AGNES Notes

Niven Achenjang

July 2023

These are notes on talks given in "AGNES Summer School on Intersection Theory on Moduli Spaces" which took place at Brown. Unfortunately for the reader, these notes are live-texed and so their quality is upper bounded both by my (quite) limited ability to understand the material in real time and by my typing speed. With that in mind, they are doubtlessly missing content/insight present in the talks and certainly contain confusions not present in the talks. Despite all this, I hope that you can still find some use of them. Enjoy and happy mathing.

The website for this seminar is available here. The website has lecture videos and exercises.

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1 Hannah Larson: Introduction to the tautological ring of $\overline{M}_{g,n}$

1.1 Lecture 1 (7/11/23): Chow Rings

Let X be a variety.

Definition 1.1.1. The `cycles` on X, $Z(X)$, is the free abelian group on irreducible subvarieties of X, $\mathbb{Z}(X) = \mathbb{Z}\{\text{irred. subvars of } X\}$. ⋄

This group is too big, so we quotient out by rational equivalence. Given a family of subvaratieis $\mathcal{Z} \subset X \times \mathbb{P}^1 \to \mathbb{P}^1$, we say any two fibers are `rationally equivalent`. That is, we define

$$\mathrm{Rat}(X) := \left\{\Phi_0 - \Phi_\infty : \Phi \subset X \times \mathbb{P}^1 \text{ flat over } \mathbb{P}^1\right\}.$$

The `Chow group` is $\mathrm{CH}_*(X) = Z(X)/\,\mathrm{Rat}(X)$, graded by dimension. Given $Y \subset X$ irreducible, we write $[Y] \in \mathrm{CH}_*(X)$ for its class in Chow. Extend this to reducible varieties in the natural way.

Assumption. Now assume that X is a *smooth* variety.

Write $\mathrm{CH}^*(X) = \mathrm{CH}_{\dim X - *}(X)$. This is a graded ring w/ intersection product satisfying

$$A[B] = [A \cap B]$$

if A, B are *generically transverse.*

Definition 1.1.2. Say A, B are `transverse` at a point p if p is a smooth point of both, and $T_pA + T_pB = T_pX$. Equivalently, $\operatorname{codim} T_pA + \operatorname{codim} T_pB = \operatorname{codim} T_p(A \cap B)$. We say A, B are `generically transverse` if they are transverse at a general point in each component of $A \cap B$. ⋄

Let's mention some basic properties of this Chow ring

(1) (`pushforward`) If $f : X \to Y$ is proper, get $f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$ via

$$f_*[A] = (\deg f|_A) \cdot [f(A)] \;\text{ if }\; \dim f(A) = \dim A$$

(otherwise, the pushforward is 0).

Example 1.1.3. When Y is a point, get the degree map $\deg : \mathrm{CH}_*(X) \to \mathbb{Z}$ on 0-cycles. △

(2) (`pullback`) If $f : X \to Y$ is a map of smooth varieties, can define $f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ via $f^*[A] = [f^{-1}(A)]$ when A is generically transverse to f.[1]

(3) (`push-pull formula`)

$$(f_*[A])[B] = f_*([A] \cdot f^*[B]).$$

Slogan. f_* is a $\mathrm{CH}(Y)$-module map (though not a ring map)

Example 1.1.4. Say $\iota : D \subset X$ is a divisor, so $[D] = c_1(\mathscr{O}(D))$. Then,

$$[D]^2 = \iota_*[1] \cdot [D] = \iota_*([1] \cdot \iota^*[D]) = \iota_*\iota^* c_1(\mathscr{O}(D)) = \iota_*(c_1(\mathscr{O}(D))|_D).$$

Note that $\mathscr{O}(D)|_D$ is the normal bundle $N_{D/X}$. △

[1] Sounds like this is always satisfied if f is flat

(4) (excision) Say $Z \subset X$ is closed and $U = X \setminus Z$. Then, we get a right exact sequence

$$\mathrm{CH}_*(Z) \longrightarrow \mathrm{CH}_*(X) \longrightarrow \mathrm{CH}_*(U) \longrightarrow 0.$$

If Z is smooth, we'll sometimes write this as

$$\mathrm{CH}^{*-c}(Z) \longrightarrow \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(U) \longrightarrow 0,$$

where $c = \mathrm{codim}_X(Z)$. The first map is a group homomorphism, but the second is a ring homomorphism.

(5) (homotopy) If $V \to X$ is an affine bundle, then $\mathrm{CH}^*(X) \to \mathrm{CH}^*(V)$ is an isomorphism.

Example 1.1.5. $\mathrm{CH}^*(\mathbb{A}^n) \simeq \mathrm{CH}^*(*) = \mathbb{Z}$. △

Theorem 1.1.6. *If $X = \bigsqcup X_i$ with $X_i \cong \mathbb{A}^{n_i}$ (X stratified by affine spaces), then* $\mathrm{CH}_*(X) = \bigoplus \mathbb{Z} \cdot [\overline{X}_i]$.

(use excision + homotopy)

Example 1.1.7. $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{A}^{n-1} \sqcup \cdots \sqcup \mathbb{A}^1 \sqcup *$. Hence, $\mathrm{CH}_*(\mathbb{P}^n)$ is freely (additively) generated by the fundamental classes of the closures $[\overline{\mathbb{A}^{n-k}}] = [\mathbb{P}^{n-k}]$. What's the ring structure? Write $\zeta = [\mathbb{P}^{n-1}]$. Intersect k transverse hyperplanes, to see that $[\mathbb{P}^{n-k}] = \zeta^k$, $[*] = \zeta^n$ and $0 = \zeta^{n+1}$. Thus,

$$\mathrm{CH}^*(\mathbb{P}^n) = \frac{\mathbb{Z}[\zeta]}{(\zeta^{n+1})}.$$ △

Example 1.1.8. Consider Grassmannian $\mathrm{Gr}(k, n)$. This admits a stratification into Schubert cells Σ_λ, where $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a tuple of integers $\lambda_i \le n - k$. Furthermore, $\mathrm{codim}\, \Sigma_\lambda = \lambda_1 + \cdots + \lambda_k$.

Hannah wrote G. I don't know why I wrote Gr

Recall $\mathrm{Gr}(k, n)$ parameterizes k-dimensional subspaces of an n-dimensional vector space. This admits the following quotient presentation

$$\mathrm{Gr}(k, n) = \{\text{full rank } k \times n \text{ matrix}\} \,/\, \mathrm{GL}_k\,.$$

Each GL_k-orbit has a unique representative in row reduced echelon form, and the Schubert cells correspond to different row reduced echelon forms. Σ_λ corresponds to matrices w/ rref having pivots prescribed by λ.

Example 1.1.9. $\mathrm{Gr}(2, 4)$.

$$\Sigma_{(0,0)} \leftrightarrow \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \cong \mathbb{A}^4$$

$$\Sigma_{(2,1)} \leftrightarrow \begin{pmatrix} 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cong \mathbb{A}^1$$

I missed how to go from λ to the description in terms of matrices... △

Each Schubert cell is an affine space, to the classes of their closures give a basis for the Chow groups of Grassmannians. What about ring structure? Something something Schubert calculus something something.

In general, can at least give a set of multiplicative generators. The special Schubert classes are those where the partition λ has just one part, $\Sigma_p = \overline{\Sigma_{(p,0,\ldots,0)}}$. Can think of these as $\{\Lambda : \Lambda \cap F_{n-k-p+1} \neq 0\}$ (F_{blah} fixed subspace of dimension blah). There's a 'Giambelli formula' saying how to write $[\overline{\Sigma}_\lambda]$ in terms of $[\Sigma_p]$.

Let
$$0 \longrightarrow S \longrightarrow \mathscr{O}^{\oplus n} \longrightarrow Q \longrightarrow 0$$
be the universal sub/quotient bundles on $\mathrm{Gr}(k,n)$. Say $F_{n-k-p+1} = \langle v_1, \dots, v_{n-k-p+1} \rangle$. Each v_i corresponds to a section of $\mathscr{O}^{\oplus n}$, so give sections $\overline{v}_i \in \mathrm{H}^0(\mathrm{Gr}(k,n), Q)$. These $\overline{v}_1, \dots, \overline{v}_{n-k-p+1}$ become dependent precisely when $\langle v_1, \dots, v_{n-k-p+1} \rangle$ meets the kernel Λ of this map, i.e. when $F_{n-k-p+1} \cap \Lambda \neq 0$. Thus,
$$[\Sigma_p] = \text{locus where } \overline{v}_1, \dots, \overline{v}_{n-k-p+1} \text{ become dependent} = c_p(Q)$$
(note $\operatorname{rank} Q = n-k$, so $n-k-p+1 = \operatorname{rank} Q - p + 1$). By the Whitney product formula, we have
$$\frac{1}{1 + c_1(Q) + \dots + c_{n-k}(Q)} = 1 + c_1(S) + \dots + c_k(S).$$
Formally expanding the LHS, all terms in degree $> k$ must vanish. These turn out to give all relations among the Chern classes of the tautological quotient bundle, so lead to a presentation for the Chow ring of Grassmannians. △

1.2 Lecture 2 (7/13): Boundary Strata & Excess Intersection

Recall the discussion on the Grassmannian and its stratification into Shubert cells. We'll talk about something similar for the moduli space of curves. We'll stratify it based on the topological type of the curve. These are encoded using `dual graphs` which have (labelled) vertices for each component of the curve (labelled w/ the genus of that component), and there is an edge for each node (connecting the components this node meets). Finally, we have "half-edges" for each marked point (labelled by which marked point it is).

Fact. The stratum with dual graph Γ has codimenion equal to the number of nodes.

Question 1.2.1. *How can we describe the closure of these strata?*

TODO: Draw an example

Write
$$\overline{M}_\Gamma := \prod_{v \in \Gamma} \overline{M}_{g(v),n(v)} \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$$
(the map is given by gluing curves together as prescribed by the dual graph).

Fact. The image of ξ_Γ is the closure of the stratum of curves of topological type Γ. Furthermore, $\operatorname{Im} \xi_{\Gamma'} \subset \operatorname{Im} \xi_\Gamma \iff$ there exists an edge contraction $\Gamma' \to \Gamma$.

Remark 1.2.2. If you contract a self-loop, the corresponding vertex has its genus go up by one. If you contract an edge between different vertices, you sum their genera. ◦

Question 1.2.3. *What is $\xi_{A*}[1] \cdot \xi_{B*}[1] \in \mathrm{CH}^*(\overline{M}_{g,n})$, for two dual graphs A, B?*

Example 1.2.4. Say A is the graph $(1)-(4)$ and B is $(2)-(3)$. In the intersection of these strata, it turns out there are two components corresponding to the dual graphs $(1)-(1)-(3)$ and $(2)-(2)-(1)$; both of these edge contract to both A, B (furthermore, any dual graph with this property is a specialization of at least one of these). △

Definition 1.2.5. An (A,B)-`structure` on a graph Γ is the data of

- An edge contraction $\Gamma \xrightarrow{\alpha} A$ (color contracted edges red)

- An edge contraction $\Gamma \xrightarrow{\beta} B$ (color contracted edges blue)

We say $(\alpha, \beta) \sim (\alpha', \beta')$ is there's an automorphism $\gamma \in \mathrm{Aut}(\Gamma)$ s.t. $\alpha' = \alpha \circ \gamma$ and $\beta' = \beta \circ \gamma$.[2] We call such a structure `generic` if no edge is colored both red and blue. $\diamond$

Fact. Generic (A, B)-graphs are in bijection with the components of the fiber product

$$\begin{array}{ccc} F_{AB} & \longrightarrow & \overline{M}_B \\ \downarrow & & \downarrow \xi_B \\ \overline{M}_A & \xrightarrow{\xi_A} & \overline{M}_{g,n}. \end{array}$$

For F_{AB}, irreducible comp. = connected comp.

Example 1.2.6 (Continuing earlier example...)**.** Note that both A, B are codimension 1 (each has 1 edge) while both of the generic (A, B)-graphs are codimension 2. Thus, this intersection is generically transverse, and so the product in the Chow ring is simply the sum of these two cycles (maybe modulo coefficient caveats). $\triangle$

Example 1.2.7. Let A be $C(2)-(1)$ and let B be $(1)-(3)$. Note that there is an edge contraction taking A to B (contract the self-edge), so their intersection won't be transverse. The generic (A, B)-graphs in this case are

$$C(2)-(1) \quad C(0)-(2)-(1) \quad (1)-\underbrace{(1)}-(1)$$

(the middle vertex in the third graph above has a self-loop, indicated by the underbrace). Multiple remarks:

- The middle graph above hst multiple (A, B)-structures, not all of which are generic
- It is possible for one graph to have multiple (A, B)-structures

  If you have two different generic (A, B)-structures on a graph, these will correspond to two different components in the fiber product.
- There's an uncolored edge in the generic (A, B)-structure on the first graph

In general, uncolored edges will be called 'excess' and correspond to non-transverse intersections. $\triangle$

Excess Intersection Let X be smooth. Say $A, B \subset X$ are subvarieties of codimension a, b. Then, the `excess intersection formula` says that

$$[A][B] = \sum_{\substack{\gamma \subset A \cap B \\ \text{conn. comp.}}} \iota_{\gamma,*}(E_Y) \text{ for some } E_Y \in \mathrm{CH}_d(Y),$$

where $d = \dim X - a - b$.

