A Note on Auto. Forms. for Quat. Algs.

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These are notes on Automorphic Forms for Quaternion Algebras, following [Gee22, Section 4], written for the MF learning seminar. They reflect my understanding (or lack thereof) of the material, so are far from perfect. They are likely to contain some typos and/or mistakes, but ideally none serious enough to distract from the mathematics. With that said, enjoy and happy mathing.

These notes (and the accompanying talk) are pretty rough.

Contents

1 Reminder on Quaternion Algebras **1**

2 Some Rep Theory **2**

3 Modular Forms + Jacquet-Langlands **3**

Our main reference is [Gee22, Section 4], though also see [Zho] (especially Lectures 13 - 17) for more details. The goal of this talk is not to cover all of [Gee22, Section 4], but to introduce enough of it for one to be able to read it on their own ahead of our last two talks after Thanksgiving.

1 Reminder on Quaternion Algebras

Note 1. For more info here, consult e.g. [Mil20, Chapter IV].

Setup 1. Let F be a field of characteristic not 2.

Definition 2. A `quaternion algebra` D/F is a 4-dimensional central simple F-algebra. ⋄

Fact. Any such D is isomorphic to an algebra of the form $H(a,b) = H_F(a,b) := F\langle i,j\rangle/(i^2 = a, j^2 = b, ij = -ji)$.

Fact. It is always the case that either $D \cong M_2(F)$ or D is a `division algebra`, i.e. every nonzero element is invertible.

Fact. $D \otimes_F \overline{F} \simeq M_2(\overline{F})$ is the algebra of 2×2 matrices over $\overline{F}$. Thus, quaternion algebras are twists of $M_2(F)$ (the converse holds to) and are classified by $H^1(F, \operatorname{Aut} M_2(F)) = H^1(F, \mathrm{PGL}_2)$.[1]

[1] All automorphisms of $M_2(F)$ are inner

Definition 3. On D, one can define a `reduced norm` $\mathrm{Nm} : D \to F$ such that $\alpha \in D$ is invertible if and only if $\mathrm{Nm}(\alpha) \neq 0$. ⋄

Example 4. If $D = M_2(F)$, then $\mathrm{Nm} : D \to F$ is simply the determinant. △

Example 5. If $D = H_F(a, b)$, then

$$\mathrm{Nm}(\alpha + \beta i + \gamma j + \delta k) = (\alpha + \beta i + \gamma j + \delta k)(\alpha - \beta i - \gamma j - \delta k) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2. \quad \triangle$$

Notation 6. Given D, we consider the associated F-algebraic group $G_D := \mathrm{Res}_{D/F}\,\mathbb{G}_m$ whose functor of points is

$$G_D(R) := (R \otimes_F D)^\times$$

for any F-algebra R.

Assumption. Now assume F is a number field.

Definition 7. For any place v of F, $D_v := D \otimes_F F_v$ is a quaternion algebra over the completion F_v. We say that D is `ramified` at v if D_v is a division algebra. We let $S(D)$ denote the set of places at which D ramifies. ⋄

The fact that $H^1(F, \mathrm{PGL}_2) \cong \mathrm{Br}(F)[2]$ along with the short exact sequence (taking 2-torsion is left-exact, $(-)[2] = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$)

$$0 \longrightarrow \mathrm{Br}(F) \longrightarrow \bigoplus_v \mathrm{Br}(F_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

of class field theory shows that $S(D)$ classifies D up to isomorphism; it also shows that $S(D)$ can be any even cardinality set of real or finite places of F.

Example 8.

- $S(D) = \emptyset \iff D \cong M_2(F)$
- For $F = \mathbb{Q}$, $S(D) = \{2, \infty\} \iff D \cong \mathbb{H} = H_{\mathbb{Q}}(-1, -1)$ is the (most obvious $\mathbb{Q}$-form of) usual Hamilton quaternions. △

2 Some Rep Theory

Skip in talk?

Definition 9. A `locally profinite group` G is a topological group where every open neighborhood of $1 \in G$ contains a compact, open subgroup. ⋄

Fact. (locally) profinite $\iff$ (locally) compact and totally disconnected.

