and then extend to some $\mathscr{F} : \mathrm{EssSm}_B^{\mathrm{op}} \to \mathrm{Sp}$ via colimts. Here, EssSm_B is the category of **essentially smooth schemes**, i.e. limits of smooth schemes w/ affine transition maps. ○

(Note that if k is perfect, then any field extension is separable and so essentially smooth)

Definition 20.12. A **homotopy module** is an object of $\mathrm{SH}(k)^{\heartsuit}$ for k perfect. ⋄

Example. For any $E \in \mathrm{SH}(k)_{\geq 0}$, get map $E \to \pi_0(E)_*$, e.g. there's a map

$$\mathrm{H}\,\mathbb{Z} \longrightarrow K^M_{-*}.$$

We'll formulate our goal in terms of this one map. △

21 Lecture 21 (11/10)

OH Today at 5:30

Let's remind ourselves of our goal.

Theorem 21.1 (Geisser-Levine). *Let $\mathscr{O}$ be a regular local $\mathbb{F}_p$-algebra. Then,*

$$\mathrm{H}^i(\mathscr{O}, \mathbb{Z}/p^r\mathbb{Z}(j)) = \begin{cases} 0 & \textit{if } i \neq j \\ W_r\Omega^j_{\log} & \textit{otherwise.} \end{cases}$$

Recall 21.2 (previous lecture)**.**

- There is a category $\mathrm{SH}(B)$ with objects $\{E(\bullet)\}_{\bullet\in\mathbb{Z}} \in \mathrm{SH}(B)$, $\mathbb{A}^1$-invariant motivic cohomology theories.
- Let k be a perfect field.

  Theorem 21.3 (Morel)**.** *There is a t-structure* $(\mathrm{SH}(k)_{\geq 0}, \mathrm{SH}(k)_{\leq 0})$ *such that*

  – $E \in \mathrm{SH}(k)_{\geq 0}$ *iff*
  $$\underline{\pi}_i(E)_*(L) = 0 \ \text{for}\ i < 0$$
  for any L/k a f.g. field extension.

  – $E \in \mathrm{SH}(k)_{\leq 0}$ *iff*
  $$\underline{\pi}_i(E)_*(L) = 0 \ \text{for}\ i > 0$$
  for any L/k a f.g. field extension.

  As a consequence, get
  $$\underline{\pi}_0(E)_* \in \mathrm{SH}(k)^{\heartsuit} \hookrightarrow \mathrm{SH}(k).$$

  Example. $\underline{\pi}_0(H\mathbb{Z})_j(L) \cong K^M_{-j}(L)$. This is a reinterpretation of Theorem 18.13 (since LHS is $\mathrm{H}^{-j}_{\mathrm{mot}}(L, \mathbb{Z}(-j))$). △

  Finally, note that it is not obvious how these t-structure conditions are only determined on fields. ⊙

Recall 21.4. Say R is a noetherian, normal domain. Then, $R \cong \bigcap_{x\in X^{(1)}} R_x \subset \mathrm{Frac}(R)$ where $X = \operatorname{Spec} R$. ⊙

Say R is smooth$/k$. A lot of people have defined Milnor K-theory on R as follows:

$$K^M_j(R) := \bigcap_{v\in X^{(1)}} \ker\left(K^M_j(k(X)) \xrightarrow{\partial_v} K^M_{j-1}(\kappa(v))\right)$$

(above, $X = \operatorname{Spec} R$ and $k(X) = \operatorname{Frac} R$). This is called **unramified Milnor K-theory**.

Gersten (spelling?) showed that $K^M_j(R) \cong \mathrm{H}^j_{\mathrm{mot}}(X, \mathbb{Z}(j))$.

Remark 21.5. Sounds like it's a somewhat recurrent phenomenon of $\mathbb{A}^1$-invariants that many things are determined by their values on fields (along with connecting maps between them). ◦

21.1 Morel's theorem

Fix k a perfect field along with $E : \mathrm{Sm}_K^{\mathrm{op}} \to \mathrm{Sp}$, an $\mathbb{A}^1$-invariant Nisnevich sheaf. These two conditions are nontrivial for the following reason (not just because they look like topology):

Lemma 21.6. *Let $X \in \mathrm{Sm}_k$ and let $\mathscr{O}_{X,x}$ be the local ring of a closed point $x \in X$. Then, for any j,*

$$\pi_j E(\mathscr{O}_{X,x}) \longrightarrow \pi_j E(\mathrm{Frac}(\mathscr{O}_{X,x}))$$

is injective.