Example 1.2.8. Say A, B are generically transverse, so $\dim Y = d = \dim X - a - b$. In this case, $E_Y = [1]$ is the fundamental class of Y. $\triangle$

Example 1.2.9. Say $A = B$, the other extreme. In this situation, one uses push-pull to get

$$[A][A] = \iota_{A,*}[1][A] = \iota_{A,*}(\iota_A^*[A]) = \iota_{A,*}c_{\mathrm{top}}(N_{A/X}) \text{ so } E_A = c_a(N_{A/X}).$$

[2]Sounds like there's some subtelty in what's meant by an automorphism of the graph

For thinking about self-intersections, imagine moving A slightly off itself and then taking intersections. If the normal bundle has global sections, then you could use one to perturb A. This is why the normal bundle appears above. △

Example 1.2.10. Say Y is l.c.i.

Slogan. Expected normal bundle over actual normal bundle

In this case,

$$E_Y = \left[\frac{c(N_{A/X}|_Y)c(N_{B/X}|_Y)}{c(N_{Y/X})}\right]_d,$$

i.e. its the degree d piece of the above expression (written in terms of *total* Chern classes). △

In our situation, we had a picture like

$$\begin{tikzcd} F_{A,B} \arrow[r] \arrow[d] & B \arrow[d, "\xi_B"] \\ A \arrow[r, "\xi_A"] & X \end{tikzcd}$$

In this case, the formula looks like

$$\xi_{A,*}[1] \cdot \xi_{B,*}[1] = \sum_{\substack{Y \subset F_{A,B} \\ \text{conn. comp.}}} \xi_{Y,*}(E_Y),$$

where

$$E_Y = \left[\frac{\xi_{Y,A}^* c(N_{\xi_A}) \cdot \xi_{Y,B}^* c(N_{\xi_B})}{c(N_{\xi_Y})}\right]_d.$$

Above, $\xi_Y : Y \to X$, $\xi_{Y,A} : Y \to A$, $\xi_{Y,B} : Y \to B$, and our "normal bundles" are $N_{\xi_A} = \xi_A^* T_X / T_A$, etc.

Remark 1.2.11. The above version holds when A, B are smooth and the component Y is also smooth. Probably you can write down a correct statement when they're not too badly singular (e.g. lci), but in our application, everything will be smooth. ○

Back to moduli of curves To apply this general formula, we need to understand the components of the fiber product (which we said something about before), and we need to understand the normal bundles of these gluing maps $\xi_A : \overline{M}_A \to \overline{M}_{g,n}$. Somehow, N_{ξ_A} corresponds to "smoothing the nodes," whatever that means.

Example 1.2.12. Given a node like X, the "smoothing parameter" for that node is the tensor product of the tangent bundles of the two branches (in the normalization). △

Sounds like the final answer is

$$N_{\xi_A} = \bigoplus_{\text{edges } e = X - X'} T_X \otimes T_{X'}$$

(or something like this? I don't really follow). Above, have tangent spaces at marked points. Then,

$$c(N_{\xi_A}) = \prod_{e \in E(A)} (1 - \psi_x - \psi_{x'}),$$

where these are the ψ-classes from Eric's (second) lecture. What about the pullback to some component Y? One will get

$$\xi_{\Gamma,A}^{*} c(N_{\xi_A}) = \prod_{\substack{e\in E(\Gamma)\\ \text{not red}}} (1-\psi_x-\psi_{x'})$$

(product over edges not being contracted). One gets a similar expression with B in place of A (and blue in place of red). Finally,

$$c(N_{\xi_\Gamma}) = \prod_{e\in E(\Gamma)} (1-\psi_x-\psi_{x'})\,.$$

Putting this all together,

$$E_\Gamma = \left[\prod_{\substack{e\in E(\Gamma)\\ \text{not colored}}} (1-\psi_x-\psi_{x'})\right]_d = \prod(-\psi_x-\psi_{x'})$$

(recall that a generic graph has no purple edges)

Example 1.2.13 (Continuing from Example 1.2.7)**.** The final answer is a sum of 4 things:

- $C(2)-(1)$ where the (1) is decorated by a $-\psi_x$

  This notation means the pull then push the ψ-class along the maps

  $$\overline{M}_{1,1} \xleftarrow{\text{pr}} \overline{M}_{1,1}\times\overline{M}_{2,3} \xrightarrow{\xi_\Gamma} \overline{M}_4.$$

  This is an example of a "decorated boundary stratum"

- $C(2)-(1)$ where the (1) is decorated by a $-\psi_{x'}$
- $C(0)-(2)-(1)$ with no decoration (pushforward of fundamental class)
- $(1) - \underbrace{(1)}\, -(1)$ w/ no decoration (pushforward of fundamental class) △

One can compute intersections of decorated boundary strata (push-pull's of polynomials in κ,ψ).

Proposition 1.2.14. *The span of decorated boundary strata in* $\mathrm{CH}^*(\overline{M}_{g,n})_{\mathbb{Q}}$ *is closed under multiplication. This subring is called the* ***tautological subring*** $R^*(\overline{M}_{g,n}) \subset \mathrm{CH}^*(\overline{M}_{g,n})_{\mathbb{Q}}$.

1.3 Lecture 3 (7/13): Tautological & Non-tautological classes

Recall 1.3.1. The tautological subring $R^*(\overline{M}_{g,n})$ is the subring generated by the decorated boundary strata. ⊙

Question 1.3.2. *What is* $\mathrm{CH}^*(\overline{M}_{g,n})_{\mathbb{Q}}$*?*

(Can even ask this with integral coefficients). This is really hard, so can break it apart into two related questions.

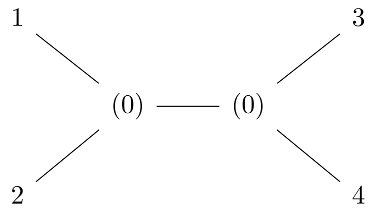
Question 1.3.3. *What is the structure of* $R^*(\overline{M}_{g,n})$*? When is* $R^*(\overline{M}_{g,n}) = \mathrm{CH}^*(\overline{M}_{g,n})_{\mathbb{Q}}$*?*

We'll spend most of our time on the second question, but first a few words on the first one...

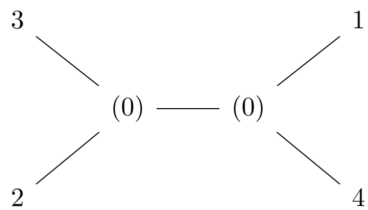
There are known relations ("Pixton's 3-spin relations") among tautological classes. These are implemented in a program called adm cycles. For some small values of g, n, it is possible to computationally verify that these relations are complete.

Open Question 1.3.4. *Are Pixton's relations all the relations?*

Example 1.3.5. On $\overline{M}_{0,4}$, one has the stable graph $>(0)-(0)<$ apparently corresponding to the class of a point



This has the same class as the graph



△

1.3.1 Question 2

It is known that $R^*(\overline{M}_{g,n}) = \mathrm{CH}^*(\overline{M}_{g,n})_{\mathbb{Q}}$ when

- $g = 0$ and $n \geq 3$ (Keel)
- $g = 1$ and $n \leq 10$ (Belorowwski)
- $[g = 2$ and $n \leq 9]$ or $[g \geq 3$ and $2g + n \leq 14]$ (Canning-L.)
- $g = 2, n = 0$ (Mumford)
- $(g, n) = (2, 1)$ and $(g, n) = (3, 0)$ due to Faber

However, equality does *not* holds when

- $2g + n \geq 24$ and $g \geq 2$ (van Zelm)
- $(g, n) = (2, 10)$
- $g = 1$ and $n \geq 11$

In all of these cases, the issue arises from the presence of odd cohomology on the moduli space.

Fact. When X is smooth, there is a `cycle class map`

$$\mathrm{CH}^*(X) \longrightarrow \mathrm{H}^*(X).$$

In general, this map is neither injective nor surjective.

Example 1.3.6. Say E is an elliptic curve. Note the Cycle class map always lands in even degree. However, $H^1(E) = \mathbb{Z} \oplus \mathbb{Z}$, so it can't be surjective. At the same time,

$$\mathrm{CH}^1(E) = \mathrm{Pic}(E) \longrightarrow \mathrm{H}^2(E) = \mathbb{Z}$$

(via taking the degree of line bundles), so this map is not injective either (the kernel is $\mathrm{Pic}^0(E)$) △

Theorem 1.3.7 (Kimor (can't read the actual spelling),Totar)**.** *If X is smooth and proper/$\mathbb{C}$ and* $\mathrm{CH}^*(X)$ *is finitely generated, then*

$$\mathrm{CH}^*(X)_{\mathbb{Q}} \xrightarrow{\sim} H^*(X).$$

So suppose $\mathrm{CH}^*(\overline{M}_{g,n}) = R^*(\overline{M}_{g,n})$. The tautological subring is finitely generated (only finitely many boundary strata and finitely many ways to decorate each). Thus, in this case, the cycle class map would be an isomorphism, so $\mathrm{H}^{\mathrm{odd}}(\overline{M}_{g,n}) = 0$. Conversely, if $\overline{M}_{g,n}$ has odd cohomology, then there must be non-tautological classes.

Question 1.3.8 (Audience)**.** *What goes into the proof of this theorem?*

Answer. The Chow ring being finitely generated implies that X has the "Chow-Künneth degeneration property" which in turn implies that you can get a decomposition of the diagonal. ⋆

Example 1.3.9 (Deligne)**.** It is known that $H^{11}(\overline{M}_{1,11}) \neq 0$. This comes from the fact that there's some holomorphic 11-form related to the weight 12 cusp form for $\mathrm{SL}_2(\mathbb{Z})$. For similar reasons, it is also known that $\mathrm{CH}_0(\overline{M}_{1,11})$ is huge, "there's not a f.dim variety which dominates it."

Open Question 1.3.10. *Are these all the non-tautological classes in this case?* △

Example 1.3.11 (Graber-Panonaipande (spelling), van Zelm)**.** Let

$$B_{g,2m} = \left\{(C, p_1, \dots, p_m, q_1, \dots, q_m) \in M_{g,2m} : \exists \pi : C \xrightarrow{2} E \text{ such that } \pi(p_i) = \pi(q_i)\right\}$$

be the bielliptic locus. Then, $[\overline{B}_{g,2m}] \notin R^*(\overline{M}_{g,2m})$ provided that $g+m \geq 12$ ($\iff 2g+n \geq 24$). △

(Sounds like G-P proved this when $g = 2$ and $n = 2m = 20$).

Remark 1.3.12. Via Andelo's second lecture, the λ classes land in the tautological ring. ○

Sounds like before the above example(s), people were under the impression that any "naturally occuring classes" would live in the tautological ring.

Sketch of Proof of van Zelm when $(g, m) = (12, 0)$. Consider the gluing map $\xi : \overline{M}_{1,11} \times \overline{M}_{1,11} \to \overline{M}_{12}$. Also recall the (closure of the) bielliptic locus $\overline{B}_{12} \subset \overline{M}_{12}$. The key claim is that the pullback of this locus to the product is

$$\Delta \cup (\text{stuff supported on boundary}) \subset \overline{M}_{1,11} \times \overline{M}_{1,11}$$

(Δ is the diagonal). Suppose that $\overline{B}_{12}$ was tautological. Then,

$$\xi^*[\overline{B}_{12}] \in R^*(\overline{M}_{1,11} \times \overline{M}_{1,11}) := \mathrm{Im}\big(R^*(\overline{M}_{1,11}) \times R^*(\overline{M}_{1,11})\big).$$

Applying the cycle class map to this expression, $\mathrm{cl}(\xi^*[\overline{B}_{12}])$ would have tautological Künneth components. However, since $\mathrm{H}^{11}(\overline{M}_{1,11}) \neq 0$, the diagonal has odd Künneth components, giving a contradiction. ■

Cases when all of the Chow ring is tautological Excision tells us that if we have an equality $R^*(\overline{M}_{g,n}) = \mathrm{Ch}^*(\overline{M}_{g,n})_{\mathbb{Q}}$, then also $R^*(M_{g,n}) = \mathrm{CH}^*(M_{g,n})_{\mathbb{Q}}$ on the interior.

