# APAW Notes

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These are notes on talks given in "APAW" which took place at the University of Oregon. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here.

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# 1 Samit Dasgupta: Ribet's Method

*Note* 1. These are slide talks, so quite likely they'll be too fast for me to take good notes... On the other hand, there are slides and lecture recordings on the workshop website.

## 1.1 Lecture 1 (7/26) – Ribet's Method I: Converse to Herbrand's Theorem

Today we want to describe Ribet's theorem. Dick Gross called this theorem a 'lightning bolt' when it first appeared.

Let $p$ be an odd prime, let $H = \mathbb{Q}(\mu_p)$, a CM field. Let $G = \mathrm{Gal}(H/\mathbb{Q}) \cong \mathbb{F}_p^\times$ and consider

$$\chi : G_{\mathbb{Q}} \longrightarrow \mathbb{F}_p^\times, \quad \sigma(\zeta_p) = \zeta_p^{\chi(\sigma)}.$$

**Notation 1.1.1.** Let $\mathrm{Cl}(H)$ denote the class group of $H$, the group of fractional ideals modulo the principal fraction ideals.

Let $A = \mathrm{Cl}(H) \otimes \mathbb{F}_p$, the $p$-part of the class group. This has a natural $G$-action (i.e. $A$ is an $\mathbb{F}_p$-linear $G$-rep), so we can write

$$A = \bigoplus_{k=0}^{p-2} A^{\chi^k}, \quad A^{\chi^k} = \left\{a \in A : \sigma(a) = \chi^k(\sigma) \cdot a, \sigma \in G\right\}$$

(since $\gcd(\#G, p) = 1$).

**Theorem 1.1.2 (Ribet).** *Let $k$ be a positive even integer, $k \not\equiv 0 \pmod{p-1}$. Then,*

$$p \mid \zeta(1-k) = -\frac{B_k}{k} \implies A^{\chi^{1-k}} \neq 0.$$

*Remark* 1.1.3. Herbrand had proven the other direction of this. ○

*Remark* 1.1.4. The class number formula (dividing the results for $H$ and $H^+$) implies that $p \mid \zeta(1-k)$ for some positive even $k$ implies that

$$A^- := \bigoplus_{k \text{ even}} A^{\chi^{1-k}} \neq 0.$$

Ribet's theorem is like a character-by-character refinement of what one would get just using the class number formula. ○

Here's a diagram of Ribet's method due to Barry Mazur (see Figure 1).

Whirlwind summary of Ribet's proof:

- **Eisenstein series**

$$E_k = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k.$$

  Above, $\sigma_{k-1}$ is the $(k-1)$st power divisor function.

- There exists $g \in M_k$ with constant term 1 (and integral Fourier coefficients). Define the *cusp form*

$$f = E_k - \zeta(1-k)/2 \cdot g \in S_k$$

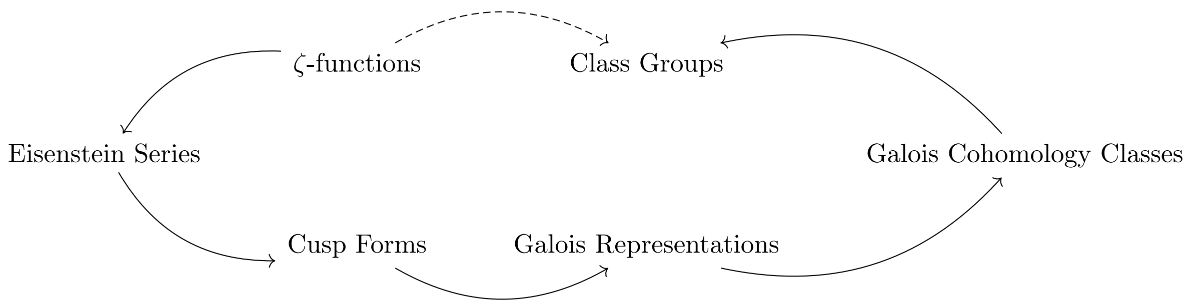


Figure 1: A diagram of Ribet's method

(In particular, *do not* just take $g = 2E_k/\zeta(1-k)$)

- If $p \mid \zeta(1-k)$, then $f \equiv E_k \pmod{p}$.

  This is because $g$ has integral Fourier coefficients.

- The Deligne-Serre Lifting Lemma implies that there is a *cuspidal eigenform* with the same congruence property.

- Galois representation associated to $f$:

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Z}_p).$$

- The congruence implies it is reducible mod $p$

$$\rho \mod p \equiv \begin{pmatrix} 1 & b(\sigma) \\ 0 & \chi^{k-1}(\sigma) \end{pmatrix}$$

- Cohomology class

$$\kappa(\sigma) = b(\sigma)\chi^{1-k}(\sigma) \in \mathrm{H}^1(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k})).$$

- Ribet shows this class is *nonzero* and *unramified*.

- Class field theory implies the existence of such a class is equivalent to $A^{\chi^{1-k}} \neq 0$.

  CFT gives the following lemma, where $F = \mathbb{Q}(\mu_p)$ from now on

  **Lemma 1.1.5.** $A^{\chi^{1-k}} \neq 0$ *if and only if there exists a Galois extension* $E/\mathbb{Q}$ *satisfying the following conditions:*

  - $E \supset F$, *and* $\mathrm{Gal}(E/F) \cong \mathbb{F}_p$
  - $E$ *is everywhere unramified over* $F$
  - *The action of* $G$ *on* $\mathrm{Gal}(E/F)$ *via conjugation is given by the character* $\chi^{1-k}$, *i.e.*

$$\sigma\tau\sigma^{-1} = \tau^{\chi^{1-k}(\sigma)} \text{ for } \sigma \in \mathrm{Gal}(E/\mathbb{Q}) \text{ and } \tau \in \mathrm{Gal}(E/F).$$

Question: Should this say irreducible?

Answer: No, See e.g. proof of Lemma 1.1.16

Remember: $A$ is a quotient of $\mathrm{Cl}(F)$

*Proof Sketch.* Let $H$ be the Hilbert class field of $F$, so $H$ is everywhere unramified over $F$, and $\mathrm{Gal}(H/F) \cong \mathrm{Cl}(F)$. Furthermore, the action of $G = \mathrm{Gal}(F/\mathbb{Q})$ on LHS via conjugation corresponds to natural action on RHS.

If $A^{\chi^{1-k}} \neq 0$, choose any nonzero functional $A^{\chi^{1-k}} \to \mathbb{F}_p$, and let $E$ be the fixed field of

$$\mathrm{Gal}(H/F) \cong \mathrm{Cl}(F) \twoheadrightarrow A \twoheadrightarrow A^{\chi^{1-k}} \to \mathbb{F}_p.$$

This $E$ has all the desired properties by construction. These steps are reversible, so get other implication as well. ■

**Recall 1.1.6.** (First) Galois cohomology is

$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k})) = Z^1/B^1,$$

where

$$Z^1 = \left\{\kappa : G_{\mathbb{Q}} \to \mathbb{F}_p : \kappa(\sigma\tau) = \kappa(\sigma) + \chi^{1-k}(\sigma)\kappa(\tau)\right\},$$

and

$$B^1 = \left\{\kappa_x(\sigma) = (1 - \chi^{1-k}(\sigma))x : x \in \mathbb{F}_p\right\}.$$

⊙

**Definition 1.1.7 (unramified classes).** For each place $v$ of $\mathbb{Q}$, choose an inertia group $I_v \subset G_v \subset G_{\mathbb{Q}}$. There is a restriction map

$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k})) \longrightarrow \mathrm{H}^1(I_v, \mathbb{F}_p(\chi^{1-k})).$$

We define $\mathrm{H}^1_{\mathrm{ur}}(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k}))$ to be the intersection of the kernels of these restriction maps. ⋄

**Lemma 1.1.8.** *There exists a field extension $E$ as in Lemma 1.1.5 iff*

$$\mathrm{H}^1_{\mathrm{ur}}(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k})) \neq 0.$$

*Proof.* Let $s : G \to \mathrm{Gal}(E/\mathbb{Q})$ ($G = \mathrm{Gal}(F/\mathbb{Q})$) denote a splitting of the canonical projection $\mathrm{Gal}(E/\mathbb{Q}) \to G$ (exists by size considerations, kernel and image have coprime sizes). Define

$$\kappa(\sigma) := \sigma \cdot s(\sigma^{-1}|_F) \in \mathrm{Gal}(E/F).$$

*Exercise.* This is a 1-cocycle, and is independent (mod coboundaries) of the section chosen.

We check $\kappa$ is unramified at all places $v$ of $\mathbb{Q}$. If $w$ is a place of $F$, then the restriction of $\kappa$ to $I_w$ is clearly trivial since $E$ is unramified over $F$. Now, we use **inflation-restriction**

$$0 \longrightarrow \mathrm{H}^1(I_v/I_w, \mathbb{F}_p(\chi^{1-k})) \longrightarrow \mathrm{H}^1(I_v, \mathbb{F}_p(\chi^{1-k})) \longrightarrow \mathrm{H}^1(I_w, \mathbb{F}_p(\chi^{1-k})).$$

Since $|I_v/I_w|$ divides $(p-1)$,[1] the left term vanishes. Hence, the image of $\kappa$ in $\mathrm{H}^1(I_v, \mathbb{F}_p(\chi^{1-k}))$ vanishes as well (since it's restriction to $I_w$ is 0).

[1] It's a subquotient of $\mathrm{Gal}(F/\mathbb{Q}) \simeq \mathbb{F}_p^\times$

We check $\kappa$ is non-trivial. Altering $\kappa$ by a coboundary amounts to choosing a different section in its definition, so if $\kappa$ is a trivial cohomology class, we may assume that it is even trivial as a cocycle (choose a different section). However, then $\sigma = s(\sigma|_F)$ which is absurd. This would imply that $s$ is surjective even though it's mapping from a smaller group.

*Exercise.* Do the reverse direction. That is, given $\kappa \in \mathrm{H}^1_{\mathrm{ur}}(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k}))$ non-trivial, consider

$$\kappa|_{G_F} \in \mathrm{H}^1(G_F, \mathbb{F}_p(\chi^{1-k})) = \mathrm{Hom}(G_F, \mathbb{F}_p).$$

Let $E$ be the fixed field of the kernel of $\kappa|_{G_F}$. Show that $E$ has the desired properties. ■

*Goal* (Reduction of Ribet's theorem). It suffices to show $\mathrm{H}^1_{\mathrm{ur}}(G_{\mathbb{Q}}, \mathbb{F}_p(\chi^{1-k})) \neq 0$.

*Remark* 1.1.9. In fact, it suffices to show $\mathrm{H}^1_{\mathrm{ur}}(G_{\mathbb{Q}}, \mathbb{F}_q(\chi^{1-k})) \neq 0$ for some $p$-power $q$. This is because, as an abelian group, $\mathbb{F}_{p^r} \cong \mathbb{F}_p^{\oplus r}$. ○

Much of the trick in using Ribet's method is observing that the desired result can be restated as the existence of some cohomology class.

Now we start on the LHS of the cycle from before, and we want to see how to get a Galois cohomology class. Start with

$$E_k = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in M_k.$$

This is basically integral (except possible for the constant term). This is an element of $M_k$, the space of weight $k$ modular forms for $\mathrm{SL}_2(\mathbb{Z})$.

**Example.** The numerators of the constant terms in $E_4, E_6$ are both $= 1$. △

**Lemma 1.1.10.** *For every even $k \geq 4$, there exists $g \in M_k$ with integer Fourier coefficients and constant term 1. Specifically,*

$$E_4 = \frac{1}{240} + \dots \quad \textit{and} \quad E_6 = -\frac{1}{504} + \dots.$$

*Proof.* Write $k = 4a + 6b$ with $a, b \geq 0$, and set $g = (240E_4)^a(-504E_6)^b$. ■

Now we let

$$g' := E_k - \frac{\zeta(1-k)}{2} g \in S_k.$$

If $p \mid \zeta(1-k)$, then $g' \equiv E_k \mod p$.

**Notation 1.1.11.** Let $R = \mathbb{Z}_{(p)} = \mathbb{Z}[1/q : q \neq p]$.

Note $g' \in S_k(R)$.

**Theorem 1.1.12** ((Consequence of?) **Deligne-Serre Lifting Theorem**). *Suppose $p \mid \zeta(1-k)$. Then there exists a number field $K$, a prime ideal $\mathfrak{p} \mid p$, and a cuspidal eigenform $f \in S_k(\mathcal{O}_K)$ such that*

$$f \equiv g' \equiv E_k \mod \mathfrak{p}.$$

*Proof.* Let $\mathbb{T}$ denote the Hecke algebra of $S_k(R)$.[2] Since $g'$ is congruent to an eigenform, the map

I guess Eisenstein series are eigenforms since cusp forms span a codim 1 subspace

[2] $\mathbb{T} = \mathbb{Z}[T_\ell : \ell]$

$t \mapsto a_1(t \cdot g') \pmod p$ is a ring homomorphism[3] $\mathbb{T} \to \mathbb{F}_p$. Let $\mathfrak{m}$ denote the kernel. Since $p \in \mathfrak{m}$ is not nilpotent, there exists a prime ideal $\mathfrak{P} \subset \mathfrak{m}$ not containing $p$. Let $K = \mathrm{Frac}(\mathbb{T}/\mathfrak{P})$, a number field by finite dimensionality of the space of cusp forms. We then define

$$f := \sum_{n \geq 1} \overline{T}_n q^n,$$

where $\overline{T}_n$ denotes the image of the Hecke operator $T_n \in \mathbb{T}$ in the integral domain $\mathbb{T}/\mathfrak{P}$. Since each $T_n$ satisfies a monic integer polynomial (by Cayley-Hamilton), it follows that $f \in S_k(\mathscr{O}_K)$. The maximal ideal $\mathfrak{m}$ gives rise to a maximal ideal $\mathfrak{p}$ of $\mathscr{O}_K$ such that $f \equiv g' \equiv E_k \pmod{\mathfrak{p}}$. ■

*Remark* 1.1.13. The eigenform $f$ is **$\mathfrak{p}$-ordinary** in the sense that $a_p(f)$ is a $\mathfrak{p}$-adic unit, since

$$a_p(f) \equiv a_p(E_k) = 1 + p^{k-1} \equiv 1 \pmod{\mathfrak{p}}. \qquad \circ$$

**Notation 1.1.14.** Let $E = K_{\mathfrak{p}}$ (completion at $\mathfrak{p}$), $\mathscr{O} = \mathscr{O}_E$, and $\mathfrak{m}$ be the maximal ideal of $\mathscr{O}$.

**Theorem 1.1.15** (Deligne, Hida, Wiles)**.** *There exists a continuous irreducible representation*

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(E)$$

*such that*

- *$\rho$ is unramified outside $p$*
- *For $\ell \neq p$,* $\mathrm{char}(\rho(\mathrm{Frob}_\ell)) = x^2 - a_\ell(f)x + \ell^{k-1}$
- *Restricting to the one place it's ramified,*

$$\rho|_{G_p} \cong \begin{pmatrix} \eta^{-1}\varepsilon^{k-1} & * \\ 0 & \eta \end{pmatrix},$$

*where $\varepsilon$ is the $p$-adic cyclotomic character, and $\eta$ is an unramified character such that $\eta(\mathrm{art}(p)) = \alpha_p(f)$. Here, $\alpha_p(f)$ is the unit root of*

$$x^2 - a_p(f)x + p^{k-1} = 0.$$

The only thing from the thrid bullet point above we need to know about $\eta$ is that it is unramified.

**Lemma 1.1.16 (Ribet's Lemma, Version 1).** *There exists a basis for $\rho$ such that*

- $\rho = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \in \mathrm{GL}_2(\mathscr{O})$,
- $a(\sigma) \equiv 1$, $c(\sigma) \equiv 0$, $d(\sigma) \equiv \chi^{k-1}(\sigma) \mod \mathfrak{m}$
- *$\kappa(\sigma) := \overline{b}(\sigma) \cdot \chi^{1-k}(\sigma) \in \mathscr{O}/\mathfrak{m}$ is a cocycle defining a non-trivial class in* $\mathrm{H}^1(G_{\mathbb{Q}}, (\mathscr{O}/\mathfrak{m})(\chi^{1-k}))$.

[3]ring homomorphism and not just linear map since $g' \equiv E_k$

*Proof.* Modulo $\mathfrak{m}$,

$$\operatorname{char}(\rho(\operatorname{Frob}_\ell)) \equiv x^2 - (\ell^{k-1}+1)x + \ell^{k-1} = (x-1)(x-\ell^{k-1})$$

is reducible. By Cebotarv density, we get

$$\operatorname{char}(\rho(\sigma)) \equiv (x-1)(x-\chi^{k-1}(\sigma)) \pmod{\mathfrak{m}}$$

for all $\sigma \in G_{\mathbb{Q}}$. Fix $\tau \in G_{\mathbb{Q}}$ such that $\chi^{k-1}(\tau) \not\equiv 1$. Such a $\tau$ exists since $k \not\equiv 1 \pmod{p-1}$, so $\chi^{k-1} \not\equiv 1$. By Hensel's lemma, $\rho(\tau)$ has two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathscr{O}$ such that

$$\lambda_1 \equiv 1 \text{ and } \lambda_2 \equiv \chi^{k-1}(\tau) \pmod{\mathfrak{m}}.$$

Choose a basis now such that

$$\rho(\tau) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We know (integrality condition by Cebotarev) that

$$\operatorname{Tr}(\rho(\sigma)) = a(\sigma) + d(\sigma) \in \mathscr{O} \text{ and } \operatorname{Tr}(\rho(\sigma\tau)) = a(\sigma)\lambda_1 + d(\sigma)\lambda_2 \in \mathscr{O}.$$

Mutiplying the first by $\lambda_1$ and subtracting, we get $(\lambda_1 - \lambda_2)d(\sigma) \in \mathscr{O}$. Since $\lambda_1 \not\equiv \lambda_2 \mod \mathfrak{m}$, we see that $d(\sigma) \in \mathscr{O}$. Then we get $a(\sigma) \in \mathscr{O}$.

At this point, we're out of time, so we'll pick up here next lecture... ∎

## 1.2 Lecture 2 (7/27)

Our goal is

**Theorem 1.2.1** (Ribet). *Let $k$ be a positive even integer, $k \not\equiv 0 \pmod{p-1}$. Then,*

$$p \mid \zeta(1-k) = -\frac{B_k}{k} \implies A^{\chi^{1-k}} \neq 0.$$

Last time we had constructed a congruence between a cuspidal eigenform $f \in S_k(\mathscr{O}_K)$ and an Eisenstein series $E_k$. Attached to the eigenform was a Galois rep $\rho : G_{\mathbb{Q}} \to \operatorname{GL}_2(E)$ which is unramified outside of $p$. We were in the middle last time of proving Ribet's Lemma

**Lemma 1.2.2** (*Ribet's Lemma, Version 1*, Lemma 1.1.16)**.** *There exists a basis for $\rho$ such that*

- $\rho = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \in \operatorname{GL}_2(\mathscr{O})$,
- $a(\sigma) \equiv 1$, $c(\sigma) \equiv 0$, $d(\sigma) \equiv \chi^{k-1}(\sigma) \mod \mathfrak{m}$
- $\kappa(\sigma) := \bar{b}(\sigma) \cdot \chi^{1-k}(\sigma) \in \mathscr{O}/\mathfrak{m}$ *is a cocycle defining a non-trivial class in* $\mathrm{H}^1(G_{\mathbb{Q}}, (\mathscr{O}/\mathfrak{m})(\chi^{1-k}))$.

*Proof continued.* We showed that

$$\operatorname{char}(\rho(\sigma)) \equiv (x-1)(x-\chi^{k-1}(\sigma)) \pmod{\mathfrak{m}}$$

for all $\sigma \in G_{\mathbb{Q}}$. Since $\chi^{k-1} \not\equiv 1 \pmod{p}$, we can choose $\tau \in G_{\mathbb{Q}}$ with $\chi^{k-1}(\tau) \not\equiv 1$. We used this to construct $\lambda_1, \lambda_2 \in \mathscr{O}$ with $\lambda_1 - \lambda_2 \in \mathscr{O}^\times$ which were used to show that $a(\sigma), d(\sigma) \in \mathscr{O}$. Since

$$\mathrm{Tr}(\rho(\sigma)) \equiv a(\sigma) + d(\sigma) \equiv 1 + \chi^{k-1}(\sigma) \pmod{\mathfrak{m}},$$

and similarly one gets

$$\mathrm{Tr}(\rho(\sigma\tau)) \equiv 1 + \chi^{k-1}(\sigma\tau) \pmod{\mathfrak{m}},$$

we can solve to obtain $a(\sigma) \equiv 1, d(\sigma) \equiv \chi^{k-1}(\sigma) \pmod{\mathfrak{m}}$.

The function $G_{\mathbb{Q}} \to E$ given by $\sigma \mapsto b(\sigma)$ has bounded image since $G_{\mathbb{Q}}$ is compact and $\rho$ is continuous. It's also not identically zero since $\rho$ is irreducible. Therefore, there exists $\tau' \in G_{\mathbb{Q}}$ for which $b(\tau') \neq 0$ and $\mathrm{ord}_{\mathfrak{p}}(b(\tau'))$ is minimized. Conjugating by

$$\begin{pmatrix} 1 & 0 \\ 0 & b(\tau') \end{pmatrix}$$

leaves $a(\sigma)$ and $d(\sigma)$ unchanged, but replaces $b(\sigma) \rightsquigarrow b(\sigma)/b(\tau')$. We may therefore assume that $b(\sigma) \in \mathscr{O}$ (since $b(\tau')$ had minimal valuation) for all $\sigma$, and $b(\tau') = 1$.

*Exercise.* Show $c(\sigma) \in \mathfrak{m}$, and so

$$\overline{\rho}(\sigma) = \begin{pmatrix} 1 & \overline{b}(\sigma) \\ 0 & \chi^{k-1}(\sigma) \end{pmatrix}.$$

Now,

$$b(\sigma\tau) = a(\sigma)b(\tau) + b(\sigma)d(\tau) \equiv \overline{b}(\tau) + \overline{b}(\sigma)\chi^{k-1}(\tau) \pmod{\mathfrak{m}}.$$

Define

$$\kappa(\sigma) = \chi^{1-k}(\sigma)\overline{b}(\sigma) \in Z^1\left(G_{\mathbb{Q}}, \mathscr{O}/\mathfrak{m}(\chi^{1-k})\right).$$

Let's show it's non-trivial as a cohomology class. If $\kappa$ is a coboundary, then

$$\kappa(\sigma) = (1 - \chi^{1-k}(\sigma))x \text{ for some } x \in \mathscr{O}/\mathfrak{m}.$$

Plug in $\sigma = \tau$ to get

$$0 = (1 - \chi^{1-k}(\tau))x \implies x = 0 \implies \kappa = 0$$

(since $\chi^{1-k}(\tau) \not\equiv 1$). This contradicts $b(\tau') = 1$, and concludes the proof of Ribet's lemma. ∎

It remains to show that $\kappa$ is everywhere unramified.

- $\kappa$ is unramified at all finite $v \neq p$ since $\rho$ is
- $\kappa$ is unramified at $\infty$ since $p$ is odd (and $2 = \#I_\infty$ is relative prime to $p$)
- To show $\kappa$ is unramified at $p$, we use **Ribet's Wrench**:

$$\overline{\rho}(\sigma) = \begin{pmatrix} 1 & \overline{b}(\sigma) \\ 0 & \chi^{k-1}(\sigma) \end{pmatrix}, \text{ but } \overline{\rho}|_{I_p} \cong \begin{pmatrix} \chi^{k-1} & * \\ 0 & 1 \end{pmatrix}$$

(since $\eta$ in Theorem 1.1.15 is unramified). Since $\chi^{k-1} \not\equiv 1$, this can only happen if $\kappa|_{I_p}$ is a coboundary. Have $1, \chi^{k-1}$ as both a sub and quotient will only happen if the representation is split (on the inertia group) which only happens if $\kappa|_{I_p}$ is a coboundary.

This concludes the proof of Ribet's theorem.

*Exercise.* Explicitly exhibit $\kappa|_{I_p}$ as a coboundary.

### 1.2.1 Generalizations

We showed

$$p \mid \zeta(1-k) \implies p \mid \#\left(\mathrm{Cl}(F) \otimes \mathbb{Z}_p\right)^{\chi^{1-k}}$$

What if $p^2 \mid \zeta(1-k)$?

This is the type of question attempted to be answer by the *Main Conjecture of Iwasawa Theory.* We should have $p^2 \mid \#(\mathrm{Cl}(F) \otimes \mathbb{Z}_p)^{\chi^{1-k}}$.

**Warning 1.2.3.** Hard to determine the group structure. •

One of the main obstracles in generalizing Ribet's argument is that after construction a cusp form $g \in S_k$ such that $g \equiv E_k \pmod{p^2}$, there is no generalization of the Deligne-Serre Lifting Theorems that yields a cuspidal eigenform $f$ such that $f \equiv g \equiv E_k \pmod{p^2}$. A typical sigutation is that there are two normalized cuspidal eigenforms $f_1, f_2 \equiv E_k \pmod{p}$ and $g = (f_1 + f_2)/2$.

Working over $\mathbb{Z}_p$, consider Hecke algebra

$$\mathbb{T} = \{(a,b) \in \mathbb{Z}_p \times \mathbb{Z}_p : a \equiv b \pmod{p}\}$$

($T_\ell \mapsto (a_\ell(f_1), a_\ell(f_2))$) acting on span of $f_1, f_2$. The (non-Eigen) cusp form $g \equiv E_k \pmod{p^2}$ yields a $\mathbb{Z}_p$-algebra homomorphism

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & \mathbb{Z}/p^2\mathbb{Z} \\ T_\ell & \longmapsto & a_1(T_\ell g) = a_\ell(E_k) = 1 + \ell^{k-1} \end{array}.$$

**Example.** If $g = (f_1 + f_2)/2$, this is $(a,b) \mapsto (a+b)/2$. △

In general, we won't be able to say much about the structure of $\mathbb{T}$: it is a Noetherian commutative ring of characteristic zero. We have a homomorphism $\varphi : \mathbb{T} \to \mathbb{Z}/p^2\mathbb{Z}$ sending $T_\ell \mapsto 1 + \ell^{k-1}$. If we use only Hecke operators away from the level, we can assume $\mathbb{T}$ is *reduced.* If we localize at the maximal ideal containing the kernel of $\varphi$, we can assume $\mathbb{T}$ is *local.*

**Example.** Recall our toy example

$$\mathbb{T} \cong \{(a,b) \in \mathbb{Z}_p \times \mathbb{Z}_p : a \equiv b \pmod{p}\},$$

with each copy of $\mathbb{Z}_p$ corresponds to one cuspidal eigenform $f_i$. Each has an associated Galois representation $\rho_i : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$. We thus obtain

$$\rho = \rho_1 \times \rho_2 : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Q}_p \times \mathbb{Q}_p),$$

and we note that $\mathbb{Q}_p \times \mathbb{Q}_p = \mathrm{Frac}(\mathbb{T})$ (total ring of fractions). △

**Theorem 1.2.4.** *There exists a continuous irreducible representation*

$$\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(K) \ \ \textit{where} \ \ K = \mathrm{Frac}(\mathbb{T}),$$

*such that*

- *$\rho$ is unramified outside $p$.*
- *For $\ell \neq p$, $\mathrm{char}(\rho(\mathrm{Frob}_\ell)) = x^2 - T_\ell x + \ell^{k-1}$.*
- $$\rho|_{G_{\mathbb{Q}_p}} \cong \begin{pmatrix} \eta^{-1}\varepsilon^{k-1} & * \\ & \eta \end{pmatrix}$$

  *($\eta$ still an unramified character).*

**Theorem 1.2.5 (Ribet's Lemma, Version 1).** *Let $\mathbb{T}$ be a complete dvr with maximal ideal $\mathfrak{m}$. Let $G$ be a compact group. Suppose we are given*

$$\rho : G \longrightarrow \mathrm{GL}_2(K) \ \ \textit{with} \ \ K = \mathrm{Frac}(\mathbb{T}),$$

*continuous, irreducible s.t.*

$$\mathrm{char}(\rho(g)) = (x - \chi_1(g))(x - \chi_2(g)) \pmod{\mathfrak{m}}$$

*for characters $\chi_1, \chi_2 : G \to \mathbb{T}^\times$ with $\chi_1 \not\equiv \chi_2 \pmod{\mathfrak{m}}$. Then, there exists a nonzero cohomology class*

$$\kappa \in \mathrm{H}^1\left(G, \mathbb{T}/\mathfrak{m}(\chi_2\chi_1^{-1})\right).$$

**Theorem 1.2.6 (Ribet's Lemma, Version 2).** *Let $\mathbb{T}$ be a reduced complete local Noetherian ring, $I \subset \mathbb{T}$ an ideal. Suppose we're given a compact group $G$ and a continuous representation*

$$\rho : G \longrightarrow \mathrm{GL}_2(K), \ \ K = \mathrm{Frac}(\mathbb{T})$$

*such that*

$$\mathrm{char}(\rho(g)) \equiv (x - \chi_1(g))(x - \chi_2(g)) \pmod{I}$$

*for characters $\chi_1, \chi_2 : G \to \mathbb{T}^\times$ with $\chi_1 \not\equiv \chi_2 \pmod{\mathfrak{m}}$. Suppose that for each projection $K \to k$ onto a field, the projection of $\rho$ is irreducible. Then, there exists a fractional ideal $B \subset K$ and a surjective cohomology class*

$$\kappa \in \mathrm{H}^1\left(G, (B/IB)(\chi_2\chi_1^{-1})\right).$$

**Definition 1.2.7.** The cohomology class above being **surjective** means that the image of every representative cocycle generates $B/IB$ as a $\mathbb{T}$-module. In other words... ⋄

We wan to prove version 2 of Ribet's lemma now

*Proof Sketch of Lemma 1.2.6.* Fix $\tau \in G$ s.t. $\chi_1(\tau) \not\equiv \chi_2(\tau) \pmod{\mathfrak{m}}$. By hensel's lemma, $\rho(\tau)$ has two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{T}$... Same argument as before shows $a(\sigma), d(\sigma) \in \mathbb{T}$. Unlike the dvr case, we will not be able to show that $b(\sigma), c(\sigma) \in \mathbb{T}$. Note that

$$b(\sigma)c(\tau) = a(\sigma\tau) - a(\sigma)a(\tau) \in \mathbb{T}.$$

The data of $(a(\sigma), d(\sigma), b(\sigma)c(\tau))$ for $\sigma, \tau \in G$ is known as a **pseudorepresentation** of $G$ valued in $\mathbb{T}$. We won't use the perspective of pseudoreps in this course.