(In particular, if you kill the values on fields, then you kill the values on stalks[22])

Lemma 21.7 (**Gabber's presentation Lemma**). *Let X be an affine, smooth connected d-dimensional scheme over an arbitrary field k (e.g. not necessarily perfect). Suppose that $Z \hookrightarrow X$ is a closed subscheme of positive codimension. Let $t_1, \dots, t_j \in X$ be closed points. Then, possibly after shrinking X around $t_1, \dots, t_j$, we can find a nonempty open $V \subset \mathbb{A}^{d-1}$ and a map*

$$\varphi = (\psi, v) : U \to V \times \mathbb{A}^1$$

such that

(1) *φ is étale*

(2) *$\varphi|_Z : Z \to V \times \mathbb{A}^1$ defines a closed immersion*

(3) *$\psi|_Z : Z \to V$ is a finite morphism*

(4) *$\varphi^{-1}(\varphi(Z)) = Z$.*

Note 5. There was a picture here to help digest the statement, but it was more involved than I could probably recreate later...

Remark 21.8. Some remarks from the missing picture

- We only really care about the t_j's not about all of X

  (eventually want to make a statement about local rings)

- Think of Z as the "bad guy." We want to isolate this, so we can remove it.

- From the lemma, one can extract

$$\begin{tikzcd} X \setminus Z \arrow[r, hook, "\text{open}"] \arrow[d] & X \arrow[d, "\varphi", "\text{ét}"'] & Z \arrow[l, hook', "\text{closed}"'] \arrow[d, "\cong"] \\ V \times \mathbb{A}^1 \setminus \varphi(Z) \arrow[r, hook, "\text{open}"] & V \times \mathbb{A}^1 & \varphi(Z) \arrow[l, hook', "\text{closed}"'] \end{tikzcd}$$

  We want to replace the top line with the bottom line. Note in particular that the left square is a Nisnevich square, and that we have a product with $\mathbb{A}^1$ showing up.

- A few remarks on the proof

  – Think of this lemma as a simultaneous Noether normalization of X, Z.

    You prove Noether normalization by embedding in $\mathbb{P}^N$ and projecting away from a hyperplane. In the dual projective space of hyperplanes in $\mathbb{P}^N$, there is a locus of those which work well for both X, Z. This is how you get started (but then need more work to get φ étale and a closed immerion on Z and yadda yadda).

  – Gabber proved this for k infinite, email to Morel for k finite

[22]You can replace $\mathscr{O}_{X,x}$ above with its henselization

– Hogadi-Kulkanni (spelling?) proved this for k finite using Poonen (presumably his Bertini over finite fields paper)

$\circ$

Definition 21.9. A k-scheme X is **essentially smooth** if $X \simeq \lim X_i$ with X_i smooth over k and with affine transition maps. $\diamond$

For any closed $Z \hookrightarrow X$ with X essentially smooth, set

$$E_Z(X) := \mathrm{fib}\,(E(X) \longrightarrow E(X \setminus Z))$$

(note $X, X \setminus Z$ both essentially smooth, so E's values on them are defined).

Warning 21.10. E_Z is not a sheaf in general. $E_Z(X)$ is just the above defined spectrum. $\bullet$

This sits in a long exact sequence

$$\pi_i E_Z(X) \longrightarrow \pi_i E(X) \longrightarrow \pi_i E(X \setminus Z) \longrightarrow \pi_{i-1} E_Z(X).$$

Lemma 21.11 (Key)**.** *Let $E : \mathrm{Sm}_k^{op} \to \mathrm{Sp}$ be a Nisnevich sheaf satisfying*

($\star$) For $V \in \mathrm{Sm}_k$ and $W \hookrightarrow V$ closed, get a commutative diagram

$$\begin{tikzcd} E_{\mathbb{P}^1_W}(\mathbb{P}^1_V) \arrow[r, "\infty^*"'] \arrow[rr, bend left, "j^*"] & E_W(V) \arrow[r, "\pi^*"'] & E_{\mathbb{A}^1_W}(\mathbb{A}^1_V) \end{tikzcd} .$$

(Note this holds e.g. is E is $\mathbb{A}^1$-invaraint. More generally, use projective bundle formula?)