Question 1.3.13. *Can we go the other way?*

Consider the excision sequence

$$\mathrm{CH}_*(\partial M_{g,n}) \xrightarrow{\iota_*} \mathrm{Ch}_*(\overline{M}_{g.n}) \longrightarrow \mathrm{CH}_*(M_{g,n}) \longrightarrow 0.$$

If we know $\mathrm{Ch}(M_{g,n})$ is tautological, then you ask: is $\mathrm{im}(\iota_*) \leq R_*(\overline{M}_{g,n})$? I missed something, but we have

$$\mathrm{CH}_*(\overline{M}_{g-1,n+2}) \oplus \bigoplus_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \mathrm{CH}_*\big(\overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2}\big) \twoheadrightarrow \mathrm{CH}_*(\partial M_{g,n}) \xrightarrow{\iota_*} \mathrm{CH}_*(\overline{M}_{g,n})$$

(implicitly using rational coefficients everywhere above, especially to get surjectivity). This looks like one might have some hope of setting up and inductive argument using this. Note that the tautological classes in the LHS above will map to tautological classes in $\mathrm{CH}_*(\overline{M}_{g,n})_{\mathbb{Q}}$. Thus, we want to know if these tauological classes on the small $M_{g,n}$'s generate the LHS above. This involves knowing whether the maps

$$\mathrm{CH}_*(\overline{M}_{g_1,n_1}) \otimes \mathrm{CH}_*(\overline{M}_{g_2,n_2}) \longrightarrow \mathrm{CH}_*(\overline{M}_{g_1,n_1} \times \overline{M}_{g_2,n_2})$$

are surjective. In general, Künneth does *not* holds in Chow rings.

Definition 1.3.14. We say X has the Chow Künneth generation Property (something like this) if for all Y, the map $\mathrm{CH}_*(Y) \otimes \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y \times X)$ is surjective. $\diamond$

Lemma 1.3.15. *If $R^*(M_{g',n'}) = \mathrm{CH}^*(M_{g',n'})$ holds for all $g' \leq g$ and $2g'+n' \leq 2g+n$, and $M_{g',n'}$ has the CKgP, then $R^*(\overline{M}_{g,n}) = \mathrm{CH}^*(\overline{M}_{g,n})$.*

Fact. CKgP implies that the cycle class map is an isomorphism.

Question 1.3.16 (Audience)**.** *Can you say a bit about how you veried CKgP in cases where it holds?*

Answer. Here are some properties which were used:

- If $U \subset X$ open and X has CKgP, then U has CKgP
- If $X = \bigsqcup X_i$ and X_i has CKgP, then X has CKgP
- $\mathbb{A}^n$ has CKgP
- (working w/ rational coeffs) if $X \twoheadrightarrow Y$ is proper + surjective and X has CKgP, then Y has CKgP

This is all they used to show all the moduli spaces they needed have CKgP. Also, every time they've shown as space has CKgP (by describing it explicitly as a union of strata related to opens in affine), they've also proven along the way that it is unirational. $\star$

Non-example. Elliptic curves don't have CKgP $\triangledown$

Question 1.3.17 (Audience)**.** *Do you know examples of spaces w/ CKgP but non-finitely generated Chow or vice versa?*

Answer. If the Chow group is finitely generated, then it must have CKgP. As a partial converse, if the variety is proper, its cohomology if finitely generated, so cycle class being isomorphism implies that Chow is finitely generated. $\star$

Question 1.3.18 (Audience)**.** *Does CKP fail (Kunneth map being an isomoprhism) in the cases you know CKgP holds?*

Answer. Unknown. CKP is much harder to get at. Most of the facts about CKP one uses follow easily from the excision sequence. $\star$

2 Andrea di Lorenzo: Introduction to equivariant intersection theory

2.1 Lecture 1 (7/11): Chern classes

Let's start w/ some intuition. Say X is a smooth variety/k and $L \to X$ is a line bundle. Then, $L \cong \mathscr{O}(D)$ for some divisor $D \subset X$. This divisor is well defined up to linear equivalence, so we get assignment $L \mapsto [D] \in \mathrm{CH}_{n-1}(X) = \mathrm{CH}^1(X)$. If L is trivial, it maps to $0 \in \mathrm{CH}^1(X)$. Thus, there is some cycle which measures how trivial/non-trivial L is. We would like to call $c_1(\mathscr{O}(D)) = D$. What about for higher rank vector bundles? For $E = \mathscr{O}^{\oplus 2}$ (really really trivial), we would like to have two cycles $C_1(E) = c_2(E) = 0$. For $E = \mathscr{O} \oplus \mathscr{L}$ (mildly trivial), maybe we want $c_2(E) = 0$ and $c_1(E) = c_1(L)$. We'd even like the same thing for $0 \to \mathscr{O} \to E \to L \to 0$.

Let X be an irreducible, separated scheme of finite type over a field k. Let $E \to X$ be a vector bundle. We look for a polynomial $c(E) = 1 + c_1(E)x + c_2(E)x^2 + \dots$ such that

I think we secretly want integral?

(1) Given flat $f : X' \to X$, then $c(f^*E) = f^*c(E)$

(2) If we have $0 \to E' \to E \to E'' \to 0$, then $c(E')c(E'') = c(E)$.

(3) $c_1(\mathscr{O}(D)) = D \in \mathrm{CH}^1(X)$

"[Chern classes were first defined in cohomology.] As we are algebraic geometers and we don't know any topology, we must work with algebraic stuff." (paraphrase)

2.1.1 Step 1: first Chern class

Let $L \to X$ be a line bundle.

Assumption. Assume X is normal.

Consider an open subset $U \subset X$ such that $L|_U \cong U \times \mathbb{A}^1$. In particular, there exists a nonzero section $s : U \to L|_U$. This defined a rational section of L/X, $s : X \dashrightarrow L$. Let's look for zeros/poles along subvarieties of codim 1. We define

$$c_1(L) = \sum_{\substack{V \subset X \\ \text{subvar, codim 1}}} \underbrace{\mathrm{ord}_V(s)}_{\in \mathbb{Z}} \cdot [V]$$

Example 2.1.1. Say L ideal sheaf of $[0:1]$ in $\mathbb{P}^1$. Let $U_1 = \{[x:y] \mid y \neq 0\}$, so $L|_{U_1}$ is trivial, i.e. has nonvanishing section s. In the other chart U_0, we have $s|_{U_0} = w^{-1}$ where w is the coordinate $w : U_0 \xrightarrow{\sim} \mathbb{A}^1$. Thus, $\mathrm{ord}_{[1:0]}(s) = -1$, so $c_1(L) = -[\infty]$, where $\infty = [1:0]$.

Exercise. Pick different rational section for the ideal sheaf and see what happens. △

We can define $c_1(L) \cap [W] =: i_*(c_1(L|_W))$ for any codim k subvar $\iota : W \to X$.

Remark 2.1.2. For non-normal spaces, pass to normalization and the pushforward. ∘

2.1.2 Step 2: Segre classes

Let $E \to X$ be a vector bundle of rank $r > 1$. Then, over $\mathbb{P}(E)$, we have $\mathscr{O}_{\mathbb{P}(E)}(1)$. Set $h := c_1\big(\mathscr{O}_{\mathbb{P}(E)}(1)\big)$. Note the projection map $\pi : \mathbb{P}(E) \to X$ is both proper and flat. Given $\xi \in \mathrm{CH}(X)$, we can define

$$\pi_*(h \cap \pi^*\xi) \in \mathrm{CH}(X).$$

Example 2.1.3. If $\xi - [X]$, this is $\pi_* h = 0$ unless $r = 2$ (look at dimensions) △

We can do better.

Definition 2.1.4. The `Segre classes` (capped with ξ) are

$$s_i(E) \cap \xi := \pi_*\left(h^{r-1+i} \cap \pi^*\xi\right).$$

In particular, $s_i(E) := s_i(E) \cap [X] = \pi_*\left(h^{r-1+i}\right)$. ◇

(This makes sense by projection formula)

2.1.3 Step 3: Chern classes

Assemble Segre classes together. The `total Segre class` is

$$s(E) = s_0(E) + s_1(E)x + \dots, \text{ where } s_0(E) = 1.$$

Definition 2.1.5. The `total Chern class` is $c(E) = s(E)^{-1}$ (formal inverse). ◇

"Now I have been talking for half an hour, but I just wrote down a definition, so I think I will stop here." (paraphrase)

This isn't the nicest definition to work with. It's not immediately clear it satisfies the properties we wanted. It's also not so clear how to compute something like $c(E \otimes L)$ for L a line bundle.

2.1.4 Step 4: splitting principle

As before, E is a rank r vector bundle over some variety X.

Proposition 2.1.6 (`Splitting principle`)**.** *There exists a flat morphism $f : X' \to X$ such that*

(1) *$f^* : \mathrm{CH}(X) \to \mathrm{CH}(X')$ is injective*

(2) *there exists a filtration*
$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_0 = 0$$
by vector bundles so that $E_i/E_{i-1} =: L_i$ is a line bundle.

Suppose we have such a filtration on X, and assume that the Chern classes satisfy the Grothendieck axioms. Then,

$$c(E) = 1 + \sigma_1(\ell_1, \dots, \ell_r)x + \sigma_2(\ell_1, \dots, \ell_r)x^2 + \dots,$$

where $\ell_i := c_1(L_i)$ and σ_i is the ith elementary symmetric polynomial. In other words,

$$c(E) = \prod_{i=1}^{r}(1 + \ell_i x).$$

These ℓ_i's are called the `Chern roots`.

Slogan. For computations, you can pretend that $E = L_1 \oplus \cdots \oplus L_r$.

Example 2.1.7.
$$c(E \otimes M) = c\Big(\bigoplus (L_i \otimes M)\Big) = \prod (1 + (\ell_i + m)x),$$

where $\ell_i = c_1(L_i)$ and $m = c_1(M)$. Expanding this out let's one expresses $c(E \otimes M)$ in terms of $c_i(E)$ and $c_j(M)$. △

Proof of Proposition 2.1.6. Consider $\pi : \mathbb{P}(E^\vee) \to X$. Get bundle $0 \to \mathscr{O}(-1) \to \pi^* E^\vee \to Q$. Dualize to get $0 \to Q^* \to \pi^* E \to \mathscr{O}(1) \to 0$.

Fact. The pullback map $\pi^* : \mathrm{CH}(X) \to \mathrm{CH}(\mathbb{P}(E^\vee))$ is injective.

This gives the first step in a filtration (set $E_{r-1} = Q^\vee$). Repeat this process (consider $\mathbb{P}(E_{r-1}^\vee) \to \mathbb{P}(E^\vee) \to X$)... ■

Lemma 2.1.8. *Assume we have $E \to X$ w/ filtration by vector bundles whose graded pieces are line bundles. Suppose that $s : X \to E$ is a global section s.t. $Z = \{s = 0\} = \emptyset$. Then, $\prod_{i=1}^r c_1(L_i) = 0$.*

By splitting principle, up to replacing X w/ X', we have such a filtration for E. Upon considering $\mathbb{P}(E) \to X$, there's an injection $\mathscr{O}(-1) \hookrightarrow \pi^* E$; equivalently, there is a non vanishing section of $\mathscr{O}(1) \otimes \pi^* E$. Hence,

$$\sum \sigma_{r-i}(\ell_1, \ldots, \ell_r) h^i = \prod (h + \ell_i) = \prod c_1(\mathscr{O}(1) \otimes L_i) = 0.$$

By the projection formula, this says

$$\sum \sigma_{r-i} \cdot s_i(E) = 0.$$

One deduces that $s(E)\big(1 + \sigma_1 x + \sigma_2 x^2 + \ldots\big) = 1$, so $c_i(f^* E) = \sigma_i(\ell_1, \ldots, \ell_r)$.

2.2 Lecture 2 (7/12): equivariant Chow groups & quotient s*****

Definition 2.2.1. Let G be a linear algebraic group acting on a variety X, and let $f : X \to Y$ be a morphism. Then, X is a G-`torsor over` Y if

- **(i)** $f : X \to Y$ is G-invariant
- **(ii)** there exists some étale cover $\{Y_i \to Y\}$ such that $G \times Y_i \simeq X \times_Y Y_i$. ◇

This is our model in algebraic geometry for a space of orbits.

Example 2.2.2. Consider $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ via scaling in the usual way. The orbits of this action correspond to lines in $\mathbb{A}^2$ and also the origin. In this case, one has $(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m = \mathbb{P}^1$ and $\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ is a $\mathbb{G}_m$-torsor. △

In general, if $X \to Y$ is a G-torsor, then we really have $Y = X/G$.

Slogan. Torsors are the quotients that we like.

However, in general, given a G-action on X, we don't always have "X/G" in the sense that there's some scheme Y s.t. $X \to Y$ is a G-torsor. Nevertheless

Slogan. if equivariant Chow groups exist, they must be the Chow groups of the quotient.