We can still show $a(\sigma) \equiv \chi_1(\tau)$ and $d(\sigma) \equiv \chi_2(\sigma) \mod I$ as before. Let $B$ be the $\mathbb{T}$-module generated by $\{b(\sigma) : \sigma \in G\}$. By continuity assumption, $B$ is a f.g. $\mathbb{T}$-submodule of $K$. The irreduciblity assumption implies that $B \otimes_{\mathbb{T}} K = K$, so $B$ is a fractional ideal. Note

$$b(\sigma\tau) = a(\sigma)b(\tau) + b(\sigma)d(\tau) \equiv \chi_1(\sigma)\bar{b}(\tau) + \bar{b}(\sigma)\chi_2(\tau) \pmod{I},$$

and define

$$\kappa(\sigma) = \chi_2^{-1}(\sigma)\bar{b}(\sigma) \in Z^1\big(G_{\mathbb{Q}}, B/IB(\chi_1\chi_2^{-1})\big).$$

Suppose

$$\kappa(\sigma) = \kappa'(\sigma) + (1 - \chi_1\chi_2^{-1}(\sigma))x$$

for some cocycle $\kappa'$ valued in a proper $\mathbb{T}$-submodule $B' \subset B/IB$ (i.e. that $\kappa$ not surjective). Plug in $\sigma = \tau$ to get

$$0 = \kappa'(\tau) + (1 - \chi_1\chi_2^{-1}(\tau))x,$$

so $x \in B'$. Hence, $\kappa(\sigma) \in B'$ for all $\sigma \in G$. However, the elements $\kappa(\sigma) = \chi_2^{-1}(\sigma)\bar{b}(\sigma)$ generate $B$ as a $\mathbb{T}$-module by definition, so we have a contradiction. ■

Even though $B$ may not be isomorphic to $\mathbb{T}$ as a $\mathbb{T}$-module, in some sense $B/IB$ is "as large as" $\mathbb{T}/I$, which is usually good enough for application. More precisely,

$$\mathrm{Fitt}_{\mathbb{T}}(B/IB) \subset \mathrm{Fitt}_{\mathbb{T}}(\mathbb{T}/I) = I,$$

where Fitt dentoes the 0th Fitting ideal.

**Definition 1.2.8.** Let $R$ be a commutative ring, and let $M$ be a finitely presented $R$-module:

$$R^n \xrightarrow{f} R^m \longrightarrow M \longrightarrow 0.$$

The **0th Fitting ideal** $\mathrm{Fitt}_R(M)$ is the ideal of $R$ generated by all $m \times m$ minors of the matrix representing the map $f$. ⋄

**Example.** If $n = m$, then we say $M$ is **quadratically presented** over $R$, and $\mathrm{Fitt}_R(M) = (\det(f))$. △

*Exercise.*

- The Fitting ideal does not depend on the chosen presentation
- $\mathrm{Fitt}_R(R/I) = I$ for any ideal $I \subset R$

- $\mathrm{Fitt}_R(M) \subset \mathrm{Ann}_R(M)$
- If $R = \mathbb{Z}$, and $M$ is a finitely generated abelian group, then $\mathrm{Fitt}_{\mathbb{Z}}(M) = 0$ if $M$ is infinite, but $\mathrm{Fitt}_{\mathbb{Z}}(M) = (\#M)$ is $M$ is finite.
- If $M \twoheadrightarrow M'$, then $\mathrm{Fitt}_R(M') \supset \mathrm{Fitt}_R(M)$
- Base change: If $S$ is an $R$-algebra, then

$$\mathrm{Fitt}_S(M \otimes_R S) = \mathrm{Fitt}_R(M) \cdot S.$$

  (Note this is false for annihilators in place of Fitting ideal)

**Corollary 1.2.9.** *If $B \subset K = \mathrm{Frac}(\mathbb{T})$ is a fractional ideal, then*

$$\mathrm{Fitt}_{\mathbb{T}}(B/IB) \subset \mathrm{Fitt}_{\mathbb{T}}(\mathbb{T}/I) = I.$$

*Proof.* $\mathrm{Ann}_{\mathbb{T}}(B) = 0$, so $\mathrm{Fitt}_{\mathbb{T}}(B) = 0$. Therefore, $\mathrm{Fitt}_{\mathbb{T}/I}(B/IB) = 0$ (use base change), so $\mathrm{Fitt}_{\mathbb{T}}(B/IB) \subset I$ (use base change again?). ■

## 1.3 Lecture 3 (7/28)

Today we want to talk about the Brumer-Stark conjecture.

**Setup 1.3.1.**

- Let $F$ be a totally real field
- Let $H$ be a finite Galois extension, and a CM field
- Let $G = \mathrm{Gal}(H/F)$, assumed abelian
- Let $S$ be the set of infinite places and ramified places
- Let $T$ be a finite set of places, disjoint from $S$
- Define

$$L_{S,T}(\chi, s) := \prod_{\mathfrak{p} \notin S} \frac{1}{1 - \chi(\mathfrak{p}) \, \mathrm{Nm}(\mathfrak{p})^{-s}} \prod_{\mathfrak{p} \in T} \left(1 - \chi(\mathfrak{p}) \, \mathrm{Nm}(\mathfrak{p})^{1-s}\right)$$

  with $\chi$ a character on $G = \mathrm{Gal}(H/F)$.

  When $\chi$ is the trivial character, the second factor should cancel the pole at $s = 1$.

Fix a prime $\mathfrak{p}$ of $F$ and a prime $\mathfrak{P}$ of $H$ above $\mathfrak{p}$. Assume that $\mathfrak{p}$ splits completely in $H$.

**Conjecture 1.3.2 (Tate-Brumer-Stark).** *There exists $u \in \mathscr{O}_H[1/\mathfrak{p}]^\times$ such that $|u| = 1$ under each embedding $H \hookrightarrow \mathbb{C}$,*

$$L_{S,T}(\chi, 0) = \sum_{\sigma \in G} \chi^{-1}(\sigma) \, \mathrm{ord}_{\mathfrak{P}}(\sigma(u))$$

*for all characters $\chi$ of $G$, and $u \equiv 1 \pmod{\mathfrak{q}\mathscr{O}_H}$ for all $\mathfrak{q} \in T$.*

This gives the existence of special units in the abelian extensions of totally real fields. Note in general, we don't know how to construct the abelian extensions of totally real fields.

This conjecture goes back to work of Stickelberger (showed Gauss sums had valuations related to zeta values), Brumer, Stark, Tate, etc. Sounds like Brumer made the conjecture for finite places, Stark for infinite places, and it was Tate who realized these were the same conjecture (just at different places).

**Theorem 1.3.3** (D.–Kakde)**.** *There exists*

$$u \in \mathscr{O}_H[1/\mathfrak{p}]^\times \otimes \mathbb{Z}[1/2]$$

*satisfying the conditions of the Brumer-Stark conjecture.*

(Remove the 1/2 above is motivation for last lecture(s). missed some stuff...)

**Theorem 1.3.4** (Deligne-Ribet, Cassou-Noguès)**.** *There is a unique* $\Theta \in \mathbb{Z}[G]$ *such that*

$$\chi(\Theta) = L_{S,T}(\chi^{-1}, 0)$$

*for all characters* $\chi$ *of* $G$*.*

(In particular, $L_{S,T}$'s actually give algebraic integers. This better be true if Brumer-Stark is). The fact you can get a (unique) such $\Theta$ in $\mathbb{C}[G]$ is obvious. The content is that you can get something in $\mathbb{Z}[G]$ here.

Let's look at class groups (to get relation to cohomology). Define (only kill principals generated by things which are 1 mod all primes of $T$)

$$\mathrm{Cl}^T(H) := \frac{I(H)}{\langle (u) : u \equiv 1 \pmod{T} \rangle}.$$

This is a $G$-module.

*Remark* 1.3.5. Brumer-Stark states that $\Theta$ annihilates $\mathrm{Cl}^T(H)$.

Saying that $\mathfrak{P}^\Theta$ is trivial in the class group is saying that $\mathfrak{P}^\Theta = (u)$ (with $u \equiv 1 \pmod{T}$). Taking valuations of this equality at prime ideals gives the equation showing up in the Brumer-Stark conjecture. There's some subtley here at $p = 2$ which goes away if you invert 2.

TODO: Work this out

To show this annihilation statement, it suffices to prove that

$$\Theta \in \mathrm{Ann}_{\mathbb{Z}_p[G]}\left(\mathrm{Cl}^T(H) \otimes \mathbb{Z}_p\right)$$

for all primes $p$. D.–Kakde are only able to do this for odd $p$. ◦

Let $c$ be complex conjugation in $G$. Define

$$\mathbb{Z}_p[G]_- = \mathbb{Z}_p[G]/(1+c), \quad M_- = M/(1+c)M$$

for any $\mathbb{Z}[G]$-module $M$. Note $c = -1$ in $\mathbb{Z}_p[G]_-$.

Below two thereoms are conjectures of Kurihara, Burns, and Sano. THey were proved by D.–Kakde.

**Theorem 1.3.6.** *For odd primes p, we have*

$$\Theta \in \mathrm{Fitt}_{\mathbb{Z}_p[G]_-}\left(\mathrm{Cl}^T(H)^\vee_-\right) \subset \mathrm{Ann}_{\mathbb{Z}_p[G]_-}(\mathrm{Cl}^T(H)^\vee) = \mathrm{Ann}(\mathrm{Cl}^T(H))$$

*where* $^\vee$ *is Pontryagin dual.*

**Theorem 1.3.7.** *For odd primes p, we have*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}\left(\nabla_S^T(H)_-\right) = (\Theta).$$

We won't define the nabla above, but it's some $G$-module made from class field theory. There is a canonical map

$$\nabla_S^T(H)_- \twoheadrightarrow \mathrm{Cl}^T(H)^\vee_-.$$

Therefore, the Kurihara-Burns-Sano conjecture implies the Brumer-Stark conjecture:

$$\begin{aligned}\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\nabla_S^T(H)_-) = (\Theta) &\implies \Theta \in \mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\mathrm{Cl}^T(H)^\vee_-)\\ &\implies \Theta \in \mathrm{Ann}_{\mathbb{Z}_p[G]_-}(\mathrm{Cl}^T(H)^\vee_-)\\ &\implies \Theta \in \mathrm{Ann}_{\mathbb{Z}_p[G]_-}(\mathrm{Cl}^T(H)_-)\end{aligned}$$

This is starting to look related to Ribet's method. Though one needs to relate nabla to Galois cohomology.

**Theorem 1.3.8.** *For a* $\mathbb{Z}[1/2][G]_-$*-module* $N$*, a surjection*

$$\nabla_S^T(H)_- \twoheadrightarrow N$$

*is equivalent to a cohomology class* $\kappa \in \mathrm{H}^1(G_F, N)$ *satisfying certain local conditions.*

That is, D.-Kakde found a functor represented by $\nabla$. At this point, they want a surjective cohomology class living in a large module (or something like this?).

**Corollary 1.3.9.** *To prove* $\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\nabla_S^T(H)_-) \subset (\Theta)$*, it suffices to construct a* $\mathbb{Z}_p[G]_-$*-module* $N$*, and a cohomology class* $\kappa \in \mathrm{H}^1(G_F, N)$ *satifying local conditions such that* $\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(N) \subset (\Theta)$*.*

**Theorem 1.3.10.** *The inclusion above implies an equality:*

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(\nabla_S^T(H)_-) = (\Theta).$$

The hypothesis of the above corollary is the type of thing that Ribet's method tries to produce.

### 1.3.1 Hilbert Modular Forms

Let $F$ be a totally real field of degree $n$. If $\#\,\mathrm{Cl}^+(F) = 1$, a **Hilbert modular form** for $F$ is a holomorphic function

$$f : \mathbb{H}^n \longrightarrow \mathbb{C}$$

such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathscr{O}_F)$ we have $f|_\gamma = f$, where

$$f|_\gamma(z_1, \ldots, z_n) = \prod_{i=1}^{n}(c_i z_i + d_i)^{-k} f\left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \ldots, \frac{a_n z_n + b_n}{c_n z_n + d_n}\right).$$

Above $a_i, b_i, c_i, d_i$ are the images of $a, b, c, d$ under the $n$ embeddings $F \hookrightarrow \mathbb{R}$.

*Remark* 1.3.11. When $n > 1$, you get the necessary growth condition automatically. This is essentially because the cusps are still points, and so have high codimension (functions have poles in codimension 1, so can't one at just a point). ∘

What about forms with level? Let $\mathfrak{n} \subset \mathscr{O}_F$ be an ideal, and define

$$\Gamma_0(\mathfrak{n}) = \left\{\gamma \in \mathrm{GL}_2^+(\mathscr{O}_F) : c \in \mathfrak{n}\right\}.$$

For $\chi : (\mathscr{O}_F/\mathfrak{n})^\times \longrightarrow \mathbb{C}^\times$, we define the space $M_k(\mathfrak{n}, \chi)$ of **nebentypus** $\chi$ to be the holomorphic $f : \mathbb{H}^n \to \mathbb{C}$ such that...

TODO: Finish definition, look at slides

In the general case $h = \#\mathrm{Cl}^+(F) \geq 1$, a Hilbert modular form will be an $h$-tuple of holomorphic functions $\mathbb{H}^n \to \mathbb{C}$, each with a modularity property with respect to a certain congruence subgroup. A more natural definition is as a function on a certain adelic space. A Hilbert modular form $f$ is described by its Fourier coefficients

$$c(\mathfrak{m}, f), \mathfrak{m} \subset \mathscr{O}_F \text{ nonzero}, \;\; c_\lambda(0, f), \lambda \in \mathrm{Cl}^+(F).$$

Each coefficient above is $\in \mathbb{C}$.

*Remark* 1.3.12. The nonzero ideals of $\mathscr{O}_\mathbb{Q} = \mathbb{Z}$ are in bijection with the positive integers. ∘

Define

$$M_k(\mathbb{Z}) := \{f \in M_k : c(\mathfrak{m}, f), c_\lambda(0, f) \in \mathbb{Z} \text{ always}\}.$$

In general, $M_k(R) := M_k(\mathbb{Z}) \otimes R$.

**Fact.** For any $R \subset \mathbb{C}$, we have

$$M_k(R) = \{f \in M_k : c(\mathfrak{m}, f), c_\lambda(0, f) \in R \text{ always}\}.$$

In particular, there's a $\mathbb{C}$-basis for $M_k$ consisting of integral forms.

A **group ring valued modular form** is an element of

$$M_k(G) = \{f \in M_k(\mathbb{Z}_p[G]) : \chi(f) \text{ has nebentypus } \chi \text{ for all characters } \chi \text{ of } G\}.$$

Over a field, this would the direct sum of the forms of various nebentypi. With integral coefficients, there's an additional congruence condition baked in.

**Example** (Eisenstein Series)**.** $E_1(G) \in M_1(G)$ defined by

Question: What is $d$?

Answer: The degree of the totally real field, i.e. $n$

$$c(\mathfrak{m}, E_1(G)) = \sum_{\substack{\mathfrak{a}\supset\mathfrak{m} \\ (\mathfrak{a},S)=1}} \sigma_{\mathfrak{a}}, \quad c_\lambda(0, E_1(G)) = \frac{1}{2^d}\Theta$$

(this must be modified in level 1). △

These Eisenstein series give the connection to $\Theta$. Now consider

$$f = E_1(G)V_k - \frac{\Theta}{2^d}H_{k+1}(G)$$

(which is cuspidal at $\infty$) where $V_k$ and $H_{k+1}(G)$ have constant term 1. This is cuspidal at infinity.

Choose $V_k \equiv 1 \pmod{p^N}$, where $\Theta \mid p^N$ away from trivial zeros.

$$f \equiv E_1(G) \pmod{\Theta}.$$

The existence of $V_k$ and $H_{k+1}(G)$ are non-trivial theorems of Jesse Silliman, generalizing reslults of Hida and Chai. This can be modified to yield a cusp form $f$ satisfying $f \equiv E$.

Apparently doing/did his postdoc with Dasgupta

The congrunce $f \equiv E_1(G) \pmod{\Theta}$ yields a homomorphism

$$\varphi : \mathbb{T} \longrightarrow \mathbb{Z}_p[G]_-/(\Theta), \quad T_\ell \mapsto 1 + \chi(\ell),$$

where $\chi : G_F \to G$ is the caononical character. Note this Hecke algebra $\mathbb{T}$ is a largely mysterious object.

**Theorem 1.3.13** (Deligne, Carayol, Hida, Wiles)**.** *There exists a continuosu irreducible representation*

$$\rho : G_F \longrightarrow \mathrm{GL}_2(K), \quad K = \mathrm{Frac}(\mathbb{T})$$

*such that*

- *$\rho$ is unramified outside $p$ (and the level)*
- *For $\ell \neq p$,*
$$\mathrm{char}(\rho(\mathrm{Frob}_\ell)) = x^2 - T_\ell x + \mathrm{Nm}(\ell)^{k-1}.$$
- $$\rho|_{G_p} \cong \begin{pmatrix} \eta^{-1}\varepsilon^{k-1} & * \\ & \eta \end{pmatrix}.$$

If $I := \ker(\varphi)$, we have

$$\mathrm{char}\,\rho(\mathrm{Frob}_\ell) = \ldots \pmod{I}.$$

This put's us in the setting of Ribet's Lemma 1.2.6. Applying it as a black box yields a fractional ideal $B \subset K = \mathrm{Frac}(\mathbb{T})$ and a cohomology class

$$\kappa \in \mathrm{H}^1\left(G_F, B/IB(\chi^{-1})\right).$$

We have $\mathrm{Fitt}_{\mathbb{T}/I}(B/IB) = 0$, so $\mathrm{Fitt}_{\mathbb{Z}_p[G]_-/\Theta}(B/IB) = 0$, so

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]_-}(B/IB) \subset (\Theta).$$

Now apply Corollary 1.3.9, hopefully...

There are local conditions we need satisfied. Pretend for now that $\kappa$ is unramified at $p$. Then all the local conditions will be satisfied, and one gets $\nabla_S^T(H)_- \twoheadrightarrow B/\Theta B$. Hence,

$$\mathrm{Fitt}(\nabla_S^T(H)_-) \subset \mathrm{Fitt}(B/\Theta B) \subset (\Theta)$$

since $B$ is a faithful $\mathbb{Z}_p[G]$-module.

How do you get a class unramified at $p$? "This is probably the deepest part of our paper."

(Step 1) There is a non-zero divisor $x \in \mathbb{Z}_p[G]$ such that we can construct a "higher congruence"

$$f \equiv E_1(G) \pmod{x\Theta}$$

- $x$ measures "trivial zeros at $p$"
- Requires detailed construction of cusp form
- Calculation of constant terms of Eisenstein series at all cusps

(Step 2) Define

$$B' = \langle b(\sigma) : \sigma \in I_{\mathfrak{p}}, \mathfrak{p} \mid p \rangle \subset B, \;\; \overline{B} = B/(x\Theta B, B'),$$

and

$$\kappa(\sigma) = [\sigma]^{-1} b(\sigma) \in \mathrm{H}^1(G_F, \overline{B}).$$

$\kappa$ is now tautologically unramified at $p$ since we killed inertia above $p$. Hence, we get

$$\nabla_S^T(H)_- \twoheadrightarrow \overline{B}.$$

(Step 3) A miracle:

$$\mathrm{Fitt}(\overline{B}) \cdot (x) \subset \mathrm{Fitt}(B/x\Theta B) \subset (x\Theta).$$

Hence, $\mathrm{Fitt}(\nabla_S^T(H)_-) \subset \mathrm{Fitt}(\overline{B}) \subset (\Theta)$ ($x$ a non-zero divisor) as before.

Last talk tomorrow we'll discuss new directions (for example, related to $p = 2$ part of Brumer-Stark).

**Question 1.3.14** (Audience)**.** *Can you say a bit about what goes wrong with $p = 2$?*

**Answer.** In application of Ribet's lemma, have hypothesis $\chi_1 \not\equiv \chi_2 \pmod{\mathfrak{m}}$. We need $\chi_1 = 1$ and $\chi_2 = \chi$. For us, $\chi(c) = -1$ ($c$ = complex conjugation), and $1 \not\equiv -1 \pmod{\mathfrak{m}}$ if $\mathfrak{m} \nmid 2$, so we're good there. However at 2, it's possible that $\chi_1$ and $\chi_2$ become congruent. Hence, need a version of Ribet's lemma which handles this case (will be stated next time). ⋆

## 1.4 Lecture 4 (7/29) – Ribet's Method IV, New Directions: The Residually Inditinguishable Case

They've been working on this argument for a couple years and apparently just came up with the idea(s) they needed a couple weeks ago.

Applications of Ribet's method

- Main Conjectecure of Iwasawa Theory

  Mazur–Wiles, Wiles

- Cases of Bloch-Kato

  Bellaiche–Chenavier (spelling?)

- Main conjecture of Iwasawa theory for elliptic curves

  Skinner–Urban

- Gross–Stark and Brumer–Stark conjectures, ETNC

  D.–Darmon–Pollack–Kakde–Ventullo

- Class grou of $\mathbb{Q}(N^{1/p})$

  Lang–Wake

**Theorem 1.4.1** (**Ribet's Lemma, Version 2**, Theorem 1.2.6)**.** *Let $\mathbb{T}$ be a reduced complete local Noetherian ring, $I \subset \mathbb{T}$ an ideal. Suppose we're given a compact group $G$ and a continuous representation*

$$\rho : G \longrightarrow \mathrm{GL}_2(K), \quad K = \mathrm{Frac}(\mathbb{T})$$

*such that*

$$\mathrm{char}(\rho(g)) \equiv (x - \chi_1(g))(x - \chi_2(g)) \pmod{I}$$

*for characters $\chi_1, \chi_2 : G \to \mathbb{T}^\times$ with $\chi_1 \not\equiv \chi_2 \pmod{\mathfrak{m}}$. Suppose that for each projection $K \to k$ onto a field, the projection of $\rho$ is irreducible. Then, there exists a fractional ideal $B \subset K$ and a surjective cohomology class*

$$\kappa \in \mathrm{H}^1\left(G, (B/IB)(\chi_2\chi_1^{-1})\right).$$

This was state of the art until recent work proving...

**Theorem 1.4.2** (**Ribet's Lemma, Version 3**, 'Residually indistinguishable case', D.–Kakde–Silliman–Wang)**.** *Let $\mathbb{T}$ be a reduced complete local Noetherian ring, $I \subset \mathbb{T}$ an ideal. Suppose given a compact group $G$ and a continuous representation*

$$\rho : G \longrightarrow \mathrm{GL}_2(K), \quad K := \mathrm{Frac}(\mathbb{T})$$

*such that*

$$\mathrm{char}(\rho(g)) \equiv (x - \chi_1(g))(x - \chi_2(g)) \pmod{I}$$

*for characters $\chi_1, \chi_2 : G \longrightarrow T^\times$. Suppose that for each projection $K \twoheadrightarrow k$ onto a field, the projection of $\rho$ is irreducible. Then there exists a f.g. $\mathbb{T}$-module $M$ and a surjective cohomology class $\kappa \in \mathrm{H}^1(G, M(\chi_2\chi_1^{-1}))$ such that $\mathrm{Fitt}_{\mathbb{T}}(M) \subset I$.*

We want to spend the rest of today going over the proof of this theorem. Warning: it is hard.

**Theorem 1.4.3** (Ophir–Weiss, Hajjar Muñoz)**.** *The previous theorem holds when $\mathbb{T}$ is a dvr*

Ophir–Weiss study versions of Ribet's lemma in the residually indistinguishable case over a dvr...

What's the $M$ appearing in the theorem? We can no longer choose a rigidifying bases and use the "$b$" coefficient. Instead, they extend $\rho$ to a $\mathbb{T}$-algebra homomorphism

$$\rho : \mathbb{T}[G] \longrightarrow M_2(K),$$

and define

$$\Delta_i = \mathbb{T}\langle \rho(g) - \chi_i(g) : g \in G\rangle \text{ and } M := \Delta_1/\Delta_1\Delta_2$$

(I guess $\rho$ acts by $\chi_2$ on $M$?)

*Exercise.* Show $\Delta_1\Delta_2 \subset \Delta_1$.

Now define

$$\kappa(g) := \chi_1^{-1}(g)\,(\rho(g) - \chi_1(g)) \in M.$$

It's possible one of the 1's below was supposed to be a 2.

*Exercise.* $\kappa \in Z^1(G, M(\chi_2\chi_1^{-1}))$.

**surjectivity of $\kappa$** Let $M' \subset M$ be a $\mathbb{T}$-submodule, and suppose there's $\kappa' \in Z^1(G, M(\chi_2\chi_1^{-1}))$ and $x \in M$ such that

$$\kappa(g) = \kappa'(g) + (\chi_2\chi_1^{-1}(g) - 1)x.$$

By version 2 of Ribet's lemma, we can and do assume $\chi_1 \equiv \chi_2 \pmod{\mathfrak{m}}$, so

$$(\chi_2\chi_1^{-1}(g) - 1)x \in \mathfrak{m}M.$$

Therefore, $\kappa(g) \in M' + \mathfrak{m}M$, but $\kappa(g)$ clearly generates $M$, so $M = M' + \mathfrak{m}M$. By Nakayama, one gets that $M = M'$.

Question: Is it clear that $M$ is finitely generated?

Answer: $G$ compact, so it will have bounded image. I think this is the key

**Fitting ideal of $M = \Delta_1/\Delta_1\Delta_2$** It remains to prove that $\mathrm{Fitt}_{\mathbb{T}}(M) \subset I$.

**Assumption** (Notational simplification)**.** Let's say $\chi_1 = \chi_2$, so $M = \Delta_1/\Delta_1^2$

Let $\rho_1, \ldots, \rho_r$ denote $\mathbb{T}$-module generates of $\Delta_1$. Write

$$\rho_i\rho_j \sum_{k=1}^{r} \delta_{ijk}\rho_k, \quad \delta_{ijk} \in \mathbb{T}.$$

There may be additional relations of the form

$$\sum_{j=1}^{r} \varepsilon_{ij}\rho_j = 0, \quad \varepsilon_{ij} \in \mathbb{T}.$$

Now, $\mathrm{Fitt}_{\mathbb{T}}(M)$ is the ideal generated by $r \times r$ determinants where each row has the form $(\delta_{ijk})_{k=1}^{r}$ or $(\varepsilon_{ij})_{j=1}^{r}$.

