18.755 (Lie Groups and Lie algebras II) Notes

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These are my course notes for "Lie Groups and Lie algebras II" at MIT. Each lecture will get its own "chapter." These notes are live-texed or whatever, so there will likely to be some (but hopefully not too much) content missing from me typing more slowly than one lectures. They also, of course, reflect my understanding (or lack thereof) of the material, so they are far from perfect.[1] Finally, they contain many typos, but ideally not enough to distract from the mathematics. With all that taken care of, enjoy and happy mathing.

The instructor for this class is Pavel Etingof. This class overlaps once a week with a seminar that I am attending, so that might cause issue in these notes.

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[1] In particular, if things seem confused/false at any point, this is me being confused, not the speaker

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1 Lecture 1 (2/16)

1.1 Class stuff

Homeworks assigned/due on Thursdays. Lecture notes here if you can acess the Canvas.

1.2 Review of material from last term

1.2.1 Lie groups

Definition 1.1. A **real (complex) Lie group** is a real (complex) manifold G which is also a group such that $G \times G \to G$ is regular (analytic). A **homomorphism of Lie groups** $G \to H$ is a group homomorphism given by a regular map.

Example. Real: $\mathbb{R}^n, U(n), \mathrm{SU}(n), \mathrm{GL}_n(\mathbb{R}), O(p,q), \mathrm{Sp}_{2n}(\mathbb{R})$
complex: $\mathbb{C}^n, \mathrm{GL}_n(\mathbb{C}), O_n(\mathbb{C}), \mathrm{Sp}_{2n}(\mathbb{C})$

Every Lie group G has a connected component of 1 denoted G°. This is a normal subgroup, and G/G° is discrete and countable.

Say G is connected. Then its universal cover $\widetilde{G}$ is a simply connected Lie group, and comes with a map $\pi : \widetilde{G} \twoheadrightarrow G$ with $\ker \pi = Z$, some central discrete subgroup, such that $\widetilde{G}/Z \cong G$.

Example. $G = S^1$ then $\widetilde{G} = \mathbb{R}$ and $Z = \mathbb{Z}$. Hence, in this case $\pi_1(S^1) = \mathbb{Z}$.

1.2.2 Lie subgroups

Definition 1.2. A **Lie subgroup** $H \subset G$ is a subgroup which is also an immersed submanifold (i.e. H is a Lie group and $H \hookrightarrow G$ is a regular map with injective differential at every point). A **closed Lie subgroup** $H \subset G$ is a subgroup which is an embedded submanifold (i.e. locally closed).

Remark 1.3. A closed Lie subgroup is equivalently a Lie subgroup which is closed in G.

Example. $\mathbb{Q} \subset \mathbb{R}$ is a Lie subgroup, but not an embedded submanifold. However, $\mathbb{Z} \subset \mathbb{R}$ is a closed Lie subgroup.

Example. $O_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ is a closed Lie subgroup.

Example. An irrational torus winding $R \subset S^1 \times S^1$ is a Lie subgroup which is not closed.

We will almost always work with closed Lie subgroups.

Fact (Did not prove last semester)**.** Any closed subgroup of G is a closed Lie subgroup.

Fact (Did prove last semester)**.** Any connected Lie group is generated by any neighborhood of 1.

Definition 1.4. Let $H \subset G$ be a closed Lie subgroup. Then, the quotient G/H is a manifold with a transitive G-action, i.e. a **homogeneous G-space**. If H is a normal subgroup, then G/H is a Lie group.

If G acts transitively on a manifold X (i.e. $G \times X \to X$ regular), then for any $x \in X$, we get a stabilizer $G_x \subset G$ (a closed Lie subgroup), and $G/G_x \cong X$. Hence, every homogeneous space is given by a quotient of G.

More generally, say G acts on X not necessarily transitively. Then, there are orbits. For any $x \in X$, $Gx \subset X$ is an immersed submanifold, and is isomorphic to G/G_x.

1.2.3 Representations

Reps are actions of G on a vector space by a linear transformations. We usually consider complex representations, i.e. maps $G \to \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$.

We get the usual notions from representation theory: homomorphisms of reps (intertwiing operators $A : V \to W$), subreps, direct sums, duals, tensor products, irreps, indecomposable reps, etc.

Lemma 1.5 (**Schur's lemma**). *Let V, W be irreps. If they are not isomorphic then any $A : V \to W$ is trivial ($A = 0$). If they are isomorphic, then any $A : V \to V$ is scalar multiplication ($A = \lambda \mathrm{Id}$).*

Example. G acts on itself by conjugation: $g \cdot x = gxg^{-1}$. This induces $g_* : T_1G \to T_1G$, and the map $\mathrm{Ad} : g \mapsto g_*$ gives the **adjoint representation** $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

Definition 1.6. A **unitary representation** is one with an invariant positive def Hermitian form (v, w), i.e. $(gv, gw) = (v, w)$, i.e. $G \to U(n) \subset \mathrm{GL}_n(\mathbb{C})$.

Recall that in general, indecompsoable (not a direct sum) is weaker than irreducible (no nontrival proper subreps). However, any unitary representation is a direct sum of irreducible representations (so unitary indecomposable = unitary irreducible).

If G is finite (or, more generally, compact), then any representation is unitary. Take a random positive Hermitian form, and then average it over the group to get an invariant one. Thus, any finite dimensional representation of G (finite or compact) is a direct sum of irreps (i.e. **completely reducible**).

1.2.4 Lie algebras

Note that G acts on itself by right translations, i.e. $g \circ x = xg$. This is a right action. Fix $a \in T_1G =: \mathfrak{g}$. Right translation gives rise to a tangent vector $ag \in T_gG$ at g. Doing this at every point gives rise to a left invariant vector field (since left multiplication commutes with right translation) R_a on G (i.e. $Ra|_g = ag$).

We know vector fields correspond to derivations of functions. We can consider the commutator

$$[R_a, R_b] = R_aR_b - R_bR_a,$$

another left-invariant derivation (vector field), so $[R_a, R_b] = R_{[a,b]}$ for some $[a, b] \in \mathfrak{g}$. Hence, for any $a, b \in \mathfrak{g}$, we get in this way a commutator $[a, b] \in \mathfrak{g}$. This is a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

Any left-invariant vector field is determined by its value at the identity

- (**skew-symmetry**) $[x, x] = 0 \implies [x, y] = -[y, x]$
- (**Jacobi identity**)
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

Definition 1.7. A **Lie algebra** over any field k is a k-vector space $\mathfrak{g}$ with a bilinear operation $[-, -] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying skew-symmetric + Jacobi identity.

Example. If G is a Lie group, then $\mathfrak{g} = T_1G$ is a Lie algebra. We also denote it by $\mathrm{Lie}(G)$.

Example. If $G = \mathrm{GL}_n(\mathbb{C})$, then $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \mathrm{Mat}_n(\mathbb{C})$ with Lie bracket $[A, B] = AB - BA$

Example. If $G = O_n(\mathbb{C})$, then $\mathfrak{g}$ = skew-symmetric $n \times n$ matrices with same Lie bracket.

Definition 1.8. A **lie subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a subspace invariant under $[-,-]$. A **Lie ideal** is a Lie subalgebra $\mathfrak{g} \subset \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Example. The center $\mathfrak{z} \subset \mathfrak{g}$ given by $\{z \in \mathfrak{g} \mid \forall x \in \mathfrak{g} : [x, z] = 0\}$ is a Lie ideal.

If $H \subset G$ is a Lie subgroup, then $\operatorname{Lie} H \subset \operatorname{Lie} G$ is a Lie subalgebra. If H is normal, then $\operatorname{Lie} H$ is a Lie ideal.

The same representations theory notions apply to Lie algebras as well, e.g. an **n-dimensional representation of $\mathfrak{g}/k$** is a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}_n(k)$ of Lie algebras, i.e. $\varphi([a, b]) = [\varphi(a), \varphi(b)]$.

1.2.5 Exponential Map

Say G a Lie group and $\mathfrak{g} = \operatorname{Lie} G$. Given $a \in \mathfrak{g}$, consider differential equation

$$g'(t) = ag(t) \text{ and } g(0) = 1.$$

This has a unique solution which we denote by $g(t) = \exp(ta)$. This defines a **1-parameter subgroup** $\varphi : \mathbb{R} \to G$, $\varphi(t) = \exp(ta)$. This satisfies

$$\exp(ta) \exp(sa) = \exp((t + s)a).$$

Example. When $G = \mathrm{GL}_n(K)$ (and $K = \mathbb{R}, \mathbb{C}$), this is usual matrix exponential

$$\exp(ta) = \sum_{n=0}^{\infty} \frac{t^n a^n}{n!}.$$

Setting $t = 1$ gives the exponential map $\exp : \mathfrak{g} \to G$. The differential of this map at the idenity is a map $\exp_* : \mathfrak{g} \to \mathfrak{g}$ which is actually the identity map $\exp_* = \mathrm{id}$. Hence, exp is invertible near identity, and the inverse map is called $\log : U \subset G \to \mathfrak{g}$ (only defined on some open neighborhood $U \subset G$ of the identity).

This allows another definition of the commutator. One has

$$\log(\exp(a) \exp(b)) = a + b + \frac{1}{2}[a, b] + \cdots .$$

Similarly (Note: $\exp(a)^{-1} = \exp(-a)$),

$$\log(\exp(a) \exp(b) \exp(-a) \exp(-b)) = [a, b] + \cdots .$$

In either case, the $\cdots$ refers to higher order terms. The commutator measure the extent to which G° is non-commutative, e.g. G° commutative $\iff$ $[-,-] = 0$ on $\mathfrak{g}$.

1.2.6 Fundamental Theorems of Lie Theory

There are 3, and we proved 2 of them last semester?

Theorem 1.9. *For any Lie group G, there is a bijection between connected Lie subgroups $H \subset G$, and Lie subalgebras $\mathfrak{g} \subset \mathfrak{g} = \operatorname{Lie} G$ given (in one direction) by $H \mapsto \operatorname{Lie}(H)$.*

Theorem 1.10. *Let G, K be Lie groups with G simply connected. Then,*

$$\mathrm{Hom}(G, K) \xrightarrow{\sim} \mathrm{Hom}(\mathfrak{g}, \mathfrak{k})$$

where $\mathfrak{g} = \mathrm{Lie}\, G$ and $\mathfrak{k} = \mathrm{Lie}\, K$. This iso is given by taking the derivative at the identity.

Theorem 1.11 (Did not prove)**.** *For any finite dimensional real or complex Lie algebra $\mathfrak{g}$, there exists a Lie group G such that $\mathfrak{g} = \mathrm{Lie}\, G$.*

We will prove this one this term.

Corollary 1.12. *Say $K = \mathbb{R}, \mathbb{C}$. Then there is an equivalence of categories between simply connected K-Lie groups and finite dimensional K-Lie algebras given by $G \mapsto \mathrm{Lie}\, G$.*

Any connected Lie group is of the form G/Γ where G is simply connected Lie group, and $\Gamma \subset G$ is a central, discrete subgroup.

These give good classification of Lie groups in terms of Lie algebras.

1.2.7 Representations of $\mathfrak{sl}_2(\mathbb{C}), \mathrm{SL}_2(\mathbb{C})$

Recall 1.13.

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} \text{ and } \mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$

Recall 1.14. $\mathfrak{sl}_2(\mathbb{C})$ has basis given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and } f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying

$$[e, f] = h, \quad [h, e] = 2e, \quad \text{and } [h, f] = -2f.$$

Let V_n be the $(n+1)$-dimensional representation on homogeneous polynomials in x, y of degree n: $a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$. $\mathfrak{sl}_2$ acts on V_n by acting on x, y in the natural way.

Theorem 1.15.

(1) *These representations (for $n \geq 0$) are all the irreps of $\mathfrak{sl}_2$.*

(2) *Every representation is a direct sum of irreps (i.e. completely reducible)*

(3)

$$V_n \otimes V_m = \bigoplus_{i=1}^{\min(m,n)} V_{|m-n|+2i-1}$$

*(**Clebsh-Gordan**, up to spelling)*

1.2.8 Universal enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra. Get tensor algebra $T\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$. The **universal enveloping algebra** is

$$U(\mathfrak{g}) = \frac{T\mathfrak{g}}{(x \otimes y - y \otimes x - [x,y])}.$$

It is easy to see that representations of $\mathfrak{g}$ are the same as reps of $U(\mathfrak{g})$.

Let x_i be a totally ordered basis of $\mathfrak{g}$. Then we can form ordered monomials $\prod_i x_i^{n_i}$ with $n_i \geq 0$ (and only finitely many nonzero). These span $U(\mathfrak{g})$.

Theorem 1.16 (PBW)**.** *Such monomials form a basis of* $U(\mathfrak{g})$ *(so they are linearly independent).*

1.2.9 Solvable and nilpotent Lie algebras

Given a Lie algebra $\mathfrak{g}$, consider $D(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$.

Definition 1.17. $\mathfrak{g}$ is **solvable** if $D^n(\mathfrak{g}) = 0$ for some n.

Define $L_1(\mathfrak{g}) = \mathfrak{g}$, $L_2(\mathfrak{g}) = [\mathfrak{g}, L_1(\mathfrak{g})]$, $L_3(\mathfrak{g}) = [\mathfrak{g}, L_2(\mathfrak{g})], \dots$. We say $\mathfrak{g}$ is **nilpotent** if $L_n(\mathfrak{g}) = 0$ for some n.

Remark 1.18. nilpotent $\implies$ solvable, but the reverse does not always hold.

Example. $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is solvable, but not nilpotent.

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

is nilpotent.

Theorem 1.19 (**Lie**)**.** *Working over* $\mathbb{C}$. *If* $\mathfrak{g}$ *is a f.d. solvable Lie algebra, then every irrep of* $\mathfrak{g}$ *is 1-dimensional (false is positive characteristic). Hence, any f.d. rep has a basis in which all elements of* $\mathfrak{g}$ *act by upper triangular matrices.*

Theorem 1.20 (**Engel's Theorem**)**.** *A f.d. Lie algebra* $\mathfrak{g}$ *is nilpotent* $\iff$ *all* $x \in \mathfrak{g}$ *are* ***nilpotent*** *(i.e.* $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ *given by* $\operatorname{ad} x(y) = [x,y]$ *is nilpotent).*

1.2.10 Semisimple and reductive Lie algebras

Everything from here on out is over $\mathbb{C}$ (and finite dimensional).

Definition 1.21. Let the **radical** $\operatorname{rad}(\mathfrak{g})$ of $\mathfrak{g}$ be the sum of all its solvable ideals (equivalently, the largest solvable ideal).

Definition 1.22. We say $\mathfrak{g}$ is semisimple if $\operatorname{rad}(\mathfrak{g}) = 0$. The **semisimplification** is $\mathfrak{g}_{ss} := \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$.

Proposition 1.23. $\mathfrak{g}_{ss}$ *is semisimple, and* $\mathfrak{g} = \mathfrak{g}_{ss} \ltimes \operatorname{rad}(\mathfrak{g})$ *(**Levi decomposition**)*

Definition 1.24. $\mathfrak{g}$ is simple if its only ideals are 0 and $\mathfrak{g}$.

Proposition 1.25. *A semisimple Lie algebra is a direct sum of simple Lie algebras.*

Really hard to classify general Lie algebras, but classifying (semi)simple Lie algebras is doable in terms of Dynkin diagrams.

Definition 1.26. We say $\mathfrak{g}$ is **reductive** if $\text{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ (radical = center). Any reducitive Lie algebra is of form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{ss}$ with $\mathfrak{h}$ abelian.

Example. $\mathfrak{sl}_n(\mathbb{C})$ is simple for $n \geq 2$.
$\mathfrak{gl}_n(\mathbb{C}) = \mathbb{C} \oplus \mathfrak{sl}_n(\mathbb{C})$ is reductive.
$\mathfrak{so}_n(\mathbb{C})$ is simple for $n \geq 3$ except $n = 4$, where $\mathfrak{so}_4(\mathbb{C}) = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is semisimple.

1.2.11 Killing form

Definition 1.27. The **Killing form** is

$$K(x, y) = \text{tr}_{\mathfrak{g}}(\text{ad}x \cdot \text{ad}y).$$

Above $\text{ad}x(z) = [x, z]$. This is symmetric bilinear form on $\mathfrak{g}$ which is ad-invariant:

$$K([x, z], y) = K(x, [z, y]).$$

Theorem 1.28 (Cartan's Criteria). $\mathfrak{g}$ *is solvable iff* $[\mathfrak{g}, \mathfrak{g}] \subset \ker K$.
$\mathfrak{g}$ *is semisimple* $\iff$ K *is nondegenerate.*

2 Lecture 2 (2/18)

Note 1. A few minutes late.

Continuing where we left off I think.

2.1 More general forms

Let V be a f.d. rep of $\mathfrak{g}$, so $\rho : \mathfrak{g} \to \text{End}\, V$ a Lie algebra homomorphism. We consider

$$B_V(x, y) = \text{Tr}_V(\rho(x)\rho(y)),$$

e.g. $B_{\mathfrak{g}} = K$ is the Killing form.

Proposition 2.1. *If* B_V *is nondegenerated for some* V*, then* $\mathfrak{g}$ *is reductive,* $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \mathfrak{h}$ *with* $\mathfrak{h}$ *abelian.*

2.2 Semisimple Lie algebras

Consider $\mathfrak{g}$ semisimple over $\mathbb{C}$.

Theorem 2.2. *Every f.d. representation of* $\mathfrak{g}$ *is completely reducible*

$$V = \bigoplus_i V_i.$$

If $\mathfrak{g}$ is semisimple, we can construct a G with $\operatorname{Lie} G = \mathfrak{g}$. Take $\operatorname{Aut}(\mathfrak{g}) \subset \operatorname{GL}(\mathfrak{g})$, let $G = \operatorname{Aut}(\mathfrak{g})^\circ$. This may not be simply connected, but $\operatorname{Lie} G \cong \mathfrak{g}$.

2.3 Jordan decomposition and Cartan subalgebras

Recall that any matrix can be written as the sum of a diagonal matrix and a semisimple matrix with the two commuting (e.g. put it in Jordan normal form).

Let $\mathfrak{g}$ be a semisimple Lie algebra. We say $x \in \mathfrak{g}$ is **semisimple** if $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is semisimple (diagonalizable).

Remember: Always consider s.s. Lie algebras in characteristic 0

Theorem 2.3. *Any $x \in \mathfrak{g}$ can be uniquely written as a sum $x = x_s + x_n$ with x_s semisimple, x_n-nilpotent, and $[x_s, x_n] = 0$.*

Definition 2.4. The **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is the maximal commutative subalgebra consisting of semisimple elements.

Cartan subalgebras are also maximal w.r.t. commutative subalgebras.

Theorem 2.5. *All Cartan subalgebras are conjugate under the action of the group G.*

Hence, all Cartan subalgebras are of the same dimension r, called the **rank** of G.

Example. $\mathfrak{g} = \mathfrak{sl}_n$. Then $\mathfrak{h} \subset \mathfrak{sl}_n$ consisting of diagonal matrices is a Cartan subalgebra.

$$\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} \text{ where } x_1 + \cdots + x_n = 0.$$

$\mathfrak{h} \cong \mathbb{C}^{n-1}$, so $\operatorname{rank} \mathfrak{sl}_n = n - 1$.

2.4 Root decomposition

Fix $\mathfrak{h} \subset \mathfrak{g}$ Cartan. $\mathfrak{h} \curvearrowright \mathfrak{g}$ via adjoint action, and we can decompose $\mathfrak{g}$ into eigenspaces:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus 0} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h) x \text{ for all } h \in \mathfrak{h}\}.$$

We call $\alpha \in \mathfrak{h}^* \setminus 0$ a **root** if $\mathfrak{g}_\alpha \neq 0$. There are only finitely many roots since $\dim \mathfrak{g} < \infty$. The set $R \subset \mathfrak{h}^*$ of roots is called the **root system** of $\mathfrak{g}$. Note that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

by Jacobi (maybe want $\alpha + \beta \neq 0$).

Let B be a non-degenerate bilinear form (e.g. Killing form). Then, $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ unless $\alpha + \beta = 0$. On the other hand,

$$B : \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$$

is a nondegenerate pairing. In fact, $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$, so

$$|R| = \dim \mathfrak{g} - \operatorname{rank} \mathfrak{g}.$$

By above nondegeneracy, $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$, so $\alpha \in R \implies -\alpha \in R$.

The span $\operatorname{span}(R) = \mathfrak{h}_{\mathbb{R}}^*$ of the roots is a real subspace of $\mathfrak{h}$ such that $\mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}$. Write E for this span (this is a Euclidean space since $K|_{\mathfrak{h}_{\mathbb{R}}^*}$ gives a positive definite inner product).

Example. A_2 is the root system of $\mathfrak{sl}_3$. Here, $\dim \mathfrak{sl}_3 = 8$ and $\operatorname{rank} \mathfrak{sl}_3 = 2$, so $|R| = 6$. One can check that these roots form the vertices of a regular hexagon.

2.5 Abstract Root Systems

Let $E \cong \mathbb{R}^n$ be a Euclidean space. Let $R \subset R \setminus 0$ be finite. If it satisfies the axioms

(1) R spans E

(2) For all $\alpha, \beta \in R$,

$$n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$$

(3) For all $\alpha, \beta \in R$,

$$s_\alpha(\beta) = \beta - n_{\alpha\beta}\alpha \in R.$$

then we call $R \subset E$ an **abstract root system**. We set $\operatorname{rank}(R) := \dim E$. We call it **reduced** if $\alpha \in R \implies 2\alpha \notin R$. We call R **irreducible** if we can not write $R = R_1 \sqcup R_2$ (with $E = E_1 \times E_2$ and $R_i \subset E_i$ root systems).

Fact. The set of roots of a semisimple Lie algebra form a reduced root system, which is irreducible iff the Lie algebra is simple.

The reflections s_α give the root system lots of symmetries.

Definition 2.6. The **Weyl group** is the subgroup $W \subset O(E)$ generated by s_α for $\alpha \in R$.

Note that the Weyl group is finite since it acts faithfully on the roots R (so subgroup of permutation group of roots).

Example. $\mathfrak{g} = \mathfrak{sl}_n$, then $E = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i = 0\}$ (coordinates on diagonal). The roots are $\alpha_{ij} = e_i - e_j$ where $i, j \in [1, n]$ and $i \neq j$, so $n^2 - n$ roots. The reflections $s_{\alpha_{ij}} = (ij)$ act by transpositions. Hence, $W = S_n$ is the symmetric group. This gives the root system A_{n-1} $(n \geq 2)$.

Example. The root system B_n $(n \geq 2)$ comes from $\mathfrak{o}(2n+1)$.

The root system C_n $(n \geq 3)$ comes from $\mathfrak{sp}(2n)$.

The root system D_n $(n \geq 4)$ comes from $\mathfrak{o}(2n)$.

Example. There are 5 exceptional root systems G_2, F_4, E_6, E_7, E_8.

2.5.1 Positive and Simple Roots

Pick some $t \in E$ such that $(t, \alpha) \neq 0$ for all $\alpha \in R$.

Definition 2.7. We say α is **positive** w.r.t to t if $(t, \alpha) > 0$ and **negative** if $(t, \alpha) < 0$. We call α a **simple root** if it is positive, but not the sum of two other positive roots.

Pavel drew a picture of A_2-root system with a choice of polarization. If you want to see a picture, track down and look at my notes from last semester...

Notation 2.8. Let R_+ be the set of positive roots, and R_- be the set of negative roots. Let Π be the set of simple roots. These all depend on the polarization (choice of t).

Fact.

(1) Every positive root is a sum of simple roots

(2) If α, β are simple roots, and $\alpha \neq \beta$, then $(\alpha, \beta) \leq 0$.

(3) $\Pi \subset R_+$ is a basis of E.

Any root can be written uniquely as

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i$$

where $n_i \in \mathbb{Z}$ and $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Furthermore, $n_i \geq 0$ for all i if $\alpha \in R_+$, and $n_i \leq 0$ for all i if $\alpha \in R_-$.

2.5.2 Dual root system

For $R \subset E$, we can attach to **coroot** $\alpha^\vee \in E^*$ defined by $s_\alpha(\alpha^\vee) = -\alpha^\vee$ and $(\alpha, \alpha^\vee) = 2$. Thus,

$$s_\alpha(x) = x - (x, \alpha^\vee)\alpha$$

for any x. Write $R^\vee = \{\alpha^\vee : \alpha \in R\} \subset E^*$.

Example. $B_n^\vee = C_n$. Other irreducible root systems are self-dual.