Definition 2.2.3. $\mathrm{CH}_*^G(X) = \mathrm{Ch}_{*-g}(X/G)$ ◇

Proposition 2.2.4. *There exists a (f.dim) G-representation V such there's some open $U \subset V$ which is G-invariant whose induced G-action is free and s.t.* $\mathrm{codim}(V \setminus U)$ *is as high as you like.*

For every f.dim G-representation V, we have that $X \times V/G \to X/G$ is a vector bundle, so we'd expect/want/dream $\mathrm{Ch}_i^G(X) = \mathrm{Ch}_{i-g}(X/G) \cong \mathrm{CH}_{i-g+r}(X \times V/G)$ (homotopy invariance). By the proposition, if we pick the representation cleverly, we can ensure $\mathrm{Ch}_{i-g+r}(X \times V/G) \cong \mathrm{CH}_{i-g+r}(X \times U/G)$.

Remark 2.2.5. Note $X \times U/G \overset{\text{open}}{\subset} X \times V/G$, with complement Z of dimension $< i - g + r$. Thus, the excision sequence will give us the isomorphism before this remark. ◦

The utility of all of this is that G acts freely on U, and so acts freely on $X \times U$. Thus, $X \times U/G$ exists as a scheme.

Definition 2.2.6 (Edidin-Graham, up to spelling)**.** $\mathrm{CH}_i^G(X) := \mathrm{CH}_{i-g+r}(X \times U/G)$ ⋄

This involved lots of choices.

Proposition 2.2.7. *Say $V_1 \supset U_1$ and $V_2 \supset U_2$ are both G-reps as above. Then, there is an isomorphism*

$$\mathrm{CH}_{i-g+r_1}(X \times U_1/G) \cong \mathrm{CH}_{i-g+r_2}(X \times U_2/G).$$

Proof. $X \times U_1 \times V_2/G$ is a vector bundle over $X \times U_1/G$ which has $X \times U_1 \times U_2/G$ as an open subset. Similarly, this is an open subset of $X \times V_1 \times U_2/G$, which is a vector bundle over $X \times U_2/G$. The maps on Chow groups between all these spaces are isomorphisms. ■

Example 2.2.8. $\mathrm{CH}_i^{\mathbb{G}_m}(pt)$. Need $\mathbb{G}_m$-representations which are free on spaces of high codimension. Well, $\mathbb{G}_m \curvearrowright \mathbb{A}^{n+1}$ in the usual way, and acts freely away from the origin. Thus,

$$\mathrm{CH}_i^{\mathbb{G}_m}(pt) = \mathrm{CH}_{i-1+n+1}\big(pt \times (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m\big) = \mathrm{Ch}_{i+n}(\mathbb{P}^n)$$

(note that the origin has codimension $n+1$). This tells us that $\mathrm{CH}_{13}^{\mathbb{G}_m}(pt) = 0$ and $\mathrm{CH}_{-2}^{\mathbb{G}_m}(pt) = \mathrm{CH}_{n-2}(\mathbb{P}^n) = \mathbb{Z}$, for example. In particular, equivariant Chow of a point is supported in nonpositive degrees:

$$\mathrm{Ch}_{-i}^{\mathbb{G}_m}(pt) \cong \mathbb{Z} \cdot h^i \text{ for all } i \geq 0.$$

Switching to the upper numbers (and accepting the existence of a ring structure), this shows that

$$\mathrm{CH}^*_{\mathbb{G}_m}(pt) = \mathbb{Z}[h] \text{ with } \deg h = 1.$$ △

Slogan ("Propostiion")**.** Every statement that you have seen on Chow groups works in the equivariant setting as well. The only difference is that the maps have to be equivariant.

Example 2.2.9. Get equivariant Chern classes attached to equivariant vector bundles. △

Example 2.2.10. Smooth schemes w/ G-action have intersection product. △

Example 2.2.11. Consider $\mathbb{A}^1 \to pt$ with $\mathbb{A}^1$ given the $\mathbb{G}_m$-action $\lambda \cdot x = \lambda^d x$. Attached to this is some first Chern class $c_1^{\mathbb{G}_m}\left(\mathbb{A}^1_d\right) \in \mathrm{CH}^{\mathbb{G}_m}(pt)$. Let's compute this. Use equivariant approximations, so replace pt with $\mathbb{P}^n = pt \times (\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m$. Then, $\mathbb{A}^1_d$ becomes the line bundle $\mathscr{O}_{\mathbb{P}^n}(d) = \mathbb{A}^1 \times (\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$. Thus, $c_1^{\mathbb{G}_m}\left(\mathbb{A}^1_d\right) = dh$. Thus,

$$\mathrm{CH}^*_{\mathbb{G}_m}(pt) = \mathbb{Z}\left[c_1^{\mathbb{G}_m}\left(\mathbb{A}^1_1\right)\right].$$ △

Recall 2.2.12. If X is a scheme, can define the functor

$$\begin{array}{rccc} h_X: & \mathrm{Sch}^{\mathrm{op}} & \longrightarrow & \mathrm{Set} \\ & Y & \longmapsto & \mathrm{Hom}(Y, X). \end{array}$$ ⊙

What functor would X/G represent? Well, $X \to X/G$ is a G-torsor, so given any $Y \to X/G$, we get a Cartesian square

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X/G, \end{tikzcd}$$

with $P \to X$ a G-torsor. Forgetting the part of this that "is only a dream" when is tempted to define

$$\mathrm{Hom}(Y, X/G) = \left\{ Y \xleftarrow{G} P \xrightarrow{G\text{-equiv}} X \right\}.$$

This is almost the definition of a quotient (something). To make things work out well in general, the target of the functor should be a groupoid instead of a set. So $[X/G](Y) = h_{[X/G]}(Y)$ is the groupoid whose objects are diagrams $Y \leftarrow P \to X$ as above, and whose (iso)morphisms are isomorphisms of torsors w/ the expected compatibilities (over the identity on Y).

Proposition 2.2.13 (Edidim-Graham, up to spelling)**.** *Given two quotient s***** $[X/G]$ and $[Y/H]$ such that $[X/G] \simeq [Y/H]$, then $\mathrm{CH}^G(X) \cong \mathrm{CH}^H(Y)$.*

Definition 2.2.14. $\mathrm{CH}([X/G]) := \mathrm{CH}^G(X)$ ⋄

2.3 Lecture 3 (7/14)

Want to introduce additional tools which can be used in our everyday lives as people computing equivariant Chow rings, as well as an application to moduli of curves.

2.3.1 Localization formula

For the sake of the lecture, we'll explain this in a relatively simple setup. For me, look at paper "localization formulas in equivariant Chow groups" (or something like this).

Setup 2.3.1. Let $T = \mathbb{G}_m^n$ be a split torus. Let V be a T-rep, $V = L_{\chi_1} \oplus \cdots \oplus L_{\chi_r}$, where L_{χ_i} is the rank 1-representation where T acts via the character $\chi_i : T \to \mathbb{G}_m$. Let $W = L_{\psi_1} \oplus \cdots \oplus L_{\psi_{r'}}$ be another T-rep ($\psi_j : T \to \mathbb{G}_m$ other characters). Furthermore, suppose we have a homogenous, T-equivariant map $f : V \to W$, i.e. $\lambda \cdot f(x) = f(\lambda \cdot x)$. Note this gives rise to an induced map $\overline{f} : \mathbb{P}(V) \to \mathbb{P}(W)$ which is T-equivariant and proper.

Assumption. You probably want to assume that all the characters are nontrivial.

Example 2.3.2. $\overline{f}$ could be $\mathbb{P}\,\mathrm{H}^0(\mathbb{P}^2, \mathscr{O}(1)) \to \mathbb{P}\,\mathrm{H}^0(\mathbb{P}^2, \mathscr{O}(2))$ via $\ell \mapsto \ell^2$. In this example, $T = \mathbb{G}_m^3 \curvearrowright \mathbb{P}^2$ via

$$(\lambda_1, \lambda_2, \lambda_3) \cdot [x_1, x_2, x_3] = [\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3]\,.$$

This induces an action of T on homogeneous polynomials g via

$$(\lambda \cdot g)(x_1, x_2, x_3) = g\big(\lambda_1^{-1} x_1, \lambda_2^{-1} x_2, \lambda_3^{-1} x_3\big)\,.$$ △

Fact (exercise)**.** I maybe copied something down incorrectly...

- $\mathrm{CH}_T^*(\mathbb{P}(V)) = \mathbb{Z}[t_1, \ldots, t_n, K]/(p(K))$. Computing the relation (whose coefficients are polynomials in the t's) is part of the exercise.

- $\mathrm{CH}_T^*(\mathbb{P}(W)) = \mathbb{Z}[t_1, \ldots, t_n, H]/(q(H))$

Hint: $[\mathbb{P}(V)/T] \to [pt/T]$ is a projective bundle, and so you can apply the projective bundle formula.

Question 2.3.3. *How do you compute $\overline{f_*}(K^\alpha)$, $\alpha \geq 0$.*

Theorem 2.3.4 (Edidin-Graham, up-to-spelling)**.**

$$\overline{f}_*(K^\alpha) = \sum_{F \subset \mathbb{P}(V)} \frac{(\iota_F^* K)^\alpha \, \overline{f_*}[F]_T}{c_{top}^T(N_F)},$$

where

(1) F is an irreducible component of the fixed locus of the T-action on $\mathbb{P}(V)$

(2) N_F is the normal bundle of $F \subset \mathbb{P}(V)$

There was some comment I couldn't hear about these irreducible components basically being points or something?

This is gonna take some unpacking. To start, where do all these terms/factors live, and what do they mean?

(1) Say F is a point, e.g. $F = [0, \dots, 0, 1, 0, \dots, 0] =: p_i$ with the 1 in the ith coordinate. In this case,

$$N_F = T_{p_i} = \left\langle \frac{x_0}{x_i}, \dots, \frac{x_r}{x_i} \right\rangle$$

(skip $x_i/x_i = 1$). Above, T acts via

$$\lambda \cdot \frac{x_j}{x_i} = \frac{\chi_j(\lambda) x_j}{\chi_i(\lambda) x_i},$$

i.e. $N_F = L_{\chi_0 - \chi_i} \oplus \cdots \oplus L_{\chi_r - \chi_i}$ (skip $L_{\chi_i - \chi_i}$), so

$$c_{\text{top}}^T(N_F) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \left(c_1^T(L_{\chi_i}) - c_1^T(L_{\chi_j}) \right) \in \mathrm{CH}^T(pt) = \mathrm{CH}^T(F).$$

Remark 2.3.5. Say $\lambda \cdot x = \lambda_1^{d_1} \cdots \lambda_n^{d_n} x$ is the action by the character χ_j. Then, $c_1^T(L_{\chi_j}) = d_1 t_1 + \cdots + d_n t_n$. ○

(2) Apparently $K = c_1^T\left(\mathscr{O}_{\mathbb{P}(V)}(1)\right)$. Thus,

$$\iota_F^* K = c_1^T(\iota_F^* \mathscr{O}(1)).$$

Remark 2.3.6. $\mathscr{O}_{\mathbb{P}(V)}(-1)|_{p_i} = \langle x_i \rangle$ ○

Hence, $\iota_F^* K = -c_1^T(\langle x_i \rangle) = -c_i^T(L_{\chi_i}) =: -c_1^T(\chi_i) \in \mathrm{CH}^T(pt)$.

(3) If p_i is a fixed point, then so is $\overline{f}(p_i) = q_{f(i)}$ (the point where the $f(i)$th coordinate is the only one which is nonvanishing).

Example 2.3.7. For $\ell \mapsto \ell^2$, the point $[1, 0, 0]$ (corresponding to the linear form z_1) maps to to the point $[1, 0, 0, 0, 0, 0, 0, 0, 0, 0]$ (corresponding to the linear form z_1^2). △

Lemma 2.3.8. *For $V(x_j) \subset \mathbb{P}(W)$, $[\{x_j = 0\}]_T = H + c_1^T(\psi_j)$.*

Observe that

$$\overline{f}_*[p_i]_T = [q_{f(i)}]_T = \left[\bigcap_{j \neq f(i)} \{x_j = 0\} \right]_T = \prod_{j \neq f(i)} [\{x_j = 0\}]_T = \prod_{j \neq f(i)} \left(H + c_1^T(\psi_j) \right).$$

These are all the ingredients. Put together, they give the equation

$$\overline{f}_*(K^\alpha) = \sum_{i=1}^{r} \frac{\left(-c_1^T(\chi_i)\right)^\alpha \prod_{j\neq f(i)}\left(H + c_1^T(\psi_j)\right)}{\prod_{j\neq i}\left(c_1^T(\chi_j) - c_1^T(\chi_i)\right)}.$$

This i a priori a rational fraction, but in fact it is a polynomial.