*Remark* 1.4.4. It's easier to deal with rows of $\varepsilon$'s, so we can ignore them. Even logically, any row of $\varepsilon$'s will be a difference of rows of $\delta$'s. ∘

Let $D = (\delta_{ijk})$, as we slect some $r$ pairs $(i, j)$ and let $k = 1, \ldots, r$. We want to show that $\det(D) \in I$. We are given $\operatorname{Tr}\rho(g) - 2\chi(g) \in I$ and $\det \rho(g) - \chi^2(g) \in I$.

*Exercise.* Use these facts to show that $\mathrm{Tr}(A), \det(A) \in I$ for all $A \in \Delta_1$.

I think at this point, I'm gonna go ahead and stop taking notes because these details are getting hairy. See the slides on the website if you wanna know what went on...

# 2 Lassina Dembélé: An Algorithmic Approach to Hilbert–Siegel Modular Forms and the Paramodularity Conjecture

## 2.1 Lecture 1 (7/25)

*Goal.*

**(1)** Computational aspect of Hilbert-Siegel modular forms

**(2)** Study congruences of these forms

**(3)** Study modularity of abelian surfaces via concrete examples

### 2.1.1 Symplectic groups and inner forms

**Setup 2.1.1.**

- Let $g \geq 1$ be an integer.
- Fix a number field $F$.
- We let $J_{2g}$ denote the matrix

$$J_{2g} = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \in \mathrm{GL}_{2g}(F)$$

  (with $I_g$ the $g \times g$ identity matrix)

**Definition 2.1.2.** The **symplectic group of genus $g$ over $F$** is the algebraic group over $F$ defined by

$$\mathrm{GSp}_{2g}(A) = \left\{ M \in \mathrm{M}_{2g}(A) : M^t J_{2g} M = \nu(M) J_{2g} \text{ for some } \nu(M) \in A^\times \right\}$$

with $A$ any $F$-algebra. The **similitude factor** is the natural homomorphism

$$\nu : \mathrm{GSp}_{2g} \longrightarrow \mathbb{G}_m.$$

We define $\mathrm{Sp}_{2g}$ via the exact sequence

$$1 \longrightarrow \mathrm{Sp}_{2g} \longrightarrow \mathrm{GSp}_{2g} \xrightarrow{\nu} \mathbb{G}_m \longrightarrow 1,$$

and call it the **special symplectic group**. ◇

**Notation 2.1.3.** Set $G = \mathrm{GSp}_{2g}/F$.

**Definition 2.1.4.** An **inner form** of $G$ is an algebraic group $G'/F$ such that $G(\overline{F}) \cong G'(\overline{F})$. ◇

Question: Is it obvious that this implies $G_{\overline{F}} \cong G'_{\overline{F}}$?

Question: Where does the word inner come into play?

We will mostly be interested in the following inner forms of $\mathrm{GSp}_{2g}/F$.

**Recall 2.1.5.** A **quaternion algebra** over $F$ is a rank 4 algebra

$$B = F \oplus Fi \oplus Fj \oplus Fk$$

with

$$i^2 = a, \quad j^2 = b, \quad \text{and} \quad k = ij = -ji.$$

We require $a, b \in F^\times$. ⊙

**Fact.** For any quaternion algebra $B/F$, there exists infinitely many quadratic extensions $K/F$ s.t. $B \otimes_F K \cong M_2(K)$. Such a $K$ is called a **splitting field** of $B$.

*Remark* 2.1.6. Let $B/F$ be a quaternion algebra. Note that

- it has an $F$-algebra involution $\bar{\ } : B \to B$ determined by $\bar{i} = -i$ and $\bar{j} = -j$.
- it has a reduced trace: $\mathrm{Tr}(\gamma) = \gamma + \bar{\gamma} = 2x$ if $\gamma = x + yi + zj + wk$.
- it has a reduced norm $\mathrm{Nm}(\gamma) = \gamma\bar{\gamma}$.

Observe that the element $\gamma$ satisfies the polynomial

$$T^2 - \mathrm{Tr}(\gamma)T + \mathrm{Nm}(\gamma),$$

which is called its minimal polynomial. ○

**Definition 2.1.7.** Let $B/F$ be a quaternion algebra. The **unitary similitude group of genus** $g$ **attached to** $B$ is the $F$-algebraic group $\mathrm{GU}_g^B/F$ whose $A$-points (for $A$ an $F$-algebra) are

$$\mathrm{GU}_g^B(A) = \mathrm{GU}_g(B \otimes_F A) = \left\{ M \in M_g(B \otimes_F A) : M^*M = \nu(M)I_g \ \text{for some} \ \nu(M) \in A^\times \right\}$$

(note this is the restriction of scalars of a group defined over $B$), where $M^* = \left(\overline{M}\right)^t$. ◇

**Lemma 2.1.8.** *$\mathrm{GU}_g^B/F$ is an inner form of $\mathrm{GSp}_{2g}/F$.*

*Proof.* Recall that $B \otimes_F \overline{F} \cong M_2(\overline{F})$. Hence, it is enough to show that

$$\mathrm{GU}_g(M_2(\overline{F})) \cong \mathrm{GSp}_{2g}(\overline{F}).$$

Note that $\mathrm{GU}_g(M_2(\overline{F})) \hookrightarrow M_{2g}(\overline{F}) \hookleftarrow \mathrm{GSp}_{2g}(\overline{F})$. The isomorphism $\mathrm{GU}_g(M_2(\overline{F})) \xrightarrow{\sim} \mathrm{GSp}_{2g}(\overline{F})$ will be of the form $M \mapsto P^tMP$ for a suitable permutation matrix $P$.

**Example.** Note that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\overline{F})$, then

$$\overline{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = J_2^{-1}AJ_2.$$

△

Let

$$\widetilde{J}_{2g} = \begin{pmatrix} J_2 & & 0 \\ & \ddots & \\ 0 & & J_2 \end{pmatrix} \in M_{2g}(\overline{F}).$$

Question: Presumably this is coming from a map $\mathrm{PGL}_2 \to \underline{\mathrm{Aut}}\,\mathrm{GSp}_{2g}$ (landing in inner automorphisms). What is this map? Presumably it's coming from some map $\mathrm{GSp}_2 \to \mathrm{GSp}_{2g}$ I guess?

I may have messed up some of the expressions in this proof

For any $M = (A_{ij})_{1\leq i,j\leq g}$ (each $A_{ij}$ a $2\times 2$ matrix), set $M^* = \widetilde{J}_{2g}^{-1}M^t\widetilde{J}_{2g}$. Let $P$ be the permutation matrix such that

$$P^t\widetilde{J}_{2g}P = J_{2g}.$$

*Exercise.* Show that

$$M^*M = \nu\cdot 1 \implies (M')^tJ_{2g}M' = \nu I_{2g} \text{ for } M' := P^tMP.$$

■

**Example.** Let $B/\mathbb{Q}$ be the Hamilton quaternion algebra (i.e. $i^2 = -1 = j^2$ and $k = ij = -ji$). If we let $K = \mathbb{Q}(i)$, one has $B\otimes_{\mathbb{Q}} K \cong M_2(K)$. Can take

$$P = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

in the above proof, i.e. $\mathrm{GU}_2^B \cong \mathrm{GSp}_4$ over $K$. △

### 2.1.2 Integral models

Recall $G = \mathrm{GSp}_{2g}/F$. Note that the matrix $J_{2g}$ appearing in the definition of $G$ has integral coefficients, so $G$ is really the base change of an algebraic group $\mathrm{GSp}_{2g,\mathbb{Z}}/\mathbb{Z}$ defined over the integers. Specifically,

$$\mathrm{GSp}_{2g,\mathbb{Z}}(A) = \left\{M\in \mathrm{M}_{2g}(A) : M^tJ_{2g}M = \nu(M)J_{2g} \text{ for some } \nu(M)\in A^\times\right\}$$

for any ring $A$. We let $\underline{G} := \mathrm{GSp}_{2g,\mathscr{O}_F}$ be our preferred choice of integral model for $G$.

## 2.2 Lecture 2 (7/26)

Let's take a digression into quaternion algebras.

### 2.2.1 Digressions on quaternion algebras

We want to cover the basic properties of quaternion algebras which we will need.

**Recall 2.2.1.** A **quaternion algebra** $B/F$ is a rank 4 $F$-algebra with a basis

$$B = F\oplus Fi\oplus Fj\oplus Fk$$

such that $i^2 = a, j^2 = b$, and $k = ij = -ji$ for some $a, b\in F^\times$. We denote this by writing

$$B = \left(\frac{a,b}{F}\right).$$

⊙

**Example.**

$$B = \left(\frac{1,1}{F}\right)$$

is the $2 \times 2$ matrix algebra $B \simeq M_2(F)$. △

**Example.** If $F = \overline{F}$ is algebraically closed, then the only quaternion algebra over $F$ is $M_2(F)$.[4] △

**Example.** Let $F$ be a local field (e.g. $\mathbb{Q}_p$ or $\mathbb{R}$). Up to isomorphism, there are only two quaternion algebras.[5] One is $M_2(F)$, and the other is the unique division quaternion algebra $B$. To construct this division quaternion algebra, let $K/F$ be the unique unramified quadratic extension, then $K \subset B$, and in fact, $B \otimes_F K \simeq M_2(K)$. Take a uniformizer $\pi \in K$. Then, there is $\theta \in B$ such that $\theta^2 = \pi$ (note $\theta \notin K$ since $K$ unramified), and you can write $B = K \oplus K\theta$. This simultaneously gives existence and uniqueness. △

Question: Why does $B$ contain an unramified quadratic extension?

Question: Why?

Let $F$ be a number field, and let $B/F$ be a quaternion algebra. Let $p$ be a rational prime. Then,

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v \mid p} F_v$$

is a product of local fields. Accordingly,

$$B \otimes_{\mathbb{Q}} \mathbb{Q}_p \simeq \prod_{v \mid p} B_v$$

with $B_v$ a quaternion algebra over $F_v$. We say that $B$ **splits** at $v$ (or is **unramified** at $v$) if $B_v \simeq M_2(F_v)$.

Here is the most important theorem about quaternion algebras over number fields.

**Theorem 2.2.2.** *Let $F$ be a number field, and let $B/F$ be a quaternion algebra. Let $S = \operatorname{Ram}(B)$ be the set of places where $B$ ramifies. Then, $S$ is finite with even cardinality. Conversely, any such $S$ is the set of ramified places for a unique quaternion algebra over $F$.*

**Notation 2.2.3.** Let $\mathcal{M}_F$ be the set of places of $F$, and let $\operatorname{QA}/F$ be the set of quaternion algebras over $F$.

To restate the theorem, we have a bijection

$$\left\{ \begin{array}{c} S \subset \mathcal{M}_F \\ \#S < \infty \text{ even} \end{array} \right\} \longleftrightarrow (\operatorname{QA}/F)_{/\simeq}$$

given by $\operatorname{Ram}(B) \leftarrow\!\shortmid B$.

**Example.** Say $B = \left(\frac{1,1}{F}\right) \simeq M_2(F)$. Then, $\operatorname{Ram}(B) = \emptyset$. △

**Example.** Say $B = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ (the Hamilton quaternions). Then, $\operatorname{Ram}(B) = \{2, \infty\}$. △

**Example.** Say $B = \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{5})}\right)$. Then, $\operatorname{Ram}(B) = \{\infty_1, \infty_2\}$. △

[4] In general, for $a, b, \lambda, \mu \in F$, one has

$$\left(\frac{a,b}{F}\right) \simeq \left(\frac{a\lambda^2, b\mu^2}{F}\right)$$

[5] CFT gives $\operatorname{Br}(F)[2] \simeq \frac{1}{2}\mathbb{Z}/\mathbb{Z}$

**Definition 2.2.4.** The **discriminant** of a quaternion algebra $B/F$ is the ideal

$$d_B = \prod_{\substack{v \in \mathrm{Ram}(B) \\ v < \infty}} v$$

(I guess this is an $\mathscr{O}_F$-ideal) ◇

**Warning 2.2.5.** Some people include $v \mid \infty$ in the definition of the discriminant. •

**Example.** $B = \left(\frac{-1,-1}{\mathbb{Q}}\right) \implies d_B = (2)$ △

**Example.** $B = \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{5})}\right) \implies d_B = (1)$ △

**Definition 2.2.6.** Let $V$ be a f.dim $F$-v.s. A **lattice** in $V$ is a f.g. $\mathscr{O}_F$-submodule $L \subset V$ which spans $V$ over $F$. ◇

(In particular, $L$ does not have to be free)

**Definition 2.2.7.** An **order** $\mathscr{O}$ in $B$ is an $\mathscr{O}_F$-lattice in $B$ which is also a subring. We say $\mathscr{O}$ is a **maximal order** if it is maximal for inclusion inside $B$. ◇

**Example.** Say $B = \left(\frac{-1,-1}{\mathbb{Q}}\right)$. Then,

$$\mathscr{O} = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$$

is an order in $B$. However, it is *not* maximal. It is contained in the order

$$\mathscr{O}_B = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{1+i+j+k}{2}$$

which is maximal.[6] △

*Remark* 2.2.8. If $\mathscr{O}$ is an order inside $B$, then every nonzero element of $\mathscr{O}$ will satisfy a monic polynomial with $\mathscr{O}_F$ coefficients. ○

Let $\mathscr{O}$ be an order inside $B$.

**Definition 2.2.9.** A **left** (resp. **right**) **ideal** $I$ is an $\mathscr{O}_F$-lattice which is also a left (resp. right) $\mathscr{O}$-module. We say that $I$ is a **integral ideal** if $I \subset \mathscr{O}$. ◇

(An 'ideal' as above is the analogue of a 'fractional ideal' in a number field)

Let $I$ be an $\mathscr{O}_F$-lattice inside $B$. Set

$$\mathscr{O}_R(I) := \{x \in B : Ix \subset I\} \text{ and } \mathscr{O}_L(I) := \{x \in B : xI \subset I\}.$$

**Proposition 2.2.10.** *$\mathscr{O}_R(I)$ and $\mathscr{O}_L(I)$ are orders in $B$. Furthermore, $I$ will be a right ideal for $\mathscr{O}_R(I)$ and a left ideal for $\mathscr{O}_L(I)$.*

[6] If I'm not mistaken, this is the subset of $\frac{1}{2}\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}i \oplus \frac{1}{2}\mathbb{Z}j \oplus \frac{1}{2}\mathbb{Z}k$ where very coefficient is in $\mathbb{Z}$ or in $\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

**Class set** Let $B/F$ be a quaternion algebra. Let $\mathscr{O}$ be an order in $B$. Let $I, J$ be left (resp. right) ideals of $\mathscr{O}$. We say that they are **equivalent** if there is some $\gamma \in B^\times$ such that $J = I\gamma$ (resp. $J = \gamma I$). Now, we define the class sets

$$\mathrm{Cl}_R(\mathscr{O}) = \{\text{right } \mathscr{O}\text{-ideals}\}_{/\sim} \quad \text{and} \quad \mathrm{Cl}_L(\mathscr{O}) = \{\text{left } \mathscr{O}\text{-ideals}\}_{/\sim}.$$

**Warning 2.2.11.** These are just pointed sets, they do not have a natural group structure. •

**Theorem 2.2.12.** *$\mathrm{Cl}_R(\mathscr{O})$ and $\mathrm{Cl}_L(\mathscr{O})$ are finite sets.*

*Remark* 2.2.13.

**(1)** $\mathrm{Cl}_R(\mathscr{O})$ and $\mathrm{Cl}_L(\mathscr{O})$ do not have any group structure.

**(2)** There are class number formulas for $\mathrm{Cl}_R(\mathscr{O})$ and $\mathrm{Cl}_L(\mathscr{O})$ as well as algorithms to compute them.

○

**Theorem 2.2.14.** *Let $\mathscr{O}_B$ be a maximal order inside $B$. Then, $\#\,\mathrm{Cl}_R(\mathscr{O}_B)$ and $\#\,\mathrm{Cl}_L(\mathscr{O}_B)$ is independent of $\mathscr{O}_B$ (i.e. it's the same for any maximal order), and $\#\,\mathrm{Cl}_R(\mathscr{O}_B) = \#\,\mathrm{Cl}_L(\mathscr{O}_B)$. This number is called the **class number** of $B$.*

**Example.** Let $B = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, and take

$$\mathscr{O}_B = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{1+i+j+k}{2}.$$

Then, $\mathrm{Cl}_R(\mathscr{O}_B) = \{[\mathscr{O}_B]\}$. Note that this $B$ supports a Euclidean algorithm. △

## 2.3 Lecture 3 (7/28)

### 2.3.1 Hermitian lattices and integral models for $\mathrm{GU}_g$

Let $B/F$ be a quaternion algebra over a number field. Recall this has a reduced trace $\mathrm{Tr} : B \to F$ as well as a reduced norm $\mathrm{Nm} : B \to F$.

Let $V$ be a free left $B$-module of finite rank $g$.

**Definition 2.3.1.** A **quaternion Hermitian form** on $V$ is an $F$-bilinear form

$$Q : V \times V \longrightarrow B$$

satisfying

**(i)** $Q(ax, y) = aQ(x, y)$ for any $a \in B$ and $x, y \in V$

**(ii)** $Q(y, x) = \overline{Q(x, y)}$

We say that $Q$ is **nondegenerate** if $Q(x, V) = 0 \implies x = 0$. ⋄

The following theorem highlights the parallels between the commutative and quaternionic settings for Hermitian form.

**Theorem 2.3.2** (Shimura)**.** *Let $Q : V \times V \to B$ be a Hermitian form. Then, there exists a $B$-basis $e_1, \dots, e_g$ of $V$ along with $\alpha_i \in F$ such that*

$$Q(e_i, e_j) = \alpha_i \delta_{ij}.$$

*If, in addition, the norm $\mathrm{Nm} : B \to F$ is surjective, then we can choose the basis $e_i$ so that $Q(e_i, e_j) = \delta_{ij}$ (i.e. so that $\alpha_i = 1$ for all $i$).*

Let $\mathscr{O}$ be an order inside $B$. A **Hermitian lattice** $L$ inside $V$ is an $\mathscr{O}_F$-lattice which is also a left $\mathscr{O}$-module ($\implies B \cdot L = V$).

**Example** (Standard lattice)**.** If $V = B^n$, can take $L = \mathscr{O}^n$. △

**Assumption.** From now on, assume $\mathscr{O} = \mathscr{O}_B$ is maximal.

(Most of what we'll say extends to the case of arbitrary order)

**Definition 2.3.3.** The **dual lattice** to $L$ (w.r.t. the given pairing $Q$) is

$$L^\vee := \{x \in V : Q(x, L) \subset \mathscr{O}_B\}.$$

The **norm** of $L$ is the following two-sided ideal:

$$\nu_Q(L) := \{Q(x, y) : x, y \in L\}.$$

◇

**Fact.** $\nu_Q(L)^{-1} L \subset L^\vee$. This is essentially by definition:

$$Q(ax, y) = aQ(x, y) \in a\nu_Q(L) \subset \mathscr{O}_B$$

for any $a \in \nu_Q(L)^{-1}$ and $x, y \in L$.

**Definition 2.3.4.** We say that $L$ is **integral** if $Q(L, L) \subset \mathscr{O}_B$, i.e. $Q$ restricts to a pairing $L \times L \to \mathscr{O}_B$. ◇

**Definition 2.3.5.** We say that $L$ is **modular** if $\nu_Q(L)^{-1} L = L^\vee$. We say that $L$ is **maximal** if it is maximal w.r.t. inclusion of lattices with the same norm. ◇

**Lemma 2.3.6.** *If $L$ is integral and maximal, then $L$ is self-dual and hence modular.*

*Proof.* Integrality tells you that $\nu_Q(L) \subset \mathscr{O}_B$. Maximality + the inclusion $\nu_Q(L)^{-1} L \subset L^\vee$ then tells you that $L = L^\vee$. ■

I think actually Lassina claimed this is an equality, but I don't see why

Let $L$ be an $\mathscr{O}_B$-lattice. Let $\mathcal{M}_F^0$ denote the set of finite places. We get a family of local lattices $(L_v)_{v \in \mathcal{M}_F^0}$. Conversely, let $(L_v)_{v \in \mathcal{M}_F^0}$ be a family of local lattices such that $L_v = \mathscr{O}_{B,v}^{\oplus g}$ for almost all $v \in \mathcal{M}_F^0$. Then, there exists a global lattice $L \subset V$ giving rise to this family.

This second sentence might not actually hold given what I wrote in the

Let $L, M$ be $\mathscr{O}_B$-lattices in $V$. We say that $L, M$ are **in the same genus** if there is some $\gamma_v \in \mathrm{GU}_g(B_v)$ (for all $v \in \mathcal{M}_F^0$) such that

$$M_v = L_v \gamma_v$$

(i.e. they are locally isomorphic). The **genus** of $L$ is simply its equivalence class under this relation, i.e. it is the set of all lattices 'in the same genus' as $L$. This genus of $L$ is denoted by $\mathrm{gen}_G(L)$ $(G = \mathrm{GU}_g)$.

**Assumption.** We fix an arbitrary choice of genus $\mathrm{gen}_G(L)$ and work only with lattices belong to it.

**Definition 2.3.7.** We say $L, M \in \mathrm{gen}_G(L)$ are **equivalent** if there is some $\gamma \in \mathrm{GU}_g(B)$ such that $M = L\gamma$. The **class set** of $\mathrm{gen}_G(L)$ is the set of equivalence classes, and is denoted $\mathrm{cl}_G(L) := \mathrm{gen}_G(L)/\simeq$. ⋄

If $L$ is a maximal integral lattice, we'll let

$$\underline{G}_L := \mathrm{Stab}_G(L) \text{ where } G = \mathrm{GU}(V).$$

Can show that this will give an integral model of $G$. We will use the notation

$$\mathrm{Cl}_{\underline{G}_L} \text{ or } \mathrm{Cl}_{\underline{G}_L}(L), \quad \mathrm{gen}_{\underline{G}_L}(L).$$

**Theorem 2.3.8.** *The class set* $\mathrm{Cl}(\underline{G}_L)$ *is finite.*

Let $\underline{G}_L/\mathscr{O}_F$ be an integral model. We can define the class set adelically as:

I guess this is why functions on $\mathrm{Bun}_G$ are related to automorphic forms.

$$\mathrm{Cl}_{\underline{G}_L} = \underline{G}_L(\widehat{\mathscr{O}}_F)\backslash \underline{G}_L(\widehat{F})/\underline{G}_L(F)$$

where $\widehat{F} = \mathbb{A}_{F,f}$ is the group of finite adeles, and $\widehat{\mathscr{O}}_F = \prod_{v\nmid\infty} \mathscr{O}_{F,v}^\times$.

**Algebraic automorphic forms** Let $U := \underline{G}(\mathscr{O}_F)$. Set

$$M_{\underline{G}_L}(U;\mathbb{C}) := \{f : \mathrm{Cl}(\underline{G}_L) \longrightarrow \mathbb{C}\}.$$

This is called the **space of algebraic automorphic forms of level $U$ and trivial weight on $\underline{G}_L(\widehat{F})$**.

Recall that $G = \mathrm{GU}(V)$ is an inner form of $\mathrm{GSp}_{2g}$. By Langlands functoriality, we should expect that this will give us a way to go from one of these simpler-seeming final objects to a Hilbert-Siegal modular form. More on this next time.

## 2.4 Lecture 4 (7/29)

Yesterday we defined algebraic automorphic forms.

*Remark* 2.4.1. In order to use material from the last few lectures to compute automorphic forms, it is important to know that the class group is finite and that it is computable. ∘

Most of what we said yesterday should be true for any Hermitian form, but for today, we'll require our forms to be positive definite.

**Setup 2.4.2.** We keep the setup/notation from last time. In particular, $F$ is a totally real field, and $Q$ is a Hermitian form.

**Assumption.** We assume that $Q$ is **positive definite**, i.e. that $Q(x, x) > 0$ if $x \neq 0$.

Under this assumption, the group $\underline{G}_L(F \otimes_Q \mathbb{R})$ will be compact. As a consequence, the group

$$\Gamma_L = \mathrm{Stab}_{\underline{G}_L}(L)/\mathscr{O}_F^\times$$

is finite.

**Definition 2.4.3.** The **mass** of the genus $\mathrm{gen}_G(L)$ is given by

$$\mathrm{Mass}_G(L) := \sum_{[M]\in \mathrm{Cl}_{\underline{G}}(L)} \frac{1}{|\Gamma_M|}$$

(This is the groupoid cardinality of the groupoid of lattices in a given genus). ◇

**Theorem 2.4.4.** *Let $L$ be a maximal integral lattice in $V$. Then, the mass of the genus of $L$ is given by*

$$\mathrm{Mass}_G(L) = \left(\frac{1}{2^{ng}} L(M)\right) \prod_{v\in \mathrm{Ram}(B)\setminus\{v|\infty\}} \lambda_v,$$

*where*

I'm not sure what $M$ is

- $$L(M) = \left|\prod_{r=1}^{g} \zeta_F(1-2r)\right|$$

- $$\lambda_v = \prod_{r=1}^{g} (q_v^r + (-1)^r) \quad \textit{where } q_v = \#\mathscr{O}_{F_v}/\mathfrak{p}_v.$$

- $n = [F : \mathbb{Q}]$

*Remark* 2.4.5. Sounds like every maximal, integral lattice belongs to the same genus, the *principal genus*? ○

**Example.** Consider $B = \left(\frac{-1,-1}{\mathbb{Q}}\right)$ with maximal order

$$\mathscr{O}_B = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\frac{1+i+j+k}{2}.$$

Recall that $\mathscr{O}_B$ has class number one, but what about higher rank lattices? Take $V = B^g$ and $L = \mathscr{O}_B^g$ (I guess with $g = 2$?). The previous theorem will give

$$\mathrm{Mass}_G(L) = \frac{1}{4} |\zeta_\mathbb{Q}(-1)\zeta_\mathbb{Q}(-3)| (2-1)(4+1).$$

Now,

$$\Gamma_L = \mathrm{Stab}_G(L) = \left\{\begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix} : u, v \in \mathscr{O}_B^\times\right\} \cup \left\{\begin{pmatrix} 0 & u \\ v & 0\end{pmatrix} : u, v \in \mathscr{O}_B^\times\right\},$$

so $\#\Gamma_L = 1152$, $\mathrm{Mass}_G(L) = 1/\#\Gamma_L$, and we have class number one again $\#\,\mathrm{Cl}_{\underline{G}}(L) = 1$. △

### 2.4.1 Algebraic automorphic forms

Let $U = \underline{G}_L(\widehat{\mathscr{O}}_F) \subset \underline{G}_L(\widehat{F})$, a compact open. Let

$$M_{\underline{G}}(U) := \left\{ f : \mathrm{Cl}_{\underline{G}}(L) \longrightarrow \mathbb{C} \right\},$$

a f.dim $\mathbb{C}$-v.s. called the **space of algebraic automorphic forms of level $U$ and trivial weight**.

**Recall 2.4.6.** Adelically, we have

$$\mathrm{Cl}_{\underline{G}}(L) = \underline{G}_L(\widehat{\mathscr{O}}_F) \backslash \underline{G}_L(\widehat{F}) / \underline{G}_L(F).$$

⊙

In this setting, it is easy to define the Hecke action. Take finite $v \notin \mathrm{Ram}(B)$, and recall

$$\underline{G}_L(F_v) = \mathrm{GSp}_{2g}(F_v).$$

These two groups will have the same (local) Hecke algebra $\mathcal{H}_v$. To get the global Hecke algebra, one combines all the $\mathcal{H}_v$ (for finite $v \notin \mathrm{Ram}(B)$)

$$\mathbb{T} := \bigotimes_{v \notin \mathrm{Ram}(B)} \mathcal{H}_v.$$

**Jacquet-Langlands** We won't describe this correspondence in full generality. In the particular case we are working in, this will tell us that $M_{\underline{G}_L}(U)$ is isomorphic to a certain subspace of Hilbert-Siegal modular forms as a Hecke module. In particular, if we can compute the Hecke module structure of this f.dim vector space, we will automagically know there is a subspace of Hilbert-Siegal modular forms which have the same Hecke structure.