Example 10. Let $K/\mathbb{Q}_p$ be a finite extension, and let D/K be a central simple algebra. Then, $\mathrm{GL}_n(K), D^\times, \mathrm{GL}_n(\mathscr{O}_K), \mathscr{O}_D^\times$ (for $\mathscr{O}_D \subset D$ a maximal order) are all locally profinite. Similarly, $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ and $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q},f})$ are locally profinite. △

Definition 11. Let V be a (possibly infinite dimensional) $\mathbb{C}$-vector space, and let G be a locally profinite group. A representation $\pi : G \to \mathrm{GL}(V)$ (often abbreviated (π, V)) is `smooth` if the stabilizer of any $v \in V$ is an open subgroup of G. It is `admissible` if it is smooth and $\dim V^U < \infty$ for all compact open $U \subset G$. ⋄

Assumption. Suppose that G supports a bi-invariant Haar measure μ. Thus, for any $\varphi \in C_c^\infty(G) := \{\text{smooth compactly supported functions } G \to \mathbb{C}\}$ – where `smooth` means there's some compact open $K \subset G$ such that $f(gk) = \varphi(g)$ for all $g \in G, k \in K$ – we have

$$\int_G \varphi(g)\mathrm{d}\mu = \int_G \varphi(gh)\mathrm{d}\mu = \int_G \varphi(hg)\mathrm{d}\mu$$

for any $h \in G$.

Definition 12. We define the `Hecke algebra` to be the associative algebra $\mathcal{H}(G) := C_c^\infty(G)$ with product given by convolution:

$$\left(\varphi * \psi\right)(x) := \int_G \varphi(g)\psi(g^{-1}x)\mathrm{d}\mu(g).$$

Sometimes, one will specify a compact open $K \subset G$ and then define $\mathcal{H}(G/K) := C_c^\infty(K\backslash G/K)$. ⋄

Fact. Let (π, V) be a smooth representation of G. Then, π induces a homomorphism $\mathcal{H}(G) \to \mathrm{End}_{\mathbb{C}}(V)$ where $\varphi \in \mathcal{H}(G)$ acts on V via

$$\pi(\varphi) \cdot v := \int_G \varphi(g)\pi(g) \cdot v\mathrm{d}\mu.$$

Remark 13. If $K \subset \mathrm{Stab}(v)$ and φ is right K-invariant (e.g. $\varphi \in \mathcal{H}(G/K)$), then

$$\pi(\varphi) \cdot v = \sum_{g \in G/K} \mu(K)\varphi(g)\pi(g) \cdot v$$

is a finite sum. You can always arrange this by taking K sufficiently small. ∘

3 Modular Forms + Jacquet-Langlands

Setup 14. Let F be a totally real number field, and let D/F be a quaternion algebra. Recall the algebraic group G_D/F and the set $S(D)$ of ramified places.

We first define our spaces of (cuspidal) modular forms.

Construction 15 (`Cusp forms of weight` (k, η)). For each (real) place $v \mid \infty$, choose some integers $k_v \geq 2$ and $\eta_v \in \mathbb{Z}$ such that $w := k_v + 2\eta_v - 1$ is independent of v. Set $k = (k_v)_{v|\infty}$ and $\eta = (\eta_v)_{v|\infty}$, both in $\mathbb{Z}^{\oplus[F:\mathbb{Q}]}$.

Warning: lots of non-canonical choices incoming...

I have no idea what the significance of this w is

As our next piece of notation, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we set $j(\gamma, z) := cz + d$. One can check that

$$j(\gamma\delta, z) = j(\gamma, \delta z) j(\delta, z). \tag{3.1}$$

For each (real) place $v \mid \infty$, define a subgroup $U_v \subset (D \otimes_F F_v)^\times = G_D(F_v)$ along with a U_v-rep (τ_v, W_v) as follows:

What's the significance of these U_v's? U_v is the center of $G_D(F_v)$ times a maximal compact

- if $v \in S(D)$ (i.e. if $D_v = D \otimes_F F_v$ is a division algebra), then we set $U_v := D_v^\times = G_D(F_v) \cong \mathbb{H}^\times$, where $\mathbb{H}$ denotes the usual Hamilton quaternions (the unique non-trivial quaternion algebra over $\mathbb{R}$).