Theorem 2.3.9 (Edidin-(can't read second name)). *Compute* $\mathrm{CH}(\mathcal{M}_0^{\leq 1})$*, the Chow ring of the moduli stack of genus 0 curves w/ at most 1 node.*

3 Eric Larson: User's guide to explicit calculations with higher Chow groups

3.1 Lecture 1 (7/11)

Let M_g be the space of smooth curves of genus g, and $\overline{M}_g \supset M_g$ be that of stable curves of genus g. We have two main questions:

Question 3.1.1. *What is a stable curve? Why do we care?*

Definition 3.1.2. We say a curve C is `stable` if

(1) C is nodal

(2) $\#\operatorname{Aut}(C) < \infty$. ◇

The key feature of this definition is that $\overline{M}_g$ is proper. Equivalently (via valuative criterion), we have the "`Stable Reduction Theorem`". Given a family of $C^* \to \Delta^*$ over a punctured disk (or generic point of $\Delta = \operatorname{Spec}(\text{dvr})$), we can uniquely complete C^* to a family of stable curves $C \to \Delta$. This is not strictly true as stated. Instead, this is true possibly after making a base change (replace Δ by a cover).

Remark 3.1.3. If there were more 'stable curves', there might be multiple ways to fill in families. If there were fewer 'stable curves', there might be no ways to fill in families. So, this definition is in the Goldilocks zone. ○

Let $M_{g,n} = \{\text{smooth curves } C \text{ of genus } g \text{ w/ } n \text{ distinct marked points } p_1, \dots, p_n \in C\}$. Similarly let $\overline{M}_{g,n} = \{\text{stable } n\text{-marked curves } (C, p_1, \dots, p_n) \text{ of (arithmetic) genus } g\}$.

Definition 3.1.4. We say that $(C, p_1, \dots, p_n)$ is `stable` if

(1) C is nodal

(2) $p_1, \dots, p_n$ are distinct smooth points

(3) $\#\operatorname{Aut}(C, p_1, \dots, p_n) < \infty$ ◇

Example 3.1.5. A genus 3 curve w/ a node is stable. △

Remark 3.1.6. The automorphism group of a genus 1 curve is a 1-dimensional infinite group. Translate by points of the curve. The automorphism group of $\mathbb{P}^1$ is PGL_2, which is 3-dimensional. ○

Example 3.1.7.

- A genus one curve meeting a genus two curve

  For each genus one curve, need 1 special point

- A genus 0 curve meeting a genus 2 curve in three points

  For each genus zero curve, need 3 special points △

Non-example.

- A genus 3 curve w/ a cusp is not stable.
- A genus 1 curve
- A genus 0 curve meeting a genus 2 curve in two points ▽

For this talk, want to work over an algebraically closed field. In fact, one of characteristic 0. In fact, over $\mathbb{C}$. We want to indicate the idea(s) behind the original proof of the stable reduction theorem. This proof fails in positive characteristic. Stable reduction holds in positive characteristic, but this came later (sounds like it was due to Deligne and Mumford, who reduced it to a similar theorem about abelian varieties).

Question 3.1.8 (Audience, missed part of it)**.** *What about semi-stable reduction theorem?*

Answer (assuming I heard correctly)**.** That says that if you work w/ families of semi-stable curves, can complete them to a family whose total space is smooth. $\star$

Example 3.1.9. Say have curve C of genus $g \geq 1$ w/ two points p, q. Let $C' = C/(p \sim q)$. This is a stable curve. What happens as q approaches p? The obvious way of filling this in would lead to a cuspidal curve "$C/(p \sim p)$". How else may we fill the family in, so that we get a stable central fiber. Consider $C \times C$ w/ diagonal Δ and constant section $\Gamma = C \times \{p\}$. Blowup the point (p, p), so get new family whose fiber over p is C glued with $\mathbb{P}^1$ at p. In the blowup, the sections $\widetilde{\Delta}, \widetilde{\Gamma}$ no longer intersect. Glue these together to get a family of curves whose central fiber is not C w/ a nodal $\mathbb{P}^1$ attached. $\triangle$

Example 3.1.10. Let's look at a family of curves acquiring a cusp, but whose general member is smooth. Further suppose that the total space is smooth, e.g. $t = y^2 - x^3$ (w/ t the coordinate on the base Δ). Blow up the cusp. You'll get the normalization C of the original central fiber w/ an exceptional divisor $E = \mathbb{P}^1$ tangent to C. Furthermore, this $\mathbb{P}^1$ occurs w/ multiplicity 2 in the central fiber (since $t = y^2 - x^3$ vanishes to order 2 at the cusp), i.e. the new central fiber is the non-reduced $C + 2E$. Blow up again. In this case, you'll get $C + 2E + 3F$ (3 b/c t vanishes w/ multiplicity 1 along C and w/ multiplicity 2 along E) meeting as pictured in $*$. Note F is a $\mathbb{P}^1$. At this point, we have distinct tangent directions at the singularity, so we can do one last blowup. The picture now looks like an E (with the spine being $G \cong \mathbb{P}^1$ appearing w/ multiplicity 6), i.e. the central fiber is $C + 2E + 3F + 6G$. The reduced central fiber is nodal, but the central fiber is not reduced. This is where we have to make a base change, say $t = s^2$.

Near a point x on E, t vanishes to order 2, so t is the square of a local coordinate z, i.e. $s^2 = t = z^2$. This factors as $(s+z)(s-z) = 0$, so after normalizing, the point x will have 2 preimages (where $s = \pm z$). Near a point y on F, we have $s^2 = t = z^3$. This is unibranched (it looks like a cusp), so in the normalization of the base change, the point y has 1 preimage. In the normalization of the basechange, the component of F is ramified (ultimately because F has odd multiplicity). The upshot is that the central fiber of the resulting family looks like

$$C + (E_1 + E_2) + 3F + 3G'$$

(note $\mathbb{P}^1 \cong G' \to G \cong \mathbb{P}^1$ is a double cover of $\mathbb{P}^1$ branched at 2 points, where G meets C and F). Next, one makes an order 3 base change. The resulting central fiber looks like

$$C + (E_1 + E_2) + (F_1 + F_2 + F_3) + G'',$$

where $g(G'') = 1$ by Riemann-Hurwitz $(2 - 2g = 3 \cdot 2 - 3 \cdot 2 = 0)$. Can check that every $\mathbb{P}^1$ above is a (-1)-curve, so contract them, and the final result is a nodal union of C and G'' (an elliptic curve w/ j-invariant 0 since it has an order 3 automorphism). $\triangle$

3.2 Lecture 2 (7/12): The dualizing sheaf

Recall 3.2.1. Let C be a smooth curve. It has a canonical bundle K whose sections are holomorphic/regular differential forms. Furthermore, Riemann-Roch tells us that

$$h^0(L) - h^0(KL^{-1}) = d + 1 - g,$$

for any degree d line bundle L on C. Furthermore, by Serre duality, $H^1(L) = H^0(KL^{-1})^\vee$. ⊙

The dualizing sheaf plans an analogous role in the theory of singular curves to that of the canonical bundle in the theory of smooth curves.

Before getting into it, let's recall: why is Riemann-Roch true? Suppose $L = \mathscr{O}_C(p_1 + \cdots + p_d)$ with the p_i's distinct, so a section of L is determined by its principal parts at the p_i's (up to addition of a global holomorphic function, i.e. a constant).

Question 3.2.2 (Audience)**.** *Can you say explicitly what principal parts are?*

Answer (paraphrase)**.** Say t is a coordinate at p_i. Then a section s of L, near p_i, looks like a holomorphic function w/ at worst a simple pole at $t = 0$, so looks like a power series $\frac{a}{t} + blah$. THe principal part at p_i is this a. ⋆

Which principal parts can be complete to a section? Well, say $\alpha \in H^0(K)$ and $\sigma \in H^0(\mathscr{O}_C(p_1 + \cdots + p_d))$, then the residue theorem tells us that

$$\sum_i \operatorname{Res}_{p_i}(\alpha\sigma) = 0.$$

This produces a linear condition on the principal parts, which is trivial iff $\alpha \in \mathrm{H}^0(K(-p_1 - \cdots - p_d))$ (this ensures all the residues vanish). Thus, one gets the bound

$$\dim \mathrm{H}^0(\mathscr{O}_C(p_1 + \cdots + p_d)) \le (d+1) - \left[\dim \mathrm{H}^0(K) - \dim \mathrm{H}^0(K(-p_1 - \cdots - p_d))\right].$$

In other words, $\dim \mathrm{H}^0(\mathscr{O}_C(p_1 + \cdots + p_d)) - \dim \mathrm{H}^0(K(-p_1 - \cdots - p_d)) \le (d+1) - g$. R-R says this is an equality.

Remark 3.2.3. The same logic as above will apply to singular curves provides we replace the canonical bundle with something different. It should be the case that if you multiply a section by a function with no poles, then it has no residue. This is what was used above: holomorphic differentials have zero residue. ◦

Say now that C is singular. Define its `dualizing sheaf` ω_C to have sections being meromorphic/rational differential forms whose product w/ any regular function has residue zero everywhere.

Example 3.2.4. What does this look like near smooth points? Choose a local coordinate x. Let $\alpha = \sum a_i x^i \mathrm{d}x$ is a meromorphic differential form. We want

$$0 = \operatorname{Res}_0\left(\alpha x^{n-1}\right) = a_{-n} \text{ for all } n \ge 1.$$

Thus, we just want α to be a holomorphic 1-form near $x = 0$. △

Example 3.2.5. What happens near a mode? Say along the two branches have local coordinates x, y. Thus, we can write $\alpha = \sum a_i x^i \mathrm{d}x + b_i y^i \mathrm{d}y$. We should have

$$0 = \operatorname{Res}_0(\alpha) = a_{-1} + b_{-1}.$$

Furthermore,[3] $0 = \text{Res}_0(\alpha x^{n-1}) = a_{-n}$ for any $n \geq 2$ (and similarly for y, i.e. $b_{-n} = 0$ for $n \geq 2$). Thus, we allow simple poles along both branches of the curve as long as the sum of their residues is 0. △

Question 3.2.6 (Audience)**.** *This is actually a line bundle on a nodal curve?*

Answer (paraphrased)**.** Yes. All fibers are 1-dimensional (at the nodes, you only need to keep track of the value of one of a_{-1}, b_{-1}). In general, the dualizing sheaf may not be a line bundle. ⋆

This definition of ω_C behaves well in families. Given a family $\mathcal{C} \to B$ of nodal curves, can produce a relative dualizing sheaf $\omega_{\mathcal{C}/B}$ which will be a line bundle on $\mathcal{C}$ whose restriction to any fiber recovers the dualizing sheaf of that fiber.

Remark 3.2.7 (Response to audience question)*.* For a nodal curve C w/ normalization $\widetilde{C}$, ω_C is naturally a subsheaf of $\nu_*\omega_{\widetilde{C}}(p+q)$, where p, q are the preimages of the node. ◦

3.2.1 Some Intersection Theory

Given a family of curves $\mathcal{C} \xrightarrow{\pi} B$, can produce Chow classes on B:

$$\lambda_i = c_i\big(\pi_*\omega_{\mathcal{C}/B}\big) \text{ and } \kappa_i = \pi_*\big(c_1(\omega_{\mathcal{C}/B})^{i+1}\big)\,.$$

Moreover, *if there is a section* $\sigma : B \to \mathcal{C}$, then we can produce the class

$$\psi = \sigma^* c_1\big(\omega_{\mathcal{C}/B}\big)\,.$$

Example 3.2.8. Consider a general pencil of plane quartics, i.e. $\mathcal{C} = V(t_0F + t_1G) \subset \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1_{[t_0,t_1]}$.

- Compute $\omega_{\mathcal{C}/B}$

  Let $\beta : \mathcal{C} \to \mathbb{P}^2$ be the other projection. Let $L = \beta^*[\text{line}]$ and let E be the sum of the 16 exceptional divisors. Note: $\mathcal{C}$ is exactly the blowup of $\mathbb{P}^2$ along the 16 points $\{F = G = 0\}$. The canonical bundle of a blowup is $K_{\mathcal{C}} = \beta^*K_{\mathbb{P}^2} + E = -3L + E$. Furthermore, $\pi^*K_{\mathbb{P}^1} = -2\pi^*[pt] = -2(4L - E) = -8L + 2E$. Finally

  $$\omega_{\mathcal{C}/\mathbb{P}^1} = \text{the difference} = (-3L + E) - (-8L + 2E) = 5L - E.$$

- κ_1. First,

  $$(5L - E)^2 = 25L^2 - 10LE + E^2.$$

  Pushing this to $\mathbb{P}^1$, we get $\kappa_1 = 25 - 16 = 9$ (really, 9 times the class of a point).