Definition 2.9. The **root lattice** if the $\mathbb{Z}$-span of the roots (equivalently, $\mathbb{Z}$-span fo simple roots), i.e. it is

$$Q = \langle R \rangle = \langle \Pi \rangle = \left\{ \sum_{i=1}^{r} n_i \alpha_i : n_i \in \mathbb{Z} \right\}.$$

The **coroot lattice** is $Q^\vee = \langle R^\vee \rangle$. The **weight lattice** is the dual lattice to $Q^\vee$

$$P = (Q^\vee)^* = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \in R\}.$$

The **coweight lattice** is $P^\vee = Q^*$.

Example. $\#(P/Q) = n$ for $\mathfrak{sl}_n$.

Inside the weight lattice are the **fundamental weights** $\omega_i \in P$ satisfying

$$(\omega_i, \alpha_j^\vee) = \delta_{ij},$$

i.e. they are the dual basis to simple coroots. A weight $\lambda = \sum x_i \omega_i$ is called **dominant** if $x_i \geq 0$ for all i. It is called **integral** if $x_i \in \mathbb{Z}$ (i.e. if $\lambda \in P$ belongs to the weight lattice).

2.5.3 Cartan matrix and Dynkin Diagrams

We have simple roots $\alpha_1, \ldots, \alpha_r$. Recall

$$\mathbb{Z} \ni n_{\alpha_i \alpha_j} = (\alpha_i^\vee, \alpha_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} =: a_{ij}.$$

The **Cartan matrix** is $A = (a_{ij})$. These satisfy

- $a_{ii} = 2$ always.
- $a_{ij} \leq 0$ if $i \neq j$.
- $a_{ij} = 0 \iff a_{ji} = 0$.
- $a_{ij} a_{ji} \in \{0, 1, 2, 3\}$.

One can reduce classifying irreducible root systems to classifying indecomposable Cartan matrices.

Example. For $\mathfrak{sl}_4$, the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

We visualize these using Dynkin diagrams. There are $r = \operatorname{rank}(R \subset E) = \dim E = |\Pi|$ vertices corresponding to the simple roots. Vertex i is connected to vertex j by a(n) (undirected) single edge if $a_{ij} = -1$. There is a (directed) double edge $i \rightrightarrows j$ if $a_{ij} = -2$ and $a_{ji} = -1$. There is a (directed) triple edge from i to j if $a_{ij} = -3$ and $a_{ji} = -1$. Set

Directed edges in a Dynkian diagram point to the longer root

$$m_{ij} = \begin{cases} 2 & \text{if } a_{ij} a_{ji} = 0 \\ 3 & \text{if } a_{ij} a_{ji} = 1 \\ 4 & \text{if } a_{ij} a_{ji} = 2 \\ 6 & \text{if } a_{ij} a_{ji} = 3 \end{cases}$$

Let $s_i = s_{\alpha_i}$ be the simple reflections. These already generated the Weyl group $W = \langle s_i \rangle$, and satisfy $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$. These are the defining relations (no other ones needed).

2.6 Serre presentations

This, among other things, let's you construct Lie algebras for the exceptional root systems.

Let $\mathfrak{g}$ be a simple Lie algebra. Let $\alpha_1, \dots, \alpha_r$ be the simple roots (choose Cartan subalgebra and polarization of root system). Then we get 1-dim spaces $\mathfrak{g}_{\alpha_i} = \langle e_i \rangle$ and $\mathfrak{g}_{-\alpha_i} = \langle f_i \rangle$. We can normalize our generates so that

$$[e_i, f_i] =: h_i = \alpha_i^\vee.$$

For fixed i, the elements e_i, f_i, h_i generate an $\mathfrak{sl}_2$ triple with normal relations.

Theorem 2.10.

(1) As i varies, these e_i, f_i, h_i generate all of $\mathfrak{g}$.

(2) They satisfy

$$[h_i, h_j] = 0, \ \ [h_i, e_j] = a_{ij} e_j, \ \ [h_i, f_j] = -a_{ij} f_j, \ \ \textit{and} \ \ [e_i, f_j] = \delta_{ij} h_i.$$

*In addition, they satisfy the **Serre relations** (for all $i \neq j$)*

$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0 \ \ \textit{and} \ \ (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0.$$

(3) $\mathfrak{g}$ is defined by these generators and relations.

(4) For any reduced, irreducible root system, this defines a simple f.d. Lie algebra.

Corollary 2.11. *Simple f.d. Lie algebras correspond bijectively to the Dynkin diagrams $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.*

2.7 Representation Theory

We can write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ where $\mathfrak{n}_+ = \bigoplus_{\alpha \in R_+} \mathfrak{g}_\alpha$ and similarly for $\mathfrak{n}_-$.

Let $\lambda \in \mathfrak{h}^*$ (we call such things **weights**). We can consider

$$M_\lambda = \langle v_\lambda \,|\, h v_\lambda = \lambda(h) v_\lambda \ \text{ for all } \ h \in \mathfrak{h}, e_i v_\lambda = 0 \rangle .$$

By PBW (PWB?), we can write $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. Then,

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda = U(\mathfrak{n}_-) v_\lambda,$$

where $\mathbb{C}_\lambda = \mathbb{C} v_\lambda$ is rep where $h v_\lambda = \lambda(h) v_\lambda$ and $e_i v_\lambda = 0$. In particular, M_λ is a fee module of rank 1 over $U(\mathfrak{n}_-)$.

Definition 2.12. A **highest weight module** for $\mathfrak{g}$ with **highest weight** λ is a quotient of M_λ.

Proposition 2.13. *Any f.d. irreducible representation of $\mathfrak{g}$ is a highest weight module.*

Any rep V of $\mathfrak{g}$ (= semisimple) splits as

$$V = \bigoplus_{\mu \in P} V[\mu].$$

If V is f.d., there exists a "highest" weight λ s.t. $\lambda + \alpha_i$ is not a weight for any i. For any $\lambda \in \mathfrak{h}^*$, there is a smallest quotient $M_\lambda / J_\lambda = L_\lambda$ which is irreducible (but in general ∞-dimensional).

Theorem 2.14. *L_λ is finite dimensional $\iff$ λ is a dominant, integral weight (i.e. $\lambda \in P_+$).*

Hence, f.d. irreps of $\mathfrak{g}$ correspond bijectively to $\lambda \in P_+$ via $\lambda \mapsto L_\lambda$.

It is hard to understand L_λ in general, but not when $\lambda \in P_+$.

Theorem 2.15. *For $\lambda \in P_+$, $\lambda = \sum_i n_i \omega_i$, we have*

$$L_\lambda = \frac{M_\lambda}{\langle f_i^{n_i+1} v_\lambda \rangle}.$$

2.8 Weyl Character Formula

Write

$$V = \bigoplus_\mu V[\mu].$$

(I missed the hypotheses on V needed to have this decomposition) with each $V[\mu]$ fin dimensional. Let

$$\chi_V = \sum_\mu \dim V[\mu] \cdot e^\mu \in \widehat{\mathbb{C}[P]}.$$

If V is finite dimensional, then this is in the usual (non-completed) group algebra $\mathbb{C}[P]$. Note that, for $h \in \mathfrak{h}$,

$$\mathrm{tr}_V(e^h) = \sum_\mu \dim V[\mu] e^{\mu(h)}.$$

This is why χ_V above is called a 'character'.

Recall W is the Weyl group and $W \subset O(E)$, so $\det W \to \pm 1$ makes sense. We can define this combinatorially. If $w = s_{i_1} \dots s_{i_m}$, then $\det(w) = (-1)^m$ (i.e. it gives the parity of the length of w). Define

$$\mathfrak{h}^* \ni \rho = \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{\alpha \in R_+} \alpha.$$

Remember: $\rho = \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$

Theorem 2.16 (**Weyl Character Formula**)**.**

$$\chi_{L_\lambda} = \frac{\sum_{w \in W} \det(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}.$$

Example. If $\lambda = 0$, then $L_\lambda = \mathbb{C}$ with trivial action, and $\chi_{L_0} = 1$. Therefore, we get the **Weyl denominator formula**

$$\sum_{w \in W} \det(w) e^{w\rho - \rho} = \prod_{\alpha \in R_+} (1 - e^{-\alpha}).$$

For $\mathfrak{sl}_n$, above becomes the Vandermonde determinant

$$\det \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ x_1^{n-1} & \dots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le j < i \le n} (x_i - x_j).$$

Next time we start discussing new material. Homework out tonight; due in a week.

3 Lecture 3 (2/23)

Today we start new material. We talked last time about the Weyl character formula, so a good place to go next is the...

3.1 Weyl dimension formula

Let $\mathfrak{g}$ be a semisimple (complex) Lie algebra. Recall that P_+ denotes the set of dominant, integral weights. For every $\lambda \in P_+$, we get a f.d. irreducible highest weight representation L_λ with highest weight λ. For $\mathfrak{h} \subset \mathfrak{g}$ Cartan and $h \in \mathfrak{h}$, we had a formula for the character

$$\chi_{L_\lambda}(e^h) = \mathrm{Tr}_{L_\lambda}(e^h) = \sum_{\beta \in P(L_\lambda)} \dim L_\lambda[\beta] e^{\beta(h)} = \frac{\sum_{w\in W} \det(w) e^{(w(\lambda+\rho)-\rho,h)}}{\prod_{\alpha\in R_+}\left(1-e^{-\alpha(h)}\right)}.$$

Note that $\dim L_\lambda = \chi_{L_\lambda}(e^h)|_{h=0}$, but this is not so easy to compute directly. In fact, both the numerator and the denominator vanish at $h=0$.

Question 3.1. *How to compute the limit as $h \to 0$?*

(We know this is possible since the character is secretly a polynomial)

Key idea: specialize to $h = th_\rho$ ($t \in \mathbb{R}$) where $h_\rho \leftrightarrow \rho$ under identification $\mathfrak{h}^* \leftrightarrow \mathfrak{h}$. Then,

$$\begin{aligned}
\chi_{L_\lambda}(th_\rho) &= \frac{\sum_{w\in W} \det(w) e^{(w(\lambda+\rho)-\rho,t\rho)}}{\prod_{\alpha\in R_+}\left(1-e^{t(\alpha,\rho)}\right)} \\
&= \frac{e^{-t(\rho,\rho)}\sum_{w\in W} \det(w) e^{(w(\lambda+\rho),t\rho)}}{\prod_{\alpha\in R_+}\left(1-e^{-t(\alpha,\rho)}\right)} \\
&= e^{-t(\rho,\rho)}\frac{\sum_{w\in W} \det(w) e^{(\lambda+\rho,tw^{-1}\rho)}}{\prod_{\alpha\in R_+}\left(1-e^{-t(\alpha,\rho)}\right)} \\
&= e^{-t(\rho,\rho)}\frac{\sum_{w\in W} \det(w) e^{t(\lambda+\rho,w\rho)}}{\prod_{\alpha\in R_+}\left(1-e^{-t(\alpha,\rho)}\right)}
\end{aligned}$$

(Above, we've used that $(,)$ is W-invariant, and we've replace $w \mapsto w^{-1}$ at one point (noting $\det w = \det w^{-1}$)). At this point, we recall the **Weyl denominator formula**:

$$\sum_{w\in W} \det(w) e^{w\rho} = e^\rho \prod_{\alpha\in R_+}\left(1-e^{-\alpha}\right).$$

Applying this to the previous displayed equation shows that

$$\chi_{L_\lambda}(e^{th_\rho}) = e^{-t(\rho,\rho)}\frac{e^{t(\rho,\lambda+\rho)}\prod_{\alpha\in R_+}\left(1-e^{-t(\alpha,\lambda+\rho)}\right)}{\prod_{\alpha\in R_+}(1-e^{-t(\alpha,\rho)}} = e^{t(\lambda,\rho)}\prod_{\alpha\in R_+}\frac{1-e^{-t(\alpha,\lambda+\rho)}}{1-e^{-t(\alpha,\rho)}}.$$

Recall 3.2. L'Hôpital let's us see that

$$\lim_{t\to 0}\frac{1-e^{ta}}{1-e^{tb}} = \frac{a}{b}.$$

Thus, we now see that

$$\chi_{L_\lambda}(1) = \dim L_\lambda = \prod_{\alpha \in R_+} \frac{(\alpha, \lambda+\rho)}{(\alpha, \rho)}.$$

This is called the **Weyl dimension formula**.

3.2 Tensor products of fundamental representations

Fix some $\lambda \in P_+$, so $\lambda = \sum m_i \omega_i$, with $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ and $m_i \in \mathbb{Z}_+$.

Definition 3.3. The representations L_{ω_i} are called **fundamental representations**.

Consider

$$T_\lambda := \bigotimes_{i=1}^r L_{\omega_i}^{\otimes m_i}.$$

Note that this contains $v_\lambda = \otimes_{i=1}^r v_{\omega_i}^{\otimes m_i}$, where v_{ω_i} is a highest weight vector of L_{ω_i}.

Proposition 3.4. *Let V be the subrepresentation of $\bigotimes_{i=1}^r L_{\omega_i}^{\otimes m_i}$ generated by v_λ. Then, $V \cong L_\lambda$.*

Proof. Recall that $P(V_\lambda) \subset \lambda - Q_+$. Since $v_\lambda \in V$ is a vector of weight λ, we can write

$$V = L_\lambda \oplus \bigoplus_{\mu \prec \lambda} m_{\lambda\mu} L_\mu,$$

where $\mu \prec \lambda$ means $\mu \in (\lambda - Q_+) \cap P_+$. Recall the **Casimir element** $C \in U(\mathfrak{g})$ (even in its center): for x_i any basis of $\mathfrak{g}$ with dual basis $x_i^* \in \mathfrak{g}$ under the Killing form[2], then

$$C = \sum x_i x_i^* = \sum_{j=1}^r y_j^2 + 2 \sum_{\alpha \in R_+} f_\alpha e_\alpha$$

(above, y_j some orthonormal basis of $\mathfrak{h}$ and e_α, f_α chosen so that $(e_\alpha, f_\alpha) = 1$). Then, $C|_{L_\mu}$ acts via multiplication by $(\mu, \mu + 2\rho)$. We have shown previously that if $\mu \prec \lambda$, then

$$(\mu, \mu + 2\rho) < (\lambda, \lambda + 2\rho).$$

However, since V is generated by v_λ, we know that $C|_V = (\lambda, \lambda + 2\rho)\,\mathrm{Id}_V$. Therefore, we must have $m_{\lambda\mu} = 0$ since C has no other eigenvalues. ■

3.3 Representations of $\mathrm{SL}_n(\mathbb{C})$

Recall 3.5. $\mathrm{Lie}\,\mathrm{SL}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) = \{A \in \mathfrak{gl}_n(\mathbb{C}) : \mathrm{Tr}\,A = 0\}$ has a natural Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$ consisting of diagonal matrices $\mathrm{diag}(x_1, \dots, x_n)$ (with $x_1 + \cdots + x_n = 0$). Hence, we may identify

$$\mathfrak{h} \cong \mathbb{C}_0^n = \left\{ (x_1, \dots, x_n) \in \mathbb{C}^n : \sum_{i=1}^n x_i = 0 \right\}.$$

[2]or any symmetric, ad-invariant, bilinear form

Note that we then have

$$\mathfrak{h}^* = \mathbb{C}^n/\mathbb{C}\,\mathrm{diag} = \{(y_1, \ldots, y_n) \in \mathbb{C}^n\} \text{ modulo shift } (y_1, \ldots, y_n) \sim (y_1 + c, \ldots, y_n + c).$$

Here, the simple coroots are

$$\alpha_i^\vee = \left(0, \ldots, 0, \underbrace{1}_{i}, \underbrace{-1}_{i+1}, 0, \ldots, 0\right) = e_i - e_{i+1} \text{ for } i = 1, \ldots, n-1.$$

The fundamental weights ω_i satisfy $(\omega_i, e_j - e_{j+1}) = \delta_{ij}$, and it is easy to see that these are

$$\omega_i = (\underbrace{1, 1, \ldots, 1}_{i \text{ times}}, 0, 0, \ldots, 0) \text{ for } i = 1, \ldots, n = 1.$$

We would like to construct representations corresponding to the fundamental weights. This turns out to be each. Let $V = \mathbb{C}^n$ be the standard/tautological representation. Let $v_1, \ldots, v_n \in V$ be the standard basis. It is not hard to see that V is irreducible. What is the highest weight? Recall $\mathfrak{g}_{\alpha_i}$ is generated by $e_i = E_{i,i+1}$. From this, it is not too hard to see that the highest weight vector (killed by all e_i) is v_1. Note that $h = (x_1, \ldots, x_n) \in \mathbb{C}_0^n$ satisfies $hv_1 = x_1 v_1$, so the highest weight is $\omega_1 = (1, 0, \ldots, 0)$. Hence, $V = L_{\omega_1}$.

Note 2. Pavel occastionally draws pictures to illustrate points, but I'm currently too lazy to draw these and add them to the notes...

Now consider exterior powers $\bigwedge^m V$ for $1 \le m < n$. This has basis $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_m}$ for $i_1 < i_2 < \cdots < i_m$. Say v is a highest weight vector. Then, $E_{12}v = 0, E_{23}v = 0, \ldots, E_{n-1,n}v = 0$. Note that

$$E_{ij} v_{i_1} \wedge \cdots \wedge v_{i_m} = \begin{cases} 0 & \text{if } i_1, \ldots, i_m \neq j \\ \pm v_{i_1} \wedge \cdots \wedge \underbrace{v_i}_{k\text{th place}} \wedge \cdots \wedge v_{i_m} & \text{if } i_k = j \end{cases}$$

Thus, the highest weight vector is $v_1 \wedge v_2 \wedge \cdots \wedge v_m$. Note that $h = (x_1, \ldots, x_n)$ satisfies $h \cdot v_1 \wedge \cdots \wedge v_m = (x_1 + x_2 + \cdots + x_m)\, v_1 \wedge \cdots \wedge v_m$, so the highest weight is

$$\omega_m = (\underbrace{1, 1, \ldots, 1}_{m \text{ times}}, 0, 0, \ldots, 0).$$

Further, $\bigwedge^m V$ is irreducible (easy exercise), so $L_{\omega_m} \cong \bigwedge^m V$.

Example. $\bigwedge^m V = 0$ for $m > n$. $\bigwedge^n V = \mathbb{C}$ is the trivial representation (basically, matrices act by both trace and determinant, but determinant = trace = 0). There is an invariant, nonndegenerate pairing

$$\bigwedge^{n-1} V \otimes V \xrightarrow{\wedge} \bigwedge^n V = \mathbb{C},$$

so $\bigwedge^{n-1} V \cong V^*$. More generally,

$$\left(\bigwedge^k V\right)^* \cong \bigwedge^{n-k} V.$$

Say $\lambda = \sum m_i \omega_i$. Then, L_λ is the subrep in

$$\bigotimes_{i=1}^{n-1} \left(\bigwedge^i V \right)^{\otimes m_i}$$

generated by tensor product of highest weight vectors. This is fairly concrete construction of L_λ.

Example. Take $\lambda = m\omega_1$. $V^{\otimes m} \ni v_1 \otimes v_1 \otimes \cdots \otimes v_1$. Say $m = 2$. To get $L_{2\omega_1}$, we start applying Lie algebra elements to this, e.g.

$$E_{21}(v_1 \otimes v_1) = v_1 \otimes v_2 + v_2 \otimes v_1.$$

One gets in the end that $L_{m\omega_1} = \operatorname{Sym}^m V$ (exercise).

Say $\lambda = m_1\omega_1 + \cdots + m_{n-1}\omega_{n-1}$. We can write this as a vector

$$m_1(1, 0, \ldots, 0) + m_2(1, 1, 0 \ldots, 0) + \cdots = (m_1 + m_2 + \cdots + m_{n-1}, m_2 + \cdots + m_{n-1}, \ldots, m_{n-1}, 0),$$

so dominant weights (of $\mathfrak{sl}_n(\mathbb{C})$) correspond exactly to nonincreasing sequences

$$p_1 \geq p_2 \geq \cdots \geq p_{n-1} \geq 0$$

of nonnegative integers.

Example. When $n = 2$, get one number $p_1 \geq 0$ which is exactly our old friend $\mathfrak{sl}_2 \curvearrowright V_{p_1}$.

Exercise. $\mathfrak{sl}_n(\mathbb{C})$ is simply connected, so it's fine to not distinguish between representations of it and of $\mathrm{SL}_n(\mathbb{C})$.

3.4 Representations of $\mathrm{GL}_n(\mathbb{C})$

Recall 3.6. $\operatorname{Lie} \mathrm{GL}_n(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{C})$.

Warning 3.7. $\mathfrak{gl}_n(\mathbb{C})$ is *not* semisimple, so our general theory does not directly apply. However, $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$ so it's close enough for us to be able to understand things.

On the Lie group side, $\mathrm{GL}_n(\mathbb{C})$ is not quite a product of $\mathrm{SL}_n(\mathbb{C})$, but instead one has

$$\mathrm{GL}_n(\mathbb{C}) = \left(\mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times \right) / \mu_n,$$

where μ_n is the group of nth roots of unity (in $\mathbb{C}$). If $\zeta^n = 1$, then

$$\begin{pmatrix} \zeta & & \\ & \ddots & \\ & & \zeta \end{pmatrix} \in \mathrm{SL}_n,$$

and we embed $\mu_n \hookrightarrow (\mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times)$ diagonally. This identification is given by

$$(\mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times)/\mu_n \ni (A, z) \mapsto zA \in \mathrm{GL}_n(\mathbb{C}).$$

Proposition 3.8. $\operatorname{Rep}\mathrm{GL}_n = \operatorname{Rep}(\mathrm{SL}_n \times \mathbb{C}^\times)$ *on which* μ_n *(embedding diagonally) acts trivially.*[3]

Example. When $n = 1$, we just have $\mathbb{C}^\times$. The Lie algebra is $\operatorname{Lie}\mathbb{C}^\times = \mathbb{C}h$, so a rep of the Lie algebra is a choice of operator $H : V \to V$ such that $e^{2\pi i H} = 1$ (since $e^{2\pi i h} = 1$). Hence, H is diagonalizable with integer eigenvalues. Thus, every representation of $\mathbb{C}^\times$ is completely reducible (since H diagonalizable), and its irreps are 1-dimensional corresponding to $n \in \mathbb{Z}$: $\chi_n(z) = z^n$.

Note that $\operatorname{Lie}\mathbb{C}^\times$ is not semisimple, and its representations are not completely reducible (e.g. think Jordan blocks), but not every rep of $\operatorname{Lie}\mathbb{C}^\times$ lifts to one of $\mathbb{C}^\times$ since it is not simply connected

For $\mathrm{SL}_n \times \mathbb{C}^\times$, all representations will be completely reducible. The irreducible representations are $L_{\lambda,N} = L_\lambda \otimes \chi_N$. What about for GL_n, i.e. when does the center act trivially? For GL_n, you get the $L_{\lambda,N}$ for which $|\lambda| + N$ is divisible by n.

$|\lambda| = \sum p_i$ from before

We can look at this from another perspective. $\mathbb{C}^n \cong \mathfrak{h} \subset \mathfrak{gl}_n$ consisting of diagonal matrices gives a Cartan subalgebra (reductive Lie algebras have these as well). The dominant weights will correspond to tuples $(p_1, p_2, \ldots, p_n)$ with $p_1 \geq p_2 \geq \cdots \geq p_n \in \mathbb{Z}$. The fundamental weights will be $\omega_1, \omega_2, \ldots, \omega_n$ with

$$\omega_i = (\underbrace{1, 1, \ldots, 1}_{i \text{ times}}, 0, 0, \ldots, 0)$$

as before. Note that $\omega_n \neq 0$ now (it gives the determinant character). Given $\lambda = m_1\omega_1 + \cdots + m_n\omega_n$, one has

$$L_\lambda \subset \bigotimes_k \left(\bigwedge^k V\right)^{\otimes m_k} \quad \text{with } V = \mathbb{C}^n,$$

and $m_1, \ldots, m_{n-1} \geq 0$ while $m_n \in \mathbb{Z}$ (possibly negative).

Remark 3.9. If χ is a 1-dim representation and $k < 0$, we can and do set

$$\chi^{\otimes k} := (\chi^*)^{\otimes(-k)}.$$

Say $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \mathbb{Z}$.

Definition 3.10. L_λ is **polynomial** if $\lambda_n \geq 0$.

Exercise. L_λ is polynomial $\iff$ it is a direct summand in a tensor power of V.

Why are these called polynomial? Given $x = (x_{ij}) \in \mathrm{GL}_n$, $v \in Y$, and $f \in Y^*$, can form the **matrix element** (f, Xv). This gives a function on $G = \mathrm{GL}_n$ which is a polynomial for polynomial representations. Note that $\mathrm{GL}_n \subset \mathrm{Mat}_n$ is an open subset. Matrix elements will extend to functions on Mat_n if they are polynomials. Note that any irreducible representation L_λ will be of the form

$$L_\lambda = \text{Poly rep} \otimes \left(\bigwedge^n V\right)^{\otimes(-k)} \quad \text{for some } k \geq 0,$$

so understanding polynomial representation will let us understand everything. We also see that general matrix elements are $\frac{P(X)}{\det(X)^k}$ so only extend to all matrices if $k \neq 0$ (need invertible determinant).