  Let $\mathbb{C}_v^2$ denote the 2-dimensional U_v-rep $U_v \hookrightarrow \mathrm{GL}_2(\overline{F}_v) \cong \mathrm{GL}_2(\mathbb{C})$ and then we let (τ_v, W_v) denote the representation

$$\left(\mathrm{Sym}^{k_v - 2} \mathbb{C}^2\right) \otimes \left(\bigwedge^2 \mathbb{C}^2\right)^{\eta_v}.$$

- if $v \notin S(D)$ (i.e. if $D_v \cong M_2(\mathbb{R})$), then $D_v^\times \cong \mathrm{GL}_2(\mathbb{R})$. In this case, we take $U_v = \mathbb{R}^\times \mathrm{SO}(2)$.

  Furthermore, we take $W_v = \mathbb{C}$ and let U_v act on it via

$$\tau_v(\gamma) = j(\gamma, i)^{k_v} (\det \gamma)^{\eta_v - 1}$$

Now, set

$$U_\infty := \prod_{v \mid \infty} U_v, \quad W_\infty := \bigotimes_{v \mid \infty} W_v, \quad \text{and} \quad \tau_\infty := \bigotimes_{v \mid \infty} \tau_v.$$

Let $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ be the adeles and let $\mathbb{A}^\infty$ be the finite adeles. Finally, we let $S_{D,k,\eta}$ denote the space of functions $\varphi : G_D(\mathbb{Q}) \backslash G_D(\mathbb{A}) \to W_\infty$ satisfying

(1) $\varphi(g u_\infty) = \tau_\infty(u_\infty)^{-1} \varphi(g)$ for all $g \in G_D(\mathbb{A})$ and $u_\infty \in U_\infty$

(2) There is a nonempty open subset $U^\infty \subset G_D(\mathbb{A}^\infty)$ such that $\varphi(gu) = \varphi(g)$ for all $u \in U^\infty$, $g \in G_D(\mathbb{A})$

I guess this is encoding the transformation law of usual modular forms?

(3) Let S_∞ denote the set of infinite places of F and fix some $g \in G_D(\mathbb{A}^\infty)$. By condition **(1)**, the function

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{R})^{S_\infty \setminus S(D)} & \longrightarrow & W_\infty \\ (\gamma_v)_{v \in S_\infty \setminus S(D)} & \longmapsto & \tau_\infty(\gamma)\varphi(g\gamma), \end{array}$$

where $\gamma = (\gamma_v)_{v \mid \infty}$ (and $\gamma_v = 1$ if $v \in S(D)$) descends[2] to a function

$$(\mathbb{C} \setminus \mathbb{R})^{S_\infty \setminus S(D)} \longrightarrow W_\infty.$$

We require the above function to be holomorphic (for all $g \in G_D(\mathbb{A}^\infty)$).

I suppose this is asking φ to be 'smooth'. I think also corresponds to the level of usual modular forms

Skip this during the talk, just write "holomorphy condition"

[2] along the map $\mathrm{GL}_2(\mathbb{R}) \twoheadrightarrow \mathbb{C} \setminus \mathbb{R}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(i) = \frac{ai+b}{ci+d}$

(4) If $S(D) = \emptyset$ (i.e. $D = M_2(F)$), then we also ask that

Cuspidality condition

$$\int_{F\backslash \mathbb{A}_F} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \mathrm{d}x = 0 \text{ for all } g \in G_D(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_F).$$

If, furthermore, $F = \mathbb{Q}$, then we also demand that the function

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{R}) & \longrightarrow & W_\infty \\ \gamma & \longmapsto & \varphi(g\gamma)\left|\mathrm{Im}(\gamma(i))\right|^{k/2} \end{array}$$

is bounded, for all $g \in G_D(\mathbb{A}^\infty)$.