- λ_1. Note $\omega_{\mathcal{C}/B} = (4L - E) + L = \pi^*\mathscr{O}_{\mathbb{P}^1}(1) \otimes \beta^*\mathscr{O}_{\mathbb{P}^2}(1)$. Projection (= push-pull) gives

  $$\pi_*\omega_{\mathcal{C}/B} = \mathscr{O}_{\mathbb{P}^1}(1) \otimes \pi_*\beta^*\mathscr{O}_{\mathbb{P}^2}(1).$$

  Furthermore, the right factor is a trivial bundle w/ fiber $\mathrm{H}^0(\mathscr{O}_{\mathbb{P}^2}(1))$, so $\pi_*\omega_{\mathcal{C}/B} = \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3}$. Thus, $\lambda_1 = c_1(\mathscr{O}_{\mathbb{P}^1}(1)^3) = 3$. △

3.3 Lecture 3 (7/14): Higher Chow Groups

Note 1 (Announcement before lecture, unrelated to material)*.* Visit https://icerm.brown.edu/collaborate to apply to be hosted at icerm for meetings between collaborators or something like this?

[3] Note $xy = 0$ on this curve

Recall 3.3.1. Say $Y \subset X$ closed of codimension c, and $U := X \setminus Y$.[4] Get an excision sequence

$$\mathrm{CH}^{*-c}(Y) \longrightarrow \mathrm{CH}^*(X) \longrightarrow \mathrm{CH}^*(U) \longrightarrow 0.$$

One difficulty in Chow rings is that this sequence is only right exact. $\odot$

Higher Chow groups are meant to allow us to continue this sequence to the left.

Definition 3.3.2. Set $\Delta^n := V(x_0 + \cdots + x_n - 1) \subset \mathbb{A}^{n+1}$. Note $\Delta^n \cong \mathbb{A}^n$, non-canonically. Define

$$Z^c(X,n) := \left\{ \begin{matrix} \text{codim } c \text{ cycles in } X \times \Delta^n \text{ transverse} \\ \text{to all coordinate hyperplanes } \{x_{i_0} = \cdots = x_{i_k} = 0\} \end{matrix} \right\}.$$

The transversality condition allows us to define a differential

$$\begin{array}{rccc} \mathrm{d}_n : & Z^c(X,n) & \longrightarrow & Z^c(X,n-1) \\ & [Y] & \longmapsto & \displaystyle\sum_{i=0}^n (-1)^i [Y \cap \{x_i = 0\}]. \end{array}$$

(note there's a natural/canonical identification[5] of $V(x_i) \subset \Delta^n$ with Δ^{n-1}) Finally, the `higher Chow groups` are the cohomology of this complex:

$$\mathrm{CH}^c(X,n) := \frac{\ker \mathrm{d}_n}{\operatorname{im} \mathrm{d}_{n+1}}. \qquad \diamond$$

Example 3.3.3. What is $\mathrm{CH}^c(X,0)$. Take group of cycles $Z^c(X,0)$ – i.e. codimension c cycles in X – and quotient it by d(codimension c cycles in $X \times \Delta^1 \cong X \times \mathbb{A}^1$) – i.e. quotient by rational equivalence. Thus, $\mathrm{CH}^c(X,0) \cong \mathrm{CH}^c(X)$ is the usual Chow group. $\triangle$

Slogan ("Proposition")**.** Higher Chow enjoy all the usual properties (think: Hannah's first lecture) of ordinary Chow groups, except the excision sequence becomes a long exact sequence.

There's a long exact sequence

$$\begin{tikzcd} & \dots \arrow[r] & \mathrm{CH}^{*-c}(X \setminus Y, 2) \arrow[dll] \\ \mathrm{Ch}^{*-c}(Y,1) \arrow[r] & \mathrm{CH}^{*-c}(X,1) \arrow[r] & \mathrm{CH}^{*-c}(X \setminus Y, 1) \arrow[dll] \\ \mathrm{Ch}^{*-c}(Y,0) \arrow[r] & \mathrm{CH}^{*-c}(X,0) \arrow[r] & \mathrm{CH}^{*-c}(X \setminus Y, 0) \arrow[r] & 0 \end{tikzcd}$$

Example 3.3.4. Let's compute $\mathrm{CH}^*(\operatorname{Spec} k, 1)$, where k is algebraically closed. Look at codimension c cycles in $\operatorname{Spec} k \times \Delta^1 = \Delta^1$ mod d(codimension c cycles in Δ^2). The only interesting case is $c = 1$ (there are no codimension ≥ 2 cycles, and the only codimension 0 cycle is Δ^1, which is zero is Higher Chow since it's $\mathrm{d}(\Delta^2)$). Thus, $\mathrm{CH}^*(\operatorname{Spec} k, 1) = 0$ if $* \neq 1$.

Claim 3.3.5. $\mathrm{CH}^1(\operatorname{Spec} k, 1) = k^\times$

The map comes from $[(x,y)] \mapsto -\frac{x}{y} \in k^\times$ (here, $(x,y) \in \Delta^1 \subset \mathbb{A}^2$).[6]

[4] Assume X, Y both smooth

[5] Relabel variables in increasing order

[6] Important: there are no coordinate points where either x or y is 0, since those points are not transverse to the hyperplanes

Exercise. Show this gives the claimed isomorphism.

In general, we don't actually know the higher Chow groups of a point. △

As a toy application of this higher stuff, we will compute the Chow ring of $\mathbb{P}^1$.

Example 3.3.6. Consider $[pt] \in \mathbb{P}^1$ and $\mathbb{P}^1 \setminus \{pt\} = \mathbb{A}^1$. Look at the excision sequence:

$$\begin{tikzcd}
\mathrm{CH}^*(\mathbb{A}^1, 1) \arrow[r] \arrow[d, equal] & \mathrm{CH}^{*-1}(pt) \arrow[r] \arrow[d, equal] & \mathrm{CH}^*(\mathbb{P}^1) \arrow[r] & \mathrm{CH}^*(\mathbb{A}^1) \arrow[r] \arrow[d, equal] & 0 \\
\mathrm{CH}^*(pt, 1) \arrow[d, equal] & \mathbb{Z}[1] & & \mathrm{CH}^*(pt) \arrow[d, equal] & \\
k^\times[1] & & & \mathbb{Z}[0] &
\end{tikzcd}$$

(The square brackets indicate the degree in which the group lives). The key point now is that $\mathrm{Hom}(k^\times, \mathbb{Z}) = 0$, so the class of a point in $\mathrm{CH}^1(\mathbb{P}^1)$ is not torsion. △

Remark 3.3.7. One can use this strategy to prove Theorem 1.1.6. In particular, to show the classes of the strata are independent. Maybe also keep in mind that if you want to show such a map (e.g. $\mathrm{CH}^{*-1}(pt) \to \mathrm{CH}^*(\mathbb{P}^1)$) is injective, you can do this after passing to an algebraic closure. ○

The rest of the lecture is focused on a non-toy application.

Theorem 3.3.8 (L.)**.** *In characteristic $\neq 2, 3$, the Chow ring of $\overline{M}_2$ is*[7]

$$\mathrm{CH}^*(\overline{M}_2) = \frac{\mathbb{Z}[\lambda_1, \lambda_2, \delta_1]}{(24\lambda_1^2 - 48\lambda_2, 20\lambda_1\lambda_2 - 4\delta_1\lambda_2, \delta_1^3 + \delta_1^2\lambda_1, 2\delta_1^2 + 2\delta_1\lambda_1)}.$$

(Eric also explicitly thanked Akhil Mathew, up to spelling, for conversations from which he benefited greatly)

Remark 3.3.9. $\delta_0 = 10\lambda_1 - 2\delta_1$ ○

Fact (exercise)**.** $|\omega|_C$ is basepoint free $\iff$ $[C] \notin \Delta_1$ (this is for $[C] \in \overline{M}_2$)

In the complement $\overline{M}_2 \setminus \Delta_1$, $|\omega|_C$ is branched at 6 points, so your curve looks like $z^2 = ax^6 + bx^5y + \cdots + gy^6$. Thus, $\overline{M}_2 \setminus \Delta_1 \simeq \left(\mathbb{A}^7 \setminus \{\text{triple roots}\}\right) / ((\mathrm{GL}_2 \times \mathbb{G}_m)/\mathbb{G}_m)$ with the $\mathbb{G}_m$ at the bottom corresponding to $(t\,\mathrm{Id}, t^3)$.[8] On the other hand, $\Delta_1 \simeq \mathrm{Sym}^2 \overline{M}_{1,1}$. In equations, the two curves looks like $y_1^2 = x_1^3 + a_1x_1 + b$ and $y_2^2 = x_2^3 + a_2x_2 + b_2$. Thus,

The equations here are the only place where the characteristic assumption enters

$$\Delta_1 \simeq \frac{\mathbb{A}^4 \setminus \{0 \times \mathbb{A}^2 \cup \mathbb{A}^2 \times 0\}}{(\mathbb{G}_m \times \mathbb{G}_m) \rtimes (\mathbb{Z}/2\mathbb{Z})}$$

Using these presentations, it's apparently not hard to compute the Chow rings of Δ_1 and its complement. The difficulty comes in patching them together, and this is where we turn to higher Chow groups:

$$\mathrm{CH}^*(\overline{M}_2 \setminus \Delta_1, 1) \longrightarrow \mathrm{CH}^{*-1}(\Delta_1) \xrightarrow{\star} \mathrm{CH}^*(\overline{M}_2) \longrightarrow \mathrm{CH}^*(\overline{M}_2 \setminus \Delta_1) \longrightarrow 0$$

(we want $\star$ to be injective).

[7] Note: this is the stack, not the coarse space. I guess integral Chow of the coarse space isn't a ring?
[8] Also, GL_2 acts on x, y and $\mathbb{G}_m$ acts on z. The hyperelliptic involution is $(\mathrm{id}, -1)$

Warning 3.3.10. One might hope that $\mathrm{CH}^*(\overline{M}_2 \setminus \Delta_1, 1)$ is divisible so the boundary map is zero (no maps from divisible to f.g. group). Unfortunately, it is not. $\bullet$

However, it is not that bad. Up to a divisible part, it is generated by two 2-torsion classes in degrees 4 and 5. There are only finitely many 2-torsion classes in degrees $3, 4$ in $\mathrm{CH}^*(\overline{M}_2 \setminus \Delta_1, 1)$. Thus, you can write down a finite list of possibilities for the kernel of $\star$. One then uses various additional tricks to rule out these possibilities.

4 Angelo Vistoli: Patching techniques for moduli problems and the integral Chow ring of $\overline{M}_{1,2}$

4.1 Lecture 1 (7/12)

"I'm not supposed to say the word stack."

Let M be a moduli space (secretly, a smooth, separated DM stack).

Example 4.1.1. $M_g, \overline{M}_g, M_{g,n}, \overline{M}_{g,n}$ △

Remark 4.1.2. In this talk, we work classically. In particular, everything is over $\mathbb{C}$. ○

All these examples have some universal deformation spaces (which give étale charts on the stack). These are local replacements for the fact that (the coarse space of) M is not a fine moduli space. If X_0 is an object over $\mathbb{C}$. We require $\mathrm{Aut}(X_0)$ to be finite. By a `universal deformation space` we mean a family $X \to U$ along with a point $u_0 \in U$ and an isomorphism $X_0 \simeq X_{U_0}$ satisfying the following universal family: étale locally, if $Y \to S$ is a family and $s_0 \in S$ s.t. $Y_{s_0} \simeq X_0 \simeq X_{u_0}$, then there's a (unique) morphism $S \to U$ (sending $s_0 \mapsto u_0$) such that $Y \simeq S \times_U X$.

Remark 4.1.3. The universal deformation spaces depends on a choice of isomorphism $X_0 \simeq X_{u_0}$. ○

Remark 4.1.4. $\mathrm{Aut}(X_0)$ acts (locally) on U. Say have $Y : S \to U$. If $g \in \mathrm{Aut}(x_0)$, the isomorphism $X_0 \xrightarrow{g} X_0$ gives (étale locally) a map $U \to U$. This gives an action of $\mathrm{Aut}(X_0)$ on U. ○

If you look at the map $U \to M$ induces by $X \to U$, it will be $\mathrm{Aut}(x_0)$-invariant, so induces a map $U/\mathrm{Aut}(X_0) \to M$. This resulting map will be étale. This let's you describe the moduli space locally as a quotient of a finite group.

Question 4.1.5 (Audience)**.** *How do you construct the universal deformation space?*

Answer. You start w/ a versal deformation space and then you slice it. For something like M_g, start with the family over some Hilbert scheme. This will give a map to M with very large fibers. Then you start slicing it down, and if you do so generically, eventually you'll get something with finite fibers. The details of the construction are non-obvious. ⋆

Remark 4.1.6. Sounds like you can remove lots of the explicit/implicit "étale locally" by passing to the henselization (and then spread things back out to U's via the phrase "by standard limit arguments."). ○

Warning 4.1.7. The moduli problem is smooth if U is smooth for all x_0. However, this *does not* imply that M is smooth. •

"I'm not supposed to work with the stack. Working with the stack would be much easier for me" (after a question clarifying whether M was the stack or the coarse space).