Then, for $x \in X$ closed (with $X \in \mathrm{Sm}_k$), the map

$$\pi_j(E(\mathscr{O}_{X,x})) \longrightarrow \pi_j(E(F))$$

is injective ($F = \mathrm{Frac}\,\mathscr{O}_{X,x}$).

Proof. Begin with an element $s \in \pi_j(E(\mathscr{O}_{X,x}))$ such that $s|_F = 0$. We want to show that $s = 0$. By possible shrinking X, we may assume that s is defined on X and vanishes away from some $Z \hookrightarrow X$ with positive codimension, i.e. there is some $\widetilde{s} \in \pi_j(E(X))$ restricting to $s \in \pi_j(E(\mathscr{O}_{X,x}))$. By definition, we have $\widetilde{s} \in \pi_j(E_Z(X)) = \ker(\pi_j(E(X)) \to \pi_j(E(X \setminus Z)))$. To prove the result, it suffices to construct an open $U \ni x$ along with a closed subscheme Z' with $Z \cap U \hookrightarrow Z' \cap U$ such that $\widetilde{s}$ vanishes on $\pi_j\left(E_{Z' \cap U}(U)\right)$. To see this, consider

$$\begin{tikzcd} \pi_j(E_Z(X)) \arrow[r] \arrow[d] & \pi_j(E(X)) \arrow[r] \arrow[d] & \pi_j(E(X \setminus Z)) \arrow[d] \\ \pi_j(E_{Z \cap U}(U)) \arrow[r] & \pi_j(E(U)) \arrow[r] & \pi_j(E((X \setminus Z') \cap U)) \end{tikzcd}$$

Moving around the left square shows that $\widetilde{s} \mapsto 0 \in \pi_j(E(U))$ and so $s = 0 \in \pi_j(E(\mathscr{O}_{X,x}))$.

By Gabber (Lemma 21.7), we may produce

$$\begin{tikzcd} U \setminus Z \arrow[r, hook, "\text{open}"] \arrow[d] & U \arrow[d, "\text{ét}"', "\varphi=(v,\psi)"] \\ \mathbb{A}^1_V \setminus \varphi(Z) \arrow[r, hook, "\text{open}"] & \mathbb{A}^1_V \end{tikzcd}$$

with $Z \xrightarrow{\sim} \varphi(Z)$ (note Z above the same Z from the start of the proof, and U is whatever open Gabber outputs). Nisnevich descent tells us that

$$\begin{tikzcd} E_{\varphi(Z)}(\mathbb{A}^1_V) \arrow[r] \arrow[d, "\wr"] & E(\mathbb{A}^1_V) \arrow[r] \arrow[d] & E(\mathbb{A}^1_V \setminus \varphi(Z)) \arrow[d] \\ E_Z(U) \arrow[r] & E(U) \arrow[r] & E(U \setminus Z) \end{tikzcd}$$

the right square above is Cartesian (both rows above are fiber sequences). Note the fibers are equivalent e.g. because the right square is Cartesian. Now, let $F = \psi(Z) \hookrightarrow V$, and define $Z' := \psi^{-1}(F)$, which contains $Z \cap U$. We get

$$\begin{tikzcd} \pi_i(E_Z(U)) \arrow[r] & \pi_i(E_{Z'}(U)) \\ \pi_i(E_{\varphi(Z)}(\mathbb{A}^1_V)) \arrow[r] \arrow[u, "\wr"] & \pi_1(E_{\mathbb{A}^1_F}(\mathbb{A}^1_V)). \arrow[u] \end{tikzcd}$$

At this point, to finish the proof, we need only show that the bottom map is zero (can compute top map by going around square the long way). We like the bottom map because is involves $\mathbb{A}^1$'s. This is where we'll use assumption $(\star)$. Unsurprisingly, we write down another diagram

$$\begin{tikzcd} E_{\varphi(Z)}(\mathbb{A}^1_V) \arrow[r, "0?"] & E_{\mathbb{A}^1_F}(\mathbb{A}^1_V) & \\ & & E_F(V) \arrow[ul] \\ E_{\varphi(Z)}(\mathbb{P}^1_V) \arrow[r] \arrow[uu, "\wr"] & E_{\mathbb{P}^1_F}(\mathbb{P}^1_V) \arrow[uu, "j^*"] \arrow[ur, "\infty^*"] & \end{tikzcd}$$

The leftmost vertical map is an iso by Nisnevich excision, and the triangle above is where we use condition $(\star)$. Note that the bottom composition is zero simply because $\varphi(Z)$ does not meet the ∞ section (it lives in $\mathbb{A}^1$). By commutativity, the top map (with the question mark) must be 0 too. ■

Proof of Lemma 21.6. Condition $(\star)$ is verified because E is $\mathbb{A}^1$-invariant, so follows from Lemma 21.11. ■

Let's give a sketch of the proof of Morel's theorem on t-structures.