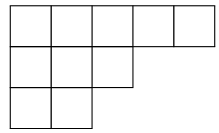
Note that $\lambda_1 \geq \cdots \geq \lambda_n$ is a **partition in** n **parts** of

$$N = |\lambda| := \lambda_1 + \cdots + \lambda_n.$$

Partitions are usually denoted uses Young diagrams.

[3] Pavel uses the notation μ_n^{diag} to emphasize the diagonal embedding

Example. The partition $(5, 3, 2)$ corresponds to the diagram



Note that L_λ occurs in $V^{\otimes|\lambda|}$, e.g. $L_{(5,3,2)}$ occurs in $V^{\otimes 10}$. Also, $|\lambda|$ is the eigenvalue of $\mathrm{id} \in \mathfrak{gl}_n$ (when acting on V^N?)

Let's look more closely at the structure of $V^{\otimes N}$. We have

$$V^{\otimes N} = \bigoplus_{\lambda:|\lambda|=N} \pi_\lambda \otimes L_\lambda \text{ where } \pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n}(L_\lambda, V^{\otimes N})$$

(consequence of complete reducibility). Note that $\pi_\lambda = 0$ if λ has more than n parts.

How should we understand π_λ? The key is note: always in rep theory, if you decompose a representation into a direct sum, in order to understand the spaces showing up, you need to understand what acts on those spaces; what acts on them normally is something that commutes with your group. Here, the symmetric group $S_N \curvearrowright V^{\otimes N}$ by permuting components. Therefore, $S_N \curvearrowright$ each π_λ.

There's a tight connection between rep theory of symmetric groups and rep theory of general linear groups. Inside $\mathrm{End}_{\mathbb{C}}(V^{\otimes N})$ there is an algebra A generated by $U(\mathfrak{gl}_n)$ and another algebra B generated by S_N. These two subalgebras commute: $[A, B] = 0$. They also satisfy a double centralizer property (one is the centralizer of the other).

Theorem 3.11 (Schur-Weyl duality).

(1) *The centralizer of A is B, and vice versa.*

(2) *If λ has at most n parts, then π_λ is an irreducible representation of B (hence of S_N)*

(3) *If $n = \dim V \geq N$, then π_λ exhaust all irreducible representations of s_N (each occurring exactly once).*

The π_λ correspond to partitions λ of N with $\leq n$ parts (this condition is meaningless if $n > N$), and this correspondence is independent of n. More on this next time.

4 Lecture 4 (2/25)

4.1 Schur-Weyl duality

We started talking about this last time. Recall we have $V = \mathbb{C}^n$ and $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ naturally acts on this space. We formed $V^{\otimes N}$ so $\mathrm{GL}(V)$ and $\mathfrak{gl}(V) = \mathfrak{gl}_n(\mathbb{C})$ have an induced action on $V^{\otimes N}$. We decompose

$$V^{\otimes N} = \bigoplus_{\lambda:|\lambda|=N} L_\lambda \otimes \pi_\lambda$$

into a direct sum of irreps L_λ each with 'multiplicity' π_λ (note λ ranges over partitions of N with $\leq n$ parts). Recall $\pi_\lambda = \mathrm{Hom}_{\mathrm{GL}_n}(L_\lambda, V^{\otimes N})$.

At the same time, S_N acts on $V^{\otimes N}$ by permuting the factors, and this action commutes with the one of $\mathrm{GL}_n(\mathbb{C})$. We write $S_N \curvearrowright V^{\otimes N} \curvearrowleft \mathrm{GL}_n(\mathbb{C})$ to emphasize that the actions commute. As a consequence, S_N acts on each π_λ.

Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes N})$, and let $B \subset \mathrm{End}_{\mathbb{C}}(V^{\otimes N})$ be the image of the group algebra $\mathbb{C}S_N$. Since the $\mathfrak{gl}_n$ and S_N actions on $V^{\otimes N}$ commute, these two subalgebras commute with each other. Beyond this...

Theorem 4.1 (**Schur-Weyl duality**).

(1) *The centralizer of B is A, and vice versa.*

(2) *π_λ is an irreducible representation of S_N, and the various π_λ's are pairwise non-isomorphic.*

(3) *If $n \geq N$ (so all partitions of N have $\leq n$ parts), then the collection $\{\pi_\lambda\}$ gives the full set of irreducible representations of S_N.*

Slogan. Symmetric groups and general linear groups have equivalent representation theories.

For the proof, we will need several lemmas.

Lemma 4.2. *Let U be a complex vector space. Then, $S^N U$ is spanned by vectors of the form $x \otimes \cdots \otimes x$, $x \in U$.*

Proof. Enough to consider finite dimensional U since any vector in $S^N U$ lies in the symmetric power of some finite-dimensional subspace of U. Then, $S^N U$ is an irreducible representation of $\mathrm{GL}(U)$ (or of $\mathfrak{gl}(U)$), and span $\{x \otimes \cdots \otimes x : x \in U\}$ is a nonzero subrepresentation, so it must be everything. ■

Secretly, $S^n U$ is an irreducible representation even if U is infinite dimensional, so no need to reduce to the finite dimensional case

Lemma 4.3. *If R is an associative $\mathbb{C}$-algebra, then the algebra $S^N R$ is a generated by elements*

$$\Delta_N(x) := (x \otimes 1 \otimes \cdots \otimes 1) + (1 \otimes x \otimes 1 \otimes \cdots \otimes 1) + (\cdots + 1 \otimes \cdots \otimes 1 \otimes x)$$

(N summands). Can think of this as $x_1 + x_2 + \cdots + x_n$ with $x_i = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ with x in ith slot.

Proof. Consider $z_1, \ldots, z_N \in \mathbb{C}[z_1, \ldots, z_N]$. By fundamental theorem on symmetric functions, there exists a polynomial P_N such that $P_N(\sum z_i, \sum z_i^2, \ldots, \sum z_i^N) = z_1 \ldots z_N$ (**Newton polynomial**). We apply

$$P_N(\Delta_N(x), \Delta_N(x^2), \ldots, \Delta_N(x^N)) = x \otimes \cdots \otimes x.$$

By the previous lemma, these span $S^N R$, so $\Delta_N(y)$, $y \in R$, generate R. ■

Lemma 4.4 (**Double Centralizer Lemma**). *Suppose $B \cong \bigoplus_{i=1}^r \mathrm{Mat}_{k_i}(F)$ is a direct sum of matrix algebras*[4] *over the field F. Suppose also we embed $B \hookrightarrow \mathrm{End}_F V$ for some f.d. F-vector space V. Let $A \subset \mathrm{End}_F V$ be the centralizer of B. Then, A is also a direct sum of r matrix algebras, and*

$$V = \bigoplus_{i=1}^r W_i \otimes U_i,$$

[4] apparently true for semisimple algebras over an algebraically closed field

where U_i ranges over the full set of irreducible representations of B, and W_i ranges over the full set of irreducible representations of A, and this decomposition is as a module over $A \otimes B$. In particular, there is a bijection between irreps of B and of A, and also B is the centralizer of A.

Proof. We can write $V = \bigoplus_{i=1}^r W_i \otimes U_i$ with the U_i irreps of B, and $W_i = \mathrm{Hom}_B(U_i, V)$. By definition, the centralizer of B is $A = \mathrm{End}_B V = \bigoplus_{i=1}^r \mathrm{End}_F(W_i)$.

Question 4.5 (Audience)**.** *Why is the number of summands in the decomposition of V equal to the number of summands in the decomposition of B?*

Answer. All U_i must occur as $B \hookrightarrow \mathrm{End}(V)$ (so $\mathrm{End}(V)$ contains regular rep) and so $W_i \neq 0$ for all $i = 1, \ldots, r$. Similarly, $A \hookrightarrow \mathrm{End}(V)$ so all of its irreps must occur, so the W_i must be all of them.

TODO: Review this answer

■

"A good mathematical theorem is one that takes one minute to state and one hour to prove, and a bad one is one that takes one hour to state but one minute to prove." – Kirillov, paraphrased.

Now we return to Schur-Weyl duality.

Proof of Theorem 4.1. B is a direct sum of matrix algebras since representations of S_N are completely reducible. We need to show that A is the centralizer of B. We know $A \subset Z(B)$ that A is contained in the centralizer. Note that

$$Z(B) = S^N(\mathrm{End}\, V)$$

is the endomorphisms of V^N which commute with the permutation action of S_N. The second lemma now implies that $Z(B)$ is generated by elements of the form $\Delta_N(x)$ for $x \in \mathrm{End}\, V = \mathfrak{gl}_n$. This is exactly the action of $x \in \mathfrak{gl}_n$ on $V^{\otimes N}$, so $\Delta_N(x) \in A$, the image of the enveloping algebra. Hence, $Z(B) \subset A$. At this point, the third lemma applies, and we obtain everything else:

$$V^{\otimes N} = \bigoplus W_i \otimes U_i$$

with $W_i = L_\lambda$ representations of A, and $U_i = \pi_\lambda$ representations of B. This establishes **(1)**,**(2)** of Theorem 4.1.

Recall that **(3)** said: if $n \geq N$, then π_λ gives full set of irreps of S_N. If $\dim V \geq N$, then we can pick N linearly independent vectors $v_1, \ldots, v_N \in V$ (and complete to a basis $v_1, \ldots, v_n$ of V). Then, $\sigma(v_1 \otimes \cdots \otimes v_N) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}$ are all linearly independent for different σ. Hence, $\mathbb{C}S_N \cdot (v_1 \otimes \cdots \otimes v_N) \cong \mathbb{C}S_N$, so $B = \mathbb{C}S_N \hookrightarrow \mathrm{End}_{\mathbb{C}}(V^{\otimes N})$. Hence double centralizer story tells us that the $\{\pi_\lambda\}$ do give all representations of S_N. ■

Corollary 4.6. Rep$S_N \leftrightarrow$ *partitions λ of N.*

Remark 4.7 (Sanity check). Number of irreps of finite group G = number of conj. classes of G. Conjugacy classes of S_N are determined by cycle types, but these exactly correspond to partitions of N.

Remark 4.8. Schur-Weyl duality gives a new proof that reps of $\mathfrak{gl}_n$ are completely reducible.

Remark 4.9. The algebra A appearing above is called the **Schur algebra**. It is always a quotient of $U(\mathfrak{gl}_n)$ since this is infinite-dimensional while A is finite-dimensional.

We've given an assignment

$$\text{partitions } \lambda \longmapsto \text{representations } \pi_\lambda \text{ of } S_N$$

making use of some GL_n for $n \geq N$.

Claim 4.10. *This assignment is independent of the choice of* n.

Proof. Say λ has $\leq n$ parts. Let $V = \mathbb{C}^n$ be basis $e_1, \ldots, e_n$. Consider $V \oplus \mathbb{C}e_{n+1}$. Then,

TODO: Make sense of this argument

$$(V \otimes \mathbb{C})^{\otimes N} = \bigoplus_\mu L_\mu^{(n+1)} \otimes \pi_\mu^{(n+1)},$$

with exponents signifying which GL_n these come from. Pick some $L_\lambda^{((n+1))} \otimes \pi_\lambda^{((n+1))}$. What happens when we restrict it to $\mathrm{GL}(V) \subset \mathrm{GL}(V \oplus \mathbb{C})$, i.e. matrices of the form

$$\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \text{ with } * \in \mathrm{GL}_n(\mathbb{C}).$$

Let $v_\lambda^{(n+1)} \in L_\lambda^{(n+1)}$ be a highest weight vector. Note that $v_\lambda^{(n+1)} \in V^{\otimes N} \subset (V \oplus \mathbb{C})^{\otimes N}$ since $\lambda = (\lambda_1, \ldots, \lambda_n, 0)$. Then, $v_\lambda^{(n+1)} \otimes x$ ($x \in \pi_\lambda^{(n+1)}$) generates $L_\lambda^{(n)}$ as a rep of $\mathfrak{gl}(V)$. This implies that it generates a copy of $L_\lambda^{(n)} \otimes \pi_\lambda^{(n)}$ as a $\mathrm{GL}_n \times S_n$-module, so $\pi_\lambda^{(n)} \cong \pi_\lambda^{(n+1)}$. ■

4.2 Schur functors

Definition 4.11. Let λ be a partition (say $|\lambda| = N$). Then, the **Schur functor** S^λ on the category of vector spaces (or of representations of a Lie group) is

$$S^\lambda V := \mathrm{Hom}_{S_N}\left(\pi_\lambda, V^{\otimes N}\right).$$

We can restate Schur-Weyl duality in terms of these functors:

$$V^{\otimes N} = \bigoplus_{\lambda \text{ part of } N} S^\lambda V \otimes \pi_\lambda.$$

If V is the standard representation of $\mathrm{GL}(V)$, then $S^\lambda V = L_\lambda$.

Example. $S^{(N)} V = L_{N\omega_1} = S^N V$.
$S^{(1^N)} = \bigwedge^N V$.

Example.

Question: What?

$$V \otimes V = S^{(2)} V \otimes \mathbb{C}_+ \oplus S^{(1,1)} V \otimes \mathbb{C}_- = S^2 V \oplus \bigwedge^2 V,$$

where $\mathbb{C}_+$ is trivial rep of S_N, and $\mathbb{C}_-$ is the alternating/sign rep.

$$V \otimes V \otimes V = S^{(3)} V \otimes \mathbb{C}_+ \oplus S^{(1,1,1)} V \otimes \mathbb{C}_- \oplus S^{(2,1)} V \otimes \mathbb{C}^2 = S^3 V \oplus \bigwedge^3 V \oplus S^{(2,1)} V \otimes \mathbb{C}^2.$$

Note that

$$V \otimes V \otimes V = (V \otimes V) \otimes V = (S^2 V \otimes V) \oplus \left(\bigwedge^2 V \otimes V \right)$$

(first factor contains $S^3, S^{2,1}$ and second contains $\bigwedge^3, S^{2,1}$ for some reason) so

$$S^2 V \otimes V = S^3 V \oplus S^{2,1} V \text{ and } \bigwedge^2 V \otimes V = \bigwedge^3 V \oplus S^{2,1} V.$$

This gives two descriptions of $S^{2,1}V$:

- 12 symmetric tensors in $V^{\otimes 3}$ whose full symmetrization is zero.
- 12 antisymmetric tensor in $V^{\otimes 3}$ where full antisymmetrization is zero.

What are $\dim S^\lambda V$ where $\dim V = N$? We have the Weyl formula. One can check that we may take $\rho = (N-1, N-2, \dots, 1, 0)$ (smth smth replace $\mathfrak{gl}$ with $\mathfrak{sl}_n$ smth smth). Then,

$$\dim S^\lambda V = \prod_{\alpha > 0} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)},$$

where as usual $\alpha_{ij} = e_i - e_j$ (for $i < j$). Note that $(\rho, \alpha_{ij}) = j - i$ and $(\lambda, \alpha_{ij}) = \lambda_i - \lambda_j$. Hence,

$$\dim S^\lambda V = \prod_{1 \le i < j \le N} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Say λ has k parts. Then above becomes

TODO: Fix typos below

$$\begin{aligned} \dim S^\lambda V &= \prod_{1 \le i < j \le k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \cdot \prod_{1 \le i \le k < j \le N} \frac{\lambda_i - \lambda_j + j - i}{j - i} \\ &= \prod_{1 \le i < j \le k} \frac{\lambda_i + j - i}{j - i} \cdot \prod_{1 \le i \le k} \frac{(N+1-i) \dots (N + \lambda_i - i)}{(k+1-i) \dots (k + \lambda_i - i)} \end{aligned}$$

Proposition 4.12. *$\dim S^\lambda V = P_\lambda(N)$ is a polynomial of degree $|\lambda|$ with $\mathbb{Q}$-coeffs, and it has integer roots all lying in $[1 - \lambda_1, k - 1]$ (and the endpoints are always roots). Further, $P_\lambda(N)$ has integer values at integers which means it's a $\mathbb{Z}$-linear combination of binomial coefficients $\binom{N}{m}$.*

See e.g. the part of chapter 1 of Hartshorne where he talks about Hilbert polynomials and numerical polynomials

Example. $\dim S^n V = \binom{N+n-1}{n} = P_{(n)}(N)$, and $\dim \bigwedge^n V = \binom{N}{n} = P_{(1,\dots,1)}(N)$.

Example. Say $a \ge b$. One can work out that

$$P_{(a,b)}(N) = \frac{a-b+1}{a+1} \binom{N+a-1}{a} \binom{N+b-2}{b} \overset{\text{when } a=b}{=} \frac{1}{a+1} \binom{N+a-1}{N-1} \binom{N+a-2}{N-2}.$$

The $a = b$ case gives **Narayana numbers** which combinatoralists apparently care about.

4.3 Characters of symmetric group

Recall

$$\operatorname{ch} L_\lambda = \operatorname{Tr}|_{L_\lambda = S^\lambda V} \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}.$$

By Weyl character formula, this is

$$\begin{aligned}
\operatorname{ch} L_\lambda &= \frac{\sum_{\sigma\in S_n} \det\sigma\sigma\left(x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\dots x_n^{\lambda_n}\right)}{\prod_{i<j}(x_i-x_j)} \\
&= \frac{\sum_{\sigma\in S_n} \det(\sigma)x_{\sigma(1)}^{\lambda_1+n-1}\dots x_{\sigma(n)}^{\lambda_n}}{\prod_{i<j}(x_i-x_j)} \\
&= \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j}}{\det\left(x_i^{n-j}\right)_{i,j}} = \frac{\det\left(x_i^{\lambda_j+n-j}\right)}{\prod_{i<j}(x_i-x_j)} =: S_\lambda(x)
\end{aligned}$$

which is called the **Schur polynomial** in $x=(x_1,\dots,x_n)$.

Let $V=\mathbb{C}^n$ as usual, and consider $V^{\otimes N}$. Act on it by the pair $(x,\sigma) = \left(\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n\end{pmatrix}, \sigma\right)$ (with $\sigma\in S_N$). Let's compute $\operatorname{Tr}|_{V^{\otimes N}}(x,\sigma)$ in two ways.

- I'm not sure how to type notes on what he's saying right now... The upshot is that if σ has m_i cycles of length i,[5] then

$$\operatorname{Tr}(x,\sigma) = \prod_i \left(\operatorname{Tr}|_V(x^i)\right)^{m_i} = \prod_i \left(x_1^i+\dots+x_n^i\right)^{m_i}.$$

- Recall Schur-Weyl $V^{\otimes N} = \bigoplus L_\lambda\otimes\pi_\lambda$ with x acting on first factor and σ acting on the second. Therefore,

$$\operatorname{Tr}(x,\sigma) = \sum_\lambda S_\lambda(x)\chi_\lambda(\sigma)$$

  with χ_λ an S_N-character and S_λ the Schur polynomial.

Thus,

$$\sum_\lambda S_\lambda(x)\chi_\lambda(\sigma) = \prod_i \left(x_1^i+\dots+x_n^i\right)^{m_i}.$$

Multiply by $\prod(x_i-x_j)$ to get

$$\sum_\lambda\left(\sum_s \det(s)x_1^{\lambda_1+n-1}\dots x_n^{\lambda_n}\right)\chi_\lambda(\sigma) = \prod_{i<j}(x_i-x_j)\prod_i\left(x_1^i+\dots+x_n^i\right)^{m_i}.$$

Thus,

Theorem 4.13 (Frobenius Character Formula). *$\chi_\lambda(\sigma)$ is the coefficient of*

$$x_1^{\lambda_1+n-1}\dots x_n^{\lambda_n}$$

in the product

$$\prod_{1\le i<j\le n}(x_i-x_j)\prod_i\left(x_1^i+\dots+x_n^i\right)^{m_i}.$$

[5] Note $\sum_{i\ge1} im_i = N$

5 Lecture 5 (3/2)

Last time we finished with the character formula for representations of the symmetric group using Schur-Weyl duality. We will next take a quick look at invariant theory; in particular, we want to prove the 'fundamental theorem of invariant theory' (due to Weyl).

5.1 Invariant Theory

Suppose we have a collection of tensors $T_i \in V^{\otimes m_i} \otimes (V^*)^{\otimes n_i}$ ('m_i times contravariant and n_i times covariant'), $i = 1, \ldots, k$, where V is a f.dim vector space. We want to classify invariant functions of $T_1, \ldots, T_k$, i.e. functions of the form $F(T_1, \ldots, T_k)$ (we'll restrict to polynomial functions).

In one perspective, we are looking for functions which we can write in a coordinate-free way. Physicists/engineers think about tensors not as elements of tensor products, but as collections of numbers which change in a specific way when you go from one basis to another. Then one can write various invariant expressions, usually obtained using 'Einstein summation'. If you have $T \in V^{\otimes m} \otimes (V^*)^{\otimes n}$ and e_ℓ a basis of V, then you can write this as

$$T = \sum T^{j_1,\ldots,j_n}_{i_1,\ldots,i_m} e_{j_1} \otimes \cdots \otimes e_{j_m} \otimes e^*_{i_1} \otimes \cdots \otimes e^*_{i_n}.$$

Note 3. Pavel said more things about how physicists think about tensors, but I didn't care enough to write it down.

We look for polynomial functions invariant under the $\mathrm{GL}(V)$-action.

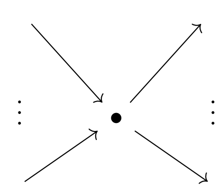
Example. If T is a linear operator, then det and Tr are both invariant functions.

It is enough to classify invariant functions of degree d_i with respect to each T_i. This is equivalent to looking for invariants in

$$\bigotimes_i S^{d_i} \left(V^{\otimes m_i} \otimes (V^*)^{\otimes n_i}\right)^*$$

(above S is symmetric power). Finding invariant functions in this space looks formidable, but in fact it isn't.

To describe such invariant functions, attach to each 'variable' T_i a vertex:



we give the vertex n_i outgoing edges and m_i incoming edges. Put on the plane d_i such vertices of each type i. Invariant functions can be built by contractions of tensors: draw a graph by connecting vertices in a way which respects directions and which makes use of each edge/stub attached to a vertex.

Example. Say $T \in V^{\otimes 2} \otimes (V^*)^{\otimes 2}$ and we want a degree 3 invariant. Then we could form a graph as in Figure 1.

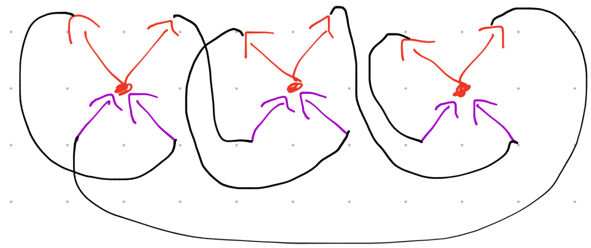


Figure 1: An example graph giving an invariant function

Apparently these graphs are related to Feynman diagrams. To every such graph Γ, one can attach an invariant F_Γ.

Theorem 5.1 (**Fundamental Theorem of Invariant Theory**). *Such functions F_Γ, as Γ varies, span the space of invariant functions.*

Note no linear independence claim above.

Example. Say you have a linear operator $T : V \to V$, so $T \in V \otimes V^*$. Say we want degree d invariant polynomials. Then we need to start with d copies of a vertex with one outgoing edge and one incoming edge. Then we need to connect them in some way. The graph in Figure 2, for example, corresponds to the function $F_\Gamma = \operatorname{Tr}(T^2) \cdot \operatorname{Tr}(T^2)$. Each cycle corresponds to the trace of T to the length of that cycle.

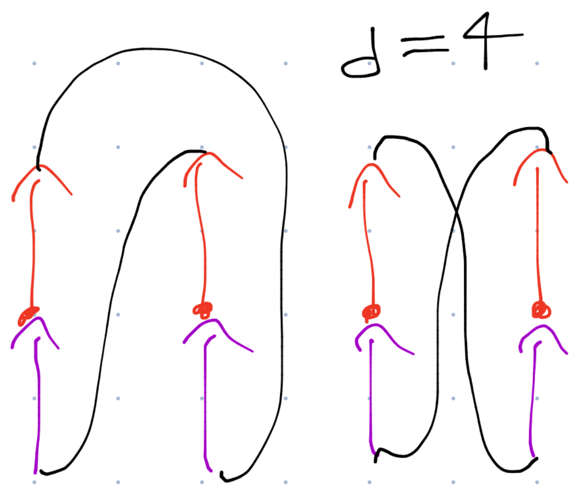


Figure 2: A graph corresponding the the invariant function $\operatorname{Tr}(T^2)^2$

Hence, in this case, the theorem says that degree 4 invariant functions are spanned by

$$\operatorname{Tr}(T)^4, \operatorname{Tr}(T^2)\operatorname{Tr}(T)^2, \operatorname{Tr}(T^2)^2, \operatorname{Tr}(T^3)\operatorname{Tr}(T), \operatorname{Tr}(T^4).$$

Hence, the algebra of invariant polynomials of T is generated by traces of powers of T, i.e. $\operatorname{Tr}(T), \operatorname{Tr}(T^2), \operatorname{Tr}(T^3), \ldots$. Observe that these are not linearly independent (e.g. characteristic polynomial can be used to get some linear dependence between them; if T is $n \times n$, then should only need to know $\operatorname{Tr}(T^i)$ for $i \leq n$).