Note that $G_D(\mathbb{A}^\infty)$ acts on $S_{D,k,\eta}$ via right translation, i.e. via

$$(g\varphi)(x) = \varphi(xg).$$ ○

Example 16 ([Gee22], Exercise 4.9)**.** Take $F = \mathbb{Q}, S(D) = \emptyset$ (so $G_D = \mathrm{GL}_{2,\mathbb{Q}}$), $k_\infty = k$, and $\eta_\infty = 1$. Define

Gee takes $\eta_\infty = 0$ instead, but I'm confused by why

$$U_1(N) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

(1) The intersection of $\mathrm{GL}_2(\mathbb{Q})^+$ and $U_1(N)$ inside $\mathrm{GL}_2(\mathbb{A}^\infty)$ is $\Gamma_1(N)$, the matrices in $\mathrm{SL}_2(\mathbb{Z})$ congruent to $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ mod N.

Proof. Say $\gamma \in \mathrm{GL}_2(\mathbb{Q})^+ \cap U_1(N) \subset \mathrm{GL}_2(\mathbb{A}^\infty)$. Then, $\det\gamma \in \mathbb{Q}^+ \cap \widehat{\mathbb{Z}}^\times = \{+1\}$ (positive rational numbers which are p-adic units for all primes p), so $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. The condition on $U_1(N)$ then becomes that $\gamma \equiv \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$ mod N, so $\gamma \in \Gamma_1(N)$. Convince yourself of the other inclusion if you don't yet see it. ■

(2) The space $S_{D,k,0}^{U_1(N)}$ of $U_1(N)$-invariant cusp forms can be identified with the usual space $S_k(\Gamma_1(N))$ of weight k holomorphic cusp forms for $\Gamma_1(N)$.

Proof Sketch. Take for granted the following facts:

$$\mathbb{A}^\times = \mathbb{Q}^\times \times \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}^\times \text{ and } \mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})U_1(N)\,\mathrm{GL}_2(\mathbb{R})^+$$

(these are related to $\mathbb{Q}$ having class number 1 and strong approximation for SL_2). Thus, the domain of any $\varphi \in S_{D,k,0}^{U_1(N)}$ can be identified with

$$\begin{aligned} \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/U_1(N) &= \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})U_1(N)\,\mathrm{GL}_2(\mathbb{R})^+/U_1(N) \\ &= \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})^+U_1(N)/U_1(N) \end{aligned}$$

$$\simeq (\mathrm{GL}_2(\mathbb{Q}) \cap U_1(N) \cap \mathrm{GL}_2(\mathbb{R})^+) \backslash \mathrm{GL}_2(\mathbb{R})^+$$
$$= \Gamma_1(N) \backslash \mathrm{GL}_2(\mathbb{R})^+,$$

where the $U_1(N)$ and the $\mathrm{GL}_2(\mathbb{R})^+$ commute because $U_1(N) \subset \mathrm{GL}_2(\mathbb{A}^\infty)$ (and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(\mathbb{R})$). Observe that $S_{D,k,0}^{U_1(N)}$ is identified with the space of functions $\varphi : \Gamma_1(N) \backslash \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C}$ ($W_\infty = \mathbb{C}$ since $v \notin S(D)$) satisfying

(1) $\varphi(gu_\infty) = j(u_\infty, i)^{-k}\varphi(g)$ for all $g \in \mathrm{GL}_2(\mathbb{R})^+$ and $u_\infty \in \mathbb{R}_{>0}^\times \mathrm{SO}(2)$.
I chose $\eta_\infty = 1$ instead of 0 in order to get no determinant appearing above.

(2) No need for an analogue of condition **(2)** in Construction 15 since the φ here are already invariant under $U_1(N)$.

(3) The function

$$\widetilde{\varphi} : \begin{array}{ccc} \mathrm{GL}_2(\mathbb{R})^+ & \longrightarrow & \mathbb{C} \\ \gamma & \longmapsto & j(\gamma, i)^k \varphi(\gamma) \end{array}$$

descends[3] (along $\mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{H}, g \mapsto g(i)$) to a *holomorphic* map $\mathbb{H} \to \mathbb{C}$ (note $\mathbb{H} = \mathrm{GL}_2(\mathbb{R})^+/(\mathbb{R}_{>0}^\times \mathrm{SO}(2))$ since $\mathbb{R}_{>0}^\times \mathrm{SO}(2) = \mathrm{Stab}_{\mathrm{GL}_2(\mathbb{R})^+}(i)$).

(4) cuspidality condition.