Theorem 4.1.8 (Mumford, V.)**.** *$\mathrm{CH}^*(M)_{\mathbb{Q}}$ has a natural ring structure. Given $S \to M$, there are pullbacks $\mathrm{CH}^*(M) \to \mathrm{CH}^*(S)$. So, in this (and other) respects, it behaves like the rational Chow ring of a smooth variety.*

The proof idea is to work w/ (rational) cycles on M, but to view them locally on U. This requires some work when M is not smooth. Locally fix a point $m_0 \in M$, corresponding to some curve X_0. Take a deformation space $X \to U$, inducing $\varphi : U \to M$. We want to figure out $Z^*(M)_{\mathbb{Q}} \xrightarrow{\varphi^*} Z^*(U)$. We have factorization

$$\begin{tikzcd} U \arrow[r, "\pi"'] \arrow[rr, bend left, "\varphi"] & U/G \arrow[r, "\text{ét}"'] & M \end{tikzcd}, \text{ where } G = \mathrm{Aut}(X_0).$$

To obtain φ^*, we want to know what multiplicites to assign to pull backs. This is easy to do for $\psi : U/G \to M$ since it's étale.

Remark 4.1.9. If M is smooth, then $U \to U/G$ is flat. $\circ$

Say $V \subset M$ irreducible, and $W \subset \psi^{-1}(V)$ is an irreducible component.

Exercise. Assume M is smooth. Say Z is a component of $\pi^{-1}(W)$, and let r be the multiplicity of $[Z]$ in $\pi^*(W)$. Show that

$$r = \frac{\# G_{Z_0}}{\# G_0} \in \mathbb{Z},$$

where G_{Z_0} is the stabilizer of a generic point of Z, and G_0 is the kernel of the action.

If M is not necessarily smooth, define r to be this quotient $\# G_{Z_0}/\# G_0$. This gives a pullback $\varphi^* : Z^*(M) \to Z^*(U)$. Note that r only depends on $V \subset M$ (and not on the choice of component of $\varphi^{-1}(V)$).

Each $V \subset M$ has a fundamental class $[V] \in \mathrm{CH}^*(M)_{\mathbb{Q}}$. The $\mathbb{Q}$-`fundamental class` of $[V]$ is $[V]_{\mathbb{Q}} = \frac{\# G_0}{\# G_{Z_0}}[V]$. Thus, $\varphi^*[V]_{\mathbb{Q}}$ is a combination of varieties in U, each w/ multiplicity 1.

Note that $U \to [U/G]$ is étale

Example 4.1.10. Take $M = \overline{M}_{1,1} = \mathbb{P}^1$, the j-line. Take a point $m_0 \in M_{1,1} \simeq \mathbb{A}^1 \subset \mathbb{P}^1$. This corresponds to some (elliptic) curve C_0 w/ j-invariant m_0. Then,

$$[m_0]_{\mathbb{Q}} = \frac{2}{\#\operatorname{Aut}(C_0)}[pt] = \begin{cases} \frac{1}{2}[pt] & \text{if } j = 1728 \\ \frac{1}{3}[pt] & \text{if } j = 0 \\ [pt] & \text{otherwise.} \end{cases}$$

$\triangle$

Theorem 4.1.11. *There exists a ring structure on $Z^*(M)_{\mathbb{Q}}$. If $V, W \subset M$ with $\operatorname{codim}(V \cap W) = \operatorname{codim} V + \operatorname{codim} W$ and Z is a component of $V \cap W$, then the multiplicity of $[Z]_{\mathbb{Q}}$ in $[V]_{\mathbb{Q}}[W]_{\mathbb{Q}}$ is*

$$\textit{multiplicity of } [Z'] \textit{ in } [V'][W']$$

if $z_0 \in Z$ is a generic point, $U \to M$ is a universal deformation space around z_0, and blah' is a (dominating) component of the inverse image (under $U \to M$) of blah.

If $X \to S$ is a family, get square

$$\begin{tikzcd} S' \arrow[r, "h"] \arrow[d, "\text{ét}"'] & U \arrow[d, "\varphi"] \\ S \arrow[r, "f"] & M \end{tikzcd}$$

Say $V \subset M$ with $\operatorname{codim} V = \operatorname{codim} f^{-1}(V)$. Write $g : S' \to S$ for above map. Then,

Something like this. I didn't really follow this part in real time...

$$g^* f^* [V]_{\mathbb{Q}} = h^*\big([\varphi^{-1}(V)_{\mathrm{red}}]\big)$$

Example 4.1.12. Say we have a 2-dimensional moduli problem which locally looks like $U = \mathbb{A}^2$ along with μ_2 acting by multiplication. The quotient U/μ_2 is a cone. Given two lines L_1, L_2 on $\mathbb{A}^2$, get two lines V_1, V_2 of the ruling in U/μ_2. Convince yourself the $\mathbb{Q}$-fundamental classes of these are the usual fundamental classes. One computes

$$[V_1][V_2] = \frac{1}{2}[L_1][L_2] = \frac{1}{2}[pt]$$

(note this is the $\mathbb{Q}$-fundamental class of the singular point). $\triangle$

4.1.1 $\overline{M}_2$ in 5 minutes

Note $\mathrm{CH}^*(M_2)_{\mathbb{Q}} = \mathbb{Q}$ because M_2 is the quotient of an open in $\mathbb{A}^3$ by S_6 (a genus 2 hyperelliptic curve has 6 Weierstrass points by R-H).

Note 2. Vistoli drew some picture of $X \to U$ that I didn't follow. Something about following how to nodes move in the family and their image in the base being a divisor w/ normal crossings?

Fact. The locus of curves in $\overline{M}_g$ w/ r nodes has codimension r.

Furthermore, $\overline{M}_g$ has a combinatorial structure we don't have time to describe.

Example 4.1.13. How to describe a curve of genus 2 with 1 node. It either looks like a nodal genus 1 curve α (giving a component Δ_0 in the moduli space) or a nodal union of two smooth genus 1 curves X (giving a component Δ_1 in the moduli space). In general, get divisors $\Delta_0, \Delta_1, \dots, \Delta_{\lfloor g/2 \rfloor} \subset \overline{M}_g$. △

Vistoli finished by drawing pictures of all (six) types of curves corresponding to components in the boundary of $\overline{M}_g$ along with which degenerate to which. There are two types each with $1, 2$, or 3 nodes. Exercise: reproduce this picture. If it helps, the strata are called $\Delta_0, \Delta_1, \Delta_{00}, \Delta_{01}, \Delta_{000}, \Delta_{001}$.

"Just to point out, you are over time." - Audience (probably an organizer) "Yes, I am over time... Let me point out very briefly *continues to lecture*"

Note 3. Missed the last thing (hopefully get reexplained next time), but Vistoli said something about getting generators for Chow using these strata.

4.2 Lecture 2 (7/13)

We want to talk about Grothendieck Riemann-Roch and relations among the canonical classes (λ_i's, κ_i's).

Recall 4.2.1. Say $C \xrightarrow{\pi} S$ is a family on $\overline{M}_g$. Get relative dualizing sheaf $\omega = \omega_{C/S}$, and set $K = c_1(\omega) \in \mathrm{CH}^1(C)$. We also define the classes

$$\lambda_i = c_i(\pi_*\omega) \text{ and } \kappa_i = \pi_*\left(K^{i+1}\right)$$

on the base S. Note

(1) We can view these classes as living in $\mathrm{CH}^*(M_g)_{\mathbb{Q}}$

(2) To prove relations, can assume $C \to S$ is `versal` (like 'universal' but w/o uniqueness). If $s_0 \in S$, the curve C_{s_0} has a universal deformation space U, so étalle locally get a map $S \to U$ (really $S' \to U$ w/ $S' \to S$ étale). $C \to S$ is versal (at s_0) if $S' \to U$ is smooth. ⊙

To justify these extra points, note that $\overline{\mathcal{M}}_g = [X/G]$ is a quotient stack of some smooth variety X. Hence,

Question: Why is it a global quotient with X smooth?

$$\mathrm{CH}^*(\overline{M}_g)_{\mathbb{Q}} = \mathrm{CH}^*(\overline{\mathcal{M}}_g)_{\mathbb{Q}} = \mathrm{CH}^*(X \times U/G).$$

Something something $C \to X$ gives rise to $(C \times U)/G \to (X \times U)/G$, a versal family. I guess $C \to X$ is the pullback of the universal family on $\overline{M}_g$.

Say X is a smooth (q.proj) variety and $E \to X$ is a vector bundle (of rank r). Then, there are Chern classes $c_i(E) \in \mathrm{CH}^i(X)$ as well as the total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \dots$. Given an exact sequence $0 \to E' \to E \to E'' \to 0$, the total Chern classes multiple

$$c(E) = c(E')c(E'').$$

Since X is smooth, the above also works for coherent sheaves. The idea is that a coherent sheaf on a smooth variety has a resolution by locally free sheaves.

Recall 4.2.2. $c_i(E) = \sigma_i(\ell_1, \dots, \ell_r)$ is the ith symmetric function in the Chern roots. This let's us define functions on Chern classes via symmetric functions on their roots. ⊙

Definition 4.2.3. The Chern character of E is the power series

$$\mathrm{ch}(E) := e^{\ell_1} + \dots + e^{\ell_r} = \operatorname{rank} E + c_1(E) + \frac{c_1(E)^2 - 2c_2(E)}{2} + \frac{c_1^3 - 3c_1c_2 + c_3}{3!} + \dots. \qquad \diamond$$

There is also the Todd class coming from the power series

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \dots = 1 + \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{(2k)!} x^{2k}.$$

Definition 4.2.4. The dual Todd class is

$$\mathrm{Td}^\vee(E) = \frac{\ell_1}{e^{\ell_1} - 1} \cdot \dots \cdot \frac{\ell_r}{e^{\ell_r} - 1} = 1 - \frac{c_1}{2} + \frac{1}{12}(c_1^2 + c_2) - \frac{1}{24} c_1 c_2 + \frac{1}{720}(c_1^3 + \dots) + \dots. \qquad \diamond$$

(the dual comes from exchanging $x \leftrightarrow -x$, so only changes the single odd term)
Given $0 \to E' \to E \to E'' \to 0$, get

$$\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'') \text{ and } \mathrm{Td}^\vee(E) = \mathrm{Td}^\vee(E')\,\mathrm{Td}^\vee(E'').$$

Theorem 4.2.5 (Grothendieck-Riemann-Roch)**.** *Let $f : X \to Y$ is a projective, surjective morphism, and $\mathscr{F}$ is a coherent sheaf on X. Then,*

$$\mathrm{ch}(Rf_*\mathscr{F}) = \mathrm{ch}(f_*\mathscr{F}) - \mathrm{ch}\big(R^1 f_*\mathscr{F}\big) + \mathrm{ch}\big(R^2 f_*\mathscr{F}\big) - \dots = f_*\big(\mathrm{ch}(F) \cdot \mathrm{Td}^\vee\big(\Omega_{X/Y}\big)\big)$$

(note $\Omega_{X/Y}$ is a coherent sheaf, but not a vector bundle, in general)

Example 4.2.6. Say Y is a point, X is a smooth projective curve, and $\mathscr{F}$ is a line bundle. Then, one gets that

$$\chi(\mathscr{F}) = \deg\left((1 + c_1(\mathscr{F}))\left(1 - \frac{K}{2}\right)\right) = \deg F - \frac{\deg K}{2}.$$

Doing this for $F = \mathscr{O}$ (to get $\deg K = 2g - 2$) and $F = \mathscr{O}(D)$ gives Riemann-Roch. △

Say $C \xrightarrow{\pi} S$ is a family of stable curves. Then, $R^1\pi_*\omega_C = \mathscr{O}_S$ and $\pi_*\omega_C =: \mathbb{E}$ (the Hodge bundle).