Proof of Theorem 5.1. An invariant function can be viewed as an element of the tensor product

$$\bigotimes_{i=1}^{k} \left(V^{\otimes m_i} \otimes V^{\otimes(-n_i)}\right)^{\otimes(-d_i)} = \mathrm{Hom}_{\mathbb{C}}\left(V^{\sum n_i d_i}, V^{\sum_i m_i d_i}\right).$$

We want $\mathrm{GL}(V)$-invariants in this space. By Schur-Weyl duality, nonzero invariants only exist if $\sum_i n_i d_i = \sum_i m_i d_i$ (i.e. same number of incoming and outgoing arrows). If so, then Schur-Weyl duality also tells us that the invariants are spanned by permutations (i.e. a matching of incoming arrows to outgoing arrows). This means that these invariants are spanned by the F_Γ's for various Γ.

Question: Why?

In order to pass from tensor products to symmetric power, simply project using symmetrization. This will cause some a priori different graphs to correspond to the same invariant, but this does not affect spanning. ■

Remark 5.2. SW duality tells us that if $\dim V \gg 0$ ($\dim V \geq \sum_i n_i d_i$, I think), then invariants corresponding to different permutations are linearly independent (in the tensor product). Symmetrization identifies some of these, but if you remove the redundant ones, what are left are still linearly independent.

From this one can deduce that for large $\dim V$ and fixed d_i, m_i, n_i, the invariants F_Γ corresponding to non-isomorphic graphs Γ are linearly independent. In this way, one gets a basis fo the space of invariant functions.

Example. Suppose $T_1, \ldots, T_k$ are operators $V \to V$, so all vertices have 2 arrows, one incoming and one outgoing. Hence, all graphs Γ are unions of cycles. Each cycle gives the trace of the product of the graphs appearing in the cycle

Remember: trace of a product is invariant under cyclic permutation

The upshot is that the algebra of polynomial invariants of k linear maps $T_1, \ldots, T_k : V \to V$ is generated by traces $\mathrm{Tr}(T_{i_1} \ldots T_{i_m})$ of *cyclic*[6] words in $T_1, \ldots, T_k$. These generators are "asymptotically algebraically independent" in the sense that for a fixed degree d and $\dim V \gg_d 0$, these generators do not satisfy any nontrivial relations in degree d.

Corollary 5.3. *There are no universal polynomial identities for (square) matrices of all sizes.*

Proof. Suppose $P(X_1, \ldots, X_n) = 0$ for all $X_1, \ldots, X_n$. Introduce another variant X_{n+1}, and consider $F = \mathrm{Tr}(P(X_1, \ldots, X_n)X_{n+1}) = 0$. Traces of words are asymptotically independent, so if it vanishes for all sizes of matrices, then $P = 0$. ■

If you fix a size, then such identities do exist, e.g. for size 1 you have $XY - YX = 0$ (i.e. multiplication of scalars is commutative). For size 2, you have $[(XY - YX)^2, Z] = 0$.[7] This fails for size 3.

In general, for size n, there is the Amitsar-Levitzk identity: for $X_1, \ldots, X_{2n}$ of size n,

$$\sum_{\sigma \in S_{2n}} \mathrm{sign}(\sigma) X_{\sigma(1)} \ldots X_{\sigma(2n)} = 0$$

(homework).

[6] i.e. words defined only up to cyclic permutation

[7] Why? $XY - YX$ has trace 0 and is generically diagonalizable, so looks like $\mathrm{diag}(\lambda, -\lambda)$. Hence, it's square looks like $\lambda^2 I$ which is in the center.

5.2 Howe Duality

Let V, W be two f.dim complex vector spaces. Then consider $S^n(V \otimes W)$ as a representation of $\mathrm{GL}(V) \times \mathrm{GL}(W)$.

Theorem 5.4 (**Howe duality**).

$$S^n(V \otimes W) = \bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes S^\lambda W$$

(if λ has $> \dim V$ or $> \dim W$ parts, then the corresponding summand is 0).

Proof. Note that

$$\begin{aligned} S^n(V \otimes W) &= \left((V \otimes W)^{\otimes n}\right)^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n} \\ &= \left(\left(\bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes \pi_\lambda\right) \otimes \left(\bigoplus_{\mu : |\mu| = n} S^\mu W \otimes \pi_\mu\right)\right)^{S_n} \\ &= \bigoplus_{\lambda, \mu : |\lambda| = |\mu| = n} S^\lambda \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n} \end{aligned}$$

Now we know that the character of π_λ is integer-valued (e.g. by Frobenius formula) so $\pi_\lambda = \pi_\lambda^*$ (character fixed by complex conjugation), so

$$(\pi_\lambda \otimes \pi_\mu)^{S_n} = \mathrm{Hom}_{\mathbb{C}}(\pi_\lambda, \pi_\mu)^{S_n} = \mathrm{Hom}_{S_n}(\pi_\lambda, \pi_\mu).$$

Schur-Weyl duality tells us that π_λ are irreducible and pairwise non-isomorphic, so we conclude that

$$S^n(V \otimes W) = \bigoplus_{\lambda : |\lambda| = n} S^\lambda V \otimes S^\lambda W$$

as desired. ∎

Not important above that V, W are finite dimensional.

Corollary 5.5 (**Cauchy identity**). *If $x = (x_1, \dots, x_r)$ and $y = (y_1, \dots, y_s)$, then*

$$\sum_\lambda s_\lambda(x) s_\lambda(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_j}$$

Proof. We use the **Molien formula**: let $A : V \to V$ be a linear operator on f.d. vector space V (over any field), and let $S^n A : S^n V \to S^n V$ be the induced action of A, then

$$\sum_{n=0}^\infty \mathrm{Tr}(S^n A) z^n = \frac{1}{\det(1 - zA)}.$$

This is easy to prove. Let $x_1, \dots, x_r$ be the eigenvalues of A (so $r = \dim V$), then the eigenvalues of $S^n A$

are $x_1^{m_1}\dots x_r^{m_r}$ s.t. $m_1+\dots+m_r=n$. Hence,

$$\mathrm{Tr}(S^nA)=\sum_{m_1+\dots+m_r=n} x_1^{m_1}\dots x_r^{m_r}=h_n(x_1,\dots,x_r),$$

the **complete symmetric function**. The generating function of these is

$$\sum_{n=0}^{\infty} h_n(x_1,\dots,x_r)z^n=\sum (x_1z)^{m_1}\dots(x_rz)^{m_r}=\prod_{i=1}^{r}\frac{1}{1-x_iz}=\frac{1}{\det(1-zA)}.$$

Now, to prove Cauchy, consider

$$g=\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_r\end{pmatrix} \text{ and } h=\begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_s\end{pmatrix}.$$

So $g\curvearrowright V=\mathbb{C}^r$ and $h\curvearrowright W=\mathbb{C}^s$. Then,

$$\begin{aligned}\mathrm{Tr}\, S^n(g\otimes h)&=\mathrm{Tr}_{S^n(V\otimes W)}(g\otimes h)\\ &=\sum_{\lambda:|\lambda|=n}\mathrm{Tr}_{S^\lambda V}(g)\,\mathrm{Tr}\, S^\lambda W(h)\\ &=\sum_{\lambda:|\lambda|=n} s_\lambda(x)s_\lambda(y).\end{aligned}$$

Hence,

$$\sum_\lambda s_\lambda(x)s_\lambda(y)z^{|\lambda|}=\sum_n \mathrm{Tr}\, S^n(g\otimes h)z^n=\frac{1}{\det(1-z(g\otimes h))}=\prod_{i,j}\frac{1}{1-zx_iy_j}.$$

■

5.3 Minuscule weights

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Definition 5.6. A dominant, integral weight $\omega\in P_+$ is **minuscule** if for all positive coroots β, we have $(\omega,\beta)\le 1$ (i.e. $(\omega,\beta)\in\{0,1\}$). This is equivalent to requiring that for all coroots β, $|(\omega,\beta)|\le 1$.

Example. $\omega=0$ is always a (trivial) minuscule weight.

Example. For $\mathfrak{sl}_n$, all fundamental weights are minuscule. Recall the fundamental weights are $\omega_i=(1,1,\dots,1,0,\dots,0)$, so we see $(\omega_i,e_j-e_k)=0,1$ (with $j<k$).

Proposition 5.7. *Every nonzero minuscule weight $\omega\neq 0$ is fundamental.*

Proof. The inner products $(\omega,\alpha_i^\vee)=0,1$ for minuscule weights. However, it can be 1 only for one i since otherwise $(\omega,\theta^\vee)\ge 2$, where the maximal coroot is $\theta^\vee=\sum m_k\alpha_k^\vee$ with $m_k>0$ for all k. ■

Warning 5.8. Not all fundamental weights are minuscule.

Proposition 5.9. *A fundamental weight ω_i is miniuscule $\iff$ $m_i = 1$, where $\theta^\vee = \sum_i m_i \alpha_i^\vee$.*

Proof. $m_i = (\omega_i, \theta^\vee)$ so for minuscule ω_i, we have $m_i = 1$. If $m_i = 1$, the for all coroot β, $(\omega_i, \beta) \le 1$ (e.g. since $\beta \in \theta^\vee - Q_+$) so ω_i is minuscule. ■

Exercise. G_2 has no minuscule weights except 0.

Here's a theorem we will discuss next time.

Theorem 5.10. *$\omega \in P_+$ is minuscule $\iff$ all weights of L_ω are in the Weyle group orbit of highest weight.*

Corollary 5.11. *When ω minuscule, $\operatorname{ch} L_\omega = \sum_{\gamma \in W\omega} e^\gamma$. In particular, all weight multiplicities are 1.*

6 Lecture 6 (3/4)

Note 4. Video for last class not up yet, so we'll see how much things make sense today...

6.1 Last Time

Recall 6.1. Let $\mathfrak{g}$ be a *simple* Lie algebra. A dominant, integral weight $\omega \in P_+$ is called **minuscule** if for all positive coroots β, we have $(\omega, \beta) \le 1$ ($\in \{0, 1\}$). Equivalently, for all coroots β, $|(\omega, \beta)| \le 1$.

Remark 6.2. Any integral weight can by conjugated to a dominate one via an element of the Weyl group.

Example. $\omega = 0$.

Example. Say $\mathfrak{g} = \mathfrak{sl}_n$. All fundamental weights are minuscule $\omega_i = (1, 1, 1, \dots, 1, 0, 0, \dots, 0)$ (with i ones).

Recall 6.3. Every nonzero minuscule weight is fundamental.

Recall 6.4. Let $\theta^\vee$ be a maximal coroot. Then, ω_i is minuscule $\iff$ $m_i = (\omega_i, \theta^\vee) = 1$.

Remark 6.5. $\theta^\vee = \sum_i m_i \alpha_i^\vee$.

6.2 This Time: minisucle weights

Lemma 6.6. *If $\omega \in Q$, and $|(\omega, \beta)| \le 1$ for all coroots β, then $\omega = 0$. Hence, there are no nonzero miniscule weights in the root lattice.*

Remember: Q is the root lattice

Proof. Suppose $\omega = \sum_i m_i \alpha_i$ (with $m_i \in \mathbb{Z}$) is a counterexample minimizing $\sum_i |m_i|$. Then,

$$0 < (\omega, \omega) = \sum_i m_i (\omega, \alpha_i),$$

so there's some index j s.t. m_j and $(\omega, \alpha_j$ are nonzero and of the same sign. Replacing ω by $-\omega$ if needed, we may assume $m_j, (\omega, \alpha_j) > 0$. Since $\alpha_j^\vee$ is a positive multiple of α_j, we have also $(\omega, \alpha_j^\vee) > 0$. By hypothesis, we know $\left|(\omega, \alpha_j^\vee)\right| \le 1$, so $(\omega, \alpha_j^\vee) = 1$. Consider

$$s_j \omega = \omega - (\omega, \alpha_j^\vee)\alpha_j = \omega - \alpha_j = \sum_i m_i' \alpha_i$$

where $m_i' = m_i$ if $i \neq j$ and $m_j' = m_j - 1$. Note that $\sum |m_i'| = \sum_i |m_i| - 1$, but $s_j\omega$ is also a counterexample (modifying by Weyl group does not affect property). ■

$\omega \neq \alpha_j$ since $(\alpha_j, \alpha_j) = 2 > 1$, I think

Example. For G_2, $P = Q$ (weight lattice = root lattice), so there are no nonzero miniscule weights.

Proposition 6.7. *A weight $\omega \in P_+$ is minuscule iff for all $\alpha \in Q_+$ ($\alpha \neq 0$), $\omega - \alpha$ is not dominant.*

Proof. ($\rightarrow$) Suppose $\omega = \omega_k$ is miniscule and $\alpha \in Q_+$ is nonzero. Suppose also that $\omega_k - \alpha$ is dominant. We can write $\alpha = \sum_i m_i\alpha_i$ with $m_i \in \mathbb{Z}_+$. If $m_j = 0$ for some $j \neq k$, then reduces to smaller rank (can delete vertex j from Dynkin diagram[8]), so we may assume $m_j > 0$ for $j \neq k$. Then, for all positive coroots β,

$$(\alpha, \beta) = (\omega_k, \beta) - (\omega_k - \alpha, \beta) \leq (\omega_k, \beta) \leq 1$$

($\omega_k - \alpha$ dominant $\implies (\omega_k - \alpha, \beta) \geq 0$) with equality if β involves $\alpha_k^\vee$. If β does not involve $\alpha_k^\vee$, then $(\alpha, \beta) \leq 0$. In particular, $(\alpha, \alpha_i^\vee) \leq 0$ if $i \neq k$ and $(\alpha, \alpha_k^\vee) \leq 1$.

Now, if $(\alpha, \alpha^\vee) \leq 0$, we'd get $(\alpha, \alpha) \leq 0$, a contradiction (α positive linear combination of positive roots). Thus, $(\alpha, \alpha_k^\vee) = 1$. As a consequence, $m_k > 0$ ($m_k = 0 \implies (\alpha, \alpha_k^\vee) \leq 0$ since only involves simple roots/coroots with different indices and those entries in Cartan matrix are always ≤ 0). Thus, $(\alpha, \theta^\vee) \geq 1$ so $(\omega_k - \alpha, \theta^\vee) = 1 - (\alpha, \theta^\vee) \leq 0$ which forces $\omega_k - \alpha = 0$. Hence, $\omega_k \in Q$, a contradiction to previous lemma.

($\leftarrow$) Suppose ω is not miniscule. We'll produce an $\alpha \in Q_+$ s.t. $\omega - \alpha$ is dominant. Since ω is not miniscule, there exists a positive root γ s.t. $(\omega, \gamma^\vee) \geq 2$. Consider[9] $\omega - \gamma$. We first claim this is not conjugate to ω. Observe

$$(\omega - \gamma, \omega - \gamma) = (\omega, \omega) - 2(\gamma, \omega) + (\gamma, \gamma).$$

Since $2(\gamma, \omega)/(\gamma, \gamma) = (\gamma^\vee, \omega) \geq 2 > 1$, we see that $2(\gamma, \omega) > (\gamma, \gamma)$, so $(\omega - \gamma, \omega - \gamma) < (\omega, \omega)$ which means $\omega - \gamma \notin W\omega$. Now, pick $w \in W$ such that $\lambda := w(\omega - \gamma) \in P_+$. Then, $\lambda \neq \omega$, but $\omega - \lambda \in Q_+$ because $\omega - \gamma$ is a weight of L_ω (for the vector[10] $f_\gamma v_\omega$). ■

Remark 6.8. We have a classification of root systems/semisimple lie algebras from last time, so in principle, we could just go through the list and check which roots are miniscule. This would be unsatisfying, so we don't do that.

Question 6.9. *Why are minuscule weights interesting?*

Proposition 6.10. *ω is minuscule $\iff$ the Weyl group W acts transitively on the weights of the irrep L_ω.*

Proof. ($\rightarrow$) Let μ be a weight of L_ω. Pick $w \in W$ such that $w\mu$ is dominant. Then, $w\mu = \omega - \alpha$ for some $\alpha \in Q_+$. This implies that $w\mu = \omega$, so $\mu = w^{-1}\omega$.

($\leftarrow$) If ω is not miniscule, take γ as in the previous proof, and consider $\omega - \gamma$, the weight of $f_\gamma v_\omega \in L_\omega$. This is nonzero so $\omega - \gamma$ is a weight not in the orbit of ω. ■

Corollary 6.11. *All weight spaces of L_ω are 1-dimensional when ω miniscule.*

Remark 6.12. The converse of this is false. Think about reps of $\mathfrak{sl}_2$, for example.

[8] Pass to root subsystem generated by α_i for $i \neq j$

[9] "Just for fun, let us use representation theory."

[10] This is nonzero since $h_\gamma v_\omega = (\gamma^\vee, \omega)v_\omega \neq 0$

Corollary 6.13. *The character of L_ω is*

$$\chi_\omega = \sum_{\lambda \in W\omega} e^\lambda.$$

You could also compute this using Weyl's character formula. Comparing the two would lead to some nontrivial identity.

Corollary 6.14. *If α is a root of $\mathfrak{g}$, then $L_\omega|_{(\mathfrak{sl}_2)_\alpha}$ is a direct sum of 1-dimensional and 2-dimensional representations of $(\mathfrak{sl}_2)_\alpha$.*

Proof. Let $v \in L_\omega$ be a highest weight vector for $(\mathfrak{sl}_2)_\alpha$, of some weight λ. Then,

$$h_\alpha v = (\lambda, \alpha^\vee)v = (w\omega, \alpha^\vee)v \in \{v, 0, -v\}.$$

It can't be -1 (since v highest weight), so it's 0 or 1. Hence, v generates a 1-d or 2-d rep of $(\mathfrak{sl}_2)_\alpha$. ∎

Note 5. Pavel worked out an example looking at B_2, but I did not pay attention. I should go back and watch the video and add it in later...

Corollary 6.15. *If ω is minuscule and $\lambda \in P_+$, then*

$$L_\omega \otimes L_\lambda = \bigoplus_{\mu \in W\omega} L_{\lambda+\mu}$$

(if $\lambda + \mu \notin P_+$, the corresponding term drops out, i.e we really sum over $\mu \in W\omega$ s.t. $\lambda + \mu \in P_+$)

Proof. We use Weyl character formula:

I possibly made some typos below

$$\begin{aligned}
\chi_{L_\omega \otimes L_\lambda} &= \left(\sum_{\mu \in W\omega} e^\mu\right) \frac{\sum_{v \in W} \det(v) e^{v(\lambda+\rho)}}{\prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})} \\
&= \frac{\sum_{\mu \in W\omega, v \in W} \det(v) e^{v(\lambda+\rho)+\mu}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})} \\
&= \frac{\sum_{\mu \in W\omega, v \in W} \det(v) e^{v(\lambda + w^{-1}\mu + \rho}}{\prod_{\alpha>0}\left(e^{\alpha/2} - e^{-\alpha/2}\right)} \\
&= \frac{\sum_{\nu \in W\omega, w \in W} \det(w) e^{w(\lambda+\nu+\rho)}}{\prod_{\alpha>0}(e^{\alpha/2} - e^{-\alpha/2})} \\
&= \sum_{v \in W\omega} \chi_{L_{\lambda+\nu}}.
\end{aligned}$$

If $\lambda + \nu \notin P_+$, then $(\lambda + \nu, \alpha_i^\vee)$, there exists i s.t. $(\lambda + \nu, \alpha_i^\vee) < 0$. But $(\lambda, \alpha_i^\vee) \geq 0$ and $|(\nu, \alpha_i^\vee)| \leq 1$, so $(\lambda + \nu, \alpha_i^\vee) = (\nu, \alpha_i^\vee) = -1$ which means $(\lambda, \alpha_i^\vee) = 0$. We know $(\rho, \alpha_i^\vee) = 1$, so

$$(\lambda + \nu + \rho, \alpha_i^\vee) = 0.$$

This means $s_i(\lambda + \nu + \rho) = \lambda + \nu + \rho$, so the terms $\det(w)e^{w(\lambda+\nu+\rho)}$ and $\det(ws_i)e^{ws_i(\lambda+\nu+\rho)}$ will cancel. This justifies ignoring the terms not in P_+. ∎

Recall all fundamental representations of $\mathfrak{sl}_n$ are miniscule.

Corollary 6.16. *Let $V = \mathbb{C}^n$ be the vector representation of* GL_n*. Then, for any partition* λ*,*

$$V \otimes L_\lambda = \bigoplus_{i=1}^{n} L_{\lambda+e_i}$$

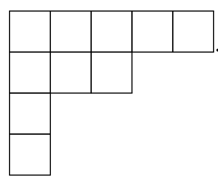
(if $\lambda + e_i$ is not nonincreasing, drop it's term).

Example. Take $L_{(1,0,\ldots,0,-1)} = \mathfrak{sl}(V)$, the adjoint representation. Then,

$$V \otimes \mathfrak{sl}(V) = L_{(2,0,\ldots,0,-1)} + L_{(1,1,\ldots,0,-1)} + L_{(1,0,1,\ldots,-1)} + \cdots + L_{(1,0,\ldots,0)},$$

but only the first two and last terms survive. Hence, $V \otimes \mathfrak{sl}(V) = V \oplus$(two other irreps).

We can give a combinatorial interpretation of the previous corollary. Recall that partitions correspond to Young diagrams, e.g. $\lambda = (5,3,1,1,0)$ (say $n = 5$) is the diagram



What is the diagram corresponding to $V \otimes L_\lambda$? The $\lambda + e_i$'s are

$$(7,3,1,1,0), (6,4,1,1,0), (6,3,2,1,0), (6,3,1,2,0), (6,3,1,1,1)$$

Note that these each correspond to adding a square a square to one row of λ. If adding the square produces another Young diagram (preserves monotonicity), we call it an **addable box**. Thus, we see that

$$V \otimes L_\lambda = \sum_{\lambda' = \lambda + \square} L_{\lambda'}$$

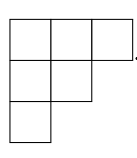
(sum over addable boxes). We can do the same thing for exterior powers.

Recall $L_{\omega_i} = \bigwedge^i V$ with weights given by $(a_1, \ldots, a_n)$ s.t. $a_j \in \{0,1\}$ and there are exactly i copies of 1. Adding $\lambda + (a_1, \ldots, a_n)$ corresponds to adding i boxes to *different* rows of λ. Since ω_i is miniscule, we have

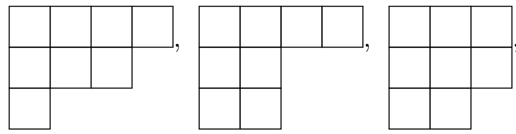
$$\bigwedge^i V \otimes L_\lambda = \bigoplus_{\substack{I \subset \{1,\ldots,n\} \\ |I| = i}} L_{\lambda + e_I}.$$

Graphically, $\bigwedge^i V \otimes L_\lambda$ is a sum over Young diagrams obtained from λ by adding i boxes in different rows.

Example. Say $n = 3$. Let's compute $\bigwedge^2 V \otimes L_{(3,2,1)}$. Note that $\lambda = (3,2,1)$ looks like



The diagrams in the sum are



i.e.

$$\bigwedge^2 V \otimes L_{(3,2,1)} = L_{(4,3,1)} + L_{(4,2,2)} + L_{(3,3,2)}.$$

Similarly, one gets

$$\bigwedge^2 V \otimes L_{(3,1,1)} = L_{(4,2,1)} + L_{(3,2,2)}.$$

If we were over GL_4, there would be extra summands, e.g. an $L_{(4,1,1,1)}$ and an $L_{(3,2,1,1)}$.