As already hinted at above, the assignment $\varphi \mapsto \widetilde{\varphi}$ (where $\widetilde{\varphi} : \mathbb{H} = \mathrm{GL}_2(\mathbb{R})^+/(\mathbb{R}_{>0}^\times \mathrm{SO}(2)) \to \mathbb{C}$ is $\widetilde{\varphi}(\gamma) = j(\gamma, i)^k \varphi(\gamma)$) identifies the space of such functions with the space $S_k(\Gamma_1(N))$ of weight k holomorphic cusp forms for $\Gamma_1(N)$. ■

If you want, fill in some of the details missing above. △

Example 17 ([Zho], Lecture 16)**.** Call D a `definite quaternion algebra` if $S_\infty \subset S(D)$. In this case, if $U \subset G_D(\mathbb{A}^\infty)$ is an open subgroup, then $S_{D,2,0}^U$ is simply the set of $\mathbb{C}$-valued functions on the finite set $G_D(\mathbb{Q}) \backslash G_D(\mathbb{A})/G_D(\mathbb{R})U$. △

Definition 18. A `cuspidal automorphic representation` of $G_D(\mathbb{A}^\infty)$ of weight (k, η) is a (smooth, admissible) irreducible subquotient of $S_{D,k,\eta}$.[4] ⋄

Fact. Any such representation is of the form $\pi = \bigotimes' \pi_v$ with $\pi_v^{\mathrm{GL}_2(\mathscr{O}_v)} \neq 0$ for almost all v, with π_v smooth, irreducible (+ admissible) rep of $G_D(F_v)$ for all v, and with the restriction in this *restricted tensor product* being that the vth component of a vector is in $\pi_v^{\mathrm{GL}_2(\mathscr{O}_v)}$ (which is 1-dimensional) for almost all v.

Fact (`global Jacquet-Langlands`)**.**

(1) The only f.dimensional cuspidal automorphic representations of $G_D(\mathbb{A}^\infty)$ are 1-dimensional representations which factor through the reduced norm; these only exist if $D \neq M_2(F)$.

[3]Use (3.1) to know that $\widetilde{\varphi}$ is invariant under right-translation by $U_\infty = \mathbb{R}_{>0}^\times \mathrm{SO}(2)$
[4]$S_{D,k,\eta}$ is already semisimple and admissible, so I think this parenthetical is technically unnecessary

(2) There is a bijection between infinite-dimensional cuspidal automorphic representations of $G_D(\mathbb{A}^\infty)$ of weight (k,η) and cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F^\infty)$ of weight (k,η) which are discrete series for all finite places $v \in S(D)$.

This bijection is compatible with (and so determined by) a local Jacquet-Langlands correspondence[5]

Remark 19. Jacquet-Langlands allows one to attach Galois reps to infinite-dimensional cuspidal automorphic representations of $G_D(\mathbb{A}^\infty)$. ∘

Remark 20. One can use cyclic base change to show that if $r : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ becomes modular when restricted to G_E, for some finite solvable Galois extension E/F of totally real fields, then r must have been modular to begin with. ∘

Fact. Let K be a number field, and let S be a finite set of (finite or infinite) places of K. For each $v \in S$, let L_v be a finite Galois extension of K_v. Then, there is a finite solvable Galois extension M/K such that, for each place w of M above a place $v \in S$, there is an isomorphism $L_v \cong M_w$ of K_v-algebras.

Recall that the absolute Galois group of a local field is solvable

These facts will allow us to reduce our modularity lifting theorem to the case where we're working a quaternion algebra D over a totally real field F such that $S(D) = S_\infty$ (in which case, $S_{D,k,\eta}$ is especially simple; see Example 17).

Remark 21. In particular, even if we only care about $F = \mathbb{Q}$, the desire to make such a reduction would lead us to want to state the final theorem for (totally real) number fields beyond $\mathbb{Q}$ (there's no quaternion algebra $D/\mathbb{Q}$ with $S(D) = S_\infty = \{\infty\}$ since this set has odd cardinality). ∘

References

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[5] If $v \notin S(D)$, then $\mathrm{JL}(\pi)_v = \pi_v$ and if $v \in S(D)$, then $\mathrm{JL}(\pi)_v = \mathrm{JL}(\pi_v)$