### 2.4.2 Computing $\mathrm{Cl}_G(L)$ and $\Gamma_G(U)$

Let $\mathfrak{p}$ be a finite prime outside of $\mathrm{Ram}(B)$. Then, there are finitely many left, simple $\mathscr{O}_B$-modules $\mathfrak{P} \subset \mathscr{O}_B$ which contain $\mathfrak{p}$. We fix one such $\mathfrak{P}$.

**Definition 2.4.7.** We say that a lattice $L'$ is a **$\mathfrak{p}$-neighbor** of $L$ if

$$\frac{L}{L \cap L'} \cong \frac{\mathscr{O}_B}{\mathfrak{p}} \cong \frac{L'}{L' \cap L}.$$

The set of $\mathfrak{p}$-neighbours is denoted $\mathcal{N}_{\mathfrak{p}}(L)$. ◇

**Example.** Say $g = 1$, so $\mathrm{GU}_1(B) = B$. Let $L = \mathscr{O}_B$. Fix some finite $v \notin \mathrm{Ram}(B)$. Lassina drew a picture of a hexagonal (in particular $(3 = 2+1)$-regular) tessellation. Apparently to pictures corresponds somehow to the lattices of $\mathrm{GL}_2(\mathbb{Q}_p)$ (for $p = 2$). Somehow this picture also corresponds to $\mathfrak{p}$-neighbors (for $\mathfrak{p} \mid p$?). △

I got distracted, but apparently there's some Hecke operator $T_1(\mathfrak{p})$ acting on $M_{\underline{G}}(V)$ corresponding to the matrix

$$\operatorname{diag}\left(\underbrace{1,\ldots,1}_{g},\underbrace{\pi,\ldots,\pi}_{g}\right).$$

Explicitly, the action is

$$(T_1(\mathfrak{p})f)([L']) := \sum_{[M]\in\mathcal{N}_\mathfrak{p}(L')} f([M]).$$

There are more Hecke operators similarly defined in terms of neighbors. In particular, if we can compute $\mathfrak{p}$-neighbors, then we should be able to compute the Hecke module $M_{\underline{G}}(U)$.

### 2.4.3 Computing $\mathfrak{p}$-neighbors

Recall that our lattice $L$ is integral and maximal. Let

$$\overline{Q} : L/\mathfrak{p}L \times L/\mathfrak{p}L \longrightarrow \mathscr{O}_B/\mathfrak{p} \xrightarrow{\mathrm{Tr}} \mathbb{F}_p$$

be the reduction of $Q$ modulo $\mathfrak{p}$. This is an $\mathbb{F}_p$-symplectic form which is nondegenerate. Let $X \subset L/\mathfrak{p}_L$ be a subspace. Then, we say that $X$ is **isotropic** if

$$Q(X,X) = 0.$$

We call it a **maximal isotropic** (or a **Lagrangian subspace**) if it's maximal w.r.t. inclusion of isotropic subspaces.

**Theorem 2.4.8.** *Let $\mathscr{L}(L/\mathfrak{p})$ be the Grassmannian of Lagrangians. For $M \in \mathrm{gen}_G(L)$, let $X_M$ be the image in $L/\mathfrak{p}L$. Then, the map*

$$\mathcal{N}_\mathfrak{p}(L) \longrightarrow \mathscr{L}(L/\mathfrak{p}L), \;\; M \mapsto X_M$$

*is a bijection.*

# 3 Sol Friedberg: Automorphic Forms and the Langlands Program: A Brief Introduction

## 3.1 Lecture 1 (7/25)

We want to start with this lecture with a big overview of the Langlands program, and then come back to fill in some of the vaguer parts over the next few lectures.

Let $K/\mathbb{Q}$ be a finite Galois extension. We want to understand $\mathrm{Gal}(K/\mathbb{Q})$. For any finite group, a natural way of understanding it is to look at representations

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}(V)$$

(say $V$ complex $n$-dimensional vector space). To such a representation, Artin attached an **Artin $L$-series** (i.e. a Dirichlet series) given by

$$L(s,\rho) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ for } \mathrm{Re}(s) \gg 0.$$

This was defined as follows

- If $p$ is a prime of $\mathbb{Z}$ which is unramified in $\mathscr{O}_K$, then there is a conjugacy class $\mathrm{Fr}_p$ in $\mathrm{Gal}(K/\mathbb{Q})$. We start with
$$\prod_{p \text{ unram}} \det\left(I_V - \rho(\mathrm{Fr}_p)p^{-s}\right)^{-1}$$
Note that each factor above is (the inverse of) a degree $n$ polynomial in $p^{-s}$.
- If $p$ is ramified, replace $V$ by $V^I$ (subspace fixed by inertia), so you get factors which are (inverses of) polynomials of degree $\le n$ in $p^{-s}$.

(I think Sol said this converges for $\mathrm{Re}(s) > 1$)

**Example.** Say $K = \mathbb{Q}$. Then, $\rho$ is trivial (and, say, 1-dimensional?), and $L(s,\rho) = \zeta(s)$ is the Riemann zeta function. This has a meromorphic continuation to all of $\mathbb{C}$ (simple pole at $s = 1$), and satisfies some functional equation. △

**Example.** Say $K = \mathbb{Q}(i)$. If $p$ is odd, then

$$\mathrm{Fr}_p = \begin{cases} \mathrm{id} & \text{if } -1 \equiv \square \mod p \\ \text{complex conjugation} & \text{otherwise.} \end{cases}$$

Consider the rep $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \mathbb{C}^\times$ sending complex conjugation to $-1$. Then,

$$L(s,\rho) = L(s,\chi) := \sum_{n\ge1} \frac{\chi(n)}{n^s}$$

is a Dirichlet $L$-function attached to $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm1\}$ (the nontrivial map). We know $L(s,\chi)$ has analytic continuation to all of $\mathbb{C}$ (no poles) + a functional equation relating $L(s,\chi)$ to $L(1-s,\overline{\chi})$. △

**Conjecture 3.1.1** (Artin)**.** *The Artin $L$-function $L(s,\rho)$ is actually entire if $\rho$ is irreducible and non-trivial.*

**Theorem 3.1.2** (Brauer)**.** *$L(s,\rho)$ always extends to a meromorphic function on $\mathbb{C}$. Moreover, if we let*

$$\Lambda(s,\rho) = \Gamma(s,\rho)L(s,\rho)$$

*(for suitable $\Gamma$-factor $\Gamma(s,\rho)$), then*

$$\Lambda(s,\rho) = W(\rho)\Lambda(1-s,\overline{\rho}),$$

*where $|W(\rho)| = 1$.*

Above, $\Gamma(s,\rho)$ is a product of $D^{s/2}$ (for some $D$, don't worry about it) as well as factors of

$$\begin{aligned} \Gamma_{\mathbb{R}}(s) &:= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \\ \Gamma_{\mathbb{R}}(s+1)& \\ \Gamma_{\mathbb{C}}(s) &:= 2(2\pi)^{-s}\Gamma(s). \end{aligned}$$

**Theorem 3.1.3** (**Artin reciprocity**)**.** *If $K/\mathbb{Q}$ is abelian and $\rho$ is one-dimensional ($\iff$ irreducible), then there exists a Dirichlet character $\chi$ s.t. $L(s,\rho) = L(s,\chi)$.*

(In particular, Artin's conjecture holds if $K/\mathbb{Q}$ is abelian, by the corresponding fact for Dirichlet characters)

**Question 3.1.4.** *What if $\rho$ is not 1-dimensional? What will Artin $L$-functions correspond to then?*

**Answer** (Langland's idea)**.** If $\rho$ is an irreducible $n$-dimensional Galois representation, then $L(s,\rho)$ should match an $L$-function 'coming from Harmonic analysis on $\mathrm{GL}_n$'. ⋆

We'll say more about what 'harmonic analysis on $\mathrm{GL}_n$' means in the coming lecture. The rough version is that the $L$-function should be attached to a cuspidal automorphic representation on $\mathrm{GL}_n$.

To say a bit about this, let's look at the Hilbert space

$$L^2\left(\mathbb{R}^\times\Gamma\backslash\,\mathrm{GL}_n(\mathbb{R})\right)$$

with $\Gamma \le \mathrm{GL}_n(\mathbb{Z})$ of finite index (so $L^2$-functions invariant under scaling and some finite index subgroup $\Gamma$). Part of the difficulty lies in this invariance (or rather, in the combination of invariance and $L^2$?).

**Example** (Later)**.** We'll see (next time?) that classical modular forms live in this space when $n = 2$. Think that classical (cuspidal) modular forms have a Petersson inner product, so they're kind of $L^2$ type of objects. △

Note that $\mathrm{GL}_n(\mathbb{R})$ acts on this space via the right regular representation, i.e. if $f \in L^2(blah)$, then so is

$$(g\cdot f)(x) := f(xg) \text{ for any } g \in \mathrm{GL}_n(\mathbb{R}).$$

**Definition 3.1.5.** An **automorphic representation** $\pi$ is an irreducible subspace. ⋄

Question: Are we saying subspace instead of subquotient because of

The adjective 'cuspidal' will refer to a growth property for the functions in $\pi$.

**Warning 3.1.6.** The definition of automorphic given above is not the one everyone uses (not necessarily even the one we'll see in other courses here), but we'll say a bit about it's relation to other defintions in later lectures. •

By the theory of automorphic forms, attached to $\pi$ will be some sort of $L$-function $L(s,\pi)$ (see Aaron's lectures). If $\pi$ on $\mathrm{GL}_n$ is cuspidal, then

**(1)** it can be expressed as an Euler product

$$L(s,\pi)=\prod_p \left(\text{degree } \le n \text{ polynomial in } p^{-s}\right)^{-1}$$

At almost all $p$, you get degree $=n$.

**(2)** it has analytic continuation and functional equation. In particular, it is entire.

*Goal* (Langlands' idea).

$$L(s,\rho)=L(s,\pi)$$

where LHS comes from arithmetic, and the RHS comes from harmonic analysis on $\mathrm{GL}_n$.

**Warning 3.1.7.** There are "more $\pi$'s than $\rho$'s." Many $\pi$'s are "transcendental." •

Langlands' vision goes further than this. In particular, he imagines that this sort of matching allows one to learn in both directions (not just from analysis to arithmetic).

**Slogan.** Harmonic analysis on algebraic groups is guided by arithmetic.

There are many natural constructions on the algebraic side. Langlands conjectures that they should extend to all (cuspidal) automorphic representations on the harmonic analysis side.

**Example.** Suppose $V,W$ are two f.dim $\mathbb{C}$-vector spaces, and that we have some homomorphism $\sigma : \mathrm{GL}(V)\to\mathrm{GL}(W)$ (e.g. $\sigma=\mathrm{Sym}^m : \mathrm{GL}(V)\to\mathrm{GL}(\mathrm{Sym}^m V)$). On the Galois side, starting with $\rho:\mathrm{Gal}(K/\mathbb{Q})\to\mathrm{GL}(V)$, we can simply compose

$$\sigma\circ\rho:\mathrm{Gal}(K/\mathbb{Q})\to\mathrm{GL}(W).$$

Langlands predicts a corresponding map

$$\mathcal{A}(\mathrm{GL}_V)\to\mathcal{A}(\mathrm{GL}_W),$$

where $\mathcal{A}$ is the space of cuspidal automorphic representations. That is, we get a relation between Harmonic analysis on two different groups. This map should satisfy (amongst other things)

$$L(s,\sigma\circ\rho)=L(s,\Pi)$$

if $\pi\mapsto\Pi$ and $L(s,\rho)=L(s,\pi)$. △

*Remark* 3.1.8. It is quite difficult to relate harmonic analysis on

$$\mathbb{R}^\times\Gamma\backslash \mathrm{GL}_n(\mathbb{R})$$

($n = \dim V$) to harmonic analysis on

$$\mathbb{R}^\times\Gamma'\backslash \mathrm{GL}_N(\mathbb{R})$$

($N = \dim W$). It's really hard to get the invariance by a discrete subgroup necessary to make these automorphic representations. ○

This sort of functoriality is known in only a few cases.

*Remark* 3.1.9. If we write

$$L(s,\rho) = {\prod_p}' \det\left(I_V - A_p p^{-s}\right)^{-1} \text{ where } A_p = \rho(\mathrm{Fr}_p),$$

(the $'$ on the $\prod$ is just meant to indicate that the factor is different at finitely many primes) then

$$L(s,\sigma\circ\rho) = {\prod_p}' \det\left(I_W - \sigma(A_p)p^{-s}\right)^{-1}. \qquad ○$$

So, Langlands predicts that if $\pi$ is an automorphic rep of $\mathrm{GL}_n$ (possibly transcendental) with

$$L(s,\pi) = {\prod_p}' (I_n - A_p p^{-s})^{-1},$$

then

$$L(s,\pi,\sigma) = {\prod_p}' \left(I_N - \sigma(A_p)p^{-s}\right)^{-1}$$

must have analytic continuation and functional equation (since it should correspond to the $L$-function attached to a cuspidal automorphic representation on $\mathrm{GL}_N$).

**Warning 3.1.10.** Even this weaker statement (AC + FE w/o explicit reference to another automorphic rep) is hard in general. •

If one could prove this is general just for $n = 2$ and $\sigma = \mathrm{Sym}^m$ ($m \geq 2$), then one would prove Ramanujan's conjecture for Maass forms (which is currently open).

*Remark* 3.1.11 (Assuming I heard correctly). Sounds like this is known for classical modular forms. ○

**Example.** Other operations from algebra should have analytic counterparts.

- Given $\rho_i : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}(V_i)$ for $i = 1,\dots,k$, then one can get

  $$\rho_1 \otimes \cdots \otimes \rho_k : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}(V_1 \otimes \cdots \otimes V_k).$$

  Hence, we should get a map

  $$\mathcal{A}(\mathrm{GL}(V_1)) \times \ldots \times \mathcal{A}(\mathrm{GL}(V_k)) \longrightarrow \mathcal{A}(\mathrm{GL}(V_1 \otimes \cdots \otimes V_k)).$$

  Even controlling and $L$-function like $L(s,\pi_1 \otimes \cdots \otimes \pi_k)$ is difficult, especially if $k > 2$.

- Can do induction and restriction of representations (needs a tower of field extension). △

**Question 3.1.12** (Audience)**.** *Is the map from automorphic forms to L-functions injective?*

**Answer** (Paraphrase)**.** At least for fixed $\mathrm{GL}_n$, you should think the answer is basically yes. The key phrase here is '(strong) multiplicity one'. ⋆

Let's end with some questions and remarks.

**(1)** What about other algebraic groups? Say $\mathrm{Sp}_{2n}, O_{2n}, U_n, \dots$ Can we do harmonic analysis on them? Also, given $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to G(\mathbb{C})$, what should we expect?

**(2)** There are lots of automorphic representations $\pi$. Which come from arithmetic?

**(3)** What is the relation between classical modular forms and harmonic analysis on $\mathrm{GL}_2$?

**(4)** How do you get $L(s, \pi)$ from $\pi$?

**(5)** If $K$ is a number field, then $K \hookrightarrow \mathbb{A}_K$ is discrete. We'll replace $\mathbb{Z} \subset \mathbb{R}$ by $K \subset \mathbb{A}_K$. In particular, we'll look at

$$L^2\left(\mathbb{A}^\times \mathrm{GL}_n(K) \backslash \mathrm{GL}_n(\mathbb{A}), \omega\right) = \left\{ f : \mathrm{GL}_n(\mathbb{A}_K) \to \mathbb{C} \;\middle|\; \begin{array}{c} f(z\gamma g) = \omega(z) f(g) \\ \displaystyle\int_{\mathbb{A}_K^\times \mathrm{GL}_n(K) \backslash \mathrm{GL}(\mathbb{A}_K)} |f(g)|^2 \, \mathrm{d}g < \infty \end{array} \right\},$$

where $\omega : \mathbb{A}_K^\times \to \mathbb{C}^\times$ is a character.

## 3.2 Lecture 2 (7/26)

The goal for today's talk is to unpack the idea of doing harmonic analysis on $\mathrm{GL}_n$. In particular, we wanna think a bit about what it includes beyond the holomorphic stuff we're used to. Ideally, we can define an automorphic form in a way that makes sense.

### 3.2.1 Maass forms (weight 0)

**Definition 3.2.1.** The **(hyperbolic?) Laplacian** is the differential operator

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

on the upper half plane. ⋄

**Fact.** $(\Delta f)|_\gamma = \Delta(f|_\gamma)$ where

$$f|_\gamma(z) = f\left(\frac{az+b}{cz+d}\right) \text{ if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

**Definition 3.2.2.** $f : \mathbb{H} \to \mathbb{C}$ is a **Maass form** (of weight 0, level $\Gamma_0(N)$, and character $\chi$) if it is a smooth function such that

**(1)** $f|_\gamma(z) = \chi(d) f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ for some Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$

**(2)** $\Delta f = \lambda f$ (i.e. $f$ is an eigenvalue for $\Delta$)

**(3)** Growth property (cusp forms have a stronger growth property) $\diamond$

"It took Maass to take us out of the ghetto of holomorphic functions" - Weil (but he said it in French)

*Remark* 3.2.3.

**(1)** Using Stoke's theorem, one can show that $\Delta$ is a positive definite operator, so any eigenvalue $\lambda$ is $\geq 0$

**(2)** Selberg conjectured that $\lambda \geq 1/4$ for $f$ on $\Gamma_0(N)$ and any $N$.

**(3)** For $N = 1$, these are all expected to be transcendental.

If $\lambda = \frac{1}{4} + r^2$, for $N = 1$, the first $r \approx 9.5337\dots$. Sounds like Selberg's conjecture is true for $N = 1$ (i.e. $r$ always real when $N = 1$).

**(4)** $\#\{\lambda \leq X\} \sim cX$ for some $c \neq 0$ as $X \to \infty$.

In particular, there are infinitely many Maass forms of a fixed level and weight, by letting $\lambda$ vary. $\circ$

**Fourier Expansion** Write $\lambda = \frac{1}{4} + r^2$. Note that a Maass form $f(z)$ is still invariant under $z \mapsto z+1$, so it will have a Fourier expansion

$$f(z) = a(0,y) + \sum_{n \neq 0} a_n \sqrt{y} K_{ir}(2\pi |n| y) e^{2\pi i n x}$$

with the particular form above derived using it's growth condition and that it's a $\Delta$-eigenvector. Above, $K$ is the $K$**-Bessel function**

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-y(t+t^{-1})} t^s \frac{\mathrm{d}t}{t}$$

(note that $K_s = K_{-s}$).

Maass forms can be split into the **even** ones where $a_n = a_{-n}$ and the **odd** ones where $a_n = -a_{-n}$.

**Question 3.2.4** (Audience)**.** *For each $\lambda$, how many Maass forms do you get?*

**Answer.** It's conjectured that you get only one (up to scaling), but this is still open. We do know they form a f.dim space. $\star$

(Sounds like this conjecture is for any fixed level $N$)

**AC + FE** We'll look at the case $N = 1$ to simplify things a bit. Let $\varepsilon = 0$ if $f$ is an even Mass form, and let $\varepsilon = -1$ if $f$ is odd. The **completed $L$-function attached to a Maass form** is

$$\Lambda(s, f) = \pi^{-s} \Gamma\left(\frac{s + \varepsilon + ir}{2}\right) \Gamma\left(\frac{s + \varepsilon - ir}{2}\right) \sum_{n \geq 1} a_n n^{-s}.$$

**Theorem 3.2.5.** *$\Lambda(s, f)$ has an analytic continuation to $s \in \mathbb{C}$ and satisfies*

$$\Lambda(s, f) = (-1)^{\varepsilon}\Lambda(1-s, f).$$

*Remark* 3.2.6. If $\lambda > \frac{1}{4}$, then $r$ is real. Then these $\Gamma$-factors involve a shift by something pure imaginary. This is not something you typically see in the world of arithmetic/algebraic geometry. Hence, we expect these to be transcendental objects. ◦

**Open Question 3.2.7.** *If $f$ is a normalized ($a_1 = 1$) Maass cusp form (with $\lambda > 1/4$), is $a_n$ transcendental for some (all?) $n > 1$.*

*Remark* 3.2.8. If $\lambda = 1/4$, then $r = 0$. In this case, the $\Gamma$ factors match those of a 2-dimensional Galois representation $\rho$ with

$$\det \rho(\text{cx conj}) = +1$$

(i.e. of **even** Galois reps). ◦

**Conjecture 3.2.9.** *If $f(z)$ is a Maass form with $\lambda = \frac{1}{4}$, then $a_n \in \overline{\mathbb{Q}}$.*

**Slogan.** Whenever the $\Gamma$-factors look like they come from arithmetic, then we *expect* to be in the world of automorphic forms which are arithmetic/cohomological.

**Question 3.2.10** (Audience)**.** *Can you get this $\Lambda(s, f)$ as some sort of transform of the Maass form?*

**Answer.** Yes. When it's even, a Mellin transform still works. When it's odd, you do something else, but things still work. ⋆

### 3.2.2 From $\mathbb{H}$ to $\mathrm{GL}_2^+(\mathbb{R})$

Given $f$ either a holomorphic modular form or a Maass form of weight $k$, can construct an associated function

$$F : \mathrm{GL}_2^+(\mathbb{R}) \longrightarrow \mathbb{C}$$

s.t.

**(1)** $F(\gamma g) = \chi(d)F(g)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

**(2)** $F(g\kappa_\theta) = e^{2\pi i k\theta}F(g)$ for all

$$\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}),$$

where we recall that $k$ is the weight of $f$.

*Remark* 3.2.11. In particular, the weight shows up in the right action of $\mathrm{SO}_2(\mathbb{R})$, a maximal compact subgroup of $\mathrm{GL}_2^+(\mathbb{R})$. ◦

**(3)** $F\left(\begin{pmatrix} a & \\ & a \end{pmatrix} g\right) = \omega(a)F(g)$ where $\omega(a) = \mathrm{sign}(a)^k$.

(See section 5.1.2 for some more details on constructing $F$).

Can also move $\Delta$ to (functions on) $\mathrm{GL}_2^+(\mathbb{R})$. It can be described using the universal envoloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = \mathrm{Lie}(\mathrm{GL}_2^+(\mathbb{R}))$. More precisely, it corresponds to the Casimir operator in the center $Z(U(\mathfrak{g}))$ of the universal envoloping algebra.

### 3.2.3 From $\mathrm{GL}_2^+(\mathbb{R})$ to $\mathrm{GL}_2(\mathbb{A})$

We again follow the recipe from Aaron's lecture. In this way, we create a function

$$\varphi : \mathrm{GL}_2(\mathbb{A}) \longrightarrow \mathbb{C}$$

using $F$.

*Remark* 3.2.12. That fact that you can do this is related to strong approximation. ○

We get (from $f$ on $\Gamma_0(N)$)

**(1)** $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_2(\mathbb{Q})$

*Remark* 3.2.13. Have left-invariance under a discrete group. ○

**(2)** Let

$$K_0(N) := \prod_{p<\infty} K_0(N)_p \text{ where } K_0(N)_p := \begin{cases} \mathrm{GL}_2(\mathbb{Z}_p) & \text{if } p \nmid N \\ \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Z}_p) & \text{if } p \mid N \end{cases}$$

That is $K_0(N)_p$ is a maximal compact subgroup at almost all primes, but we tweak it at the primes dividing the level. Then,

$$\varphi(gk) = \chi(k)\varphi(g) \text{ for all } k \in K_0(N) \times \mathrm{SO}_2(\mathbb{R}),$$

where $\chi$ is some character of $K_0(N) \times \mathrm{SO}_2(\mathbb{R})$

*Remark* 3.2.14. The level and the weight have both moved to the right, to the way the function behaves under a compact subgroup. In particular, the weight is reflected in the action of $\mathrm{SO}_2(\mathbb{R})$, the real maximal compact subgroup. Note that maximal compacts in other groups are non-abelian, so the 'weights' for them can be higher dimensional representations. ○

**(3)** $\varphi$ is $Z(U(\mathfrak{g}))$-**finite** under right action, i.e. the $Z(U(\mathfrak{g}))$-translates of $\varphi$ span a f.dim vector space.

*Remark* 3.2.15. This captures the differential equation satisfied by $f$ (Cauchy-Riemann or being a Laplace eigenvector) ○

**(4)**

$$\varphi\left(\begin{pmatrix} a & \\ & a \end{pmatrix} g\right) = \omega(a)\varphi(g)$$

(the character $\omega$ here comes from $\chi$ from **(2)**)

*Remark* 3.2.16. I think this $\omega$ is related to the $\chi$ in Definition 3.2.2. ○

**(5)** Growth condition

**Definition 3.2.17.** Let $F$ be a number field (or any global field?) with adeles $\mathbb{A}$. An **automorphic form** on $G(\mathbb{A})$ ($G$ an algebraic group) is a function

$$\varphi : G(\mathbb{A}) \longrightarrow \mathbb{C}$$

satisfying

**(1)** $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(F)$

**(2)** $\varphi$ is **right $K$-finite** where $K$ is a maximal compact subgroup of $G(\mathbb{A})$, i.e. the right $K$-translates of $\varphi$ span a f.dim vector space.

**(3)** $\varphi$ is right $Z(U(\mathfrak{g}))$-finite ($\mathfrak{g} = \operatorname{Lie} G$)

**(4)** $\varphi(zg) = \omega(z)\varphi(g)$ for all $z \in Z(G(\mathbb{A}))$ ($\omega$ some character)

**(5)** Growth condition ◇

**Recall 3.2.18.** We had an alternative version last time where we looked at $L^2(G(\mathbb{F})\backslash G(\mathbb{A}), \omega)$. ⊙

Let's consider its cuspidal subspace $L_0^2(G(F)\backslash G(\mathbb{A}), \omega)$. For $G = \mathrm{GL}_n$, the cuspidal condition is

$$\int \varphi\left(\begin{pmatrix} I_r & X \\ 0 & I_{n-r} \end{pmatrix} g\right) \mathrm{d}X = 0 \text{ for } r = 1, \ldots, n-1$$

(for any $g$?).

Let $\rho$ be the right regular representation of $G(\mathbb{A})$ on $L_0^2$.

**Theorem 3.2.19.** $L_0^2(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}), \omega)$ *is a Hilbert direct sum of irreducible $\rho$-invariant subspaces.*

For $\mathrm{GL}_2$, these spaces are generated by Maass forms and holomorphic cusp forms which are Hecke eigenforms.

## 3.3 Lecture 3 (7/27)

Let's continue thinking about automorphic forms. Recall that if we have some $\varphi \in \mathcal{A}(\mathrm{GL}_n)$, then it is, among other things, $K$-finite for $K = \prod_p \mathrm{GL}_n(\mathbb{Z}_p) \times \mathrm{SO}_n(\mathbb{R})$ a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{A})$.

**Definition 3.3.1.** $\varphi$ being **$K$-finite** means that

$$\dim_{\mathbb{C}} \operatorname{span}_{k \in K} \{g \mapsto \varphi(gk)\} < \infty.$$ ◇

How is this possible?

- For almost all $p$, $\varphi$ is fixed by $\mathrm{GL}_n(\mathbb{Z}_p)$.
- For the remaining finite primes $p$, $\varphi$ is fixed by a congruence subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ (which, in particular, has finite index).

  *Remark* 3.3.2. Note $K$-finiteness captures level information. ○

- For the archimedean place, $K_\infty := \mathrm{SO}_n(\mathbb{R})$ acts by a representation of a compact Lie group (note: irreps of compact Lie groups are finite dimensional).

*Idea* (Profound idea). Create a vector space of $\varphi$'s with a group action.

For the group action, try right translation:

$$g \mapsto \varphi(gh) \text{ with } h \in \mathrm{GL}_n(\mathbb{A}).$$

This works in the $L^2$-world, but for automorphic forms, things are more subtle.

- If $p < \infty$ and $h \in \mathrm{GL}_n(\mathbb{Z}_p)$, get something $K$-finite.
- If $p < \infty$ in general, $\varphi$ will be fixed by some $K_1$. Our translate will be fixed by

  $$h^{-1}K_1 h \cap K_1,$$

  which is still a finite index subgroup in $\mathrm{GL}_n(\mathbb{Z}_p)$. This still preserves $K$-finiteness.
- At an archimedean place, we run into trouble.