Proposition 6.17. *Every coset of P/Q contains a unique minuscule weight, so there's a bijection between P/Q and minuscule weights.*

Proof. Consider $C_q = a + Q \subset P$, the coset of a. Look at the intersection $C_a \cap P_+$. Take $\omega \in C_+ \cap P_+$, an element with smallest $2(\omega, \rho^\vee) \in \mathbb{Z}$. Weights of L_{ω_a} are all in C_a. If ω_a is not minisucle, there exists $\lambda \in P_+$ s.t. $0 \neq \omega_a - \lambda \in Q_+$ which implies $2(\lambda, \rho^\vee) < 2(\omega, \rho^\vee)$. This contradicts minimality, so ω_a is minuscule.

If $\omega_1, \omega_2 \in C_a$ are both miniscule, consider there different $\omega_1 - \omega_2 \in Q$. If it is nonzero, by a previous lemma, there exists a coroot β s.t. $(\omega_1 - \omega_2, \beta) \geq 2$. This forces $(\omega_1, \beta) = 1$ and $(\omega_2, \beta) = -1$. But the first forces beta to be positive while the second forces beta to be negative, and this is a contradiction. ∎

Corollary 6.18. *The number of miniscule weights is $\#P/Q = \det A$, where A is the Cartan matrix.*

Example. B_n ($\mathfrak{o}(2n+1)$) has $\det = 2$ so there's only one (nonzero) miniscule weight. The corresponding representation here is called the 'spinor representation'.

C_n ($\mathfrak{sp}(2n)$) has $\det = 2$ so there's again one minuscule weight. One can check that it corresponds to the vector representation.

D_n ($\mathfrak{o}(2n)$) has $\det = 4$, so 3 miniscule weights. These are the vector representation V and two spinor representations $S^\pm$.

More on rep theory of orthogonal and symplectic groups next time. After that, we will start looking at the theory of compact Lie groups.

7 Lecture 7 (3/11)

7.1 Fundamental weights/representations for classical Lie algebras

7.1.1 Type C_n

We begin with the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $B = \sum_{i=1}^n x_i \wedge x_{i+n}$ be our symplectic form. A natural choice of Cartan subalgebra is

$$\mathfrak{h} = \begin{pmatrix} \mathrm{diag}(a_1, \ldots, a_n) & \\ & -\mathrm{diag}(a_1, \ldots, a_n) \end{pmatrix} \cong \mathbb{C}^n.$$

In $\mathbb{R}^n$, the roots are

$$\alpha_i = e_i - e_{i+1} = \alpha_i^\vee \text{ for } i = 1, \dots, n-1 \text{ and } \alpha_n = 2e_n \text{ with } \alpha_n^\vee = e_n = \frac{1}{2}\alpha_n.$$

The fundamental weights are $\omega_i = (1, \dots, 1, 0, \dots, 0)$ with i copies of 1.

Recall 7.1. The Dynkin diagram for C_n is

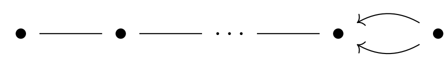


Figure 3: The Dynkin Diagram C_n

Recall 7.2. The number of miniscule weights equals the determinant of the Cartan matrix.

Here, the determinant is 2 so there is 1 nonzero miniscule weight. It is the weight 1 corresponding to the vertex on the left end of the Dynkin diagram.

Example. Consider the representation $V = \mathbb{C}^{2n}$. Its weights are $e_1, \dots, e_n$ and $-e_1, \dots, -e_n$. The Weyl group is $W = S_n \rtimes (\mathbb{Z}/n\mathbb{Z})^n$ (S_n permutes while $(\mathbb{Z}/n\mathbb{Z})^n$ changes signs). Note $V \simeq V^*$. The highest weight here is $e_1 = \omega_1$ which is the miniscule weight.

What about other fundamental representations? Well, $\mathfrak{g}$ has the same fundamental weights as GL_n, so maybe we should expect $L_{\omega_i} = \bigwedge^i V$? This is true for $i = 1$.

But it is not true for $i = 2$! $\bigwedge^2 V$ is not irreducible (but has correct highest weight). The symplectic form B is non-degenerate, so we can invert it

$$B^{-1} = \sum_i x_{i+n}^* \wedge x_i^* \in \bigwedge^2 V$$

and $B^{-1} \in \bigwedge^2 V$ generates a copy of the trivial representation $\mathbb{C}$. We can write

$$\bigwedge^2 V = \mathbb{C}\left\langle B^{-1} \right\rangle \oplus \bigwedge\nolimits_0^2 V \text{ where } \bigwedge\nolimits_0^2 V = \left\{ y \in \bigwedge^2 V : (y, B) = 0 \right\}.$$

Exercise. $\bigwedge_0^2 V$ is irreducible.[11]

Hence, $L_{\omega_2} = \bigwedge_0^2 V$ so our intuition was not bad.

Example. For $\mathfrak{sp}(4)$, only two fundamental weights, $V = \mathbb{C}^4$ and $\bigwedge^2 V = \mathbb{C} \oplus \bigwedge_0^2 V$ is 6-dimensional. Recall $\mathfrak{sp}(4) \simeq \mathfrak{o}(5)$ which has a 5-dimensional vector representation.

What happens for other i?

They will be contained in exterior powers, but some pieces will fall off. Consider the exterior algebra

$$\bigwedge V := \bigoplus_{i=0}^{2n} \bigwedge^i V.$$

[11] Look at weights, or write character formula, or show directly

Recall we have $B \in \bigwedge^2 V^*$. Hence, given $T \in \bigwedge^{i+1} V$, we can form $\iota_B T \in \bigwedge^{i-1} V$. In the other direction, we may wedge with B^{-1} to move from $m_B : \bigwedge^{i-1} \to \bigwedge^{i+1}$.

Proposition 7.3.

(1) *The operators m_B, ι_B, and h ($hT = (i-n)T$ with $T \in \bigwedge^i$) form an $\mathfrak{sl}_2$-triple.*

(2) *The operator*

$$\iota_B : \bigwedge^{i+1} \to \bigwedge^{i-1}$$

is surjective for $i \leq n$ and injective for $i \geq n$ (so an iso for $i = n$). Let $\bigwedge_0^i V := \ker \iota_B$. This is irreducible if $1 \leq i \leq n$, and $\bigwedge_0^i V \cong L_{\omega_i}$.[12]

(3)

$$\bigwedge V = \bigoplus_{i=0}^{n} L_{\omega_i} \otimes L_{n-i}$$

where $\omega_0 = 0$ and L_{n-i} is the $\mathfrak{sl}(2)$-rep with highest weight $n-i$ and dimension $n-i+1$.

This is another instance of the double centralizer property

(4) *Every irreducible representation of $\mathfrak{sp}_{2n}$ occurs in $V^{\otimes N}$ for some N (since all fundamental reps do).*[13]

Proof. Homework.[14] ■

Remark 7.4. Note $\dim \bigwedge^i V = \binom{2n}{i}$ so these dimensions form a Bell curve shape.

7.1.2 Type B_n

We now have the Lie algebra $\mathfrak{g} = \mathfrak{o}_{2n+1}$. The roots here are $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$ and $\alpha_n = e_n$. The dual roots for $\alpha_i^\vee = \alpha_i$ for $i = 1, \dots, n-1$ and $\alpha_n^\vee = 2e_n = 2\alpha_n$. The fundamental weights are

$$\omega_i = (\underbrace{1, \dots, 1}_{i}, 0, \dots, 0) \text{ for } i = 1, \dots, n-1$$

and

$$\omega_n = \left(\frac{1}{2}, \dots, \frac{1}{2}\right).$$

The Weyl group is $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ (permute coordinates and change signs). The Cartan matrix

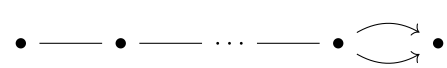


Figure 4: The Dynkin Diagram B_n

again has determine 2 (transpose of previous one?), so only one nontrivial miniscule weight. This time it is ω_n.

[12]This should follow from rep theory of $\mathfrak{sl}_2$

[13]This will not be true for the orthogonal groups and is "the reason our world exists" (something about spin in physics)

[14]Should be "easy" after establishing you have an $\mathfrak{sl}_2$-rep. Irreducibility should come from looking at characters (correct highest weight and correct dimension)

Warning 7.5. $L_{\omega_1} = V = \mathbb{C}^{2n+1}$ is the vector representation, but is *not* miniscule.

Example. The weights of $\mathbb{C}^5$ (for $\mathfrak{so}(5)$) include 0, but 0 is not a weight of miniscule representations (Weyl group acts transitively on weights).

Exercise. $\bigwedge^i V$ are irreducible for $1 \leq i \leq n$, so

$$\bigwedge^i V = L_{\omega_i} \text{ for } i \leq n-1 \text{ and } \bigwedge^n V = L_{2\omega_n}.$$

Remark 7.6. For C_n, we have an invariant skew-symmetric form. For B_n, we now have an invariant symmetric form. Hence something falls off for symmetric powers instead of for exterior powers.

Definition 7.7. L_{ω_n} is called the **spinor representation** and denoted S.

What are its weights? It's miniscule, so it's weights should be an orbit under Weyl group. Hence, they'll be $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \dots, \pm\frac{1}{2}\right)$ for any choice of signs. Hence, $\dim S = 2^n$. What is the character of S?

We are using the quadratic form $Q = x_1 x_{n+1} + \cdots + x_n x_{2n} + x_{2n+1}^2$, so $\mathfrak{o}(2n+1)$ fixes this form. The natural Cartan subalgebra is

$$\mathfrak{h} = \{\operatorname{diag}(a_1, \dots, a_n, -a_1, \dots, -a_n, 0)\} \cong \mathbb{C}^n.$$

Note that for $h \in \mathfrak{h}$, its exponential is

$$e^h = \operatorname{diag}(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}, 1).$$

We want to compute it's trace on S. This gives the character

$$\chi_S = \sum x_1^{\pm\frac{1}{2}} \dots x_n^{\pm\frac{1}{2}} = \left(x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}}\right) \dots \left(x_n^{\frac{1}{2}} + x_n^{-\frac{1}{2}}\right).$$

What's up with these 1/2 powers? Square root of a complex number is only defined up to sign, so does this make sense? What does this mean. It means this representation does not lift to the orthogonal group $\mathrm{SO}(2n+1)$. The point is that the orthogonal group is not simply connected, so not all Lie algebra representations lift to it.[15] For the same reason, S does not occur in $V^{\otimes N}$. Elements of S are called **spinors**.

If it did, the exponents would all be integers

Definition 7.8. The universal cover of $\mathrm{SO}_{2n+1}(\mathbb{C})$ is called the **Spin group**, denoted $\mathrm{Spin}_{2n+1}(\mathbb{C})$.

Now, S gives a representation of $\mathrm{Spin}_{2n+1}(\mathbb{C})$.

Theorem 7.9. *$\pi_1(\mathrm{SO}_n(\mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$ for any $n \geq 3$.*[16]

Example. When $n = 3$, $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Spin}_3$ is a double cover of $\mathrm{SO}(3)$. What is the map $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}(3)$? Take the 3-dimensional representation of SL_2; this is the adjoint representation which has an invariant form, the Killing form. The kernel of this map is $\mathbb{Z}/2\mathbb{Z} \cong \{\pm I\}$, the center of SL_2, so we see the exact sequence

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{SL}_2 \longrightarrow \mathrm{SO}_3 \longrightarrow 1.$$

[15] It will however, lift to its universal cover.

[16] $\pi_1(\mathrm{SO}_2(\mathbb{C})) = \mathbb{Z}$, apparently.

Lemma 7.10. *Let* $X_n = \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 1\right\}$*. Then,* X_n *is simply connected for* $n \geq 3$*, and* $\pi_2(X_n) = 1$ *for* $n \geq 4$*.*

Proof. Consider $X_n^{\mathbb{R}} = X_n \cap \mathbb{R}^n = S^{n-1} \subset \mathbb{R}^n$. This is simple connected for $n \geq 3$, so it suffices to show that X_n deformation retracts onto $X_n^{\mathbb{R}}$. Consider some $z \in X_n$ and write $z = x + iy$ with $x, y \in \mathbb{R}^n$. Then,

$$1 = z^2 = x^2 - y^2 + 2ix \cdot y \implies x^2 - y^2 = 1 \text{ and } x \cdot y = 0.$$

Now, consider the homotopy $f_t : X_n \to X_n$ given by

$$f_t(x + iy) = \frac{x + ity}{\sqrt{x^2 - t^2 y^2}}$$

(note $(x + ity)^2 = x^2 - t^2 y + 2ix \cdot y = x^2 - t^2 y^2 \geq x^2 - y^2 = 1$). Note that $f_1 = \mathrm{Id}$ while $f_0(x + iy) = x/|x| \in X_n^{\mathbb{R}}$. Furthermore, $f_t|_{X_n^{\mathbb{R}}} = \mathrm{Id}_{X_n^{\mathbb{R}}}$ for all t, so $X_n^{\mathbb{R}}$ is a deformation retract of X_n, finishing the proof. ■

Note that when $n = 4$, we have $X_4 \cong \{ad - bc = 1 : a, b, c, d \in \mathbb{C}\} = \mathrm{SL}_2(\mathbb{C})$, so this lemma recovers the fact that $\mathrm{SL}_2(\mathbb{C})$ is simply connected.

Proof of Theorem 7.9. We will induct in n. Note that we already know the theorem when $n = 3$. Note that $\mathrm{SO}_n \curvearrowright X_n$ transitively[17], so X_n is a homogeneous space. What is the stabilizer?

$$\mathrm{Stab}_{\mathrm{SO}_n}(e_1) = \mathrm{SO}_{n-1}$$

since if it preserves e_1 it'll also preserve the orthocomplement of e_1. Hence, $X_n = \mathrm{SO}_n / \mathrm{SO}_{n-1}$, so we have a fiber sequence $\mathrm{SO}_{n-1} \twoheadrightarrow \mathrm{SO}_n \twoheadrightarrow X_n$. Thus, we get an exact sequence

$$\pi_2(X_n) \to \pi_1(\mathrm{SO}_{n-1}) \to \pi_1(\mathrm{SO}_n) \to \pi_1(X_n).$$

The previous lemma shows $\pi_1(X_n) = 1$ for $n \geq 3$ and $\pi_2(X_n) = 1$ for $n \geq 4$. Hence, we win. ■

Corollary 7.11. $\mathrm{Spin}_n(\mathbb{C})$ *is a double cover of* $\mathrm{SO}_n(\mathbb{C})$ *for* $n \geq 3$*.*

7.1.3 Type D_n

Finally, we consider the Lie algebra $\mathfrak{g} = \mathfrak{o}_{2n}$. As usual, $V = \mathbb{C}^{2n}$ is the vector representation. The simple roots are $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$. The fundamental weights are $\omega_1 = (1, 0, \ldots, 0), \omega_2 = (1, 1, 0, \ldots, 0)$ up to $\omega_{n-2} = (1, \ldots, 1, 0, 0)$ and then

$$\omega_{n-1} = \left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right) \text{ and } \omega_n = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\right).$$

We know have two **spinor representations**, $L_{\omega_{n-1}} = S_-$ and $L_{\omega_n} = S_+$. In this case, the Cartan matrix has determinant $\det A = 4$, so there are 3 miniscule fundamental weights. These are $\omega_{n-1}, \omega_n, \omega_1$.

Example. The weights of $L_{\omega_1} = V$ are $e_1, \ldots, e_n$ and $-e_1, \ldots, -e_n$.

[17] Can always move any vector to e_1

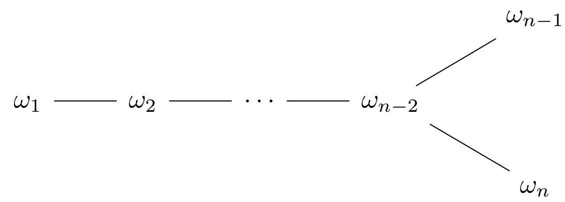


Figure 5: The Dynkin Diagram D_n

Our quadratic form is $Q = x_1x_{n+1} + \cdots + x_nx_{2n}$, so our Cartan subalgebra is

$$\mathfrak{h} = \{\text{diag}(a_1, \ldots, a_n, -a_1, \ldots, -a_n)\}.$$

The Weyl group here is $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})_0^n$ where the 0 subscript means elements whose coordinates sum to 0.

Remark 7.12. Exterior powers are irreducible for $i \leq n-1$ still. Hence, for $i \leq n-2$,

$$\bigwedge^i V = L_{\omega_i}.$$

Some aspects of orthogonal groups are uniform and some depend on even or odd. Some even depend on residue mod 4, and some even depend on residue mod 8. This is related to Bott periodicity. More on this on a homework.

Example. $S_+^* = S_+$ or $S_+^* = S_-$ depending on n mod 4. When $S_+^* = S_+$ it has an invariant inner product. Is it symmetric or skew-symmetric? This depends on n mod 8.

What do the spinor representations $S_\pm$ look like? The Weyl group allows us to permute factors and change an even number of signs. Thus, the weights of S_+ are the vectors

$$\left(\pm\frac{1}{2}, \ldots, \pm\frac{1}{2}\right)$$

with an even number of $+$'s while the weights of S_- are those with an odd number of $+$'s ($\iff$ an odd number of $-$'s). Thus, we get the characters

$$\chi_{S_\pm} = \left(\prod_{i=1}^n \left(x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}}\right)\right)_\pm,$$

so S_+, S_- don't occur in $V^{\otimes N}$ and they don't lift to SO_{2n}. We again define Spin_{2n} to be the universal cover of SO_{2n} (again a double cover by previous theorem).

8 Lecture 8 (3/16)

8.1 Last time

We talked about representations of of $\mathfrak{o}(V)$. When $V = \mathbb{C}^{2n}$ this is type D_n. When $C = \mathbb{C}^{2n+1}$, this is type B_n. We also talked about spinor representations.

For $\mathfrak{o}_{2n+1}$, the spinor representation is associated to $\omega_n = (1/2, 1/2, \ldots, 1/2)$ and has dimension $\dim S = 2^n$.

For $\mathfrak{o}_{2n}$, there are two spinor representations $S_+ = L_{\omega_n}$ and $S_- = L_{\omega_{n-1}}$ where $\omega_{n-1} = (1/2, \ldots, 1/2, -1/2)$ and $\omega_n = (1/2, \ldots, 1/2, 1/2)$.

Question 8.1. *How can we construct these explicitly?*

We know they don't occur in the tensor products of vector representations, so we have to do something new.

8.2 Clifford algebra

We know S does not occur in $V^{\otimes n}$ (its weights have half-integer coordinates), but $S \otimes S^*$ does occur (its weights have integer coordinates). One can show that

$$S \otimes S^* = \bigwedge^{\text{even}} V = \mathbb{C} + \bigwedge^2 V + \bigwedge^4 V + \ldots.$$

We need to "extract a square root" roughly in the sense that the space of vectors is a "square root" of the space of matrices. This is the idea behind Clifford algebras.

Definition 8.2. Let V be a f.d. k-vector space ($k = \overline{k}$ and $\operatorname{char} k \neq 2$) with a symmetric (non-degenerate) inner product $(-,-)$. The **Clifford algebra** $\mathrm{Cl}(V)$ of V is generated by V with defining relations $v^2 = \frac{1}{2}(v, v)$ for $v \in V$.

Remark 8.3. Given $a, b \in v$, one has

$$ab + ba = (a+b)^2 - a^2 - b^2 = \frac{1}{2}\left[(a+b, a+b) - (a,a) - (b,b)\right] = (a,b).$$

Thus, an equivalent set of defining relations is

$$ab + ba = (a,b) \cdot 1.$$

Can we describe this in terms of a basis?

Example. When $\dim V = 2n$, we can find a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ of V so that

$$(a_i, a_j) = (b_i, b_j) = 0 \text{ and } (a_i, b_j) = \delta_{ij}.$$

Then the relations are

$$a_i a_j + a_j a_i = 0, \;\; b_i b_j + b_j b_i = 0, \;\text{ and }\; a_i b_j + b_j a_i = \delta_{ij}.$$

This is a deformation of $\bigwedge V$; we'll make this more precise later.

Example. When $\dim V = 2n+1$, we can find a basis $a_1, \ldots, a_n, b_1, \ldots, b_n, z$ with a_i, b_j as above and $(z, a_i) = 0 = (z, b_i)$ while $(z, z) = 2$. The relations here are all the ones from before along with

$$z a_i + a_i z = 0 = z b_i + b_i z \;\text{ and }\; z^2 = 1.$$

This is again a deformation of $\bigwedge V$.

What is this deformation business we're claiming/alluding to?

There is a filtration on $\mathrm{Cl}(V)$ obtained by setting $\deg v = 1$ for $v \in V$. In general, for $x \in \mathrm{Cl}(V)$, $\deg(x)$ is the smallest $d \in \mathbb{Z}_+$ such that x is a sum of monomials of degrees $\leq d$. We filter by degree; $F_0 \mathrm{Cl}(V) \subset F_1 \mathrm{Cl}(V) \subset \dots$. The associated graded

$$\mathrm{gr}\,\mathrm{Cl}(V) = \bigoplus \underbrace{F_{i+1}/F_i}_{\mathrm{gr}_i \mathrm{Cl}(V)}$$

fits into a natural surjective homomorphisms

$$\varphi : \bigwedge V \longrightarrow \mathrm{gr}\,\mathrm{Cl}(V).$$

This is because the RHS's of the defining relations for $\mathrm{Cl}(V)$ all have degree strictly smaller than the LHS's so all vanish in the associated graded.

Theorem 8.4. *φ is an isomorphism.*

Equivalently, φ is injective ($\iff \dim \mathrm{Cl}(V) = 2^{\dim V}$).

Remark 8.5. This is similar to the PBW theorem for Lie algebras:

$$U(\mathfrak{g}) = \langle a \in \mathfrak{g} : ab - ba[a,b] \rangle .$$

The natural surjection

$$\varphi : S\mathfrak{g} \twoheadrightarrow \mathrm{gr} U(\mathfrak{g})$$

is an isomorphism.

In fact, PBW generalizes to "Lie superalgebras" and this theorem about $\mathrm{Cl}(V)$ is a special case of this generalization.

Theorem 8.6. $\mathrm{Cl}(V) \cong \mathrm{Mat}_{2^n}(k)$ *if* $\dim V = 2n$, *and* $\mathrm{Cl}(V) \cong \mathrm{Mat}_{2^n}(k) \oplus \mathrm{Mat}^{2\mathrm{n}}(k)$ *if* $\dim V = 2n+1$.

Note this is even stronger than Theorem 8.4.

Proof. (Even case) Let us start with the even case. Pick a basis $a_1, \dots, a_n, b_1, \dots, b_n$ of V as before. Let $m = \bigwedge(a_1, \dots, a_n)$. Define a representation

$$\rho : \mathrm{Cl}(V) \to \mathrm{End}\, M, \;\; \rho(a_i)v = a_i w \;\text{ and }\; \rho(b_i)w = \frac{\partial}{\partial a_i} w.$$

Above,

$$\frac{\partial}{\partial a_i}(a_{k_1} \dots a_{k_r}) = \begin{cases} 0 & \text{if } i \neq k_j \,\forall j \\ (-1)^{j-1} a_{k_1} \dots \widehat{a_{k_j}} \dots a_{k_r} & \text{if } i = k_j \end{cases}$$

(this is a (graded) derivation: $\partial/\partial a_i(f \cdot g) = (\partial f/\partial a_i)g + (-1)^{\deg f} f(\partial g/\partial a_i))$. In addition to making this a derivation, having the sign term above makes ρ a representation, e.g.

$$\rho(a_i)\rho(b_i) + \rho(b_i)\rho(a_i) = 1$$

(exercise). Note that this a natural spanning set for $\mathrm{Cl}(V)$: given $I = (i_1 < \cdots < i_k)$ and $J = (j_1 < \cdots < j_m)$, set

$$c_{IJ} := a_{i_1} \dots a_{i_k} b_{j_1} \dots b_{j_m} \in \mathrm{Cl}(V)$$

(the defining relations allow us to order the monomials at worse at the cost of some δ_{ij} which will introduce lower degree terms.). It is not immediately clear that these form a basis, but note that there are 2^{2n} of them, so the theorem is equivalent to showing they are linearly independent.

Like in the proof of PBW

For this, consider $\rho(c_{IJ}) = a_{i_1} \dots a_{i_k} \frac{\partial}{\partial a_{j_1}} \dots \frac{\partial}{\partial a_{j_m}} : M \to M$.

Exercise. Show that these operators are linearly independent.

Hint: Take any relation $\sum \alpha_{IJ} c_{IJ} = 0$. Pick $c_{I_0 J_0}$ with $\alpha_{IJ} \neq 0$ and $|J|$ largest. Then show

$$\sum \alpha_{IJ} c_{IJ} \cdot \prod_{j \in J_0} a_j = \alpha_{I_0, J_0} \prod_{i \in I} a_i$$

(the products in decreasing order of the j's and increasing order of the i's). This forces $\alpha_{I_0 J_0} = 0$.