Example 4.2.7. If $C \xrightarrow{\pi} S$ is smooth (so $\omega = \Omega_{C/S}$), then (note $1 = \mathrm{ch}(\mathscr{O}_S)$)

$$\mathrm{ch}(\mathbb{E}) - 1 = \mathrm{ch}\big(R\pi_*\omega_{C/S}\big) = \pi_*\big(\mathrm{ch}(\omega)\,\mathrm{Td}^\vee(\omega)\big) = \pi_*\left(e^K \cdot \frac{K}{e^K - 1}\right) = \pi_*\left(\frac{K}{1 - e^{-K}}\right).$$

This expands to ($\pi_*(1) = 0$)

$$\mathrm{ch}(\mathbb{E}) - 1 = \pi_*\left(1 + \frac{K}{2} + \frac{K^2}{12} - \frac{K^4}{720} + \dots\right) = \frac{\pi_*(K)}{2} + \frac{\pi_*(K^2)}{12} - \frac{\pi_*(K^4)}{720} = \frac{\deg K}{2} + \frac{\kappa_1}{12} - \frac{\kappa_3}{720} + \dots$$

(note $\deg K/2 = g-1$). Thus,

$$\mathrm{ch}(\mathbb{E}) = g + \sum_{k\geq 1}(-1)^{k-1}\frac{B_k}{(2k)!}\kappa_{2k-1}. \qquad \triangle$$

Example 4.2.8. What if $C \xrightarrow{\pi} S$ is not necessarily smooth. Then, $\omega \ncong \Omega_{C/S}$, so things become trickier. Let $Z \subset C$ be the singular locus. A "local calculation" (i.e. deformation theory) shows that ($\Omega_{C/S}$ is torsion-free b/c the family is versal)

$$0 \longrightarrow \Omega_{C/S} \longrightarrow \omega_{C/S} \longrightarrow \omega_{C/S}\otimes \mathscr{O}_Z \longrightarrow 0$$

(if I heard correctly, $\omega_{C/S} = \left(\Omega_{C/S}^\vee\right)^\vee$?). Somehow the point is that near a node, the situation looks like $xy = t$. Accepting this,

$$\mathrm{Td}^\vee(\Omega_{C/S}) = \frac{K}{e^K-1}\cdot \mathrm{Td}^\vee\left(\omega_{C/S}\otimes\mathscr{O}_Z\right) = \frac{K}{e^K-1} + \left(\mathrm{Td}^\vee(\omega_{C/S}\otimes \mathscr{O}_Z)-1\right)\frac{K}{e^K-1}.$$

The first factor gives the same thing as what we got in the smooth case. The second factor lives on the singular locus. To compute this, one forms a double cover $Z' \to Z$ so that a point of Z' is a node along w/ a choice of branch. In particular, $\omega_{C/S}|_Z$ pulled back to Z' becomes trivial. As a consequence of this (b/c we're working in rational Chow?), one gets

$$\frac{K}{e^K-1} + \left(\mathrm{Td}^\vee(\omega_{C/S}\otimes\mathscr{O}_Z)-1\right)\frac{K}{e^K-1} = \frac{K}{e^K-1} + \left(\mathrm{Td}^\vee(\mathscr{O}_Z)-1\right).$$

Since Z has codimension 2 in C, one gets that $c_1(\mathscr{O}_Z) = 0$. Thus,

$$\mathrm{Td}^\vee(\mathscr{O}_Z) - 1 = \frac{1}{12}c_2(\mathscr{O}_Z) + \dots.$$

Exercise. $c_2(\mathscr{O}_Z) = [Z]$ is the fundamental class of Z.

The final result is

$$\mathrm{ch}(\mathbb{E}) = g + \pi_*\left(\frac{K^2}{12} + \frac{1}{12}[Z] + \frac{k^4}{720} + ? + \dots\right) = g + \frac{\kappa_1 + [\Delta]}{12} + \dots,$$

where $\Delta \subset S$ is the `discriminant locus`; this is the image of Z (which is birational to Z, apparently, and sounds like this is why $\pi_*[Z] = [\Delta]$). One usually writes $\delta = [\Delta] = \delta_0 + \dots + \delta_{\lfloor g/2\rfloor}$. In codimension one, one gets `Mumford's formula`

$$\lambda_1 = \frac{1}{12}(\kappa_1 + \delta). \qquad \triangle$$

Recall that Eric did a calculation (Example 3.2.8) when $C \to \mathbb{P}^1$ is a general pencil of quartics in $\mathbb{P}^2$. We showed that, in this case,

$$\lambda_1 = 3, \ \ \kappa_1 = 9, \ \text{ and } \ \delta = 27.$$

This satisfies Mumford's formula, as it must. Another corollary of the above computation is

Corollary 4.2.9. $\mathrm{ch}(\mathbb{E})_k = 0$ *if k is even.*

Example 4.2.10. $\mathrm{ch}_2(\mathbb{E}) = \frac{1}{2}\lambda_1^2 - 2\lambda_2$, so $\lambda_2 = \lambda_1^2/2$ always. $\triangle$

4.3 Lecture 3 (7/14)

Let $\mathcal{M}$ be a moduli stack (e.g. $\mathcal{M}_g, \overline{\mathcal{M}}_g, \ldots$). These are typically quotient stacks $\mathcal{M} = [X/G]$. In such cases, we can describe the Chow ring using equivariant Chow rings: $\mathrm{CH}(\mathcal{M}) = \mathrm{CH}_G(X)$. Understanding these integral Chow rings is hard in general, but has been done in some cases.

- Say X is an open in a representation V of G. Let $Z := V \setminus X$. Then, you get a localization sequence

$$\mathrm{CH}_*^G(Z) \longrightarrow \mathrm{CH}_G^*(V) \longrightarrow \mathrm{CH}_G^*(X) \longrightarrow 0$$

  (note $\mathrm{CH}_G^*(V) = \mathrm{CH}_G^*(pt)$ and $\mathrm{CH}_G^*(X) = \mathrm{CH}^*(\mathcal{M})$). Computing the image of the first map is still hard, but this at least gives an approach that has been successfully carried out in some cases.

- In many cases, have a closed substack $\mathcal{Y} \subset \mathcal{M}$ and are able to understand $\mathrm{CH}^*(\mathcal{Y})$ and $\mathrm{CH}^*(\mathcal{M} \setminus \mathcal{Y})$. However, it can still be hard to get a handle on the excision sequence

$$\mathrm{CH}^*(\mathcal{Y}) \xrightarrow{i_*} \mathrm{CH}^*(\mathcal{M}) \xrightarrow{j^*} \mathrm{CH}^*(\mathcal{M} \setminus \mathcal{Y}) \to 0$$

  Let $N = N_{\mathcal{Y}/\mathcal{M}}$ be the normal bundle. Note/recall that i^*i_* is multiplication by the top Chern class $c_{\mathrm{top}}(N)$. Hence, if you have some class whose product w/ the top Chern class of the normal bundle is nonzero, then i_* of this class is nonzero.

  Remark 4.3.1. If $c_{\mathrm{top}}(N)$ is not a zero divisor, then i_* is injective. ○

  Warning 4.3.2. This basically never happens for these moduli space. Why? Rationally, $\mathrm{CH}^i(\mathcal{Y})_{\mathbb{Q}} = \mathrm{CH}^i(\mathrm{cms})_{\mathbb{Q}} = 0$ if $i > \dim Y$. Since the top Chern class will live in positive degree, some power of it will vanish. •

  I missed something, but there's some idea to extend $\mathcal{M}$ to a stack $\widetilde{\mathcal{M}}$ w/ infinite stabilizers. Apparently, if you also extend $\mathcal{Y}$ to some $\widetilde{\mathcal{Y}} \hookrightarrow \widetilde{\mathcal{M}}$, you can *sometimes* get $c_{\mathrm{top}}\left(N_{\widetilde{\mathcal{Y}}/\mathcal{M}_{\text{ét}}}\right)$ to not be a zero divisor. I think I missed something else, but somehow it can useful to consider

$$\mathrm{CH}^*(\widetilde{\mathcal{M}}) \xrightarrow{(j^*,i^*)} \mathrm{CH}^*(\widetilde{\mathcal{M}} \setminus \widetilde{\mathcal{Y}}) \times \mathrm{CH}^*(\widetilde{\mathcal{Y}})$$

  (which is injective?). Finally, one wants to compute the quotient

$$\mathrm{CH}^*(\widetilde{\mathcal{M}}) \twoheadrightarrow \mathrm{CH}^*(\mathcal{M})$$

For moduli of curves, one usually gets $\widetilde{\mathcal{M}}$ by adding some (non-stable) singular curves, e.g. cuspidal curves. In fact, there's a notion of "stable cuspidal curves."

Definition 4.3.3. Let $(C, p_1, \ldots, p_n)$ be an n-pointed cuspidal curve (p_i's distinct and smooth). This is `stable` if $\omega_C(p_1 + \cdots + p_n)$ is ample. ◇

Example 4.3.4. Consider $\overline{\mathcal{M}}_{1,1}$. This is $[\{y^2 = x^3 + ax + b : (a,b) \neq 0\}/\mathbb{G}_m]$. Note that $\mathbb{G}_m$ acts on $\mathbb{A}^4$ parameterizing (x, y, a, b) with weights $(2, 3, 4, 6)$. Write this space as $V_{2,3,4,6}$ Inside here is a universal curve $C \subset V_{2,3,4,6}$.

Remark 4.3.5. The presentation $\overline{\mathcal{M}}_{1,1} = [(V_{4,6} \setminus \{0\})/\mathbb{G}_m]$ gives a way of computing the Chow ring of $\overline{\mathcal{M}}_{1,1}$. One gets

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,1}) = \frac{\mathrm{CH}_{\mathbb{G}_m}(V_{4,6})}{([0])} = \frac{\mathbb{Z}[t]}{24t^2}$$

(note $\overline{\mathcal{M}}_{1,1} = \mathcal{P}(4,6)$ is a weighted projective stack). ○

To enlarge to stable cuspidal curves, consider $\widetilde{\mathcal{M}}_{1,1} = [V_{4,6}/\mathbb{G}_m]$ which has Chow group $\mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}) = \mathbb{Z}[t]$. △

Note 4. Had technicality difficulties for a few minutes, so missed stuff.

Summary of what was said while overleaf was not working

- Strategy developed to compute Chow ring of $\overline{\mathcal{M}}_2$. Method did not work. Ran into some issues, which were later resolved, but not before being scooped by Eric.
- It worked successfully to compute Chow ring of $\overline{\mathcal{M}}_{2,1}$ by Andrea Di Lorenzo, Michele P., and V.
- Sounds like they also computed Chow ring of $\overline{\mathcal{M}}_{1,2}$.

  We'll say more about this.

Steps, sorta kinda

(1) View $\overline{\mathcal{M}}_{1,2}$ as the universal curve $\overline{\mathcal{C}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ (non-trivial, due to Knutsen)

There are a couple natural classes in $\mathrm{CH}(\overline{\mathcal{C}}_{1,1})$, called λ and μ_1. I missed what λ_1 is (I'd guess first Chern of dualizing sheaf?) is, but μ_1 is $[\overline{\mathcal{M}}_{1,1}] \in \mathrm{CH}^1(\overline{\mathcal{C}}_{1,1})$.[9]

Theorem 4.3.6.

$$\mathrm{CH}^*(\overline{\mathcal{M}}_{1,2}) = \mathrm{CH}^*\left(\overline{\mathcal{C}}_{1,1}\right) = \frac{\mathbb{Z}[\lambda_1, \mu_1]}{(24\lambda_1^2, \mu_1(\lambda_1 + \mu_1))}.$$

(2) Extend to $\widetilde{\mathcal{M}}_{1,1}$ w/ corresponding universal family $\widetilde{\mathcal{C}}_{1,1} \to \widetilde{\mathcal{M}}_{1,1}$. Apply strategy to $\widetilde{\mathcal{M}}_{1,1} \subset \widetilde{\mathcal{C}}_{1,1}$. One knows $\mathrm{Ch}(\widetilde{\mathcal{M}}_{1,1}) = \mathbb{Z}[\lambda_1]$. Furthermore,

$$\widetilde{\mathcal{C}}_{1,1} \setminus \widetilde{\mathcal{M}}_{1,1} = \underbrace{\left\{(x, y, a, b) \in V_{2,3,4,6} : y^2 = x^3 + ax + b\right\}}_{C} / \mathbb{G}_m.$$

To understand C, consider the projection $C \to V_{2,3}$, $(x, y, a, b) \mapsto (x, y)$. This is a $\mathbb{G}_m$-equivariant affine bundle, and so induces an isomorphism on Chow rings. Thus,

$$\mathrm{CH}(\widetilde{\mathcal{C}}_{1,1} \setminus \widetilde{\mathcal{M}}_{1,1}) \simeq \mathrm{CH}([V_{2,3}/\mathbb{G}_m]) \simeq \mathrm{CH}(B\mathbb{G}_m) \simeq \mathrm{CH}(\widetilde{\mathcal{M}}_{1,1}) \simeq \mathbb{Z}[\lambda_1].$$

The upshot is that $\mathrm{CH}(\widetilde{\mathcal{C}}_{1,1})$ is generated by λ_1, μ_1. Apparently, one can directly show the relation $\mu_1(\mu_1 + \lambda_1) = 0$. To show this is the only relation, one uses that $c_{\mathrm{top}}(\text{normal bundle}) = \pm\lambda_1$ is not a zero divisor, and so concludes $\mathrm{CH}(\widetilde{\mathcal{C}}_{1,1}) = \mathbb{Z}[\lambda_1, \mu_1]/(\mu_1(\lambda_1 + \mu_1))$. Finally, one needs to throw away the cuspidal locus. Doing this let's them arrive at the expression in the theorem statement.

[9]This is a substack via the tautological section

5 List of Marginal Comments

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