  **Example.**
  $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathrm{SO}_2(\mathbb{R}) \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cap \mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right\}. \qquad \triangle$$

In summary, we get an action by $\mathrm{GL}_n(\mathbb{A}_f)$, but one does not get a $\mathrm{GL}_n(\mathbb{A})$-action on the space of automorphic forms (you would over a function field, but not over a number field). Instead, at the archimedean places, we introduce a different structure, that of a $(\mathfrak{g}, K_\infty)$-module, where $\mathfrak{g} = \mathrm{Lie}\,\mathrm{GL}_n(\mathbb{R})$.

**Definition 3.3.3.** An **automorphic representation** is a subquotient of $\mathcal{A}(G)$ which carries this structure.[7] $\diamond$

**Definition 3.3.4.** An **automorphic representation in $L^2$-sense** is an (irreducible) subspace of

$$L^2_0(\mathbb{A}^\times \mathrm{GL}_n(\mathbb{Q}) \backslash \mathrm{GL}_n(\mathbb{A}_\mathbb{Q}), \omega)$$

under the right regular representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$. $\diamond$

(only considering cuspidal representations above)

**Theorem 3.3.5.** *Let $(\pi, V)$ be an automorphic representation in $L^2$-sense. Let*

$$V^\infty = \{\varphi \in V : \varphi \text{ is } K\text{-finite}\}.$$

*Then, $V^\infty$ is an automorphic representation.*

*Remark* 3.3.6. Key advantage if $\pi$ is an irreducible automorphic rep of $\mathrm{GL}_n(\mathbb{A})$, then

$$\pi = \bigotimes_v{}' \pi_v,$$

where $\pi_p$ is a representation of $\mathrm{GL}_n(\mathbb{Q}_p)$, and $\pi_\infty$ is a $(\mathfrak{g}, K_\infty)$-module. Almost all $\pi_p$ are unramified. $\circ$

[7] A $G(\mathbb{A}_f)$-action and a $(\mathfrak{g}, K_\infty)$-module structure

*Remark* 3.3.7.

- Local Langlands correspondence is roughly a parametrization of representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ via Galois theory.
- Given $\pi$, get $\{\pi_v\}$, but what about turning it around? The converse is hard, i.e. given $\{\pi_v\}$, it is difficult to show there is an automorphic representation $\pi \cong \bigotimes' \pi_v$.

  Sounds like we know much about the LLC at almost all places, and one of the big obstructions to deducing information about GLC is putting local information together. ○

Last time Aaron explained some aspects of Hecke theory. If $\pi$ is an (irred) automorphic representation of $G(\mathbb{A})$, then at unramified places, one can attach a semisimple conjugacy class $C_p$ (think: diagonal matrix) in a group $\widehat{G}(\mathbb{C})$, the Langlands dual group.

**Example.** For $G = \mathrm{GL}_n$, $\widehat{G}(\mathbb{C}) = \mathrm{GL}_n(\mathbb{C})$. △

**Example.** We want to show that $G$ and $\widehat{G}$ are not always the same. Start with a modular form $f \in S_k(\gamma_0(N), 1)$ with trivial Dirichlet character. If $p \nmid N$, its local $L$-factor classically is

$$\left(1 - a_p p^{-s} + p^{k-1-2s}\right)^{-1}.$$

Translating to the automorphic normalization, this becomes

$$L(s, (\pi_f)_p) = \left(1 - \frac{a_p}{p^{\frac{k-1}{2}}} p^{-s} + p^{-2s}\right)^{-1} = \left(1 - \alpha_p p^{-s}\right)^{-1}\left(1 - \beta_p p^{-s}\right)^{-1}$$

($\pi_f$ is the automorphic representation attached to $f$). Note above that $\alpha_p \beta_p = 1$. Here, the diagonal matrix/semisimple conjugacy class is

Question: The one generated by $\varphi_f$?

Answer: yes

$$C_p = \begin{pmatrix} \alpha_p & \\ & \beta_p \end{pmatrix}.$$

Note here that

$$\pi_f \leftrightarrow L_0^2(\mathbb{A}^\times \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A})),$$

i.e. $\pi_f$ is something on $\mathrm{PGL}_2$. But Hecke theory gave $C_p \in \mathrm{SL}_2(\mathbb{C})$. Note $\widehat{\mathrm{PGL}_2} = \mathrm{SL}_2(\mathbb{C})$. △

If $\pi$ is automorphic on $G(\mathbb{A})$, then Hecke theory will give $C_p \in \widehat{G}(\mathbb{C})$ for unramified $p$. See Table 1 for examples of groups and their duals. Note there is a Lie theoretic description for getting from a group to its Langlands dual.

Let's continue with Langlands idea. Suppose we have $\rho : \mathrm{Gal}(K/\mathbb{Q}) \to \widehat{G}(\mathbb{C})$. Then, there is an automorphic representation $\pi$ (sounds like $\pi$ should be cuspidal if $\rho$ doesn't contain the trivial rep) on $G(\mathbb{A})$ s.t.

$$L(s, \rho) = L(s, \pi, \mathtt{Std})$$

(Standard representation I guess makes sense for one of these classical matrix groups).

One consequence of this is what's called *Endoscopic lifting*. Suppose we have

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{C}) \subset \mathrm{GL}_{2n}(\mathbb{C}).$$

| $G$ | $\widehat{G}(\mathbb{C})$ |
|---|---|
| $\mathrm{GL}_n$ | $\mathrm{GL}_n(\mathbb{C})$ |
| $\mathrm{PGL}_n$ | $\mathrm{SL}_n(\mathbb{C})$ |
| $\mathrm{SL}_n$ | $\mathrm{PGL}_n(\mathbb{C})$ |
| $\mathrm{SO}_{2n+1}$ | $\mathrm{Sp}_{2n}(\mathbb{C})$ |
| $\mathrm{Sp}_{2n}$ | $\mathrm{SO}_{2n+1}(\mathbb{C})$ |
| $\mathrm{SO}_{2n}$ | $\mathrm{SO}_{2n}(\mathbb{C})$ |

Table 1: Some groups and their Langlands duals

Then, $\rho$ should have two attached automorphic representations for different groups! It should give an automorphic representation $\pi \subset \mathcal{A}(\mathrm{SO}_{2n+1})$ as well as a $\pi' \subset \mathcal{A}(\mathrm{GL}_{2n})$. For both of these, we should have

$$L(s, \pi, \mathtt{Std}) = L(s, \rho) = L(s, \pi', \mathtt{Std}).$$

**Slogan.** If you can do it on the Galois side, you should be able to do it in general.

This suggests that there is a lifting

$$\mathcal{A}(\mathrm{SO}_{2n+1}) \longrightarrow \mathcal{A}(\mathrm{GL}_{2n})$$

which preserves (standard) $L$-functions. This map is called an **endoscopic lift**.

*Remark* 3.3.8. Locally, it's easy to start with $\pi$ and write down $(\pi_1)_v$ for almost all $v$ (where $\pi \mapsto \pi_1$ under this lifting). This is because we know almost all the local $L$-factors (or something). The hard part here is to show that $\bigotimes' (\pi_1)_v$ (including the bad places) is automorphic. ○

Such a lift has been proven to exist by Arthur using the trace formula. There is an alternative approach using $L$-functions and the Rankin-Selberg method (involving work of Cogdell-P.S.-Kim-Shakidi (spelling?) and Cai-F.-Givzburg-Kaylan (spelling?)).

### 3.4 Lecture 4 (7/28)

*Note* 2. Last lecture. Sol has to take a taxi to the airport at the end, so at most 1 question when it's over.

Let's continue our discussion of endoscopy.

**Recall 3.4.1.** If $\pi = \bigotimes' \pi_v \in \mathcal{A}(G)$ is an irreducible automorphic representation of $G$, then at places $p$ s.t. $\pi_p$ is unramified, Hecke theory gives an assignment $\pi_p \rightsquigarrow C_p$, a semi-simple conjugacy class in $\widehat{G}(\mathbb{C})$. Recall Table 1 (and also Table 2).

We saw last time that Langlands used this to predict a map (coming from $\widehat{\mathrm{SO}_{2n+1}} = \mathrm{Sp}_{2n} \hookrightarrow \mathrm{GL}_{2n}$)

$$\mathcal{A}(\mathrm{SO}_{2n+1}) \to \mathcal{A}(\mathrm{GL}_{2n}).$$

This exists, but there's no known simple way to product this map. ⊙

Similarly, other maps (*functorial lifts*), e.g.

$$\mathcal{A}(\mathrm{SO}_{2n}) \longrightarrow \mathcal{A}(\mathrm{GL}_{2n})$$

$$\mathcal{A}(\mathrm{Sp}_{2n}) \longrightarrow \mathcal{A}(\mathrm{GL}_{2n+1})$$

**Question 3.4.2.** *Given $\pi \in \mathcal{A}(\mathrm{GL}_{2n})$, how can one tell if is a lift e.g. from $\mathrm{SO}_{2n+1}$ or $\mathrm{SO}_{2n}$?*

The answer will be to look at Langlands $L$-functions.

**Warning 3.4.3.** We're implicitly thinking of $\mathcal{A}(-)$ as the set of (irreducible) automorphic representations, not as the space of forms. We're implicitly identifying a form with the representation it generates. •

Consider

$$\bigwedge^2 : \mathrm{GL}_N(\mathbb{C}) \longrightarrow \mathrm{GL}_{\binom{N}{2}}(\mathbb{C}).$$

Note that if $N = 2$, then $\bigwedge^2 = \det$.

**Example.** With a suitable choice of bases

$$\bigwedge^2 \begin{pmatrix} \alpha & & & \\ & \beta & & \\ & & \gamma & \\ & & & \delta \end{pmatrix} = \begin{pmatrix} \alpha\beta & & & & & \\ & \alpha\gamma & & & & \\ & & \alpha\delta & & & \\ & & & \beta\gamma & & \\ & & & & \beta\delta & \\ & & & & & \gamma\delta \end{pmatrix}$$

△

Given $\pi = \bigotimes' \pi_v \in \mathcal{A}(\mathrm{GL}_N)$, the **partial $\Lambda^2$ $L$-function** is

$$L\left(s, \pi, \Lambda^2\right) = \prod_{\text{unram } p} L_p\left(s, \pi_p, \bigwedge^2\right),$$

where

$$L_p\left(s, \pi_p, \bigwedge^2\right) = \det\left(I_{\binom{N}{2}} - \bigwedge^2 (C_p) p^{-s}\right)^{-1}$$

(see also Construction 5.2.5).

Let's suppose that $\pi \in \mathcal{A}(SO_{2n+1})$ comes from a Galois representation

$$\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \widehat{\mathrm{SO}_{2n+1}}(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{C}).$$

Then, we can form

$$\bigwedge^2 \circ \rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow \mathrm{GL}_N(\mathbb{C}) \text{ where } N = \binom{2n}{2}.$$

By linear algebra, this composition has a fixed vector[8] (it's not irreducible). That is, it contains the trivial representation, so

$$L\left(s, \bigwedge^2 \circ \rho\right) = \zeta(s) \cdot (\text{remaining part}),$$

[8]If $n = 1$, then $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$, so the composition itself is trivial.

with Riemann zeta coming from the trivial rep. Hence, $L(s, \bigwedge^2 \circ \rho)$ will have a pole at $s = 1$ (you have to be a little careful to show that the pole doesn't get cancelled out by the remaining side). Langlands tells us that what's true on the Galois side ought to be true in general.

**Heuristic 3.4.4.** *We predict $\pi \in \mathcal{A}(\mathrm{GL}_{2n})$ should be a lift from $\mathcal{A}(\mathrm{SO}_{2n+1})$ iff $L(s, \pi, \bigwedge^2)$ has a pole at $s = 1$.*

Question: Is it clear that if $\bigwedge^2 \circ \rho$ has a copy of the trivial representation, then $\rho$ must factor through $\mathrm{Sp}_{2n}$?

**Heuristic 3.4.5** (by a similar argument). *$\pi \in \mathcal{A}(\mathrm{GL}_{2n})$ is a lift from $\mathcal{A}(\mathrm{SO}_{2n})$ iff $L(s, \pi, \mathrm{Sym}^2)$ has a pole at $s = 1$.*

### 3.4.1 General Langlands functoriality

(for split reductive algebraic groups)

**Conjecture 3.4.6** (**Langlands functoriality**, Version 1). *If $\varphi : \widehat{H}_1(\mathbb{C}) \to \widehat{H}_2(\mathbb{C})$ is a complex analytic homomorphism of dual groups, then there exists a map*

$$\begin{array}{ccc} \mathcal{A}(H_1) & \longrightarrow & \mathcal{A}(H_2) \\ \pi & \longmapsto & \Pi \end{array}$$

*such that if $\pi_p \leftrightarrow C_p$, then $\Pi_p \leftrightarrow \varphi(C_p)$.*

(we're ignoring both the archimedean and the ramified places in the above conjecture)

This conjecture is based on the usual heuristic of looking at the Galois side, where you can simply compose

$$\mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{\rho} \widehat{H}_1(\mathbb{C}) \xrightarrow{\varphi} \widehat{H}_2(\mathbb{C}).$$

**Warning 3.4.7.** Conjecture 3.4.6 is mostly unproved. •

**Question 3.4.8.** *Can we include Artin $L$-functions in the picture directly?*

The answer is yes, and we'll do it in such a way that gives Langlands original conjecture as a special case of functoriality.

**Answer.** The **$L$-group** of a split reductive algebraic group $G/F$ over a number field $F$ is

$${}^L G = \widehat{G}(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F).$$

*Remark* 3.4.9. For non-split groups, this direct product becomes a semi-direct product, and it's a little more technical to define. ◦

Given an unramified $\pi_p$, we associate to it the pair $(C_p, \mathrm{Fr}_p)$, a conjugacy class in ${}^L G$.

**Definition 3.4.10.** An $L$-homomorphism $\varphi : {}^L H_1 \to {}^L H_2$ is a continuous homomorphism (of topological groups) such that

$$\begin{array}{ccc} \widehat{H}_1(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) & \longrightarrow & \widehat{H}_2(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F) \\ & \searrow \quad \swarrow & \\ & \mathrm{Gal}(\overline{F}/F) & \end{array}$$

commutes, and such that it restricts to a complex analytic homomorphism

$$\widehat{H}_1(\mathbb{C}) \times e \longrightarrow \widehat{H}_2(\mathbb{C}) \times e.$$

$\diamond$

(Note above that $H_1, H_2$ are defined over the same ground field)

**Conjecture 3.4.11** (**Laglands functoriality**, Version 2)**.** *If $\varphi : {}^L H_1 \to {}^L H_2$ is an L-homomorphism, then there is a corresponding map*

$$\mathcal{A}(H_1) \longrightarrow \mathcal{A}(H_2)$$

*(usually cuspidal goes to cuspidal, but not always). Note this is required to respect semisimple conjugacy classes in the expected way.*

**Example.** Take $H_1 = \{e\}$ so ${}^L H_1 = e \times \mathrm{Gal}(\overline{F}/F)$, and let $H_2 = \mathrm{GL}(V)$. Then, an $L$-homomorphism is a continuous map

$$\mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_n(\mathbb{C}) \times \mathrm{Gal}(\overline{F}/F)$$

(whose composition with projection onto $\mathrm{Gal}(\overline{F}/F)$ is the identity). Since it is continuous, it will factor through $\mathrm{Gal}(K/F)$ for $K/F$ some finite Galois extension (by the 'no small subgroup argument'). Thus, such a map is equivalent to the data of a representation

$$\rho : \mathrm{Gal}(K/F) \longrightarrow \mathrm{GL}_n(\mathbb{C}) \text{ where } n = \dim_{\mathbb{C}} V$$

(+ the choice of finite Galois extension $K/F$). Lifting the trivial function in $\mathcal{A}(H_1) = \mathcal{A}(e)$, Conjecture 3.4.11 predicts an automorphic representation $\pi \in \mathcal{A}(\mathrm{GL}_n)$ with

$$C_p = \rho(\mathrm{Fr}_p)$$

for almost all $p$. That is, we predict $L(s, \pi, \mathtt{Std}) = L(s, \rho)$. $\triangle$

From the above example, we see that this more general version of Langlands functoriality recovers the correspondence between Galois reps and automorphic reps we began with. $\star$

# 4 Elena Mantovan: Introduction to Shimura Varieties

## 4.1 Lecture 1 (7/25)

Today we'll focus on Shimura curves (and their theory of canonical models?).

**Proposition 4.1.1.** *Let $X$ be a complex manifold. Let $\Gamma \curvearrowright X$ be a free action by a discrete group. Then, the quotient $\Gamma\backslash X$ has the structure of a complex manifold.*

**Notation 4.1.2.** We let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ denote the **upper half plane**. This has an action by $\mathrm{SL}_2(\mathbb{R})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

*Remark* 4.1.3. $\operatorname{Im} z > 0$ and $\det \gamma > 0 \implies \operatorname{Im}(\gamma z) > 0$ (while $\det \gamma < 0 \implies \operatorname{Im}(\gamma z) < 0$ if $\operatorname{Im} z > 0$). ○

**Fact.** $\mathrm{Hol}(\mathbb{H}) \simeq \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$

To apply this proposition from complex geometry, we need a discrete group. First think about

$$\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{SL}_2(\mathbb{R}).$$

Note that $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$ is discrete, as is any finite index $\Gamma \subset \Gamma(1)$. We'll focus on the examples of

$$\Gamma(m) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})) = \{A \in \mathrm{SL}_2(\mathbb{Z}) : A \equiv I \pmod m\}.$$

**Fact.** $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices $S, T$ where

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $S^2 = -I$ has finite order while $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Furthermore,

$$ST = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

has order 6 (in particular, $(ST)^3 = -I$). For the action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{H}$, one has

$$\mathrm{Stab}_{\Gamma(1)}(z) = \begin{cases} \{\pm I\} & \text{if } z \notin \Gamma(1)i \cup \Gamma(1)\rho \\ \langle S \rangle & \text{if } z \in \Gamma(1)i \\ \langle ST \rangle & \text{if } z \in \Gamma(1)\rho \end{cases}$$

where $\rho = e^{2\pi i/3}$.

**Corollary 4.1.4.** *For $m \geq 3$, the group $\Gamma(m)$ acts freely on $\mathbb{H}$. Hence,*

$$Y(M)^{an} = \Gamma(m)\backslash\mathbb{H}$$

*is a complex manifold.*

If $m = 2$, still have $\pm 1$ acting everywhere, but that's not too bad (imagine acting by the image of $\Gamma(2)$ is $\mathrm{PSL}_2(\mathbb{Z})$ instead), so $Y(2)^{\mathrm{an}} = \Gamma(2)\backslash\mathbb{H}$ also has a complex structure. If $m = 1$, one has to do more work, but can still give $Y(1)^{\mathrm{an}} := \Gamma(1)\backslash\mathbb{H}$ the structure of a complex manifold.

*Exercise.* If $m \mid n$, we get an inclusion $\Gamma(n) \subset \Gamma(m)$ and a corresponding holomorphic map

$$Y(n)^{\mathrm{an}} \twoheadrightarrow Y(m)^{\mathrm{an}}.$$

**Question 4.1.5.** *Are these algebraic?*

**Fact.** Every compact Riemann surface is algebraic.

Hence, the easiest thing to do would be to compactify these. Let $\mathbb{D}^* := \mathbb{D} \sqcup \mathbb{P}^1(\mathbb{Q})$ which inherits an action of $\mathrm{SL}_2(\mathbb{Z})$. The quotient

$$X(m)^{\mathrm{an}} := \Gamma(m)\backslash\mathbb{D}^*$$

is now a compact Riemann surface (= algebraic curve over $\mathbb{C}$) which contains $Y(m)^{\mathrm{an}}$ as an open.

**Question 4.1.6.** *Does $X(m)^{an}$ have a model over a number field?*

Answering this will be related to the theory of modular forms. Say $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $2k$ and level $\Gamma = \Gamma(m)$. In particular,

$$f(\gamma z) = (cz + d)^{2k} f(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. If you fix $k, \Gamma$ and take a bunch of modular forms $f_0, \dots, f_N$, they will define a map

$$[f_0, \dots, f_N] : Y(m)^{\mathrm{an}} \longrightarrow \mathbb{P}^N_{\mathbb{C}}.$$

Hence, if you have some notion of algebracity for modular forms, you'll get some algebraicity result for $Y(m)^{\mathrm{an}}$. Note that if $k \gg 0$, you can get a projective embedding like this.

*Remark* 4.1.7.

$$T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in \Gamma(m).$$

Hence, letting $q = q_m = e^{2\pi i z/m}$, any modular form $f$ for $\Gamma(m)$ has a Fourier expansion $f(q) \in \mathbb{C}[\![q]\!]$. We will say that $f$ is **algebraic** if $f(q) \in \overline{\mathbb{Q}}[\![q]\!]$. ○

**Fact.** For cusp forms of level $\Gamma(m)$ and weight $k \gg 0$, there exists a basis $f_i$ with $f_i \in \mathbb{Q}(\zeta_m)[\![q]\!]$ for all $i$. This gives a (canonical) model of $Y(m)$ defined over $\mathbb{Q}(\zeta_m)$.

Note we have done all this so far without needing to mention the connection to moduli of elliptic curves.

**Example.** Say $m = 1$ and $2k = 12$. We get an embedding $Y(1)^{\mathrm{an}} \to \mathbb{P}^1_{\mathbb{C}}$ via $\left[E_4^3, E_6^2\right]$. These Eisenstein series have rational coefficients, so $X(1) \cong \mathbb{P}^1_{\mathbb{Q}}$. Note that we have a rational parameter given by $j = E_4^3/(4E_4^3 - 27E_6^2)$. △

I guess you know this is $\mathbb{P}^1_{\mathbb{Q}}$ (i.e. that it has a rational point) since the cusp is

Let's try and consider other discrete groups acting on $\mathbb{H}$. These will need to be related to $\mathrm{SL}_2(\mathbb{R})$ is some way.

**Example.** Say $B/\mathbb{Q}$ is a quaternion algebra which is indefinite at $\infty$ (i.e. $B$ splits over $\mathbb{R}$).[9] Attached to this is an algebraic group $G^B/\mathbb{Q}$ defined by

$$G^B(A)\left\{\gamma \in (B \otimes_{\mathbb{Q}} A)^\times : \mathrm{Nm}(\gamma) = 1\right\}$$

for $A$ any $\mathbb{Q}$-algebra. This is an inner form of $\mathrm{SL}_2$ (which is split over $\mathbb{R}$). Let $\mathscr{O}_B \subset B$ be a maximal order. This gives an integral model $\underline{G}^{\mathscr{O}_B}$ defined similarly. In particular,

$$\underline{G}^{\mathscr{O}_B}(\mathbb{Z}) = \mathscr{O}_B^{\mathrm{Nm}=1} = \Gamma_B(1) \supset \Gamma_B(m).$$

This give other discrete groups acting on $\mathbb{H}$ (via $\Gamma_B(1) \subset G(\mathbb{R}) \simeq \mathrm{SL}_2(\mathbb{R})$). For $m \gg 0$, the action will be free, and if $B$ is not split over $\mathbb{Q}$, then the quotient $X_B(m) := \widetilde{\Gamma}(m)\backslash\mathbb{H}$ will already be compact (and so is obviously an algebraic curve over $\mathbb{C}$). △

If I heard correctly, for these, the best way to get nice models over number fields is to go through to modular interpretation.

### 4.1.1 modular interpretation

We start by looking at $Y(1)^{\mathrm{an}} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. For any $z \in \mathbb{H}$, one can consider the lattice $\Lambda_z = \langle z, 1\rangle \subset \mathbb{C}$. This gives an elliptic curve $E_z = \mathbb{C}/\Lambda_z$. This correspondence actually gives an interpretation of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ as the (coarse) moduli space of elliptic curves $E/\mathbb{C}$.

If you were looking at $Y(m)^{\mathrm{an}}$ instead, this would parameterize elliptic curves $E_z$ with a fixed isomorphism $E_z[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$. This isn't quite right (I believe since we really only want symplectic isomorphisms $E_z[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$ and not just any isos).

The point of this modular interpretation is that if we let $Y(1)$ be the moduli space of elliptic curves over $\mathbb{Q}$, this it gives a rational model for $Y(1)^{\mathrm{an}}$. You can similarly define a $Y(m)/\mathbb{Q}$ as the moduli space of elliptic curves with full level $m$-structure (i.e. with iso $\alpha_m : (\mathbb{Z}/m\mathbb{Z})^2 \cong E[m]$).

**Warning 4.1.8.** $Y(m)_{\mathbb{C}} \ncong Y(m)^{\mathrm{an}}$ since the former space is not connected. This is because elliptic curves have a Weil pairing

$$E[m] \times E[m] \longrightarrow \mu_m,$$

so one gets a decomposition

$$Y(m)_{\mathbb{Q}(\zeta_m)} = \bigsqcup_{\zeta} Y(m)^{\zeta}$$

with $\zeta$ ranging over primitive $m$th roots of unity. Each $Y(m)^\zeta$ above is geometrically connected, and is isomorphic to $Y(m)^{\mathrm{an}}$ over $\mathbb{C}$. •

**Theorem 4.1.9** (Shimura (+ Taniyama?))**.** *Let $A/\mathbb{C}$ be a simple abelian variety w/ CM by $(K, \Phi)$ with $K$ a CM field and $K \subset \mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$. If $\dim A = g$, we require $[K : \mathbb{Q}] = 2g$. Then, $A$ can be defined over $\overline{\mathbb{Q}}$.*

Question: What's $\Phi$?

[9] We want (inner) forms of $\mathrm{SL}_2(\mathbb{R})$

(You can be more precise with the field of definition. Maybe the right answer is $A$ defined over the Hilbert class field of $K$?)

*Remark* 4.1.10. An elliptic curve $E_z$ (corresponding to $z \in \mathbb{H}$) is CM iff $z$ satisfies some quadratic equation (one can check this explicitly), i.e. $[\mathbb{Q}(z) : \mathbb{Q}] = 2$. ○

Shimura suggested calling a modular form $f$ **algebraic** if $f(\tau) \in \overline{\mathbb{Q}}$ for all quadratic $\tau \in \mathbb{H}$. This definition turns out to be equivalent to the one given earlier, but no longer uses the $q$-expansion. Hence, it's a definition which can be adapted to other settings (e.g. to the algebraic curves coming from quaternion algebras).

**Example.** The $X_B(m)$ from before roughly parameterize abelian surfaces with action by $\mathscr{O}_B$.[10] △

What if we want more quotients? Note we got discrete subgroups via a chain like

$$\underline{G}^{\mathscr{O}_B}(\mathbb{Z}) \subset G_B(\mathbb{Q}) \subset G_B(\mathbb{R}) \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{R}).$$

However, we don't really need the last map to be an isomorphism. We only need a surjective with compact kernel.

**Example.** Say $B/F$ a quaternion algebra over a totally real field $F$. Say that $B$ is indefinite over one place $\tau_1 : F \hookrightarrow \mathbb{R}$ (this will give a copy of $\mathrm{SL}_2(\mathbb{R})$), but is definite at all other real places $\tau_i : F \hookrightarrow \mathbb{R}$ (get something compact here) for $i \geq 2$. Note that we will get a decomposition

$$G_B(\mathbb{R}) = \prod_{\tau_i : F \hookrightarrow \mathbb{R}} G_B(\mathbb{R})_{\tau_i}$$

coming from the decomposition $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\tau_i : F \hookrightarrow \mathbb{R}} B \otimes_F \mathbb{R}$. △

*Remark* 4.1.11 (**Albert's classification**).

$$\mathrm{End}^0(A) = \begin{cases} \text{totally real field} & \text{if} \\ \begin{array}{l} B/F_0 \text{ quaternion algebra over totally real field which is} \\ \text{indefinite at all real places or definite at all of them} \end{array} & \text{if} \\ B/F \text{ division algebra over CM field} & \text{otherwise.} \end{cases}$$

○

I think maybe Elena said something about the moduli interpretation of the curves arising from the previous example, but if she did, I did not follow. Based on audience questions, it sounds like the point was that the example given before will give some compact Riemann surface, but it won't be a moduli space of abelian varieties (since to $B$ in that example don't show up as endomorphism algebras of abelian varieties). I think she said something about Shimura getting around this issue somehow (do you get moduli space of certain complex torii?), but I didn't quite hear it.

## 4.2 Lecture 2 (7/27)

*Note* 3. Roughly 9 minutes late

Missed what appears to be some amount of recap of last time.