Proof Sketch of Theorem 21.3. Begin with category $\mathcal{C} := \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_\kappa, \mathrm{Sp})$ of Nisnevich sheaves. This has a t-structure where

$$\mathcal{C}_{\geq 0} = \{E : \underline{\pi}_i E = 0 \text{ for } i > 0\} \text{ and } \mathcal{C}_{\leq 0} = \{E : \underline{\pi}_i E = 0 \text{ for } i > 0\}.$$

This t-structure is induced by the one on spectra.

Remark 21.12. If $\mathcal{D}$ is a stable presentable (think: all limits, colimits) ∞-cat and $\mathcal{C} \subset \mathcal{D}$ is a presentable subcategory, then we can arrange for $\mathcal{C} = \mathcal{D}_{\geq 0}$. That is, can generate t-structures out of subcategories. However, the corresponding $\mathcal{D}_{\leq 0}$ is not easily determined. $\circ$

If we look at $\mathrm{Sh}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k, \mathrm{Sp}) \subset \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp})$, we can formally declare

$$\mathrm{Sh}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_\kappa)_{\geq 0} = \mathcal{C}_{\geq 0} \cap \mathrm{Sh}_{\mathrm{Nis},\mathbb{A}^1}(\ldots).$$

To compute the ≤ 0 part as $\mathcal{C}_{\leq 0} \cap \mathrm{Sh}_{\mathrm{Nis},\mathbb{A}^1}$, one would need to prove that the localization functor

$$\mathrm{L}_{\mathbb{A}^1} : \mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp}) \longrightarrow \mathrm{Sh}_{\mathrm{Nis},\mathbb{A}^1}(\mathrm{Sm}_k, \mathrm{Sp})$$

satisfies $\mathrm{L}_{\mathbb{A}^1}(\mathcal{C}_{\geq 0}) \subset \mathcal{C}_{\geq 0}$. Morel proved that if $E \in \mathcal{C}_{\geq 0}$, then $\mathrm{L}_{\mathbb{A}^1} E(L) \in \mathrm{Sp}_{\geq 0}$ for any field L. One can combine this with the injectivity Lemma 21.6 to prove the for $\mathbb{A}^1$-invariant Nisnevich sheaves. Other formal maneuvers then gives you the desired t-structure on $\mathrm{SH}(k)$. ■

Having injectivity lemma + Morel's t-structure reduced G-L (Theorem 21.1) to

Theorem 21.13 (Geisser-Levine reformulated)**.** *Let k be a perfect field of characteristic $p > 0$. Then,*

$$H\mathbb{Z}/p\mathbb{Z} \longrightarrow \pi_0(H\mathbb{Z}/p\mathbb{Z})_*$$

is an isomorphism on fields.

Remark 21.14. The LHS is $\mathrm{H}^i_{\mathrm{mot}}(-, \mathbb{Z}/p\mathbb{Z}(j))$. The RHS is

$$\begin{cases} 0 & \text{if } i \neq j \\ \mathcal{K}^M_j & \text{otherwise.} \end{cases} \quad i = j$$

So we only need to prove that this Milnor K-theory object is $\mathcal{K}^M_j \simeq \Omega^j_{\log}$. $\circ$

21.2 Effectivity

Definition 21.15. Let B be a base scheme. Then, $E \in \mathrm{SH}(B)$ is **effective** if it can be written as a colimit of $M(X)[j]$ ($j \in \mathbb{Z}$). In other words, it can be built out of schemes w/o objects like $M(\mathbb{G}_m)^{\otimes < 0}$. $\diamond$

22 Lectures 22, 23, 24 (11/15,17,22): Didn't go (seems these were the last three lectures)

References

[BLM21] Bhargav Bhatt, Jacob Lurie, and Akhil Mathew. Revisiting the de Rham–Witt complex. *Astérisque*, (424):viii+165, 2021. ii, 30, 31, 34, 37

23 List of Marginal Comments

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