This completes the proof in the even case.

(Odd case) In the odd case, we also have some element $z \in \mathrm{Cl}(V)$ with $z^2 = 1$. We sill have an action $\mathrm{Cl}(V) \curvearrowright M = \bigwedge(a_1, \dots, a_n)$. In addition to $\rho(a_i)w = a_i w$ and $\rho(b_i)w = \frac{\partial}{\partial a_i} w$, we need to say how z acts. There are two options:

$$\rho(z)w = \pm(-1)^{\deg w} w;$$

these two representations are called M_+ and M_- (they are not isomorphic[18]). We can consider the direct sum

$$\rho = \rho_+ \oplus \rho_- : \mathrm{Cl}(V) \to \operatorname{End} M_+ \oplus \operatorname{End} M_-.$$

Exercise. This is an isomorphism.

∎

We want to use this to construct the spinor representations.

Proposition 8.7. *Define a linear map*

$$\xi : \mathfrak{o}(V) = \bigwedge^2 V \to \mathrm{Cl}(V)$$

via

$$\xi(a \wedge b) = \frac{1}{2}(ab - ba) = ab - \frac{1}{2}(a, b).$$

Then, ξ is a Lie algebra homomorphism.

Proof. For skew-symmetric matrices in this form, one can work out that the commutator is

$$[a \wedge b, c \wedge d] = (b, a)a \wedge d - (b, d)a \wedge c + (a, c)d \wedge b - (a, d)c \wedge b$$

(exercise, using $a \wedge b = \frac{1}{2}(a \otimes b - b \otimes a)$). Now compute

$$[\rho(a \wedge b), \rho(c \wedge d)] = \left[ab - \frac{1}{2}(a, b), cd - \frac{1}{2}(c, d)\right]$$

[18]The eigenvalue of z on the space of $v \in M_\pm$ s.t. $b_i v = 0$ is ± 1

$$
\begin{aligned}
&= [ab, cd] \\
&= abcd - cdab \\
&= (b,c)ad - acbd - cdab \\
&= (b,c)ad - (b,d)ac + acdb - cdab \\
&= (b,c)ad - (b,d)ac + (a,c)db - cadb - cdab \\
&= (b,c)ad - (b,d)ac + (a,c)db - (a,d)cb \\
&= \rho\left([a \wedge b, c \wedge d]\right)
\end{aligned}
$$

You might worry about error terms in the last equality, but they are

$$(b,c)(a,d) - (b,d)(a,c) + (a,c)(d,b) - (a,d)(c,b) = 0,$$

so we win. ■

We can now use this map $\xi : \mathfrak{o}(V) \to \mathrm{Cl}(V)$ to pull back the representations M (in the even case) and $M_\pm$ (in the odd case) from before.

Exercise. When $\dim V = 2n$,

$$\xi^* M = S_+ \oplus S_-.$$

More precisely, $S_+ = \bigwedge^{\text{even}}(a_1, \ldots, a_n)$ and $S_- = \bigwedge^{\text{odd}}(a_1, \ldots, a_n)$.

Exercise. If $\dim V = 2n+1$,

$$\xi^* M_+ \cong S \cong \xi^* M_-.$$

For these, you'll want to find highest weight vectors, compute their weights, and then compare dimensions.

So we realize the spinor representations as exterior algebras where $\mathfrak{o}(V)$ acts by some (0th, 1st, or 2nd order) differential operators.

8.3 Duals of irreps

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Consider some f.d. irrep L_λ with $\lambda \in P_+$. How do we determine L_λ^*?

Let μ be the lowest weight of L_λ. Then the highest weight of L_λ^* is $-\mu$, so $L_\lambda^* = L_{-\mu}$. Hence we only need compute μ. For this, recall the Weyl group. We know (from last semester) that the Weyl group contains a unique element w_0 such that w_0(dominant weights) = (antidominant weights) (**antidominant** means negative of dominant weight). Hence $w_0(R_+) = R_-$. This is called the **maximal element** (**element of maximal length**) w_0. Note that if $-1 \in W$ (thought of as linear transformations of Cartan subalgebra), then $w_0 = -1$.

Note that $-w_0$ always maps positive roots/weights to positive roots/weights, so it permutes ω_i, α_i. Hence, it gives a graph automorphism of the Dynkin diagram. Thus, if the Dynkian diagram has no (nontrivial) symmetries, then $-w_0 = 1$. This happens for $A_1, B_n, C_n, G_2, F_4, E_7$, and E_8.

Proposition 8.8. *The highest weight of the dual representation is* $-\mu = w_0\lambda$*, so* $\mu = -w_0\lambda$*. Thus,*

$$L_\lambda^* = L_{-w_0\lambda}.$$

Corollary 8.9. *For* $A_1, B_n, C_n, G_2, F_4, E_7, E_8$*, all representations are self-dual.*

Example. For type A_n, $n \geq 2$, there are nontrivial symmetries. In particular, there's the flip symmetry

Figure 6: The Dynkin Diagram A_n

and in fact, $-w_0(\alpha_i) = \alpha_{n+1-i}$ in this case. How do you see this? Consider the vector representation $V = L_{\omega_1} = \mathbb{C}^{n+1}$ of $\mathfrak{sl}_{n+1}$. Its dual is

$$V^* = \bigwedge^n V = L_{\omega_n}$$

so ω_n gets exchanged with ω_1. Thus, $-w_0$ must be the flip since it's the only nontrivial automorphism.

Example. Let's look at type E_6 now. Again, $-w_0$ is the flip. We won't show this rigorously right now,

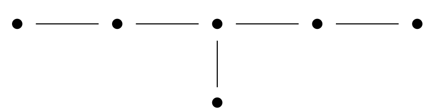


Figure 7: The Dynkin Diagram E_6

but maybe will later.

For type D_n, the action of $-w_0$ depends on the parity of n.

Proposition 8.10. *For* D_{2n}*,* $S_+^* = S_+$ *and* $S_-^* = S_-$*. For* D_{2n+1}*,* $S_+^* = S_-$*.*

How do you remember this? Note that $D_3 = A_3 = \mathfrak{sl}(4)$ so in this case it is the flip. $D_2 = \mathfrak{o}(4) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, so duality is trivial. How do you prove the proposition in general? For type D_n, the Weyl

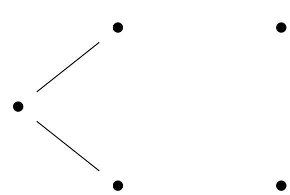


Figure 8: The Dynkin Diagrams D_3 (left) and D_2 (right)

group is $W = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})_0^n$ (vectors with sum 0). For n even, $-1 \in W$ so $-w_0 = 1$ (no flip). For n odd, $-1 \notin W$ and $w_0 = (-1, -1, \ldots, -1, 1)$ (exercise) with trivial permutation $\sigma = \mathrm{id}$. Then,

$$w_0\omega_n = w_0\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) = \left(-\frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}\right),$$

so $-w_0\omega_n = \omega_{n-1}$. Hence $S_+^* = S_-$ and $S_-^* = S_+$. Alternatively, you can see that the lowest weight of S_+ is $-\omega_{n-1}$.

Remark 8.11. This gives some mod 4 periodic phenomena. To observe mod 8 periodicity, ask yourself, "When do S, S_+, S_- have symmetric invariant forms, and when do they have skew symmetric invariant forms?"

Definition 8.12. A f.d. representation V of a group G or Lie algebra $\mathfrak{g}$ is said to be of **complex type** if $V \not\cong V^*$. It is **real type** if $V \cong V^*$ and there exists a **symmetric isomorphism** $\varphi : V \xrightarrow{\sim} V^*$, i.e. $\varphi^* = \varphi$ ($\iff$ φ has a symmetric, invariant bilinear form). It is **quaternionic type** if $V \cong V^*$ and $\exists$ **skew-symmetric isomorphism** $\varphi : V \xrightarrow{\sim} V^*$, i.e. $\varphi^* = -\varphi$

Theorem 8.13. *For D_{2n}, $S_+ = S_+^*$. For D_{4n}, S_+ has a symmetric form (real type) while D_{4n+2} has a skew-symmetric form (quaternionic type).*

No lecture on Tuesday. Next week Thursday lecture at MIT.

9 Lecture 9 (3/18)

Last time we ended while discussing reps of real, complex, and quaternionic type.

Recall 9.1. A f.d. irrep V of a group G or Lie algebra $\mathfrak{g}$ is said to be *complex type* if $V \not\cong V^*$. It is *real type* if $V \cong V^*$ and there exists a symmetric isom $\varphi : V \to V^*$ ($\varphi^* = \varphi$), and it is *quaternionic type* if $V \cong V^*$ and there exists a skew-symmetric isom $\varphi : V \xrightarrow{\sim} V^*$ ($\varphi^* = -\varphi$).

Remark 9.2. Schur says all isos $V \to V^*$ are proportional to each other, to $\varphi^* = c\varphi$ for some $c \in \mathbb{C}$. Taking double dual shows $\varphi = c^2\varphi$, so $c = \pm 1$ (i.e. real and quaternionic type are only possibilities).

Recall 9.3. For D_{2n}, $S_+ = S_+^*$ (and same for S_-). For D_{4n}, S_+ has a symmetric form (real type). For D_{4n+2}, it has a skew-symmetric form (quaternionic type).

There will be a similar statement for odd orthogonal groups. In order to prove these, we need to understand when self-dual reps are real or quaternionic type.

9.1 Principal $\mathfrak{sl}_2$-subalgebra, exponents of $\mathfrak{g}$

We have seen root $\mathfrak{sl}_2$-subalgebras before, but there are actually more copies of $\mathfrak{sl}_2$ inside other Lie algebras.

Definition 9.4. Let $\mathfrak{g}$ be a (semisimple?) Lie algebra. Let $e = \sum_{i=1}^r e_i$, and choose $h \in \mathfrak{h}$ s.t.

$$[h, e] = 2e \iff [h, e_i] = 2e_i \text{ for all } i \iff \alpha_i(h) = 2 \text{ for all } i \iff h = 2\rho^\vee$$

(any above equiv condition true). In any case, $h = \sum_{i=1}^r (2\rho^\vee, \omega_i) h_i$ where $h_i = \alpha_i^\vee$. We now take

$$f := \sum_{i=1}^r (2\rho^\vee, \omega_i) f_i.$$

Then, $[h,f]=-2f$ and

$$[e,f]=\left[\sum_i e_i, \sum_j (2\rho^\vee,\omega_j)f_j\right]=\sum_i (2\rho^\vee,\omega_i)h_i = h.$$

Then, (e,h,f) defined as above genera an $\mathfrak{sl}_2$-subalgebra inside $\mathfrak{g}$, called the **principal $\mathfrak{sl}_2$-subalgebra**.

Example. If $\mathfrak{g}=\mathfrak{sl}_n$ and $V=\mathbb{C}^n$, then $V|_{\mathfrak{sl}_2 \text{ principal}}=L_n$ (the irred $\mathfrak{sl}_2$ rep with highest weight n). One can check that

$$e=\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \quad f=\begin{pmatrix} 0 & & & \\ * & 0 & & \\ & \ddots & \ddots & \\ & & * & 0 \end{pmatrix}, \quad \text{and} \quad h=\begin{pmatrix} n & & & \\ & n-2 & & \\ & & \ddots & \\ & & & -n \end{pmatrix}.$$

On the other hand, if you restrict V to a root subalgebra, then you can see that $V|_{\mathfrak{sl}_2 \text{ root}}=\mathbb{C}^2\oplus(n-2)\mathbb{C}$. Hence, the principal $\mathfrak{sl}_2$-subalgebra is essentially different (not conjugate) to root subalgebras (at least for $n\geq 3$).

A natural thing to look at is the restriction of the adjoint representation to the principal $\mathfrak{sl}_2$-subalgebra.

Write $\mathfrak{g}|_{\mathfrak{sl}_2\text{-principal}}=\bigoplus_i L_{N_i}$. What are these N_i? Recall we can recover the decomposition of an $\mathfrak{sl}_2$-rep if we know the dimensions of its weight spaces, so what are the eigenvalues of $h=2\rho^\vee$ acting adjointly on $\mathfrak{g}$. Write $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$. Given $x\in\mathfrak{g}_\alpha$, we have

$$[h,x]=\alpha(h)x=(\alpha,2\rho^\vee)x.$$

Writing $\alpha=\sum_{i=1}^r k_i\alpha_i$ with $k_i\in\mathbb{Z}$, we see

$$(\alpha,2\rho^\vee)=\sum_{i=1}^r k_i(\alpha_i,2\rho^\vee)=2\sum_{i=1}^r k_i.$$

Recall that when $\alpha\in R_+$, its **height** is defined to be

$$\mathrm{ht}(\alpha)=\sum_{i=1}^r k_i =: |\alpha|,$$

so the weights are (twice) the heights of the roots. Thus,

$$\mathfrak{g}|_{\mathfrak{sl}_2}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}[n] \text{ with } \mathfrak{g}[0]=\mathfrak{h}.$$

Thus, $\dim\mathfrak{g}[0]=r$ and, for $n>0$, $\dim\mathfrak{g}[2n]=\#$roots of height n.

How many roots are there of each height?

- There are exactly r roots of height 1. These are the simple roots.
- What about height 2. Picture a Dynkin diagram. We need to add two distinct roots (twice a root

is not a root), and they need to be connected (sum of orthogonal roots is not a root[19]. Thus, the number of height 2 roots equals the number of edges in the Dynkin diagram. Since the diagram is a tree, this is $r-1$.

- There are zero roots of height N for $N \gg 0$.

Notation 9.5. Let $r_m := \#$roots of height m.

Definition 9.6. An **exponent** of $\mathfrak{g}$ is a positive integer m such that $r_{m+1} < r_m$. The **multiplicity of** m is $r_m = r_{m+1}$.

We have r exponents

$$1 \leq m_1 \leq m_2 \leq \cdots \leq m_r$$

This is because $\dim \mathfrak{g}[0] = r$ and each exponent corresponds to an irrep in $\mathfrak{g}|_{\mathfrak{sl}_2}$

with multiplicities. We know that $m_1 = 1$, $m_2 > 1$, and $m_r = (\theta, \rho^\vee) = h - 1$ where h is called the **Coxeter number of** $\mathfrak{g}$.

Warning 9.7. We earlier encountered the dual Coxeter number $h^\vee = (\theta, \rho)$. Do not confuse this with h.

Proposition 9.8.

$$\mathfrak{g}|_{\mathfrak{sl}_2 \text{ principal}} = \bigoplus_{i=1}^{r} L_{2m_i}$$

where the m_i are the exponents.

Proof. This follows from rep theory of $\mathfrak{sl}_2$ (exercise). ■

Example. $\mathfrak{g} = \mathfrak{sl}_n$. The roots are $\alpha_{ij} = \alpha_i + \cdots + \alpha_j$ where $i \leq j$. This has height $i+j-1$. If you look at the Dynkin diagram, roots will be connected pieces and the height will be the number of vertices in that piece. Hence, the # of root of height k is $r_k = n-k$. Thus the exponents are $\{m_i\} = \{1, 2, \ldots, n-1\}$, so

$$\mathfrak{sl}_n|_{\mathfrak{sl}_2 \text{ principal}} = L_2 \oplus L_4 \oplus \cdots \oplus L_{2n-2}.$$

Sanity Check 9.9. Consider $\mathfrak{gl}_n = V \otimes V^* = L_{n-1} \otimes L_{n-1}$. Clebsch-Gordan tells us that this is

$$\mathfrak{gl}_n = L_n \otimes L_n = L_0 \oplus L_2 \oplus L_4 \oplus \cdots \oplus L_{2n-2}.$$

Since $\mathfrak{gl}_n = \mathfrak{sl}_n \oplus \mathbb{C}$, things are agreeing which is a good sign.

What does all of this have to do with Spinor representations being real and quaternionic type?

9.2 Back to Real, Complex, Quaternionic Type

Let's first discuss what happens for $\mathfrak{sl}_2$. Recall L_n has dimension $n-1$, and $L_1 = \mathbb{C}^2$. $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C})$ preserves skew-symmetric form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathbb{C}^2$, so L_1 is quaternionic type.

More generally, $L_n = \mathrm{Sym}^n L_1$ and $L_n \cong L_n^*$ with the form on L_n being the symmetric power of the form on L^1 (or tensor power if you want to view $L_n \subset L_1^{\otimes n}$). Now, $\omega : L_1 \times L_1 \to \mathbb{C}$ is skew-symmetric, so $\omega^{\otimes n}$ is skew-symmetric for odd n, and symmetric for even n. Thus,

[19]Serre relations says $(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0$. If $a_{ij} = 0$, then this says $(\alpha_i, \alpha_j) = 0 \implies [e_i, e_j] = 0$ so $\alpha_i + \alpha_j$ is not a root

Proposition 9.10. *L_n is real for even n, and quaternionic for odd n.*

Now let $\mathfrak{g}$ be any simple Lie algebra. Choose $\lambda \in P_+$, so $L_\lambda^* = L_{-w_0\lambda}$.

Assumption. Let's assume $\lambda = -w_0\lambda$

(this is e.g. always the case if Dynkin diagram has no nontrivial automorphisms so $-w_0 = 1$).

Question 9.11. *Is L_λ real or quaternionic?*

Restrict it to principal $\mathfrak{sl}_2$-subalgebra. The weights will be $(\mu, 2\rho^\vee)$ where μ is a weight of L_λ. This will be largest when $\mu = \lambda$, and this eigenvalue $(\lambda, 2\rho^\vee)$ occurs just once. This is because any other weight of L_λ is of the form $\lambda - \beta$ with $\beta = \sum k_i\alpha_i$, $k_i \geq 0$ and $\beta \neq 0$. Hence,

$$(\mu, 2\rho^\vee) = (\lambda, 2\rho^\vee) - 2\sum k_i < (\lambda, 2\rho^\vee).$$

Thus,

$$L_\lambda|_{\mathfrak{sl}_2} = L_m \oplus \bigoplus_{n<m} c_n L_n \text{ with } m = (\lambda, 2\rho^\vee).$$

If we have an invariant nondegenerate form on L_λ, B, then $L_\lambda = L_m \oplus L_m^\perp$, so $B|_{L_m}$ is a nondegenerate invariant form. Clearly, B is symmetric $\iff$ $B|_{L_m}$ is symmetric. Thus, we obtain.

Proposition 9.12. *Assume L_λ is not complex type. Then, L_λ is real type if $(\lambda, 2\rho^\vee)$ is even, and is quaternionic type if it is odd.*

Application to Spinor representations Let $\mathfrak{g} = \mathfrak{o}(2n)$. Recall the fundamental weights are $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_{n-2} = (1, 1, \ldots, 1, 0, 0), \omega_{n-1} = \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2}\right), \omega_n = \left(\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}\right)$. Hence,

$$\rho = \rho^\vee = \sum \omega_i = (n-1, n-2, \ldots, 1, 0) \text{ and } (2\rho^\vee, \omega_n) = \frac{n(n-1)}{2} = (2\rho^\vee, \omega_{n-1}).$$

Fact. $\frac{n(n-1)}{2}$ is odd if $n \equiv 2, 3 \pmod 4$ and is even if $n \equiv 0, 1 \pmod 4$.

Corollary 9.13. *If $n = 0 \pmod 4$ then $S_\pm$ have symmetric forms. If $n \equiv 2 \pmod 4$, then $S_\pm$ have skew forms.*

(they are not self-dual when n odd).

We can do the same analysis for $\mathfrak{g} = \mathfrak{o}(2n+1)$. Here, $\omega_1 = (1, 0, \ldots, 0), \ldots, \omega_{n-1} = (1, \ldots, 1, 0), \omega_n = (1/2, \ldots, 1/2)$. And $\omega_1^\vee = \omega_1, \ldots, \omega_{n-1}^\vee = \omega_{n-1}, \omega_n^\vee = (1, 1, \ldots, 1)$. Then $\rho^\vee = \omega_1^\vee + \cdots + \omega_n^\vee = (n, n-1, \ldots, 1)$, so $(2\rho^\vee, \omega_n) = n(n+1)/2$. Hence we obtain

Proposition 9.14. *The Spinor rep S is real $\iff$ $n \equiv 0, 3 \pmod 4$, and is quaternionic $\iff$ $n \equiv 1, 2 \pmod 4$.*

Theorem 9.15 **("Bott Periodicity").** *The behavior of spinor representations of $\mathfrak{o}(m)$ depend on the remainder r of $m \bmod 8$.*

- *$r = 1, 7$: S is of real type*
- *$r = 3, 5$: S is of quat type*

- $r = 0$: S_+, S_- *are of real type*
- $r = 2, 6$: $S_+^\vee = S_-$ *are of complex type*
- $r = 4$: S_+, S_- *are of quat type*

"Now, let's move on. We've done enough representation theory, so let's switch to another subject: integration of Lie groups. This will help us do representation even better." (paraphrase)

9.3 Review of differential forms and integration on manifolds

Let M be a smooth, real n-dimensional manifold. Recall

- TM is the tangent bundle (vectors)
- T^*M is the cotangent bundle (covectors)
- A **differential k-form** on M is a smooth section of $\bigwedge^k T^*M$ (a skew-symmetric n-covariant and 0-contravariant tensor field on M)

"covariant" here counts duals while "contravariant" counts non-duals.

In local coords $x_1, \dots, x_n$ a k-form ω looks like

$$\omega = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} f_{i_1, \dots, i_k}(x_1, \dots, x_n) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

If you change coordinates $x_i = x_i(y_1, \dots, y_n)$, then

$$\omega = \sum_{j_1 < j_2 < \cdots < j_k} \sum_{i_1 < \cdots < i_k} f_{i_1 \dots i_k}(x_1, \dots, x_n) \det \left(\frac{\partial x_{i_r}}{\partial y_{i_s}} \right)_{r,s} \mathrm{d}y_{j_1} \wedge \cdots \wedge \mathrm{d}y_{j_k}.$$

Example. If $f \in C^\infty(M)$, get differential $\mathrm{d}f \in \Omega^1(M)$ (section of T^*M) so that, for $v \in T_pM$ (a derivation), $\mathrm{d}f(v) = \partial_v f$. In coordinates,

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mathrm{d}x_i.$$

Note $\Omega^0(M) = C^\infty(M)$, and $\Omega^k(M) = 0$ for $k > n$. We have a graded-commutative algebra

$$\Omega^\bullet(M) = \bigoplus_{k=0}^n \Omega^k(M)$$

of differential forms with multiplication given by wedge product $\wedge$. The operation d in the previous example extends to a degree 1 derivation of $\Omega^*(M)$, i.e. we have $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. It is defined by

$$\mathrm{d}\left(f(x_1, \dots, x_n) \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right) = \mathrm{d}f \wedge \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}.$$

It is a **graded derivation** in the sense that given homogeneous $a \in \Omega^k(M)$ and $b \in \Omega^\ell(M)$, one has

$$\mathrm{d}(a \wedge b) = \mathrm{d}a \wedge b + (-1)^k a \wedge \mathrm{d}b.$$

A **closed form** ω is one for which $\mathrm{d}\omega = 0$. It is an **exact form** if $\omega = \mathrm{d}\eta$. Also, $\mathrm{d}^2 = 0$ (exact implies closed), but the coverse is not always true.

Example. Consider $S^1 = \mathbb{R}/\mathbb{Z}$ with coordinate x (mod 1). Then, $\mathrm{d}x \in \Omega^1(S^1)$ is a closed form, but is not exact: $\nexists f \in C^\infty(S^1)$ s.t. $\mathrm{d}x = f$ (would need $f = x + c$ but x not well-defined on circle, only up to adding integers).

Definition 9.16. De Rham Cohomology of M is

$$\mathrm{H}^k(M) = \frac{\Omega^k_{\text{closed}}(M)}{\Omega^k_{\text{exact}}(M)}.$$

If $f : M \to N$ is a C^∞-map, and $\omega \in \Omega^k(N)$ is a k-form, then you get a pullback $f^*\omega \in \Omega^k(M)$. Given $v_1, \dots, v_k \in T_pM$, one has

$$f^*\omega(v_1, \dots, v_k) = \omega(f_*v_1, \dots, f_*v_k)$$

where $f_* : T_pM \to T_{f(p)}N$. Note that pullback commutes with $\wedge$ and d. Also, $(f \circ g)^* = g^* \circ f^*$.