[10] It seems the correspondence here sends $z \in \mathbb{H}$ to the quotient of $\mathbb{C}^2$ by the $\mathscr{O}_B$-lattice generated by $(z, 1) \in \mathbb{C}^2$

### 4.2.1 Unitary Shimura Curves

Consider SU(1,1) as a real Lie group. There is an isomorphism $\mathrm{SU}(1,1) \simeq \mathrm{SL}_2(\mathbb{R})$. This map is

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$\alpha = x + iy$$
$$\beta = z + iw$$
$$a = x + w$$
$$b = z - y$$
$$c = x + y$$
$$d = x - w$$

Fix $E/\mathbb{Q}$ a quadratic imaginary field (e.g. $E = \mathbb{Q}(i)$). Let $V$ be a 2-dimensional $E$-vector space. Consider a Hermitian form $(-,-)$ on $V$. If you choose a nice base, can take

$$(x, y) = x_1\overline{y}_1 - x_2\overline{y}_2.$$

This leads to the group $\mathrm{SU}(V, (\ ,\ ))$.

Can also put a **skew-Hermitian**[11] form on $V$ by taking something like $\mathrm{Tr}_{E/\mathbb{Q}}(i(x,y))$, this satisfies $\langle bx, y\rangle = \langle x, b^*y\rangle$. Can replace $i$ with any $\beta \in E$ for which $\beta^* = -\beta$. Let $\Lambda \subset V$ be an $\mathscr{O}_E$-lattice (so $\Lambda \cong \mathscr{O}_E^2$). From this, can get an integral model $\underline{G}_E$ of $G_E = \mathrm{SU}(V, (\ ,\ ))$.

Now say $F$ is a CM field, so have $F/F_0$ quadratic imaginary extension of a totally real $F_0$. Let $*$ be complex conjugation on $F$ (so $F_0$ is the fixed field of $*$). Let $V$ be a 2-dim $F$-vector space. Consider

$$(x, y) = x_1u_1\overline{y}_1 + x_2u_2\overline{y}_2$$

for some $u_1, u_2 \in F_0^\times$. This will be a Hermitian form. We want to set things up so that

$$G_F(\mathbb{R}) \simeq \mathrm{SU}(1,1) \times \prod_{\tau_i : F_0 \hookrightarrow \mathbb{R} | i \geq 2} \mathrm{SU}(2)$$

(right factors are compact, while left factor is $\mathrm{SL}_2(\mathbb{R})$). The condition to achieve this is that

$$\tau_1(u_1u_2) < 0 \ \text{ while } \ \tau_j(u_1u_2) > 0 \ \text{ for } \ j > 1.$$

Let $\Lambda$ be an $\mathscr{O}_F$-lattice in $V$. Choose $u_1, u_1 \in \mathcal{U}_{F_0}^\times$, and let $\beta$ be a generator of the different of $F$ (we'll want certain traces to have integral values). We can use these choices to get an integral model $\underline{G}_F$, and then consider

Question: What is $\mathcal{U}$?

$$\Gamma_F(1) = \underline{G}_F(\mathbb{Z}) \subset \underline{G}_F(\mathbb{Q}) \subset G(\mathbb{R}) \xrightarrow{\pi} \mathrm{SL}_2(\mathbb{R})$$

[11] alternating + $\langle bx, y\rangle = \langle x, b^*y\rangle$

($\pi$ above will have compact kernel). This leads to the construction of quotients

$$\Gamma_F(m)\backslash\mathbb{H}.$$

Sounds like these are compact if $[F:\mathbb{Q}]>2$. They will also have a moduli interpretation.

Say $z\in\mathbb{H}$. We'll want to produce an abelian variety $A_z$ with a polarization $\lambda_z$, and an action $i_z$ of $\mathscr{O}_F$. This will be the moduli problem associated to $\Gamma_F(1)$.[12] Recall we had a 2-dim $F$-vector space $V$ previously fixed (as well as an $\mathscr{O}_F$ lattice $\Lambda$). The abelian variety is

$$A_z=\Lambda\backslash V_{\mathbb{R}}$$

which has real dimension $4d$. Once we give it a complex structure (depending on $z$), it will be a complex torus of dimension $2d$. Note that[13]

$$V_{\mathbb{R}}=V\otimes_{\mathbb{Q}}\mathbb{R}=\prod_{\tau:F_0\hookrightarrow\mathbb{R}}\underbrace{V\otimes_{F_0,\tau}\mathbb{R}}_{V_\tau}.$$

Let $e,f$ be an isotopic basis of $V_{\tau_1}$, so $(e,e)=0=(f,f)$ and $(e,f)\neq 0$.[14] If $z\in\mathbb{H}$, can consider the line

$$\langle e+zf\rangle\subset V_{\tau_1}$$

which is *negative* in the sense that $(\omega_z,\omega_z)<0$ (here, $\omega_z=e+zf$). Write $V_{\tau_1}=\omega_z\oplus\omega_z^{\perp}$. We now want to define a complex structure, which we do via ($a\in\mathbb{C}$)

$$a\cdot_z v:=\begin{cases}\bar{a}v & \text{if } v\in\omega\\ av & \text{if } v\in\omega^{\perp}\\ av & \text{if } v\in V_{\tau_i}\text{ with signature }(2,0)\\ \bar{a}v & \text{if } v\in V_{\tau_i}\text{ with signature }(0,2)\end{cases}$$

These choices force $\langle i\cdot_z x,x\rangle>0$ to be positive definite, so $A_z$ will be an abelian variety, not just a complex torus.

**Notation 4.2.1.** Instead of writing $a\cdot_z v$, we may write $h_z:\mathbb{C}\to\operatorname{End}(V_{\mathbb{R}})$ and then write $h_z(a)v=a\cdot_z v$.

*Remark* 4.2.2. The $\mathscr{O}_F$-action is it's original action on $V/\Lambda$. The polarization (via generalities on complex torii) comes from $\langle h(i)-,-\rangle$ being positive definite. ○

*Remark* 4.2.3. Note that the parameter on the upper half plane is only used to give a choice of complex structure. ○

### 4.2.2 Something

Sounds like if $X$ is a connected symmetric hermitian domain, and $\Gamma$ is a discrete group acting freely ($\Longleftarrow$ $\Gamma$ torsion-free), then $\Gamma\backslash X$ has a unique structure as a q.proj algebraic variety over $\mathbb{C}$. Maybe such

[12] This a PEL-type Shimura variety, P = polarization, E = endormorphism, L = level structure

[13] Also note $F\otimes_{\mathbb{Q}}\mathbb{R}\simeq\prod_\tau\mathbb{C}$

[14] e.g. $e=\begin{pmatrix}1\\1\end{pmatrix}$ and $f=\begin{pmatrix}1\\-1\end{pmatrix}$

$X$ satisfy $\mathrm{Hol}(X)^+ =$ semisimple real Lie group (which acts transitively on $X$)?

**Example.** The **Siegal upper half space**

$$\mathbb{H}_g = \left\{ z \in M_g(\mathbb{C}) : {}^t z = z \text{ and } \operatorname{Im} z > 0 \right\}.$$

This has an action of $\mathrm{Sp}_{2g}(\mathbb{R})$ which looks like an action by linear fraction transformations if you write elements of $\mathrm{Sp}_{2g}(\mathbb{R})$ has $2 \times 2$ block matrices. The quotient is the moduli sace of principally polarized abelian varieties. △

Part of the utility of this is to be able to write $X \simeq G(\mathbb{R})/K_\infty$ for $G(\mathbb{R})$ some real Lie group with $G(\mathbb{R}) \twoheadrightarrow \mathrm{Hol}(X)^+$.

**Example.**

$$\mathrm{GL}_2(\mathbb{R}) / \mathrm{SO}_2(\mathbb{R})\mathbb{R}^\times \xrightarrow{\sim} \mathbb{H}$$

△

We want $G(\mathbb{R})/K_\infty$ to parameterize complex structures $h : \mathbb{C} \to \mathrm{End}(V_\mathbb{R})$ (and something about 'conjugacy classes'?).[15]

In the language of Deligne, a **Shimura data** is a pair $(G/\mathbb{Q}, X)$ with $G$ a reductive algebraic group, and $X$ is a conjugacy class of $h$'s?

Maybe I should read Deligne's "Traveux de Shimura"?

**Recall 4.2.4** (from Aaron's lectures?)**.**

$$\Gamma(1)\backslash\mathbb{H} \simeq \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K,$$

where $K = \mathrm{GL}_2(\widehat{\mathbb{Z}}) \cdot \mathbb{R}^\times \cdot \mathrm{SO}_2(\mathbb{R})$. ⊙

Choose $K \subset G(\mathbb{A}_f)$ compact. Can consider

$$\mathcal{W}_K := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f.$$

These will be q.proj algebraic varieties over $\mathbb{C}$. They will not be connected in general. It will be defined over some number field $E = E(G, X)$.

**Recall 4.2.5.** $Y(m)_\mathbb{C} = \bigsqcup \mathcal{Y}(m)^\gamma$ ⊙

## 4.3 Lecture 3 (7/28)

Let's try today to give an algebraic definition of Shimura varieties. Last time we mentioned Deligne's definition in terms of pairs $(G, X)$ with $G/\mathbb{Q}$ a reductive algebraic group, and $X$ some $G(\mathbb{R})$-conjugacy class of maps $h : \mathbb{S}^1 \to G(\mathbb{R})$.[16] Under certain conditions, $X$ should be a complex manifold. Deline has certain axioms. Under them, for any $K \subset G(\mathbb{A}_f)$ a compact open, the space

$$\mathcal{M}_K := G(\mathbb{A})\backslash X \times G(\mathbb{A}_f)/K$$

[15] Maybe you implicitly fix an embedding $G \hookrightarrow \mathrm{GL}_V$, choose one complex structure $h : \mathbb{C} \to \mathrm{End}(V_\mathbb{R})$, and then look at the orbit of this $h$ under $G(\mathbb{R})$?

[16] I think $\mathbb{S}^1 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m$ is the Deligne torus, and you can think of $h$ as a algebraic group homomorphism $\mathbb{S}^1 \to G_\mathbb{R}$. Alternatively, $\mathbb{S}^1$ is the unit circle and you really want $G(\mathbb{R})$ on the RHS

becomes a q.proj algebraic variety (in a unique way?) defined over a number field $E(G, X)/\mathbb{Q}$ (note $E$ independent of $K$). This perspective makes two aspects of the theory very natural

**(1)** As $K$ varies, these form a projective system. For $K \subset K'$, we get

$$\mathcal{M}_K \xrightarrow{\pi} \mathcal{M}_{K'}.$$

**(2)** This system $\{\mathcal{M}_K\}_K$ has an action of $G(\mathbb{A}_f)$. In particular, for $g \in G(\mathbb{A}_f)$, we get

$$g : \mathcal{M}_K \longrightarrow \mathcal{M}_{gKg^{-1}}$$

(via multiplication on the right). This is compatible with changing level, e.g. have commutative squares

$$\begin{array}{ccc} \mathcal{M}_{K'} & \xrightarrow{g} & \mathcal{M}_{gK'g^{-1}} \\ \pi\downarrow & & \downarrow\pi \\ \mathcal{M}_K & \xrightarrow{g} & \mathcal{M}_{gKg^{-1}}. \end{array}$$

*Remark* 4.3.1. Think of as analogue of Hecke operators, e.g. $T_p \leftrightarrow \begin{pmatrix} p & \\ & 1 \end{pmatrix}$. ○

Note you can get correspondences of the form

$$\mathcal{M}_K \xleftarrow{\pi} \mathcal{M}_{g^{-1}Kg \cap K} \xrightarrow{g} \mathcal{M}_K$$

if you want to think of actions at a single level. However, $G(\mathbb{A}_f)$ really does act on the whole system, so no need to restrict to a single level.

**Theorem 4.3.2** (Deligne?). *If you have a map $(G, X) \xrightarrow{\varphi} (G', X')$ of Shimura data, then you get a corresponding map*

$$\mathcal{M}_K(G, X) \longrightarrow \mathcal{M}_{K'}(G', X')$$

*of systems. $\varphi$ in particular includes the data of a group homomorphism $\varphi : G \to G'$ so for any open compact $K$, you can find some open compact $K' \supset \varphi(K)$ and then get a map as above. Again, best to think of this as a map*

$$\{\mathcal{M}_K(G, X)\} \xrightarrow{f_\varphi} \{\mathcal{M}_{K'}(G, X')\}$$

*of projective systems. Furthermore, if $\varphi$ is injective, then $f_\varphi$ is a closed embedding.*

**Warning 4.3.3.** To have a closed embedding at a single level, you'll need to take $K'$ so that $K = \varphi^{-1}(K')$. •

A **morphism of Shimura data** is a group homomorphism $\varphi : G \to G'$ such that

$$(h : \mathbb{S}^1 \to G(\mathbb{R})) \in X \implies (\mathbb{S}^1 \xrightarrow{h} G(\mathbb{R}) \xrightarrow{\varphi} G'(\mathbb{R})) \in X'.$$

### 4.3.1 Siegel varieties?

Consider Shimura data $(G_0, X_0)$ with $G_0 = \mathrm{GSp}_{2g}(V, \langle\, , \rangle)/\mathbb{Q}$. We can describe the $h \in X_0$ as certain $\mathbb{R}$-algebra maps $h : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(V_{\mathbb{R}})$. To get something satisfying Deligne's axioms (which we have not specified here), we'll need

- $h(\overline{z}) = h(z)^*$ (with $^*$ denoting adjoint, i.e. $\langle fv, w\rangle = \langle v, f^* w\rangle$)
- The induced pairing $\langle h(i)-, -\rangle$ should be positive definite.

This should give polarization showing that you're getting a moduli space of abelian varieties, not just tori

Given such an $h$, we'll get a decomposition $V_{h,\mathbb{C}} = V_h^+ \oplus V_h^-$ (where $z \in \mathbb{C}$ acts as $z$ on $V^+$ and as $\overline{z}$ on $V^-$) with $V^+$ a $g$-dimensional complex vector space.

**Fact.** If $h \sim h'$ (conjugate under $G(\mathbb{R})$) $\iff$ $V_h^+ \cong V_{h'}^+$ as $\mathbb{C}$-vector spaces (i.e. have same $\mathbb{C}$-dimension).

If I am hearing/following correctly, all the $h$'s with the two given properties form one big conjugacy class.

This gives $(G_0, X_0)$. Let $\Lambda \subset V$ be the standard lattice. Then, $\mathcal{M}(G_0, X_0)_K$ will be the moduli space of $g$-dimensional abelian varieties (w/ principal polarization?).

*Note* 4. I keep on getting distracted for a few seconds, and then missing stuff...

Looks like now we want to consider starting with data $F, *, V, \langle -, -\rangle$ with $*$ some sort of involution on $F$ and $\langle a, b\rangle = \mathrm{Tr}_{F/\mathbb{Q}}(\beta(a, b))$ for some $\beta \in F^\times$ with $\beta^* = -\beta$. Note this satisfies $\langle bx, y\rangle = \langle x, b^* y\rangle$. One often denotes the resulting group $G = \mathrm{GU}^F(V, \langle -, -\rangle)$ and calls it a fake unitary group. This group embeds into $G_0$ by forgetting the $F$-action. To define a Shimura data on $G$, consider the $h$ of the form

$$h : \mathbb{C} \longrightarrow \mathrm{End}_{F_{\mathbb{R}}}(V_{\mathbb{R}}) \subset \mathrm{End}(V_{\mathbb{R}})$$

(composition a map of $\mathbb{R}$-algebras) satisfying

- $h(\overline{z}) = h(z)^*$
- $\langle h(i)-, -\rangle$ positive definite.

In this way, we get a natural morphism $(G, X) \hookrightarrow (G_0, X_0)$ of Shimura data. Almost, anyways. We first need to show $X_0$ is a single conjugacy class under $G_0(\mathbb{R})$, I think. Note $h$ factoring through $\mathrm{End}_{F_{\mathbb{R}}}(V_{\mathbb{R}})$ will cause the decomposition $V_{h,\mathbb{C}} = V_h^+ \oplus V_h^-$ to be $F$-linear, i.e. to be a decomp of $(F \otimes \mathbb{C})$-vector spaces. Note LHS has decomposition

$$\prod_{\tau : F \hookrightarrow \mathbb{C}} V_\tau = V_{h,\mathbb{C}}$$

with each factor of the same dimension (call it $n$). One gets

$$V_\tau = (V_\tau^+ \oplus V_\tau^-).$$

Before, we have $V^+$ dual to $V^-$. Here, I think $V_\tau^+$ will be dual to $V_{\overline{\tau}}^-$. Define $f(\tau) = \dim V_\tau^+$. Note the iso. type of $V_{h,\mathbb{C}}$ as an $(F \otimes \mathbb{C})$-vector space is equivalent to the data of the decomp $V^+ = \bigoplus V_\tau^+$ which is equivalent to the data of the partition $f$ (note $g = \sum f(\tau)$). Note $f(\tau) + f(\tau^*) = n$ since $\dim_\tau^- = f(\tau^*)$, I think (I'm getting behind). Something above the pairs $(f(\tau), f(\tau^*))$.

**Fact** (Upshot?). $h \sim h'$ under $G(\mathbb{R}) \iff V_h^+ \cong V_{h'}^+$ as $(F \otimes \mathbb{C})$-modules.

One one unpacks what was supposed to happen above, you can apparently see that $\mathcal{M}_K(G,X)$ is the moduli space of abelian varieties $A$ of dimension $nd$ (where $n = \dim_F X$ and $d = [F_0 : \mathbb{Q}]$) with

- an $\mathscr{O}_F$-action $\mathscr{O}_F \to \mathrm{End}(A)$; and
- $\mathrm{Lie}\, A_{\mathbb{C}} \cong V_X^+$ as $(F \otimes \mathbb{C})$-modules

  Note there's a decomposition $\mathrm{Lie}\, A_{\mathbb{C}} = \bigoplus (\mathrm{Lie}\, A)(\tau)$ (mirroring $F \otimes \mathbb{C} = \prod_\tau F$), and the condition here is equivalent to $\dim(\mathrm{Lie}\, A)(\tau) = f(\tau)$.

  A maybe more algebraic way to write this is to say

  $$\mathrm{char}(b, \mathrm{Lie}\, A) = \prod_\tau (x - \tau(b))^{f(\tau)}$$

  for any $b \in F$. Note that this polynomial will be defined by a number field, generated by all these $\tau(b)$'s, and the variety will be defined over this number field. You can do even better than that if you're more careful (e.g. imagine something like $(x-i)(x+i)$ which is defined over $\mathbb{Q}$, not just $\mathbb{Q}(i)$).

Note you can think of this $G$ naively as $G = \mathrm{GL}_F \cap G_0 \subset G_0$. Let $B$ be a central simple $\mathbb{Q}$-algebra (so $Z(B) = F$). Then can consider

$$G := \mathrm{GL}_B \cap G_0 \subset G_0.$$

Say we have a **positive involution**[17] $*$ on $B$ as well as a free $B$-module $V$ with pairing $\langle -, - \rangle$ s.t. $\langle bx, y \rangle = \langle x, b^* y \rangle$. Consider the $h : \mathbb{C} \to \mathrm{End}_{B_{\mathbb{R}}}(V_{\mathbb{R}})$ with same two properties from before.

*Remark* 4.3.4. Apparently $B$ has a positive involution $\iff$ $V_{\mathbb{R}}$ has a positive definite form (which is required by the second of our two conditions from before). $\circ$

What are the $\mathrm{GU}^B(V, \langle -, - \rangle)$ conjugacy classes of these $h$? As before, $V_{\mathbb{C}} = V^+ \oplus V^-$ as $(B \otimes \mathbb{C})$-modules. The conjugacy classes will correspond to those with $V^+$ lying in a fixed isom class of $(B \otimes \mathbb{C})$-modules. Sounds like you get a moduli space $\mathcal{M}_K(G,X)$ of abelian varieties of dimension $nd$ ($n = \dim_{\mathbb{Q}} V$ and $d = \dim_{\mathbb{Q}} B$) equipped with

- $\mathscr{O}_B \to \mathrm{End}(A)$
- $\mathrm{Lie}\, A_{\mathbb{C}} \cong V_h^+$ as $(B \otimes \mathbb{C})$-modules.

  Something about traces... $\mathrm{Tr}(b, \mathrm{Lie}\, A) = \mathrm{Tr}(b, V^+) \in E$ for all $b \in \mathscr{O}_B$. Note sure if this is a separate, additional condition, or a rephrasing of the above...

Sounds like this sort of construction only gives rise to PEL-type Shimura varieties.

What other groups are there? Fix some $t \in T^*(V)$, and let $G$ be any reductive group sitting inside $G_0$? Consider

$$T^*(V) = \bigoplus_{m,n} V^{\otimes m} \otimes (V^*)^{\otimes n}.$$

[17] i.e. $(xy)^* = y^* x^*$, $x^{**} = x$, and $\mathrm{Tr}_{B_{\mathbb{R}}/\mathbb{R}}(xx^*) > 0$ for all $x \neq 0$

Since $G$ is reductive, it'll turn out that $G = \mathrm{Stab}_G\left(T^*(V)^G\right)$. Something something Hodge type something something. Note that a choice of $b \in \mathrm{End}(V)$ is a choice of $t_b \in V \otimes V^*$. I'm not so sure what's happening right now...

Something about looking at $(F = Z(B), *)$ and breaking into cases

- $F$ totally real: symplectic or orthogonal
- $F$ CM (with $*$ complex conjugation): unitary

## 4.4 Lecture 4 (7/29)

**Recall 4.4.1.** Say $(G, X)$ is a Shimura datum. Then we get a projective system $\{\mathcal{M}_K(G, X)\}_{K \subset G(\mathbb{A}_f)}$ indexed by compact opens. Each variety in this system is quasi-projective and smooth over some number field $E(G, X)/\mathbb{Q}$ independent of $K$. $\odot$

Consider the étale cohomology groups

$$\mathrm{H}^1_{\text{ét}}\left(\mathcal{M}_{K,\overline{E}}, \overline{\mathbb{Q}}_\ell\right).$$

This will live in a system, so we can take a limit

$$\mathrm{H}^i := \varinjlim_K \mathrm{H}^i\left(\mathcal{M}_{K,\overline{E}}, \overline{\mathbb{Q}}_\ell\right).$$

Since $G(\mathbb{A}_f)$ acts naturally on the projective system for the Shimura data, these limit cohomology groups obtain a action of $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{E}/E)$.

*Remark* 4.4.2. $G(\mathbb{A}_f)$ will act smoothly, i.e. everything is fixed by some open compact. In particular, the part of $\mathrm{H}^i$ fixed by $K$ will exactly recover $\mathrm{H}^i_{\text{ét}}(\mathcal{M}_{K,\overline{E}}, \overline{\mathbb{Q}}_\ell)$ apparently. $\circ$

**Assumption.** Assume $\mathcal{M}_K$ is projective for all $K$. In particular, get iso

$$\mathrm{H}^i_{\text{ét}}(\mathcal{M}_{K,\overline{E}}, \overline{\mathbb{Q}}_\ell) \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{H}^i_{\text{sing}}(\mathcal{M}_K, \mathbb{C})$$

This cohomology group having actions by both $G(\mathbb{A}_f)$ and $\mathrm{Gal}(\overline{E}/E)$ give you a way of corresponding some automorphic reps to some Galois reps. That is, one can use this to product something like

$$\left\{\begin{matrix}\text{some algebraic}\\ \text{auto reps of } G\end{matrix}\right\} \rightsquigarrow \left\{\begin{matrix}\text{Some (f.dim, a.e. unram) } \ell\text{-adic}\\ \text{Galois reps}\end{matrix}\right\}.$$

To get such a correspondence, one needs to fix an isomorphism $\widehat{\overline{\mathbb{Q}}_\ell} \simeq \mathbb{C}$ of abstract fields. This correspondence will also satisfy $L(\pi, s) = L(\rho, s)$ if $\pi \mapsto \rho$.

*Remark* 4.4.3. There was something about a 'component at $\infty$' $\mathscr{L}_\omega$ that I missed...

Apparently $\mathscr{L}_\omega$ is some lisse étale sheaf attached to certain data at infinity? $\circ$

To get correspondence, write $\pi = \pi_f \otimes \pi_\infty$ and then consider the Galois rep (the data of $\pi_\infty$ is absorbed by the choice of $\mathscr{L}_\omega$)

$$\mathrm{Hom}_{\mathbb{G}(\mathbb{A}_f)}\left(\pi_f, \mathrm{H}^*(\mathcal{M}(G, X), \mathscr{L}_\omega)\right)$$

*Remark* 4.4.4. Think of $\mathrm{H}^* = \sum(-1)^i \mathrm{H}^i$ in the Grothendieck group of representations ○

This give a Galois rep attached to $\pi$; it may not always be *the* Galois rep attached to $\pi$, but it should often be 'close enough' in some way.

**Question 4.4.5.** *How could you prove something like $L(\pi, s) = L(\rho_\pi, s)$ for a correspondence coming out of this?*

Write $\pi = \bigotimes'_p \pi_p$. Recall each unramified $\pi_p$ gets some s.s. conjugacy class $C_p$ which is meant to be $\rho(\mathrm{Frob}_p)$ (which would give $L_p(\pi_p, s) = L_p(\rho_\pi, s)$ if I'm following), at least if $\rho$ is unramified by $p$. We have unramifiedness almost everywhere on both sides. Using Chebeotarev one the Galois side and strong multiplicity one on the automorphic side, matching these up at the jointly unramified places suffices to win (I think).

**Recall 4.4.6.** $\pi_p$ is **unramified** if $\pi_p^{K_p} \neq 0$, where $K_p = G(\mathbb{Z}_p)$ (choose a nice integral model) is a maximal compact (something something hyperspecial something something). This is equivalent to $\pi^{K_pK^p} \neq 0$ for the global $\pi$ for some $K^p \subset G(\mathbb{A}_f^{\not p})$. ⊙

This suggests that $\rho$ should arise in the cohomology of $\mathcal{M}_K(G, X)$ for some $K$ of the form $K = K_pK^p$. We want the reps in this cohomology group to be unramified, and for this we use a theorem of Deligne.

**Theorem 4.4.7** (Deligne). *If $\mathcal{M}_K(G, X)$ has good reduction at $p$, then $\mathrm{H}^i_{\acute{e}t}(\mathcal{M}_K, \mathscr{L})$ is an unramified Galois representation at $p$.*[18]

This is coming from smooth base change? I can never remember

This still leaves showing that the constructed correspondence satisfies $C_p = \rho(\mathrm{Frob}_p)$ (but at least we now know both sides are unramified at the same places).

### 4.4.1 Something else?

Fix $\dim A = n[F_0 : \mathbb{Q}]$ as well as $\mathscr{O}_F \overset{i}{\hookrightarrow} \mathrm{End}(A)$ satisfying $i(\alpha^*) = i(\alpha)^\dagger$. There was also the trace condition

$$\mathrm{Tr}(b, \mathrm{Lie}\, A) = \mathrm{Tr}(b, V^+) \in E \subset \mathbb{C} \text{ for all } b \in \mathscr{O}_F.$$

This condition is no good in char $p$ for reasons I did not catch. In general, one replaces this with a certain determinant condition:

$$\det(b, \mathrm{Lie}\, A) = \det(b, V^+) \in E[x] \text{ where } b = \sum x_i b_i.$$

This will lead to a model over $\mathscr{O}_{E,(p)}$. We also want to the model to be smooth.

**Theorem 4.4.8.** *Assume $p$ is unramified in $F$, $B$ is split at (all primes above) $p$, the chosen order $\mathscr{O}_B$ is maximal in $B \otimes \mathscr{O}_p$, and for the $\mathscr{O}_B$-lattice $\Lambda$, the restricted pairing*

$$\langle -, - \rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$$

*is perfect at $p$. These conditions are satisfied at all but finitely many primes.*

*Now, if $K = K_pK^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ (with $K_p = \mathrm{Stab}(\Lambda)$), then the constructed model $\mathcal{M}_K/\mathscr{O}_{E,(p)}$ is smooth (note $\mathscr{O}_{E,(p)} \subset \mathscr{O}_{E_p}$).*

[18] Note $\mathrm{H}^i_{\text{ét}}(\mathcal{M}_{\overline{E}}, \overline{\mathbb{Q}}_\ell) \cong \mathrm{H}^i_{\text{ét}}(\mathcal{M}_{\overline{E}_p}, \overline{\mathbb{Q}}_\ell)$

Apparently you should be able to predict from the Langlands correspondence which primes are smooth for your model (I guess by looking at where your rep is unramified). Something something canonical models something something.

*Remark* 4.4.9. Sounds like automorphic forms can be realized in some cohomology groups $\mathrm{H}^0(\mathcal{M}_K, \omega^\kappa)$ with $\omega^\kappa$ a vector bundle depending on a weight $\kappa$. Sounds like this $\omega$ is still the Hodge bundle $\omega = \pi_*\Omega^1_{A/X}$ in the case of PEL-type Shimura varieties. ◦

**Theorem 4.4.10.** *The line bundle* $\det \omega$ *is ample.*

*Remark* 4.4.11. As the above shows, étale cohomology is not the only way to realize automorphic forms algebraically.

You can also, for example, look at étale $p$-adic cohomology (or better yet complete cohomology). Consider a system like $\{\mathcal{M}_{KK^p}\}_{K\subset K_p}$ with $K^p$ fixed, and then look at

$$\varprojlim_m \varinjlim_K \mathrm{H}^i_{\text{ét}}(\mathcal{M}_K, \mathbb{Z}/p^m\mathbb{Z})$$

which has a $G(\mathbb{Q}_p)$-action. ◦

What if we're interested in ramified primes? There should be a local Langlands correspondence where if $\pi \rightsquigarrow \rho$ via GLC, then $\pi_p \rightsquigarrow \rho_p$ via LLC. However, since $\rho$ was defined globally, it's not even clear that $\rho_p$ depends only on $\pi_p$ (as it should if the two correspond via local Langlands). Sounds like studying these sorts of questions leads to the notion of local Shimura varieties.

Say $(G, X)$ is a Shimura datum over a number field, and say $p$ is unramified (in the sense that it satisfies hypotheses of Theorem 4.4.8). Think of $X$ as a set of $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to G_{\mathbb{R}}$. From this, can get data

$$\left(G/\mathbb{Q}_p, \mu_h : \mathbb{G}_m \longrightarrow G_{\overline{\mathbb{Q}}_p}\right)$$

which should be a 'local Shimura datum'. In analogy with what Deligne did, should exact to be able to construct a projective system $G(\mathbb{Q}_p) \curvearrowright \{\mathcal{M}_K\}_{K\subset G(\mathbb{Q}_p)}$ with each space q.proj var. defined over the same finite extension $E(G, \mu_h)/\mathbb{Q}_p$. This sort of idea is exactly how Harris-Taylor proved local Langlands.

**Warning 4.4.12.** Sounds like such spaces are not known to be defined for arbitrary $G$. Harris-Taylor worked with $G = \mathrm{GL}_n/\mathbb{Q}_p$. •

There was more said that I did not write down...

For examples where such spaces have been constructed, look at [Rapoport-Zink] (PEL type), [W. Kim] (Hodge type), [Scholze-Weinstein], [Lubin-Tate], ...

Also, the definition of local Shimura datum we gave is apparently insufficient. Need also some $b \in B(G, \mu)$, but don't ask me what $B$ is. The spaces also inherit an action of some other algebraic group depending on $b$.

Even more stuff I did not write down...

# 5 Aaron Pollack: The Rankin-Selberg method

## 5.1 Lecture 1 (7/25)

*Note* 5.

**(1)** Notes by Aaron on his website.

**(2)** Series of lectures about computing w/ $L$-functions

**(3)** Loosely connected to Friedberg's lectures.

Lecture series is about the *Rankin-Selberg method*. This is related to integral representations of $L$-functions.[19] Suppose $f$ is a modular form (or a generalization). Let $L(f,s)$ be some $L$-function associated to $f$. Typically, we can define $L(f,s)$ for $\mathrm{Re}(s) \gg 0$. However, we may not know if it has a meromorphic continuation (in $s$) to all of $\mathbb{C}$, if it satisfies a functional equation, or of the significance of its special values (values at special points).

**Example.** One expects that $L(f,s)$ having a pole at some $s = s_0 \implies f$ is special in a precise sense. △

We also typically do not know an expression of $L(f,s)$ in terms of 'Fourier coefficients' of $f$.

When the Rankin-Selberg method applies, it can help answer all these questions. We likely won't actually give a precise definition of this method until lecture 2.

### 5.1.1 Hecke's integral

Suppose $f \in S_k(\mathbb{C})$ is a normalized Hecke eigenform (H.E.F) of level one and weight $k$, so

$$f(z) = \sum_{n\geq 1} a_n q^n$$

with $a_n \in \mathbb{C}$ and $q = e^{2\pi i z}$. Note that $a_1 = 1$ since $f$ is normalized.

**Question 5.1.1** (Audience). *Are special values also interesting for transcendental $L$-functions?*

**Answer.** The answer should usually be yes. For example, you can sometimes tell if the $L$-function of an automorphic form on a big group is a lifted from one on a small group by checking if it has certain poles. ⋆

**Example.** The **classical $L$-function associated to a modular form** $f$ is

$$L_{\text{class}}(f,s) := \sum_{n\geq 1} a_n n^{-s}$$

and converges for $\mathrm{Re}(s) \gg 0$. This has an associated completed $L$-function

$$\Lambda_{\text{class}}(f,s) = \Gamma_{\mathbb{C}}(s) L_{\text{class}}(f,s) \text{ where } \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s). \qquad △$$

**Theorem 5.1.2** (Hecke).

[19] Think, for example, writing the $L$-function of a modular form as an integral via taking a Mellin transformation (I think).

*(1) Analytic continuation:* $\Lambda_{\text{class}}(f,s)$ *has analytic continuation to* $s \in \mathbb{C}$

*(2) Functional equation:* $\Lambda_{\text{class}}(f,s) = (-1)^{k/2}\Lambda_{\text{class}}(f,k-s)$

*(note* $k$ *is even since* $f$ *is of level* 1*)*

*(3) Euler product:*

$$L_{\text{class}}(f,s) = \prod_p \left(1 - a_p p^{-s} + p^{k-1-2s}\right)^{-1}.$$

*Remark* 5.1.3. In the theory of automorphic forms, one usually normalizes $L$-functions so as to get a functional equation relating $s \leftrightarrow 1-s$. This is why we call the one above 'classical'. ∘

We will prove Hecke's theorem, but not using the usual proof.[20] We'll see a proof that generalizes better to other situations. The way we have defined things, it would be relative easy to prove **(1)**,**(2)**, but a lot of work to prove **(3)**. More often, you define an $L$-function as an Euler product, and then proving AC + FE is hard.

*Proof of **(1)**,**(2)** of Theorem 5.1.2.* Consider the function[21]

$$I(f,s) = \int_0^\infty f(iy) y^s \frac{\mathrm{d}y}{y}.$$

Since $f(iy) = \sum_{n\geq 1} a_n e^{-2\pi n y}$ is rapidly converging, one can integrate the above term-by-term. First note that

$$\int_0^\infty e^{-2\pi n y} y^s \frac{\mathrm{d}y}{y} \overset{u=2\pi n y}{=} (2\pi n)^{-s} \int_0^\infty e^{-u} u^s \frac{\mathrm{d}u}{u} = n^{-s}(2\pi)^{-s}\Gamma(s).$$

Hence,

$$I(f,s) = \frac{1}{2}\sum_{n\geq 1} a_n n^{-s} \Gamma_{\mathbb{C}}(s) = \frac{1}{2}\Lambda_{\text{class}}(f,s).$$

Since $f$ is a modular form, $f(-1/z) = z^k f(z)$, so $f(i/y) = (-1)^{k/2} y^k f(iy)$. Hence,

$$\int_0^1 y^s f(iy) \frac{\mathrm{d}y}{y} \overset{y=1/y}{=} \int_1^\infty y^{-s} f(i/y) \frac{\mathrm{d}y}{y} = (-1)^{k/2} \int_1^\infty y^{k-s} f(iy) \frac{\mathrm{d}y}{y}.$$

Thus,

$$I(f,s) = \int_1^\infty \left(y^s + (-1)^{k/2} y^{k-s}\right) f(iy) \frac{\mathrm{d}y}{y}.$$

In particular, we have an integral representation starting at $s=1$ instead of $s=0$. Since $f(iy)$ decays $\sim e^{-2\pi y}$ as $y \to \infty$, the above expression actually defined a holomorphic function on all of $\mathbb{C}$ (giving the desired analytic continuation). Furthermore, by staring at this expression, we immediately get a functional equation $I(f,k-s) = (-1)^{k/2} I(f,s)$. These imply the corresponding properties for $\Lambda_{\text{class}}(f,s) = 2I(f,s)$. ■

This wasn't too bad. One way of saying way is that we've proved some function has an analytic continuation and functional equation, but because we haven't shown it has an Euler product, we haven't prove that it's the *right function* to be looking at. We'll obtain the Euler product next time.

[20] I think the phrase 'usual proof' here refers to the usual proof of **(3)** by Hecke theory?
[21] This is the Mellin transform of $f$ if I'm not too mistaken

Thus far, we've assumed we all know what 'Hecke eigenforms' are. Let's actually go ahead and talk about/define these just to make sure we're all on the same page. We'll take a group theoretic approach to this.

### 5.1.2 Group Theory and Hecke Eigenforms

We'll transition to adele groups and representations.

"There are two reasons. These are the definitions I remember, and these are the definitions that help you, in practice, prove things."

(Step 1) Given $f \in S_k(\mathbb{C})$, we will (with some work) product a function on $\mathrm{GL}_2(\mathbb{A})$. We will then define Hecke eigenforms in terms of that associated function on $\mathrm{GL}_2(\mathbb{A})$. We first define a function on positive determinant real matrices, i.e. we define

$$\varphi_f : \mathrm{GL}_2^+(\mathbb{R}) \longrightarrow \mathbb{C}$$

as follows. Let $j(g,z) = cz+d$ if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, we set

$$\varphi_f(g) := \det(g)^{k/2} j(g,i)^{-k} f(g \cdot i) \text{ where } g \cdot i = \frac{ai+b}{ci+d}$$

(note that $g \cdot i \in \mathbb{H}$ since $\det(g) > 0$). The point is that $\varphi_f$ satisfies an easier functional equation (partially because it incorporates the factor of automorphy into the definition):

$$\varphi_f(\gamma g) = \varphi_f(g) \qquad \text{for all } \gamma \in \mathrm{GL}_2^+(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$$
$$\varphi_f\left(\begin{pmatrix} z & \\ & z \end{pmatrix} g\right) = \varphi_f(g) \qquad \text{for all } z \in \mathbb{R}^\times.$$

We want a function of $\mathrm{GL}_2(\mathbb{A})$.

**Proposition 5.1.4.** *The inclusion*

$$\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{GL}_2^+(\mathbb{R}) \hookrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}) / \prod_{p<\infty} \mathrm{GL}_2(\mathbb{Z}_p)$$

*is a bijection.*

Thus, for free, we get a function $\varphi : \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_( \mathbb{A}) \longrightarrow \mathbb{C}$ corresponding to $\varphi_f$ using this proposition. Note that $\varphi$ is right invariant under $\mathrm{GL}_2(\mathbb{Z}_p)$ for all $p$.

We can now define Hecke eigenforms.

*Remark* 5.1.5. $\mathrm{GL}_2(\mathbb{A}_f) \curvearrowright C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ via the right regular representation, i.e.

$$(g \cdot \xi)(x) = \xi(xg)$$

(this is a left action). ∘

**Warning 5.1.6.** We are being a little sloppy by writing $C^\infty$. Usually people look at $L^2$ or at smooth functions with some growth property. We don't wanna get too into the weeds, so we ignore this growth property. For what we want to say (today), it shouldn't make a difference. •

**Definition 5.1.7.** We set

$$\mathcal{H}_p := C_c^\infty\left(\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/\mathrm{GL}_2(\mathbb{Z}_p), \mathbb{C}\right),$$

compactly supported complex valued functions on $\mathrm{GL}_2(\mathbb{Q}_p)$ which are $\mathrm{GL}_2(\mathbb{Z}_p)$-invariant on both sides. If $\eta \in \mathcal{H}_p$ and $\xi$ is a function on $\mathrm{GL}_2(\mathbb{A})$ or $\mathrm{GL}_2(\mathbb{Q}_p)$, then we define

$$(\eta * \xi)(x) = \int_{\mathrm{GL}_2(\mathbb{Q}_p)} \xi(xg)\eta(g)\mathrm{d}g.$$

If $\xi$ is invariant under some discrete subgroup, then this integral will just be a finite sum. We call $\mathcal{H}_p$ the **local Hecke algebra at** $p$. ◇

**Theorem 5.1.8.** *Let $\pi_\varphi$ be the $\mathrm{GL}_2(\mathbb{A}_f)$-submodule of $C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ generated by $\varphi$ (with $\varphi$ coming from a modular form $f$ as before). Then, TFAE*

*(1)* *$\pi_\varphi$ is irreducible as a $\mathrm{GL}_2(\mathbb{A}_f)$-module*

*(2)* *For all finite primes $p$ and all $\eta \in \mathcal{H}_p$, there exists some $\lambda_p(\eta) \in \mathbb{C}$ s.t.*

$$\eta * \varphi = \lambda_p(\eta)\varphi.$$

*If $\varphi$ satisfies these equivalent conditions, we say $\varphi$ (or $f$) is a* ***Hecke eigenform****.*

**Question 5.1.9** (Audience)**.** *What happens if we replace that $\mathrm{GL}_2(\mathbb{Z}_p)$ by a larger or smaller group?*

**Answer.** Everything goes away. $\mathcal{H}_p$ is an algebra under convolution. By taking $\mathrm{GL}_2(\mathbb{Z}_p)$, this algebra is actually commutative. If you take another group, you can lose commutativity and then you wouldn't expect to be able to find such simultaneous eigenvectors. The important thing is that $\mathrm{GL}_2(\mathbb{Z}_p)$ is a maximal compact subgroup. ⋆

*Remark* 5.1.10 (Response to audience question, didn't hear the question)*.* Let $T_p$ be the characteristic polynomial of $\mathrm{GL}_2(\mathbb{Z}_p)\begin{pmatrix} p & \\ & 1\end{pmatrix}\mathrm{GL}_2(\mathbb{Z}_p)$. Then, up to making the appropriate normalization, you get a "commuative square"

$$\begin{array}{ccc} f & \rightsquigarrow & \varphi \\ {\scriptstyle T_p^{\mathrm{class}}}\downarrow & & \downarrow{\scriptstyle T_p} \\ f' & \rightsquigarrow & \varphi' \end{array}$$

This gives relation to classical theory. ○

**Hecke's integral, group-theoretically** Let $\varphi$ be associated to $f$ as above. Consider the integral

$$I(\varphi, s) = \int_{\mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})} |t|^s \varphi\begin{pmatrix} t & \\ & 1\end{pmatrix}\mathrm{d}t$$

(using a Haar measure).

**Proposition 5.1.11.**

$$I(\varphi, s) = I\left(f, s + \frac{k}{2}\right).$$

*Proof.* One can write

$$\mathrm{GL}_1(\mathbb{A}) = \mathrm{GL}_1(\mathbb{Q}) \times \prod_{p<\infty} \mathrm{GL}_1(\mathbb{Z}_p) \times \mathrm{GL}_1^+(\mathbb{R})$$

(using that $\mathbb{Q}$ has class number one). If $t \in \mathrm{GL}_1(\mathbb{A})$, we may write

$$t = t_Q\left(\prod_p t_p\right) t_\infty.$$

Now,

$$\varphi\begin{pmatrix} t & \\ & 1 \end{pmatrix} = \varphi\begin{pmatrix} t_\infty & \\ & 1 \end{pmatrix} = \det\begin{pmatrix} t_\infty & \\ & 1 \end{pmatrix}^{k/2} j\left(\begin{pmatrix} t_\infty & \\ & 1 \end{pmatrix}, i\right)^{-k} f(t_\infty i) = t_\infty^{k/2} f(t_\infty i).$$

The first equality is since $\varphi$ is left invariant under $\mathrm{GL}_1(\mathbb{Q})$ and right invariant under $\mathrm{GL}_2(\mathbb{Z}_p)$ for all $p$ (+ $\mathrm{GL}_1$ being commutative I guess), while the second is by definition. Finally we choose Haar measure $\mathrm{d}t$ whose restriction to $\mathrm{GL}_1^+(\mathbb{R})$ is $\frac{\mathrm{d}t_\infty}{t_\infty}$, and then we win. ■

Note this proposition does not require $f$ to be a Hecke eigenform. That is only needed for the Euler product.

## 5.2 Lecture 2 (7/26)

**Recall 5.2.1** (Automorphic forms, see Definition 3.2.17)**.** Let $G/\mathbb{Q}$ be a split reductive algebraic group.[22] The set of **automorphic forms**

$$\mathcal{A}(G) \subset C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$$

is the set of left $G(\mathbb{Q})$-invariant functions $\varphi : G(\mathbb{A}) \longrightarrow \mathbb{C}$ that are

**(1)** $K = \prod_v K_v$-finite

Here, $K_v \subset G(\mathbb{Q}_v)$ is a maximal compact.

**Question 5.2.2** (Audience)**.** *Should we really allow $K$ to be finite index in a maximal compact?*

**Answer.** Doesn't make a difference. If you're finite for a finite-index subgroup, then you're finite for $K$ itself maximal compact. ⋆

**(2)** $Z(U(\mathfrak{g}))$-finite

**(3)** of moderate growth

The space of **cusp forms** $\mathcal{A}_0(G) \subset \mathcal{A}(G)$ is defined as those with certain Fourier coefficients (= 'constant terms') equal to 0. ⊙

[22]Being split implies that it can be defined over $\mathbb{Z}$

*Remark* 5.2.3. For our purposes in the next couple lectures, can assume $G = \mathrm{GL}_n$ or even $G = \mathrm{GL}_2$. ◦

*Goal.*

**(1)** Define Langlands $L$-functions

**(2)** Hecke's integral:

$$I(f,s) = \int_0^\infty y^s f(iy)\frac{\mathrm{d}y}{y}$$

We proved AC + FE last time. We still need to prove that it has an Euler product.

**Recall 5.2.4.** Associated to a Hecke eigenform $f$, we obtained a function $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$. From this, we wrote down the integral

$$I(\varphi,s) = \int_{\mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})} |t|^s \varphi\begin{pmatrix} t & \\ & 1\end{pmatrix}\mathrm{d}t,$$

and showed that

$$I(\varphi,s) = I(f, s+\frac{k}{2}).$$ ⊙

### 5.2.1 $L$-functions

Let $(\pi, V_\pi) \subset \mathcal{A}_0(G)$ be an irreducible representation (I think $G(\mathbb{A}_f)$-rep, ignoring infinite places).

**Fact.** $\pi = \bigotimes_v{}' \pi_v$, where $\pi_v$ is an irreducible representation of $G(\mathbb{Q}_v)$.

Let $K_p := G(\mathbb{Z}_p)$, and let $V_p$ be the space of $\pi_p$ (for $p < \infty$).

**Fact.**

- $V_p^{K_p} \neq 0$ for almost all $p$. When this happens, we say $\pi_p$ is **unramified**.
- $\dim V_p^{K_p} \leq 1$ for all $p$.

**Fact.** Associated to $G$, there exists a complex Lie group $\widehat{G}(\mathbb{C})$ s.t. associated to an unramified representation $\pi_p$ of $G(\mathbb{Q}_p)$ is a conjugacy class $c_p \subset \widehat{G}(\mathbb{C})$. See Table 2 for some examples.

| $G$ | $\widehat{G}$ |
|---|---|
| $\mathrm{GL}_n$ | $\mathrm{GL}_n(\mathbb{C})$ |
| $\mathrm{Sp}_{2n}$ | $\mathrm{SO}_{2n+1}(\mathbb{C})$ |
| $\mathrm{SO}_{2n+1}$ | $\mathrm{Sp}_{2n}(\mathbb{C})$ |
| $\mathrm{SO}_{2n}$ | $\mathrm{SO}_{2n}(\mathbb{C})$ |

Table 2: A table of some groups with their Langlands duals

This last fact in particular is what we need to define $L$-functions.

*Construction* 5.2.5 (due to Langlands). Suppose $r : \widehat{G}(\mathbb{C}) \longrightarrow \mathrm{GL}_N(\mathbb{C})$ is a representation. Let $\pi = \bigotimes_v{}' \pi_v$ be a cuspidal automorphic representation. We define the **local $L$-function associated to $\pi_p$** to be

$$L(\pi_p, r, s) = \det\left(\mathbf{1}_N - r(c(\pi_p))p^{-s}\right)^{-1}$$

if $\pi_p$ is unramified. Note that this is the reciprocal of a degree $N$ polynomial in $p^{-s}$.

*Warning* 5.2.6. Defining the local $L$-factor for ramified $\pi_p$ is much more difficult. In particular, we only asserted the existence of the associated conjugacy class $c(\pi_p)$ when $\pi_p$ is unramified. •

Suppose $\pi$ is as above, and let $S$ be a finite set of places (including the archimedean places) such that $p \notin S \implies \pi_p$ unramified. We then define the **partial $L$-function associated to** $\pi$ to be

$$L^S(\pi, r, s) := \prod_{p \notin S} L(\pi_p, r, s).$$

This converges to a holomorphic function for $\mathrm{Re}(s) \gg 0$.

**Conjecture 5.2.7.** *$L^S(\pi, r, s)$ has meromorphic continuation and functional equation relating $s \leftrightarrow 1-s$.*

You may worry that we can't product an $L$-function with the right local factors since we never explicitly said how to get the conjugacy classes $c(\pi_p)$.

Suppose $G$ as above and that $(\pi_p, V_p)$ is unramified. Choose some nonzero $v_0 \in V_p^{K_p}$, so $\mathbb{C}v_0 = V_p^{K_p}$. For $v \in V_p$ and

$$\eta \in \mathcal{H}_p = C_c^\infty\left(K_p \backslash G(\mathbb{Q}_p)/K_p\right),$$

we define

$$\eta * v := \int_{G(\mathbb{Q}_p)} \eta(g)\pi_p(g)v\mathrm{d}g.$$

In essence, we've extended the action $\pi_p$ of $G(\mathbb{Q}_p) \curvearrowright V_p$ to one of the group algebra $\mathcal{H}_p \curvearrowright V_p$.

**Claim 5.2.8.** $\eta * v \in V_p^{K_p}$

*Proof.*

$$\begin{aligned}\pi_p(k)\cdot(\eta * v) &= \int_{G(\mathbb{Q}_p)} \eta(g)\pi_p(kg)\cdot v\mathrm{d}g \\ &= \int_{G(\mathbb{Q}_p)} \eta(k^{-1}g)\pi_p(g)\cdot v\mathrm{d}g \\ &= \eta * v\end{aligned}$$

■

Consequently, $\eta * v_0 = \lambda(\eta)v_0$ for some $\lambda(\eta) \in \mathbb{C}$.

**Proposition 5.2.9.** *Let $G$ and $r : \widehat{G}(\mathbb{C}) \longrightarrow \mathrm{GL}_N(\mathbb{C})$ be as above. Then,*

$$\exists! \Delta_r^s \in C^\infty\left(K_p \backslash G(\mathbb{Q}_p)/K_p\right)$$

*s.t. for $\mathrm{Re}(s) \gg_{\pi_p} 0$, one has*

$$\Delta_r^s * v_0 = \int_{G(\mathbb{Q}_p)} \Delta_r^s(g)\pi_p(g)\cdot v_0\mathrm{d}g = L(\pi_p, r, s)v_0$$

*for all unramified $\pi_p$.*

This sorta sounds like some Tate's thesis $L$-function is a gcd of $\zeta$-functions type of thing?

(In particular, you use the same $\Delta_r^s$ for all unramified $\pi_p$)

**Example.** Say $G = \mathrm{GL}_2$ and $r = \mathtt{Std} : \mathrm{GL}_2(\mathbb{C}) \xrightarrow{=} \mathrm{GL}_2(\mathbb{C})$ the standard rep. Then,

Here, $\mathrm{char}(E)$ denote the characteristic function of member-

$$\Delta_r^s(g) = \mathrm{char}\left(g \in M_2(\mathbb{Z}_p)\right) |\det(g)|^{s+\frac{1}{2}}.$$

Set

$$T_{p^n} := \mathrm{char}\left(g \in M_2(\mathbb{Z}_p) : |\det(g)| = |p^n|\right),$$

so $T_{p^n} \in \mathcal{H}_p = C_c^\infty(\mathrm{GL}_2(\mathbb{Z}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)/ \mathrm{GL}_2(\mathbb{Z}_p))$. Furthermore,

$$\Delta_r^s(g) = \sum_{n\geq 0} T_{p^n}(g) \, |p^n|^{s+\frac{1}{2}}.$$

Thus,

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} \Delta_r^s(g)\pi_p(g)v_0 \mathrm{d}g = \sum_{n\geq 0} |p^n|^{s+\frac{1}{2}} (T_{p^n} * v_0) = \sum_{n\geq 0} |p^n|^{s+\frac{1}{2}} \lambda(T_{p^n})v_0 = L(\pi_p, \texttt{Std}, s)v_0. \qquad \triangle$$

### 5.2.2 Hecke's integral again

*Goal.* Suppose $f \in S_k(\mathbb{C})$ is a weight $k$ normalized Hecke eigenform (or even a Maass form if you want, see Definition 3.2.2). Let $\varphi \in C^\infty(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}))$ be the function associated to $f$ as in section 5.1.2. Then,

$$I(\varphi, s) := \int_{\mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})} |t|^s \varphi \begin{pmatrix} t & \\ & 1 \end{pmatrix} \mathrm{d}t = \Lambda(\pi, \texttt{Std}, s + 1/2) = (\Gamma\text{-factors})L(\pi, \texttt{Std}, s + 1/2).$$

Above,

$$L(\pi, \texttt{Std}, s + 1/2) = \prod_p L(\pi_p, \texttt{Std}, s + 1/2).$$

Note that since $f$ is level 1, there are no ramified $p$.

(Step 1) The Fourier Expansion of $\varphi$ adellically.

Let $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ be the **standard additive character**, i.e.

$$\psi(x) = \prod_v \psi_v(x_v),$$

where $\psi_\infty(x_\infty) = e^{2\pi i x_\infty}$. Furthermore, if $x_p = x_p^0 + x_p^1$ with $x_p^0 \in \mathbb{Z}_p$ and $x_p^1 = \frac{m}{p^r}$ ($m \in \mathbb{Z}$), then,

$$\psi_p(x_p) = \psi_p(x_p^1) = e^{-2\pi i m/p^r}.$$

For $\mu \in \mathbb{Q}$, we define the **$\mu$th Fourier coefficient** of $\varphi$ to be

$$\varphi_\mu(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(\mu x)\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x.$$

Then,

$$\varphi(g) = \sum_{\mu\in\mathbb{Q}} \varphi_\mu(g)$$

(we will not prove this).

*Remark* 5.2.10. In the case of $\mathrm{GL}_2$, $\varphi$ cuspidal $\iff$ $\varphi_0(g) = 0$ for all $g$. ○

Define

$$W_\varphi(g) := \varphi_1(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(x)\varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right) \mathrm{d}x,$$

the **Whittaker function of** $\varphi$.