9.3.1 Top degree forms

Every element of $\Omega^n(M)$ looks like $\omega = f(x_1, \dots, x_n)\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n$ (in local coordinates). Say $M = \mathbb{R}^n$ and ω has compact support (i.e. f has compact support). Then we define

$$\int_M \omega := \int_{\mathbb{R}^n} f(x_1, \dots, x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n.$$

We want this to be independent of coordinates. If we change $x_i = x_i(y_1, \dots, y_n)$, then

$$\omega = f(x_1, \dots, x_n) \det\left(\frac{\partial x_i}{\partial y_j}\right) \mathrm{d}y_1 \wedge \dots \wedge \mathrm{d}y_n,$$

but

$$\int_{\mathbb{R}^n} f(x_1, \dots, x_n)\mathrm{d}x_1 \dots \mathrm{d}x_n = \int_{\mathbb{R}^n} f(x_1, \dots, x_n) \left|\det\left(\frac{\partial x_i}{\partial y_j}\right)\right| \mathrm{d}y_1 \dots \mathrm{d}y_n,$$

so there is a slight discrepancy (in one case have absolute values; in the other case we don't). Hence, integration of top forms only invariant under changes of variable that preserve orientation (det(jacobian) > 0). As a result, we will only be able to integrate differential forms on oriented manifolds.

On a general manifold, you cannot integration differential forms. You can, however, integrate densities. These multiply by absolute value of determinant instead of by the determinant itself.

10 Lecture 10 (3/25)

Note 6. Haven't seen last Thursday's lecture yet...

10.1 Last time

Last time we talked about integration of (top degree) differential forms on manifolds. Say we have a (real/smooth) manifold M of dimension $\dim M = n$ and we have $\omega \in \Omega^n(M)$. To define the integral

$$\int_M \omega,$$

we need an orientation on M, i.e. a consistent way to say which bases of $T_x M$ are 'right-handed' in the tangent space. Say we have some charts U, V of the manifold with coordinates x_i, y_i. Then the manifold is orientable if always $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$.

Say M is an oriented n-dim manifold, and suppose $\omega \in \Omega^n(M)$ is a top degree form with compact support. We won't actually need to compact support condition, but good to know the integral will converge. How do we define $\int_M \omega$?

Let $K = \overline{\operatorname{supp} \omega}$. Cover K by finitely many balls B_i, and choose a $B_i' \subset B_i$ for each i, so these B_i''s already cover K. For a containment of balls $B' \subsetneq B$, we can define a **hat function** $f \in C^\infty(B)$ satisfying $f > 0$ on B', $f \geq 0$ on B, and f has compact support in B.

Example. In the one-dimensional case, just want some bump function. Can start with

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

Can then do something like multiply this by a parabola (?) to get a hat function in the 1-d case. Then use this to get hat functions in any dimension.

Let f_i be a hat function on B_i, and consider

$$g_i := \frac{f_i}{\sum_j f_j}$$

which is well-defined in a neighborhood of K, and has support instead B_i. Note that $\sum g_i = 1$, so these give a **partition of unity**. We can now use these to define

$$\int_M \omega := \sum_i \int_{B_i} g_i \omega$$

with the RHS a sum of integrals in $\mathbb{R}^n$.

Claim 10.1. *This is well-defined and independent of choices.*

For independence, given two partitions of unity, consider their refines obtained by taken pairwise products (i.e. consisting of functions $g_i h_j$).

Remark 10.2. Integration like this also makes sense for manifolds with boundary. The only difference is that at boundary points, the local model is $\mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ instead of $\mathbb{R}^n$. Integration also makes sense for non-compactly supported forms; the integral just might diverge in these cases.

Remark 10.3. If you have an oriented manifold with boundary, then it induces a canonical orientation on the boundary (a basis of tangent space at boundary is right-handed iff basis of whole thing obtained by extending the given basis by a single vector pointing inwards is right-handed, or something like this).

10.2 Volume Forms

Definition 10.4. A differential form $\omega \in \Omega^n(M)$ is called **nonvanishing** if $\omega_x \in \bigwedge^n T_x^* M$ is not zero for all $x \in M$.

A nonvanishing top form gives rise to an orientation on M: say $v_1, \dots, v_n \in T_x M$ is a right-handed basis if $\omega(v_1, \dots, v_n) > 0$. Note that ω also defines a (Borel) measure, given on open sets U via

$$\mu_\omega(U) = \int_U \omega$$

(or $+\infty$ if integral diverges). Now say f is any measurable function on M. Then,

$$\int_M |f|\, \omega = \int_M |f|\, \mathrm{d}\mu_\omega.$$

We call f **integrable**, denoted $f \in L^1(M, \mu)$, if this integral is $< \infty$. In such cases, can define $\int_M f \mathrm{d}\mu_\omega$ just as in measure theory.

Remark 10.5. Above discussion shows that there are no non-vanishing forms on non-orientable manifolds.

Example. $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ on $M \overset{\text{open}}{\subset} \mathbb{R}^n$ is nonvanishing. The corresponding measure is the usual Lebesgue measure μ, so $\mu(U) = \mathrm{vol}(U)$ is the usual volume of $U \overset{\text{open}}{\subset} M$.

Inspired by above example, nonvanishing forms are often called **volume forms**. Given a volume form ω, $\mathrm{vol}(M) = \int_M \omega \in \mathbb{R}_+ \cup \{\infty\}$.

Proposition 10.6. *If M is compact, then it has finite volume, and any continuous function on M belongs to $L^1(M, \mu)$, i.e. is integrable.*

Proof. Cover $M = \bigcup_{x \in M} U_x$ with U_x a neighborhood of x so small that $\mu(U_x) < \infty$. Since M is compact, this has a finite subcover $U_1, \dots, U_N$. Thus, $\mu(M) \leq \sum_i \mu(U_i) < \infty$, so M has finite measure. If f is continuous, then $\max |f| < \infty$, so

$$\int |f|\, \mathrm{d}\mu \leq \max |f| \cdot \mu(M) < \infty.$$

■

10.3 Stoke's Theorem

Theorem 10.7 (**Stoke's Theorem**). *Let M be a compact orientable manifold with boundary, and let $\omega \in \Omega^{n-1}(M)$ (so restricts to a top form of ∂M). Then,*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega.$$

In particular, if M is ***closed*** *(no boundary), then $\int_M \mathrm{d}\omega = 0$. Also, if $\mathrm{d}\omega = 0$, then $\int_{\partial M} \omega = 0$.*

Notation 10.8. We let $\overline{M}$ denote M with the opposite orientation. Note that $\int_{\overline{M}} \omega = -\int_M \omega$.

Remark 10.9. When $n = 1$, we can consider an interval $M = [a, b]$ with boundary consisting of two points. Then Stoke's theorem says

$$\int_a^b \mathrm{d}f(x) = f(b) - f(a)$$

which is exactly the **fundamental theorem of calculus**.

Remark 10.10. Applying this inside $\mathbb{R}^2$ should recover Green's formula. Applying it to a surface in $\mathbb{R}^3$ recovers Stoke's classical formula. Applying it to a region in $\mathbb{R}^3$ gives Gauss's theorem.

10.4 Integration on (Real) Lie groups

For complex Lie groups the story is the same. To integrate on them, just forget the complex structure.

Let G be a real Lie group of dimension n with Lie algebra $\mathfrak{g} = \operatorname{Lie} G = T_1 G$. Note that $\bigwedge^n \mathfrak{g}^*$ is one-dimensional. Fix some nonzero $\xi \in \bigwedge^n \mathfrak{g}^*$. We can use left-translations to spread ξ over $\mathbb{G}$ in order to get a left-invariant (nonvanishing) top differential form ω_ξ.

Remark 10.11. Translating should show that $TG = \mathfrak{g} \times G$ is a trivial bundle.

This $\omega = \omega_\xi$ gives us an orientation and a measure μ_ω. Note that μ_ω so defined is left-invariant, so gives a (left-invariant) **Haar measure**.

Remark 10.12. ξ is well-defined up to scaling. Changing $\xi \rightsquigarrow \lambda\xi$ ($\lambda \in \mathbb{R}^\times$) changes the top form $\omega \rightsquigarrow \lambda\omega$ and so changes the meausre $\mu_\omega \rightsquigarrow |\lambda|\,\mu_\omega$. Hence, this Haar measure is well-defined up to positive scalar.

Notation 10.13. We use μ_L to denote a choice of left-invariant Haar measure. We similarly define μ_R as a choice of right-invariant Haar measure.

A natural question is does $\mu_L = \mu_R$, up to scaling at least? They will for abelian groups since left/right translations are the same. What about for non-abelian groups?

Suppose V is a 1-dimensional real representation of a group G, so have $\rho_V : G \to \operatorname{Aut}(V) = \mathbb{R}^\times$. We can then define a rep $|V|$ on the same underlying space with map $\rho_{|V|} = |\rho_V|$. This is still a representation since $|\cdot|$ is a character (i.e. homomorphism) on $\mathbb{R}^\times$.

Proposition 10.14. $\mu_L = \mu_R \iff |\bigwedge^n \mathfrak{g}^*|$ *is a trivial representation of* G*.*

Proof. $\mu_L = \mu_R \iff \mu_L$ is also right invariant $\iff \mu_L$ is invariant under conjugation $\iff \omega$ is invariant under conjugation, up to sign $\iff \omega_1 \in \bigwedge^n \mathfrak{g}^*$ is invariant under G, up to sign $\iff |\bigwedge^n \mathfrak{g}^*|$ is a trivial representation. ■

Above, keep in mind that conjugation is what induces the adjoint action on $\mathfrak{g}$.

Definition 10.15. G is **unimodular** if $\mu_L = \mu_R$.

Example. When G is discrete and countable $\iff \mathfrak{g} = 0 \implies G$ unimodular. Up to scaling, $\mu_L = \mu_R$ is simply the **counting measure** $\mu_L(U) = \#U$.

Definition 10.16. Say $\mathfrak{g}$ is a f.d. real Lie algebra. We say $\mathfrak{g}$ is **unimodular** if $\bigwedge^n \mathfrak{g}^*$ is a trivial $\mathfrak{g}$-module.

Proposition 10.17 (Homework)**.**

(1) *A connected Lie group* G *is unimodular* $\iff G$ *is unimodular.*

*(2) If $\mathfrak{g}$ is **perfect**, i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}$ is unimodular.*

Corollary 10.18. *Semisimple Lie algebras are unimodular.*

(3) A nilpotent (e.g. abelian) Lie algebra is unimodular.

(4) If $\mathfrak{g}_1, \mathfrak{g}_2$ are unimodular, then so its $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

Corollary 10.19. *A **reductive Lie algebra** (which is a direct sum of an abelian and a semisimple Lie algebra) is unimodular.*

(5) The Lie algebra $\mathfrak{t}_n$ of upper triangular matrices is not unimodular when $n \geq 2$.

Corollary 10.20. *Being unimodular is not closed under extensions.*

$\mathfrak{t}_2$ sits in an exact sequence with strictly upper triangular matrices and diagonal matrices.

Remark 10.21. If G has no nontrivial 1-dim reps, then G is unimodular.

If G is unimodular, then it has a bi-invariant Haar measure $\mu = \mu_L = \mu_R$. Integration with respect to this measure is denoted

$$\int_G f \mathrm{d}\mu =: \int_G f(g) \mathrm{d}g.$$

Proposition 10.22. *Compact Lie groups are always unimodular.*

Proof. Consider the representation $|\bigwedge^n \mathfrak{g}^*|$ which is given by a continuous map $\rho : G \to \mathbb{R}_{>0}$. The image $\rho(G) \subset \mathbb{R}_{>0}$ must be a compact subgroup, but there's only one of these. ■

Note that if G is compact, it has finite volume $\int_G \mathrm{d}g < \infty$, so we may normalize $\mathrm{d}g$ so that this integral is 1, i.e. require our Haar measure to be a probability measure. This gives us an actually unique choice of measure for compact G.

Example. When G is finite, the unique Haar probabality measure is the **averaging measure** $\mu(U) = \#U/\#G$.

10.5 Representations of compact Lie groups

Proposition 10.23. *Every f.d. representation V of a compact Lie group G is unitary.*

Proof. Pick a positive Hermitian form B on V. We would like an invariant form, so consider the average

$$B_{av}(v, w) = \int_G B(gv, gw) \mathrm{d}g$$

which is invariant by construction (using right-invariant of $\mathrm{d}g$) and well-defined since $\int_G \mathrm{d}g = 1$ is finite! Note that $B_{av}(v, v) > 0$ (for $v \neq 0$) since $B(w, w) > 0$ for $w \neq 0$. This gives a unitary structure on our representation, completing the proof. ■

Corollary 10.24. *Any finite dimensional representation of a compact Lie group is completely reducible.*

Proof. Unitary reps are always completely irreducible. If $W \subset V$ is a subrep, then so is $W^\perp \subset V$ and $W \oplus W^\perp = V$ (then induct). ■

Example. $G = \mathrm{SU}(n)$ is a simply-connected, compact Lie group ($U(n)$ compact since rows are vectors of unit length). It is simply-connected (and even doubly-connected) since $\mathrm{SU}(n)/\mathrm{SU}(n-1) = S^{2n-1}$. Thus, $\operatorname{Rep}\mathrm{SU}(n) = \operatorname{Rep}\mathfrak{su}(n) = \operatorname{Rep}(n)$ (when $\mathfrak{sl}(n)$ is the complexification of $\mathfrak{su}(n)$). Thus, we relearn that f.d. reps of $\mathfrak{sl}(n)$ are completely reducible. This proof strategy is called the **Weyl unitary trick**.

In fact we will show that every semisimple Lie algebra has a Lie group whose real form is a compact Lie group.

10.6 Matrix coefficients

Let G be a compact Lie group, and let V be a f.d. irreducible representation of G. Let $(-,-)$ be a unitary form on V, which is unique up to scaling by a positive number (ultimately a consequence of Schur). Choose an orthonormal basis $v_1, \dots, v_n$ of V. We can consider the expression

$$\rho_V(g)_{ij} = (\rho_V(g)v_i, v_j) = \psi_{V,ij}$$

computing the ij entry of the matrix for g in the given basis. This is a smooth function $\psi_{V,ij} : G \to \mathbb{C}$ called a **matrix coefficient**. Note that this is independent of the normalization of the form (since scaling the form by λ divides the orthonormal basis by $\sqrt{\lambda}$), so it only depends on a choice of orthonormal basis.

Let W be another irrep of G. Say $\{w_k\}$ form an orthonormal basis for W.

Theorem 10.25 (**Orthogonality of matrix coefficients**)**.**

$$\int_G \psi_{V,ij}(g)\overline{\psi_{W,k\ell}(g)}\mathrm{d}g = \frac{\delta_{VW}\delta_{ik}\delta_{j\ell}}{\dim V}.$$

Remark 10.26. $\delta_{VW} = 0$ if $V \ncong W$. If $\delta_{VW} = 1$, take $V = W$ and require $v_i = w_i$ (i.e. use same basis).

Proof. We're after the expression

$$\left(\left(\int_G \rho_V(g) \otimes \rho_{\overline{W}}(g)\right)(v_i \otimes \overline{w}_k), v_j \otimes \overline{w}_\ell\right).$$

Note that

$$\left(\int_G \rho_V(g) \otimes \rho_{\overline{W}}(g)\right) = \underbrace{\int_G \rho_{V\otimes W^*}(g)\mathrm{d}g}_{P} \in \operatorname{End}(V \otimes \overline{W}) = \operatorname{End}(V \otimes W^*).$$

We want to compute this operator. For $x \in V \otimes W^*$, we claim $Px \in (V \otimes W^*)^G$. This is because

$$\rho_{V\otimes W^*}(h)Px = \int_G \rho_{V\otimes W^*}(h)\rho_{V\otimes W^*}(g)\mathrm{d}g = \int_G \rho_{V\otimes W^*}(hg)\mathrm{d}g = Px.$$

Thus, $\operatorname{im} P \subset (V \otimes W^*)^G$ but this whole space is 0 if $V \ncong W$.

We will handle the case $V = W$ next time... ■

11 Lecture 11 (3/30)

Last time we talked about matrix coefficients.

11.1 Matrix coefficients + Peter-Weyl

Recall 11.1. Let V be an irrep of a compact Lie group G. Let $(-,-)$ be an invariant, positive Hermitian form on V. Let $v_1,\dots,v_n$ be an orthonormal basis w.r.t this form. The *matrix coefficients* are the smooth functions $\psi_{V,ij} : G \to \mathbb{C}$ given by

$$\psi_{V,ij}(g) = (\rho_V(g)v_i, v_j).$$

Hence, $\psi_{V,ij}(g)$ is the ijth coefficient of the matrix of $\rho_V(g)$ written in the basis $v_1,\dots,v_n$.

Recall 11.2.

$$\int_G \psi_{V,ij}(g)\overline{\psi_{W,k\ell}(g)}\mathrm{d}g = \frac{\delta_{VW}\delta_{ik}\delta_{j\ell}}{\dim V}.$$

We were in the middle of proving this last time. We showed this integral is 0 when $V \ncong W$ by making use of the operator

$$P := \int_G \rho_V(g)\otimes\rho_{\overline{W}}(g)\mathrm{d}g$$

on $V\otimes\overline{W} = V\otimes W^*$. We showed that $P : V\otimes W^* \to (V\otimes W^*)^G$ maps into the space $(V\otimes W^*)^G = \mathrm{Hom}_G(W,V)$ which is 0 if $V \ncong W$. The integral we are interested in is simply $(P(v_i\otimes w_k), v_j\otimes w_\ell)$, so it must vanish when $V\ncong W$. Let's now wrap up the rest of the proof.

Proof of Theorem 10.25 when $V = W$. In this case,

$$P := \int_G \rho_{V\otimes V^*}(g)\mathrm{d}g.$$

Note that $V\otimes V^* = \mathbb{C}\oplus U$ with $U^G = 0$, so

$$P = \int_G \rho_{\mathbb{C}}(g)\mathrm{d}g \oplus \int_G \rho_U(g)\mathrm{d}g.$$

The right summand takes values in U^G, so must be 0. At the same time, $\rho_{\mathbb{C}}(g) = 1$, so the left factor is 1. Hence, $P = \mathbf{1}_{\mathbb{C}}\oplus\mathbf{0}_U$ is the projection to the trivial representation (the span of the identity operator $\mathrm{id}_V \in V\otimes V^*$). From this we see that

$$P(x\otimes y) = \frac{(x\otimes y, \sum_{i=1}^n v_i\otimes v_i)}{(\sum_{i=1}^n v_i\otimes v_i, \sum_{i=1}^n v_i\otimes v_i)}\sum_{i=1}^n v_i\otimes v_i = \frac{(x,y)}{\dim V}\sum v_i\otimes v_i.$$

In particular, $P(v_i\otimes v_k) = \frac{\delta_{ik}}{\dim V}\sum_{i=1}^n v_i\otimes v_i$, so

$$(P(v_i\otimes v_k), v_j\otimes v_\ell) = \frac{\delta_{ik}\delta_{j\ell}}{\dim V},$$

which completes the proof. ∎

Corollary 11.3. *$\{\psi_{V,ij} : V\in\mathrm{Irrep}(G)$ and $i,j = 1,\dots,\dim V\}$ given an orthogonal set in $L^2(G)$.*

We can actually say something stronger.

Theorem 11.4 (Peter-Weyl). *This system is complete, i.e. the $\psi_{V,ij}$'s form an orthogonal basis of $L^2(G)$.*

Notation 11.5. Let $L^2_{\text{alg}}(G) :=$ usual linear span of the $\psi_{V,ij}$'s.

Peter-Weyl says that $L^2_{alg}(G)$ is dense in $L^2(G)$.

Recall 11.6. $L^2(G) = \left\{ f : G \to \mathbb{C} \text{ measurable} \mid \int_G |f|^2 \, \mathrm{d}g < \infty \right\}$ is the Hilbert space of square-integrable measurable functions with inner product

$$(f_1, f_2) = \int_G f_1(g)\overline{f_2(g)}\mathrm{d}g.$$

Example. Say $G = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ is the circle. Then the irreps of G are the usual characters $\psi_n(\theta) = e^{in\theta}$ for $n \in \mathbb{Z}$. PW says that these give an orthonormal basis for $L^2(S^1)$ with inner product $(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\overline{f_2(\theta)}\mathrm{d}\theta$. This recover the main theorem of Fourier analysis.

Peter-Weyl is the first step of non-abelian harmonic analysis.

Corollary 11.7 (of Theorem 10.25, **character orthogonality**)**.** *For the* ***characters***

$$\chi_V(g) = \operatorname{Tr} \rho_V(g) = \sum_i \psi_{V,ii}(g),$$

one has

$$\int_G \chi_V(g)\overline{\chi_W(g)}\mathrm{d}g = \delta_{VW}.$$

Proof.

$$\int_G \chi_V(g)\overline{\chi_W(g)}\mathrm{d}g = \sum_i \sum_k \frac{\delta_{VW}\delta_{ik}\delta ik}{\dim V} = \frac{\delta_{VW}}{\dim V}\sum_i \delta_{ii}^2 = \delta_{VW}.$$

■

Corollary 11.8 (of Peter-Weyl)**.** *The characters $\chi_V(g)$ ($V \in \operatorname{Irrep}(G)$) give an orthonormal basis of $L^2(G)^G$, the conjugation-invariant L^2-functions.*

Before proving this, we reformulate the Peter-Weyl theorem.

Theorem 11.9 (**Peter-Weyl**, reformulated)**.** *The $G \times G$-invariant map*

$$\xi : \bigoplus_{V \in \operatorname{Irrep}(G)} V \otimes V^* \hookrightarrow L^2(G) \text{ where } \xi(v_i \otimes v_j) = \psi_{V,ij}$$

has dense image in $L^2(G)$.

Proof of Corollary 11.8. Note that $\xi_G : \bigoplus_{V \in \operatorname{Irrep}(G)}(V \otimes V^*)^G \to L^2(G)^G$ satisfies (and is determined by) $\sum_i v_i \otimes v_i \mapsto \chi_V(g)$. Thus, its image is the linear span of the $\chi_V(g)$'s. Hence, it suffices to show that $L^2_{alg}(G)^G$ is dense in $L^2(G)^G$. For this, take some $\psi \in L^g(G)^G$, so there's a sequence $\psi_n \in L^2_{alg}(G)$ with $\psi_n \to \psi$ as $n \to \infty$ (by Peter-Weyl). Let $\psi'_n := \int_g \psi_n(gxg^{-1})\mathrm{d}g \in L^2_{alg}(G)^G$. Furthermore,

$$\|\psi'_n - \psi\| = \left\| \int_G (g\psi_n - \psi)\mathrm{d}g \right\| = \left\| \int_G g(\psi_n - \psi)\mathrm{d}g \right\| \le \int_G \|g(\psi_n - \psi)\| \,\mathrm{d}g = \int_G \|\psi_n - \psi\| \,\mathrm{d}g = \|\psi_n - \psi\| \xrightarrow{n \to \infty} 0,$$

so $\psi'_n \to \psi$ which completes the proof. ■

Let's now prove Peter-Weyl.

11.2 Proving Peter-Weyl

11.2.1 Analytic Background

Before we can prove PW, we need some more background in analysis. In particular, we need the know about compact operators on Hilbert spaces.

Definition 11.10. Let H be a Hilbert space. A **bounded operator** $A : H \to H$ is a linear map s.t. there exists some $C \geq 0$ s.t. for all $v \in H$, $\|Av\| \leq C\|v\|$. The set of such C is closed, so the minimal such C is called the **norm** $\|A\|$ of A. The space of bounded operators is denoted $B(H)$ and is a Banach space (Banach algebra even) with this norm.

Remark 11.11. $\|A + B\| \leq \|A\| + \|B\|$ and $\|AB\| \leq \|A\|\|B\|$.

Definition 11.12. A bounded operator A on a Hilbert space H is called **self-adjoint** if $(Av, w) = (v, Aw)$ for all $v, w \in H$. We say A is **compact** if it is the limit of a sequence of **finite rank operators** (i.e. $\dim \operatorname{im}(A_n) < \infty$) $A_n : H \to H$, i.e. $\|A_n - A\| \xrightarrow{n\to\infty} 0$. We let $K(H)$ denote the space of compact operators, the closure of the space $K_f(H)$ of finite rank operators.

Remark 11.13. $K_f(H) \subset B(H)$ is a 2-sided ideal, so $K(H)$ is also a 2-sided ideal in $B(H)$.

Lemma 11.14. *If A is compact, then it maps bounded sets to* ***pre-compact sets****, i.e. sets with compact closure.*

Remark 11.15. A bounded operator will map bounded sets to bounded sets. A compact operator will map bounded sets into compact sets.

If $\{v_n\}$ is a bounded sequence in H and A is a compact operator, then Av_n will have a convergence subsequence (with limit possible outside $\operatorname{im} A$).

Not every bounded set in a Hilbert space has a convergent subsequence.

Example. Let $e_1, e_2, \ldots$ be orthonormal vectors in H. Then this is a bounded sequence with no convergent subsequence (distance between any two vectors is $\sqrt{2}$).