**Lemma 5.2.11.** *If* $\mu \in \mathbb{Q}^\times$, *then*

$$\varphi_\mu(g) = W_\varphi\left(\begin{pmatrix}\mu & \\ & 1\end{pmatrix} g\right).$$

*Proof.* Perform a change of variables:

$$\begin{aligned}
W_\varphi\left(\begin{pmatrix}\mu & \\ & 1\end{pmatrix} g\right) &= \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(x)\varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\mu & \\ & 1\end{pmatrix} g\right) \mathrm{d}x \\
&= \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(x)\varphi\left(\begin{pmatrix}\mu & \\ & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\mu & \\ & 1\end{pmatrix} g\right) \mathrm{d}x \quad \text{by left-invariance of } \varphi \text{ under } \mathrm{GL}_2(\mathbb{Q}) \\
&= \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(x)\varphi\left(\begin{pmatrix}1 & \mu^{-1}x\\ & 1\end{pmatrix} g\right) \mathrm{d}x \\
&= \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(\mu x)\varphi\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix} g\right) \mathrm{d}x \\
&= \varphi_\mu(g)
\end{aligned}$$

■

**Corollary 5.2.12.** *If* $\varphi$ *is a cusp form, then*

$$\varphi(g) = \sum_{\mu\in\mathbb{Q}^\times} \varphi_\mu(g) = \sum_{\mu\in\mathbb{Q}^\times} W_\varphi\left(\begin{pmatrix}\mu & \\ & 1\end{pmatrix} g\right).$$

## 5.3 Lecture 3 (7/28)

**Recall 5.3.1.**

- $f \in S_k(\mathbb{C})$ level one cuspidal Hecke eigenform
- $f \rightsquigarrow \varphi \in \mathcal{A}_0(\mathrm{GL}_2)$, a cuspidal automorphic form on $\mathrm{GL}_2(\mathbb{A})$
- $\varphi$ generates an irreducible representation $\pi = \bigotimes'_v \pi_v$. This is a $\mathrm{GL}_2(\mathbb{A}_f) \times (\mathfrak{gl}_2, O(2))$-module
  $\pi_v$ is a $\mathrm{GL}_2(\mathbb{Q}_v)$-representation
- Let $V_\pi$ denote the space of $\pi$, and $V_p$ the space of $\pi_p$.
- $V_p^{\mathrm{GL}_2(\mathbb{Z}_p)} = \mathbb{C}v_0^p$. $v_0^p$ is called a **spherical vector**.
  ($\pi$ unramified everywhere since $f$ was level one) ⊙

**Question 5.3.2** (Audience)**.** *Is it clear that* $V_p$ *is infinite dimensional?*

**Answer.** It's true and not a difficult fact, but not easy enough to come up with a proof on the spot. ⋆

### 5.3.1 $L$-functions

Recall

$$L(\pi, \mathtt{Std}, s) = \prod_p L(\pi_p, \mathtt{Std}, s).$$

The completed $L$-function is

$$\Lambda(\pi, \mathtt{Std}, s) = \Gamma_{\mathbb{C}}\left(s + \frac{k-1}{2}\right) L(\pi, \mathtt{Std}, s).$$

To access the local $L$-functions, we appeal to

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{char}(g \in M_2(\mathbb{Z}_p)) \left|\det(g)\right|^{s+\frac{1}{2}} \pi_p(g) v_0^p \mathrm{d}g = L(\pi_p, \mathtt{Std}, s) v_0^p.$$

We note that the LHS above is

$$\sum_{n \geq 0} p^{-n\left(s+\frac{1}{2}\right)} T_{p^n} \cdot v_0^p$$

where $T_{p^n} \in \mathcal{H}_p$ is

$$T_{p^n} = \mathrm{char}(g \in M_2(\mathbb{Z}_p) : |\det(g)| = |p^n|).$$

That is, the LHS is a power series in $p^{-s}$ with coefficients that are Hecke eigenvalues of the representation $(\pi_p, V_p)$. The Hecke eigenvalues come up here since $T_{p^n} \cdot v_0^p = \lambda(p^n) v_0^p$.

**Recall 5.3.3.** In classical theory, the coefficients of the $L$-function of a modular form are its $a_p$'s, it's Hecke eigenvalues for $T_p$. ⊙

**Question 5.3.4** (Audience)**.** *Are these Hecke eigenvalues the same ones that appear in the classical situation?*

**Answer.** Up to some normalization factor (possibly depending on $p$), they are. ⋆

Recall that we are trying to evaluate

$$I(\varphi, s) = \int_{\mathrm{GL}_1(\mathbb{Q}) \backslash \mathrm{GL}_1(\mathbb{A})} |t|^s \varphi \begin{pmatrix} t & \\ & 1 \end{pmatrix} \mathrm{d}t.$$

Last time we had started understanding the Fourier expansion of $\varphi$ adelically. We obtained (the *Whittaker expansion of* $\varphi$)

$$\varphi(g) = \sum_{\mu \in \mathbb{Q}^\times} \varphi_\mu(g) = \sum_{\mu \in \mathbb{Q}^\times} W_\varphi\left(\begin{pmatrix} u & \\ & 1 \end{pmatrix} g\right),$$

where

$$\varphi_\mu(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(\mu x) \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \mathrm{d}x \text{ and } W_\varphi(g) = \varphi_1(g).$$

Note $\varphi_0 = 0$ since $\varphi$ is cuspidal. Above, $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^\times$ is the standard additive character from last time.

Now,

$$\begin{aligned}
I(\varphi, s) &= \int_{\mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})} |t|^s \left( \sum_{\mu\in\mathbb{Q}^\times} W_\varphi\left(\begin{pmatrix}\mu & \\ & 1\end{pmatrix}\begin{pmatrix}t & \\ & 1\end{pmatrix}\right)\right) \mathrm{d}t \\
&= \int_{\mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})} \sum_{\mu\in\mathbb{Q}^\times} \left( |\mu t|^s W_\varphi \begin{pmatrix}\mu t & \\ & 1\end{pmatrix}\right) \mathrm{d}t \qquad \text{product formula} \implies |\mu| = 1 \\
&= \int_{\mathrm{GL}_1(\mathbb{A})} |t|^s W_\varphi \begin{pmatrix}t & \\ & 1\end{pmatrix} \mathrm{d}t.
\end{aligned}$$

We've said that we will evaluate Hecke's integral in a non-standard way. What's the standard way?

*Remark* 5.3.5 (standard argument, quickly). First appeal to the following fact

*Fact.* $W_\varphi(g)$ is itself an Euler product, i.e. for all places $v$, there exists $W_v : \mathrm{GL}_2(\mathbb{Q}_v) \to \mathbb{C}$ s.t.

**(1)** $W_\varphi(g \in \mathrm{GL}_2(\mathbb{A})) = \prod_v W_v(g_v)$.

**(2)** In above infinite product, almost all factors are $= 1$.

This will cause the whole integral to split up as a product

$$I(\varphi, s) = \prod_v I_v(s) \text{ where } I_v(s) = \int_{\mathrm{GL}_1(\mathbb{Q}_v)} |t_v|^s W_v \begin{pmatrix}t_v & \\ & 1\end{pmatrix} \mathrm{d}t_v.$$

Note that $W_v$ depends (only) on the representation $\pi_v$ even though this isn't indicated in the notation.

To continue along this line, to evaluate the global $I$ in terms of $L$-functions, one would need to evaluate the $I_v$'s in terms of local $L$-functions. ◦

In what way will we evaluate this integral? For $\mu \in \mathbb{Q}^\times$, consider $L_\mu : V_\pi \to \mathbb{C}$ given as

$$L_\mu(\varphi^v) = \varphi^v_\mu(1) = \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(\mu x)\varphi^v \begin{pmatrix}1 & x \\ 0 & 1\end{pmatrix} \mathrm{d}x,$$

where $v \in V_\pi$, $\varphi^v$ is the automorphic form associated to $v$.

**Warning 5.3.6.** We're really using the smooth notation of automorphic forms in order to be able to evaluate at 1 and get a number. You can't do this with an $L^2$ 'function'. •

Note (recall group element acting by right regular representation)

$$\begin{aligned}
L_\mu\left(\begin{pmatrix}1 & x \\ & 1\end{pmatrix} v\right) &= \left(\begin{pmatrix}1 & x \\ & 1\end{pmatrix}\varphi^v\right)_\mu (1) \\
&= \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(\mu x')\varphi^v\left(\begin{pmatrix}1 & x' \\ & 1\end{pmatrix}\begin{pmatrix}1 & x \\ & 1\end{pmatrix}\right)\mathrm{d}x' \\
&\overset{x' \rightsquigarrow x'-x}{=} \psi(\mu x)\int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(\mu x')\varphi^v \begin{pmatrix}1 & x' \\ & 1\end{pmatrix}\mathrm{d}x' \\
&= \psi(\mu x) L_\mu(v).
\end{aligned}$$

We want to use these linear functionals to rewrite our global integral:

$$I(\varphi, s) = \int_{\mathrm{GL}_1(\mathbb{A})} W_\varphi \begin{pmatrix} t & \\ & 1 \end{pmatrix} |t|^s \,\mathrm{d}t = \int_{\mathrm{GL}_1(\mathbb{A})} L_1 \left( \begin{pmatrix} t & \\ & 1 \end{pmatrix} \varphi \right) |t|^s \,\mathrm{d}t$$

"Here comes the leap of faith"

Imagine we can prove the following proposition.

**Proposition 5.3.7.** *Suppose $\ell : V_p \to \mathbb{C}$ is a linear functional satisfying*

$$\ell \left( \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v \right) = \psi(x)\ell(v) \text{ for all } x \in \mathbb{Q}_p, v \in V_p.$$

*Then,*

$$\int_{\mathrm{GL}_1(\mathbb{Q}_p)} \ell \left( \begin{pmatrix} t & \\ & 1 \end{pmatrix} v_0^p \right) |t|^s \,\mathrm{d}t = L\left(\pi_p, \mathtt{Std}, s + \frac{1}{2}\right) \ell(v_0^p).$$

**Claim 5.3.8.** *Such a proposition will imply*

$$I(\varphi, s) = \frac{a_f(1)}{2} \Lambda \left( \pi, \mathtt{Std}, s + \frac{1}{2} \right)$$

*($a_f(1)$ is the first Fourier coefficient of the original form)*

*Proof.* If $\Omega$ is a finite set of places of $\mathbb{Q}$, let $\mathbb{A}_\Omega = \prod_{v \in \Omega} \mathbb{Q}_v$, and define

$$I(\Omega, \varphi, s) := \int_{\mathrm{GL}_1(\mathbb{A}_\Omega)} |t|^s L_1 \left( \begin{pmatrix} t & \\ & 1 \end{pmatrix} \varphi \right) \mathrm{d}t.$$

Basically by definition,

$$I(\varphi, s) = \varinjlim_{\Omega} I(\Omega, \varphi, s).$$

(Step 1) $I(\{\infty\}, \varphi, s) = \frac{a_f(1)}{2} \Gamma_{\mathbb{C}} \left( s + \frac{k}{2} \right)$.

(Step 2) Use Proposition 5.3.7 to induct on the size of $\Omega$:

$$I(\Omega \cup \{p\}, \varphi, s) = L(\pi_p, \mathtt{Std}, s + 1/2) I(\Omega, \varphi, s).$$

These two steps imply the desired result:

$$I(\varphi, s) = \frac{a_f(1)}{2} \Lambda \left( \pi, \mathtt{Std}, s + \frac{1}{2} \right).$$

■

We will prove the above steps in reverse order (and prove the prop next time).

*Proof of (Step 2) of Claim 5.3.8.* By definition

$$I(\Omega \cup \{p\}, \psi, s) = \int_{\mathrm{GL}_1(\mathbb{A}_\Omega)} |t_\Omega|^s \int_{\mathrm{GL}_1(\mathbb{Q}_p)} |t_p|^s L_1 \left( \begin{pmatrix} t_\Omega & \\ & 1 \end{pmatrix} \begin{pmatrix} t_p & \\ & 1 \end{pmatrix} \varphi \right) \mathrm{d}t_p \mathrm{d}t_\Omega.$$

Define a linear function $\ell = \ell_{t_\Omega} : V_p \to \mathbb{C}$ via

$$\ell(v) := L_1\left(\begin{pmatrix} t_\Omega & \\ & 1 \end{pmatrix} \varphi^v\right).$$

*Remark* 5.3.9. We should say what we mean by this. Given $v \in V_p$, since we are unramified eveywhere, we can form

$$v \otimes \bigotimes_{q \neq p} v_0^q \otimes v^\infty =: v \in V_\pi$$

(accept there's a canonical choice of $v^\infty$). The $\varphi^v$ above is the automorphic form associated to $v$. ◦

Note that this linear functional depends upon $t_\Omega$. Then, we may write

$$\begin{aligned} I(\Omega \cup \{p\}, \psi, s) &= \int_{\mathrm{GL}_1(\mathbb{A}_\Omega)} |t_\Omega|^s \int_{\mathrm{GL}_1(\mathbb{Q}_p)} |t_p|^s L_1\left(\begin{pmatrix} t_\Omega & \\ & 1 \end{pmatrix}\begin{pmatrix} t_p & \\ & 1 \end{pmatrix}\varphi\right) \mathrm{d}t_p \mathrm{d}t_\Omega \\ &= \int_{\mathrm{GL}_1(\mathbb{A})} |t_\Omega|^s \left(\int_{\mathrm{GL}_1(\mathbb{Q})} \ell\left(\begin{pmatrix} t_p & \\ & 1 \end{pmatrix} v_0^p\right) |t_p|^s \mathrm{d}t_p\right) \mathrm{d}t_\Omega \\ &= \int_{\mathrm{GL}_1(\mathbb{A}_\Omega)} |t_\Omega|^s \left(\ell(v_0^p) L(\pi_p, \texttt{Std}, s + 1/2)\right) \mathrm{d}t_\Omega && \text{by Proposition 5.3.7} \\ &= L(\pi_p, \texttt{Std}, s+1/2) \int_{\mathrm{GL}_1(\mathbb{A}_\Omega)} |t_\Omega|^s L_1\left(\begin{pmatrix} t_\Omega & \\ & 1 \end{pmatrix}\varphi\right) \mathrm{d}t_\Omega \\ &= L(\pi_p, \texttt{Std}, s+1/2) I(\Omega, \varphi, s). \end{aligned}$$

■

To get the induction off the ground, we still need to understand the archimedean piece.

*Proof of (Step 1) of Claim 5.3.8.* We'll get the actually use the classical Fourier expansion of $\varphi$ here. Say $g \in \mathrm{GL}_2(\mathbb{R})$. Then,

$$L_1(g\varphi) = \int_{\mathbb{Q}\backslash\mathbb{A}} \psi^{-1}(x)\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \mathrm{d}x = \int_{\mathbb{Z}\backslash\mathbb{R}} e^{-2\pi i x}\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \mathrm{d}x.$$

The last equality above should not be immediately obvious. To see it, first observe that

$$\mathbb{Z}\backslash\mathbb{R} \xrightarrow{\sim} \mathbb{Q}\backslash\mathbb{A}/\prod_p \mathbb{Z}_p.$$

Furthermore, $\varphi$ is right invariant under the matrices

$$\begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \subset \mathrm{GL}_2(\mathbb{Q}_p),$$

which allows you to compute this adelic integral at just the real place. By definition of $\varphi$ in terms of $f$, we have

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & \\ & 1 \end{pmatrix}\right) = t^{k/2} f(x + it) \text{ for } x \in \mathbb{R}, t \in \mathbb{R}_{>0}^\times.$$

This is true since $\varphi$ is of level 1. Sounds like it really only requires $\varphi$ being of level $\Gamma \supset \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}$. In the level one case, recall Proposition 5.1.4

The upshot is that

$$L_1\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\varphi\right) = a_f(1)t^{k/2}e^{-2\pi t} \text{ if } t\in\mathbb{R}^\times_{>0}.$$

*Exercise.* If $t<0$, then $L_1\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}\varphi\right) = 0.$

Thus,

$$I(\{\infty\},\varphi,s) = a_f(1)\int_{\mathbb{R}^\times_{>0}} t^{s+\frac{k}{2}}e^{-2\pi t}\frac{\mathrm{d}t}{t} = \frac{a_f(1)}{2}\Gamma_{\mathbb{C}}\left(s+\frac{k}{2}\right)$$

as desired. ■

## 5.4 Lecture 4 (7/29)

Recall we have been trying to evaluate Hecke's integral

$$I(\varphi,s) = \int_{\mathrm{GL}_1(\mathbb{Q})\backslash\mathrm{GL}_1(\mathbb{A})}\varphi\begin{pmatrix} t & \\ & 1\end{pmatrix}|t|^s\,\mathrm{d}t.$$

We used the functional equation of $\varphi$ to rewrite this as

$$\int_{\mathrm{GL}_1(\mathbb{A})} W_\varphi\begin{pmatrix} t & \\ & 1\end{pmatrix}|t|^s\,\mathrm{d}t.$$

We mentioned (without proof) that this Whittaker function is an Euler product, which can be used to deduce the same for this integral. However, we decided to take a different approach to arrive at this same result.

**Theorem 5.4.1** (Our Goal)**.**

$$I(\varphi,s) = \frac{a_f(1)}{2}\Lambda(\pi,\boldsymbol{Std},s+1/2).$$

Last time we reduced the proof of this statement to the following purely local result.

**Proposition 5.4.2** (Proposition 5.3.7)**.** *Write* $\pi = {\bigotimes_v}' \pi_v$ *for the representation generated by* $\varphi$*, and let* $V_p$ *be the (infinite-dimensional) space associated to* $\pi_p$*. Suppose* $\ell : V_p \to \mathbb{C}$ *is linear and satisfies*

$$\ell\left(\begin{pmatrix} 1 & x\\ & 1\end{pmatrix}v\right) = \psi_p(x)\ell(v) \text{ for all } x\in\mathbb{Q}_p, v\in V_p.$$

*Then,*

$$I_p(\ell,v_0^p,s) := \int_{\mathrm{GL}_1(\mathbb{Q}_p)}\ell\left(\begin{pmatrix} t & \\ & 1\end{pmatrix}v_0^p\right)|t|^s\,\mathrm{d}t = L(\pi_p,\boldsymbol{Std},s+1/2)\ell(v_0^p),$$

*where* $v_0^p$ *is a* ***spherical vector*** *for* $\pi_p$*, i.e.* $\mathbb{C}v_0^p = V_p^{\mathrm{GL}_2(\mathbb{Z}_p)}$.

*Remark* 5.4.3. Note that $v_0^p$ is only well-defined up to scalar multiple. With this in mind, the above formula is at least reasonable since both sides scale when $v_0^p$ scales. ○

**Recall 5.4.4.** We know that

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{char}\,(g \in M_2(\mathbb{Z}_p))\, |\det g|^{s+1}\, \pi_p(g) v_0^p \mathrm{d}g = L\left(\pi_p, \mathtt{Std}, s + \frac{1}{2}\right) v_0^p.$$

Also, the above integral is a power series in $p^{-s}$ with coefficients that are Hecke eigenvalues. ⊙

This implies that

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} \mathrm{char}(g \in M_2(\mathbb{Z}_p))\, |\det(g)|^{s+1}\, \ell(g \cdot v_0^p) \mathrm{d}g = L(\pi_p, \mathtt{Std}, s + 1/2) \ell(v_0^p),$$

which is the RHS of the desired equality in Proposition 5.4.2. Hence, we only need to know that the LHS above is also the LHS of the desired equality in Proposition 5.4.2.

### 5.4.1 Short Digression: Iwasawa decomposition

This is a useful tool for computing integrals over $\mathrm{GL}_2(\mathbb{Q}_p)$ (or other groups). Set

$$N = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Q}_p) \qquad \text{upper triangular unipotent matrices}$$

$$T = \left\{ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \right\} \subset \mathrm{GL}_2(\mathbb{Q}_p) \qquad \text{diagonal matrices}$$

$$K = \mathrm{GL}_2(\mathbb{Z}_p) \subset \mathrm{GL}_2(\mathbb{Q}_p)$$

Then,

$$\mathrm{GL}_2(\mathbb{Q}_p) = NTK.$$

Moreover, if $f \in C^\infty(\mathrm{GL}_2(\mathbb{Q}_p))$ is integrable, then

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} f(g) \mathrm{d}g = \int_T \int_N \int_K f(ntk) \left| \frac{t_1}{t_2} \right|^{-1} \mathrm{d}k \mathrm{d}n \mathrm{d}t.$$

**Question 5.4.5** (Audience)**.** *Does the order of integration matter?*

**Answer.** The factors inside of $f$ must be in that order, e.g. it must be $ntk$, not $tkn$. However, if you keep the insidemost integrand and change the order of the integrals, this will be the same by Fubini. ⋆

**Question 5.4.6** (Audience)**.** *Do we have to be careful about choosing measure in a consistent way?*

**Answer.** Yes. Above, we never specified which Haar measures on each group we're taking. To be more careful, choose

- $\mathrm{d}n$ such that $\begin{pmatrix} 1 & \mathbb{Z}_p \\ & 1 \end{pmatrix}$ has measure 1
- $\mathrm{d}k$ such that $\mathrm{GL}_2(\mathbb{Z}_p)$ has measure 1
- $\mathrm{d}t$ such that $\begin{pmatrix} \mathbb{Z}_p^\times & \\ & \mathbb{Z}_p^\times \end{pmatrix}$ has measure 1

Once these are chosen, $\mathrm{d}g$ is pinned down by the desired equality. ⋆

### 5.4.2 Back to evaluating Hecke's integral

By the above digression, we have

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} \operatorname{char}(g \in M_2(\mathbb{Z}_p)) \left|\det(g)\right|^{s+1} \ell(g \cdot v_0^p) \mathrm{d}g$$

$$\overset{g=ntk}{=} \int_T \int_N \int_K |t_1 t_2|^{s+1} \operatorname{char}\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} k \in M_2(\mathbb{Z}_p)\right) \ell\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} k v_0^p\right) \left|\frac{t_1}{t_2}\right|^{-1} \mathrm{d}k \mathrm{d}x \mathrm{d}t$$

$$= \int_T \int_N \int_K |t_1 t_2|^{s+1} \operatorname{char}\left(\begin{pmatrix} t_1 & x t_2 \\ & t_2 \end{pmatrix} \in M_2(\mathbb{Z}_p)\right) \psi_p(x) \ell\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} v_0^p\right) \left|\frac{t_1}{t_2}\right|^{-1} \mathrm{d}k \mathrm{d}x \mathrm{d}t$$

$$= \int_T \int_N |t_1|^s |t_2|^{s+2} \operatorname{char}\left(\begin{pmatrix} t_1 & x t_2 \\ & t_2 \end{pmatrix} \in M_2(\mathbb{Z}_p)\right) \psi_p(x) \ell\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} v_0^p\right) \left|\frac{t_1}{t_2}\right|^{-1} \mathrm{d}x \mathrm{d}t.$$

Now we're in good shape. The inner integral above is simply (assuming $t_2 \in \mathbb{Z}_p$ b/c of bottom left corner of the matrix)

$$\int_{\mathbb{Q}_p} \operatorname{char}(x t_2 \in \mathbb{Z}_p) \psi_p(x) \mathrm{d}x = \begin{cases} 0 & \text{if } t_2 \notin \mathbb{Z}_p^\times \\ 1 & \text{otherwise.} \end{cases}$$

(usual trick of integrating a character over $t_2^{-1}\mathbb{Z}_p = p^{-v_p(t_2)}\mathbb{Z}_p$. Note $\psi_p|_{\mathbb{Z}_p} = 1$). Plugging this in above, we arrive at

$$\int_{\mathrm{GL}_2(\mathbb{Q}_p)} \operatorname{char}(g \in M_2(\mathbb{Z}_p)) \left|\det(g)\right|^{s+1} \ell(g \cdot v_0^p) \mathrm{d}g$$

$$= \int_{t_1 \in \mathbb{Z}_p} \int_{t_2 \in \mathbb{Z}_p^\times} |t_1|^s |t_2|^{s+1} \ell\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} v_0^p\right) \mathrm{d}t_2 \mathrm{d}t_1$$

$$= \int_{t_1 \in \mathbb{Z}_p} |t_1|^s \ell\left(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix} v_0^p\right) \mathrm{d}t_1 \qquad \text{use } v_0^p \text{ invariant under matrix } \begin{pmatrix} 1 & \\ & t_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$$

$$\overset{?}{=} \int_{t_1 \in \mathbb{Q}_p^\times} |t_1|^s \ell\left(\begin{pmatrix} t_1 & \\ & 1 \end{pmatrix} v_0^p\right) \mathrm{d}t_1$$

$$= I_p(\ell, v_0^p, s).$$

**Lemma 5.4.7.** *If $t \in \mathbb{Q}_p^\times \setminus \mathbb{Z}_p$, then*

$$\ell\left(\begin{pmatrix} t & \\ & 1 \end{pmatrix} v_0^p\right) = 0.$$

*Proof.* See Aaron's notes or work it out yourself. ■

This completes the proof of Proposition 5.4.2 which in turn completes the proof of Theorem 5.1.2 (see Claim 5.3.8). We did this by computing the simplest example of a 'Rankin-Selberg integral'.

### 5.4.3 More Examples of Rankin-Selberg Integrals

Let's suppose $K/\mathbb{Q}$ is a quadratic field, and let $H/\mathbb{Q}$ be the algebraic group with functor of points

$$H(A) = \{(\lambda, g) \in \mathrm{GL}_1(A) \times \mathrm{GL}_2(K \otimes A) : \lambda = \det(g)\}$$

(subset of $\mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_2$ with determinant landing in $\mathbb{G}_m$ (instead of $\mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m$)).

Let $E(h, s)$ be an Eisenstein series on $H(\mathbb{A})$. We won't specify this completely precisely, but we'll say a bit about it.

- $s \in \mathbb{C}$ is a complex variable
-
$$E(h, s) = \sum_{\gamma \in B_H(\mathbb{Q}) \backslash H(\mathbb{Q})} f(\gamma h, s) \text{ where } B_H(\mathbb{Q}) = H \cap \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \right\},$$

  and $f(-, s) : H(\mathbb{A}) \to \mathbb{C}$ is some function which satisfies

$$f\left( \begin{pmatrix} a & b \\ & d \end{pmatrix} h, s \right) = \left| \frac{a}{d} \right|_K^{s/2} f(h, s) \text{ for all } \begin{pmatrix} a & b \\ & d \end{pmatrix} \in H(\mathbb{A}), h \in H(\mathbb{A}).$$

**Proposition 5.4.8.** *$f$ can be chosen so that $E(h, s)$ has meromorphic continuation in $s$ with simple poles at $s = 0, 2$ and so that it satisfies $E(h, s) = E(h, 2 - s)$.*

We'll use this Eisenstein series to write down a couple more integrals which represent $L$-functions. Suppose $\pi \subset \mathcal{A}_0(\mathrm{GL}_2)$ and choose some nonzero $\varphi \in V_\pi$. Consider

$$I(\varphi, s) = \int_{\mathrm{GL}_2(\mathbb{Q}) Z(\mathbb{A}) \backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g) E(g, s) \mathrm{d}g,$$

where we've used the naturally embedding $\mathrm{GL}_{2,\mathbb{Q}} \hookrightarrow H$.

**Theorem 5.4.9.**

$$I(\varphi, s) \approx L^S \left( \pi, \mathtt{Std}, s - \frac{1}{2} \right).$$

*(Have a partial L-function on the RHS)*

(note RHS depends on both $K, E$ in ways which have been absorbed into the $\approx$ and into the unspecified finite set $S$ of places)

*Remark* 5.4.10. The finite set $S$ comes e.g. from ramification in $K$ and from ramification in $\pi$. ◦

**The Rankin-Selberg method, roughly** Suppose $G/\mathbb{Q}$ is a reductive group, and suppose $\pi \subset \mathcal{A}_0(G)$ is a cuspidal automorphic representation. Say $H \subset G$ is an algebraic subgroup. Say we also have $H \subset G'$ along with an Eisenstein series $E(g', s)$ on $G'(\mathbb{A})$.

*Remark* 5.4.11. Frequently in practice, one will take $H = G \subset G'$ or $H = G' \subset G$. ◦

If $\varphi \in V_\pi$, can consider the integral

$$\int_{H(\mathbb{Q}) Z(\mathbb{A}) \backslash H(\mathbb{A})} E(h, s) \varphi(h) \mathrm{d}h =: I(\varphi, s)$$

(Above, $Z = Z(G) \cap H \cap Z(G')$). In very special circumstances, this integral will give some partial Langlands $L$-function

$$I(\varphi, s) \approx L^S(\pi, r, s - s_0)$$

for some $s_0 \in \mathbb{C}$ and some $r : \widehat{G}(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$. When this happens, one calls

$$I(-, s) : \mathcal{A}_0(G) \longrightarrow \mathbb{C}$$

a **Rankin-Selberg integral**.

**Warning 5.4.12.** It's unclear which $L$-functions you can get in this way. It's also hard in general to tell when such an integral is a R-S integral. •

*Remark* 5.4.13. Part of the utility of these things is that, from general facts about Eisenstein series, you know that these integral have meromorphic continuations in $s$ (and so you can get the same conclusion for $L$-series). ○

**Example.** Hecke's integral comes from $G' = H = \mathrm{GL}_1 \subset \mathrm{GL}_2 = G$. △

**Example.** The example above was $H = G = \mathrm{GL}_2 \subset G'$ with $G'$ this subgroup of $\mathrm{Res}_{K/\mathbb{Q}} \mathrm{GL}_2$ (the thing we previously called $H$). △

# 6 List of Marginal Comments

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