As a consequence, we see that $\mathrm{id} : H \to H$ is compact $\iff \dim H < \infty$. Let's prove the lemma now.

Proof of Lemma 11.14. Let $v_n \in H$ with $\|v_n\| \leq 1$, and say $A : H \to H$ is compact. Choose A_n of finite rank with $\|A_n - A\| \leq 1/n$ for all n. We do a usual diagonal trick. Note that, since A_n has finite rank, $\{A_n v_k\}_{k\geq 1}$ lies in a compact set (a ball in a finite dim space).

Let v_n^1 be a subsequence of v_n s.t. $A_1 v_n^1$ converges. Let v_n^2 be a subseq of v_n^1 s.t. $A_2 v_n^2$ converges, and so on and so forth. Define $w_n := v^n$ which (away from the first k elements) is a subseq of v_n^k. Note that

$$\begin{aligned} \left\|Av_i^k = Av_j^k\right\| &\leq \left\|A_k v_i^k - A_k v_j^k\right\| + \left\|(A - A_k)(v_i^k - v_j^k)\right\| \\ &\leq \left\|A_k v_i^k - A_k v_j^k\right\| + \|A - A_k\| \left\|v_i^k - v_j^k\right\| \\ &\leq \left\|A_k v_i^k - A_k v_j^k\right\| + 2\|A - A_k\| \\ &< \left\|A_k v_i^k - A_k v_j^k\right\| + \frac{2}{k}. \end{aligned}$$

Hence, for $i, j \gg 0$, we have $\left\|Av_i^k = Av_j^k\right\| < \frac{3}{k}$ since the first summand above vanishes in the limit. Since (a tail of) w_n is a subseq of v_n^k, we see that $\|Aw_i - Aw_j\| \leq 3/k$ when $i, j \gg_k 0$. Hence, Aw_i is Cauchy, so it converges. ■

Proposition 11.16. *Let K be a continuous function on $[0,1]^{2n}$. Define the operator B_K on $L^2([0,1]^n)$ by*

$$(B_K\psi)(y) := \int_{[0,1]^n} K(x,y)\psi(x)\mathrm{d}x.$$

This operator is compact.

Proof. Cover $[0,1]^{2n}$ by pixels of size $\frac{1}{m}$. Approximate K in every pixel by its maximal value on that pixel, and call the resulting function $K_m(x,y)$. Then, the corresponding B_{K_m} is a finite rank operator of rank $\leq m^n$ (functions constant on each pixel). Finally, $\|B_K - B_{K_m}\| \leq \max|K - K_m| \to 0$ as $m \to \infty$ by **uniform continuity** of K, i.e. $\forall\varepsilon > 0 \exists\delta > 0$ s.t. if $|(x,y)-(x',y')| < \delta$, then $|K(x,y)-K(x',y')| < \varepsilon$. Hence, B_K is compact. ∎

Corollary 11.17. *If M is a compact manifold with positive smooth measure $\mathrm{d}x$, then for any continuous K on $M \times M$, the operator*

$$\left(B_K\psi\right)(y) := \int_M K(x,y)\psi(x)\mathrm{d}x$$

is compact.

Proof. If $f_1,\ldots,f_m$ is a partition of unity on M, then $K(x,y) = \sum_{i,j} f_i(x)f_j(y)K(x,y)$. Defining $K_{ij}(x,y) := f_i(x)f_j(y)K(x,y)$, we have $B_K = \sum_{i,j} B_{K_{ij}}$ so it suffices to show $B_{K_{ij}}$ is compact, but K_{ij} has support in (a space homeomorphic to) $[0,1]^n$ so we win. ∎

Fact. A bounded operator $B : H \to H$ is compact $\iff$ it maps bounded sets to precompact sets.

We won't actually need this fact (the direction we haven't proved). On the other hand, we will need to below fact.

Theorem 11.18 (Hilbert-Schmidt Theorem). *Let $A : H \to H$ be a self-adjoint compact operator. Then, there exists an orthogonal decomposition*

$$H = \ker A \oplus \widehat{\bigoplus_\lambda} H_\lambda$$

where λ runs over nonzero eigenvalues of A, and

- $A|_{H_\lambda} = \lambda \cdot \mathrm{Id}$
- *H_λ are finite dimensional*
- *λ are real and either form a finite set or a sequence converging to 0.*

(Generalizes uses spectral theorem for Hermitian operators in f.dim linear algebra).

Example. When A is finite rank, this is just the spectral theorem for Hermitian operators in a f.d. space. It says there exists an orthonormal basis in which A is diagonal with real eigenvalues.

Remark 11.19. Bounded operators in a Hilbert space do not have to have eigenvalues at all. For example, consider multiplication by x on $L^2([0,1])$ (recall objects here are functions up to equality away from null sets).

Proof of Hilbert-Schmidt. We first prove the theorem for the positive operator A^2. The idea is to find the largest eigenvalue, take its orthocomplement, and then keep going...

Let $\beta = \|A\|^2 = \sup_{\|v\|=1}(A^2v, v) = \sup_{\|v\|=1}(Av, Av)$. WLOG we may assume $\beta \neq 0$ (otherwise $A = 0$). Fix a sequence A_n of self-adjoint finite rank operators converging to A.[20] Let $\beta_n = \|A_n\|^2$ which is in fact the maximal eigenvalue of A_n^2.[21] Choose v_n s.t. $A_n^2 v_n = \beta_n v_n$ and $\|v_n\| = 1$. Note that $A^2 v_n$ has a convergent subsequence, so we may assume wlog $A^2 v_n \to w \in H$. At the same time, $A_n^2 v_n \to w$ since $\|A^2 v_n - A_n^2 v_n\| \leq \|A^2 - A_n^2\| \to 0$ as $n \to \infty$. Since $A_n^2 v_n = \beta_n v_n$ and $\beta_n \to \beta$, we conclude that $v_n \to \beta^{-1} w$ so $A^2 w = \beta w$. Also, we know $\|w\| = 1$. Now replace H with $\langle w \rangle^\perp$ and continue.

In this way, we get a sequence of numbers $\beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots \geq 0$ which either terminates ($\beta_n = 0$ for $n \gg 0$) or it's infinite but tends to 0 (using compactness of A^2). We have eigenvectors w_j of norm 1 so that $A^2 w_j = \beta_j w_j$. This has a convergent subseq so $\beta_j \to 0$ as $\|\beta_j w_j - \beta_k w_k\| = \sqrt{\beta_j^2 + \beta_k^2}$. Take a vector v orthogonal to all w_k. Then $\|Av\| \leq \beta_k \|v\|$, so $\|Av\| = 0 \implies v \in \ker A$. This implies

$$H = \widehat{\bigoplus}_{k \geq 1} \mathbb{C} w_k \oplus \ker A^2.$$

This completes the proof for A^2.

Finally, A acts on $\ker A^2$ by 0 and on H_{β_k} with eigenvalues $\pm\sqrt{\beta_k}$. ■

We'll deduce Peter-Weyl next time.

12 Lecture 12 (4/1)

12.1 Peter-Weyl, Proved

Let G be a compact Lie group. Recall we want to show that

Theorem 12.1 (Peter-Weyl)**.**

$$L^2(G) = \widehat{\bigotimes}_{V \in \mathrm{Irrep}(G)} V \otimes V^*.$$

Proof. We want to make use of the Hilbert-Schmidt theorem from last time. We start by constructing a '**δ-like sequence**' of continuous function $h_N(x)$ on G, supported on small neighborhoods of 1 which shrink to 1 as $N \to \infty$. We require $h_N \geq 0$, h_N is conjugation invariant, and $\int_G h_N(x)\mathrm{d}x = 1$. Note that, if φ is a continuous function on G, then

$$\int_G h_N(x)\varphi(x)\mathrm{d}x \xrightarrow{N \to \infty} \varphi(1).$$

How do we actually construct such a sequence?

Note that $\mathfrak{g} = \mathrm{Lie}\, G$ has a positive, invariant inner product. Start with a function $h(x)$ supported on $[-\varepsilon, \varepsilon]$. Then define $h_{\mathfrak{g}}(\vec{x}) = h(|x|^2)$ where $\vec{x} \in \mathfrak{g}$. Then define $\widetilde{h}_N(g) = h_{\mathfrak{g}}(N \log g)$ (supported in a neighborhood of the identity on which log is defined). Then let $c_N = \int_G \widetilde{h}_N(g)\mathrm{d}g$, and set $h_N = \frac{1}{c_N}\widetilde{h}_N$. Note this is invariant under conjugation since it only depends on $|\log g|$, so we have our sequence.

[20] Replace with $\frac{1}{2}(A_n + A_n^t)$

[21] This is a statement about matrices. Diagonalize to see this

Next, we define the **convolution operator**

$$(B_N\psi)(g) = \int_G h_N(x)\psi(x^{-1}g)\mathrm{d}x = \int_G h_N(gy^{-1})\psi(y)\mathrm{d}y.$$

Note that this is compact by Corollary 11.17 (applied to $K(g,h) = h_N(gy^{-1})$). Furthermore, B_N is self-adjoint since $K(g,y) = K(y,g)$ (since h_N invariant under inversion). Further, it commutes with both left and right multiplication by G, so

$$L^2(G) = \ker B_N \oplus \widehat{\bigoplus}_{\lambda\neq 0} H_\lambda$$

with H_λ the (f.dim) λ-eigenspace of B_N. Each H_λ is G-invariant (say under left action) so $H_\lambda \subset L^2_{alg}(G) = \bigoplus_V V \otimes V^*$. Hence, for all N and any $b \in \operatorname{Im} B_N$ and $\varepsilon > 0$, there exists $f \in L^2_{alg}(G)$ s.t. $\|b - f\| < \varepsilon$. Note that for $\varphi \in C(G)$ continuous, $B_N\varphi \to \varphi$ as $N \to \infty$ ($\|(B_N\varphi - \varphi)\| \to 0$). We can pick $f_N \in L^2_{\mathrm{alg}}(G)$ so that $\|B_N\varphi - f_N\| < \frac{1}{N}$ and so see that $\|B_N\varphi - f_N\| \to 0$ as $N \to \infty$. Hence, L^2_{alg} is dense in L^2, so we win. ■

Question: Why is H in this space?

Answer: Ever element of H_λ generates a f.dim representation

Lemma 12.2. *Let G be a compact Lie group, and let $G = G_0 \supset G_1 \supset G_2 \supset \dots$ be a descending sequence of closed subgroups. Then it must stabilize, i.e. $G_n = G_{n+1}$ for $n \gg 0$.*

Proof. We may assume the sequence has no repetitions, and then show it is finite. Assume not. The dimensions have to stabilize, so we may assume $\dim G_i$ is the same for all i. Then, $K = G_n^0$ is the same for all n (since Lie algebras must be the same), and is normal in each of them. Then, $G_1/K \supset G_2/K \supset \dots$ is a sequence of finite groups, so it must stabilize. ■

Non-example. $\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset 8\mathbb{Z} \supset 16\mathbb{Z} \supset \dots$

Corollary 12.3. *Any compact Lie group has a faithful, finite dimensional representation.*

Proof. Pick a f.d. rep V_1 of G, and let $G_1 = \ker \rho_{V_1}$. Then pick a rep V_2 of G s.t. $V_2|_{G_1}$ is nontrivial, and take $G_2 = \ker(\rho_{V_1} \oplus \rho V_2) = \ker \rho_{V_2}|_{G_2}$. Continue in this way... By the lemma, this process can only produce a finite sequence of non-isomorphic groups, so there's a k s.t. every f.dim rep of G is trivial on G_k. By Peter-Weyl, G_k acts trivially on $L^2(G)$ which forces $G_k = 1$. Hence, $V_1 \oplus \dots \oplus V_k$ is a faithful (unitary) representation of G, so $G \hookrightarrow U(V_1 \oplus \dots \oplus V_k)$. ■

Conversely, if a compact topological group has a faithful f.dim rep, then it's a closed subgroup of $U(n)$ which implies that it is itself a (compact) Lie group.

Notation 12.4. We let $C(G,\mathbb{C})$ denote the Banach space of continuous $\mathbb{C}$-valued functions on G. This is complete w.r.t the norm $\|f\| = \max|f|$.

Theorem 12.5 (Stone-Weierstrass Theorem). *Let X be a compact metric space. Let $\mathcal{A} \subset C(X,\mathbb{C})$ be a unital subalgebra s.t.*

(1) $\mathcal{A}$ is closed.

(2) $\overline{\mathcal{A}} = \mathcal{A}$ (invariant under complex conjugation)

*(3) $\mathcal{A}$ **separates points**, i.e. for distinct $x, y \in X$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.*

then $\mathcal{A} = C(X, \mathbb{C})$.

Theorem 12.6. *$L^2_{alg}(G)$ is dense in $C(G, \mathbb{C})$ with this norm, so every continuous function on G can be uniformly approximated by matrix coefficients of f.dim reps.*

Proof. Let $\mathcal{A} = \overline{L^2_{\text{alg}}(X)}$. It is obviously unital, closed, and closed under complex conjugation. Hence, it suffices to check that it separates points. Fix any $x, y \in G$ s.t. $f(x) = f(y)$ for all $f \in L^2_{\text{alg}}(G)$. Then, for any $f \in L^2_{\text{alg}}(G)$, one has $f(1) = f(x^{-1}y)$, so $g := x^{-1}y$ acts trivially on $L^2(G)$ which forces $g = 1$, i.e. $x = y$. ∎

12.2 Compact (2nd countable) topological groups

Recall 12.7. A topological space is called **2nd countable** iff it has a countable base. A compact space is 2nd countable $\iff$ it is separable $\iff$ it is metrizable.

Lots of what we said for Lie groups didn't really need the smooth structure; it mainly just needed integration. So we'll make sense of integration on compact, 2nd countable topolgoical groups, and then reprove things in this more general setting.

Implicitly, we're assuming all our spaces are Hausdorff

Example. Let

$$\dots \twoheadrightarrow G_3 \overset{\varphi_2}{\twoheadrightarrow} G_2 \overset{\varphi_1}{\twoheadrightarrow} G_1$$

be a chain of surjective homomorphisms of finite groups. Then, the inverse limit

$$G := \varprojlim_{n \to \infty} G_n = \left\{ (g_i)_{i \geq 1} \in \prod_{i \geq 1} G_i : \varphi_i(g_{i+1}) = g_i \text{ for all } i \right\}$$

is a **profinite group**. It is visibly an abstract group. To topologize it, we give it the weakest topology in which all the projections $p_n : G \to G_n$ are continuous (with G_n discrete). Hence, a base of nbhds of 1 is given by $\ker p_n$.

Here, a sequence $\vec{a}^n = (a_1^n, a_2^n, \dots)$ converges to $\vec{a} \iff \forall k : a_k^n$ eventually stabilizes to a_k. Further, this topology is metrizable with metric

$$d(\vec{a}, \vec{b}) = C^{\inf_k (a_k \neq b_k)}$$

for some fixed $0 < C < 1$. Note that the natural map $G \hookrightarrow \prod_{k \in \mathbb{Z}_+} G_k$ is a closed embedding (using the product topology on the target), so we see that G is compact.

Example. The **p-adic integers**

$$\mathbb{Z}_p := \varprojlim_{n \to \infty} \mathbb{Z}/p^n\mathbb{Z}$$

form a profinite group. In fact, $\mathbb{Z}_p$ is a profinite ring. It's unit group $\mathbb{Z}_p^\times = \varprojlim_{n \to \infty} (\mathbb{Z}/p^n\mathbb{Z})^\times$ is also profinite, as are $\text{GL}_n(\mathbb{Z}_p), O_n(\mathbb{Z}_p), \text{Sp}_{2n}(\mathbb{Z}_p)$, etc.

Example. Absolute Galois groups (e.g. $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) are also profinite.

Note can also take inverse limits of compact Lie groups.

12.3 Integration theory on compact top. groups

Let $C(X,\mathbb{R})$ denote the space of $\mathbb{R}$-valued continuous functions on X (X some compact 2nd countable topological group). Note that this is a Banach space, complete w.r.t $\|f\| = \max|f|$.

Fact (**Riesz representation theorem**)**.** Finite volume Borel measures on X are the same thing as non-negative[22], continuous linear functionals $C(X,\mathbb{R}) \to \mathbb{R}$. Given a measure μ, the corresponding functional is $I(f) = I_\mu(f) := \int_X f \mathrm{d}\mu$.

(A measure is just a thing that let's you integrate functions). In the above correspondence, μ is a probability measure iff $I(1) = 1$. Any nonzero μ has positive, finite value and can be normalized to be a probability measure.

Theorem 12.8 (Haar, von Neumann)**.** *Let G be a second countable compact group. Then G admits a unique left-invariant probability measure which is also right-invariant.*

Don't need to second countable assumption above. In fact, for any locally compact topological group, there's some Haar measure (unique up to scaling) which is left-invariant or right-invariant, but usually not both. We won't prove that, but will prove the weaker version stated above.

Proof. Let $\{g_i\}_{i\geq 1} \subset G$ be a dense sequence in G (exists since G 2nd countable). Fix $c_i > 0$ s.t. $\sum_{i=1}^\infty c_i = 1$ (e.g. $c_i = 2^{-i}$). We use these to build an *averaging operator*

$$\begin{array}{rccc} A: & C(G,\mathbb{R}) & \longrightarrow & C(G,\mathbb{R}) \\ & f & \longmapsto & \left[x \mapsto \displaystyle\sum_{i=1}^\infty c_i f(xg_i) \right] \end{array}$$

(absolutely convergent since f bounded on compact G). Note that $\|A\| = 1$ and that A is left-invariant. Let $L \cong \mathbb{R} \subset C(G,\mathbb{R})$ be the constant functions, so $A|_L = \mathrm{Id}_L$. The distance from $f \in C(G,\mathbb{R})$ to L (the "spread of f") is $\nu(f) = \frac{1}{2}(\max f - \min f)$.

We claim that $\nu(Af) \leq \nu(f)$ with equality iff $f \in L$. Indeed, choose some $f \notin L$. For any $x \in G$, we can pick j s.t. $f(xg_j) < \max f$. Then, $(Af)(x) = \sum c_i f(xg_i) \leq (1-c_j)\max f + c_j(fxg_j) < \max f$. Thus, $\max(Af) < \max f$ (since G compact). One similarly checks that $\min f < \min(Af)$, so $\nu(Af) < \nu(f)$.

We now iterate. For $f \in C(G,\mathbb{R})$, let $f_n = A^n f$. This sequence is uniformly bounded by $\max|f|$ and is **equicontinuous**, i.e. for all $\varepsilon > 0$ there is a neighborhood $1 \ni U = U_\varepsilon \subset$ s.t. for all $x \in G$ and $u \in U$,

$$|f_n(x) - f_n(ux)| < \varepsilon.$$

To show this, it suffices to show that f is uniformly continuous, i.e. to find U s.t. for all $x \in G$ and $u \in U$, $|f(x) - f(ux)| < \varepsilon$. This would then imply

$$\left|\sum c_i f(xg_i) - \sum c_i f(uxg_i)\right| \leq \sum c_i |f(xg_i) - f(uxg_i)| < \varepsilon.$$

Hence, assume to the contrary that $\exists u_i \to 1$ and $x_i \in G$ s.t. $|f(x_i) - f(u_i x_i)| \geq \varepsilon$. Since G is compact, the x_i have a convergent subsequence, so we may assume $x_i \to x$. Taking limits then shows that $0 = |f(x) - f(1\cdot x)| \geq \varepsilon$, a contradiction.

[22] i.e. $I(f) \geq 0 \impliedby f \geq 0$

Now we appeal to **Ascoli-Arzela**: A sequence f_n in $C(X)$ (X compact) which is uniformly bounded and equicontinuous has a convergent subsequence.[23]

Hence we get $f_{n(m)} = A^{n(m)} f$ converging to some $h \in C(G, \mathbb{R})$. Consider the spread

$$\nu(f_{n(m)}) \geq \nu(f_{n(m)+1}) = \nu(Af_{n(m)}) \geq \nu(f_{n(m+1)}).$$

Taking the limit as $m \to \infty$, we have $\nu(h) \geq \nu(Ah) \geq \nu(h)$, so $\nu(Ah) = \nu(h)$. Hence h is a constant, so the assignment $f \mapsto h \in L \cong \mathbb{R}$ is a continuous linear function. It is clearly left-invariant, nonnegative, and satisfies $1 \mapsto 1$. Thus, it gives our desired Haar probability measure/integral $I : C(G, \mathbb{R}) \to \mathbb{R}$.

This just leaves uniqueness. We can similarly construct a right invariant integral $I_* : C(G, \mathbb{R}) \to \mathbb{R}$. For any left-invariant integral J, we have $J(f) = J(I_*(f))$. If $J(1) = 1$, then this says $J(f) = I_*(f)$, so we get uniqueness. We also see that $I(f) = I_*(f)$, so I if bi-invariant. ■

Question: Why?

Next time we'll generalize facts about compact Lie groups to these more general compact 2nd countable groups, and then we'll talk about hydrogen atoms I guess. Tuesday lecture at MIT.

13 Lecture 13 (4/6)

Today we learn some physics.

13.1 Hydrogen Atom

This is really a quantization of Kepler's work on planetary motion.

Let's start with the classical take on things. Imaginae planetary motion. There's a sun with planets orbiting it.

Notation 13.1. The configuration space is $\mathbb{R}^3$ (with sun at the origin) and let's call the coordinates $x, y, z \in \mathbb{R}$. We put these together to form $\vec{r} = (x, y, z)$ whose length is $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$. There's also momentum $\vec{p} = (p_x, p_y, p_z)$ and **kinetic energy** $\frac{1}{2}\vec{p}^2$ as well as **potential energy** $U(r) = -\frac{1}{r}$. The total energy is given by the **Hamiltonian**

$$H = \frac{1}{2}\vec{p} - \frac{1}{r}.$$

Remark 13.2. We normalize all units so that constants (e.g. mass) is 1.

Define the **Poisson bracket**

$$\{f, g\} := \frac{\partial f}{\partial \vec{p}} \frac{\partial g}{\partial \vec{r}} - \frac{\partial f}{\partial \vec{r}} \frac{\partial g}{\partial \vec{p}}.$$

Motion is described by **Hamilton's equations** $\dot{f} = \{H, f\}$. One (e.g. Kepler) can plug this in and solve this differential equation; we're lucky that this potential energy function $U(r)$ is simple; this cannot be solved by hand in general.

[23] Find nested subsequences converging at each point in a countable sequence (using uniform boundedness), and then take the diagonal.

13.1.1 Quantum version

In quantum theory, classical observables become operators on some Hilbert space. In the present case, this space is $L^2(\mathbb{R}^3)$. We view x, y, z as operators given by multiplication by x, y, z.

Warning 13.3. These aren't literally operators on $L^2(\mathbb{R}^3)$, e.g. multiplication by x can move a function outside L^2. In reality, these are only operators on some dense subspace of $L^2(\mathbb{R}^3)$. We won't worry about this too much.

What about momentum? $p_x \rightsquigarrow -i\partial_x$. The minus is a convention, but the i is important; smth smth real classical observables should give rise to self-adjoint operators (i.e. $\int Af \cdot \overline{g} = \int f \cdot A\overline{g}$ which we sometimes write by saying $A^\dagger = A$). Also, the classical Hamiltonian gets replaced by the quantum Hamiltonian

$$H + -\frac{1}{2}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right) - \frac{1}{r} = -\frac{1}{2}\Delta - \frac{1}{r}.$$

Hamilton's equation now becomes $\dot{f} = [H, f]$ (usual commutator) and called **Schrödinger's equation**.

Classical states were pairs $(\vec{r}, \vec{p})$ (6 coordinates), but quantum states are elements of a Hilbert space $\psi \in L^2(\mathbb{R}^3)$ (∞ coordinates) normalized so $\|\psi\| = 1$. We consider this ψ modulo 'phase factors'[24] (so we're looking at lines in $L^2(\mathbb{R}^3)$). Classical states transform non-linearly, but these quantum states will translate linearly. Then we have **Schrödinger's equation** (for states)

$$i\partial_t\psi = H\psi.$$

More explicitly,

$$i\partial_t\psi = -\frac{1}{2}\left(\partial_x^2 + \partial_y^2 + \partial_z^2\right)\psi - \frac{1}{r}\psi.$$

How do you solve this? If H was just a matrix, the solution would be $\psi(t) = e^{itH}\psi(0)$ with exponential given by the usual power series. If H is some infinite-dimensional operator, we can still take inspiration from this. If we have an eigenvalue $H\psi(0) = \Lambda\psi(0)$, then $\psi(t) = e^{-it\Lambda}\psi(0)$ is a solution; more generally, we can take superpositions of these. Hence, we'd like an eigenbasis for H (note H is symmetric and even self-adjoint[25]).

We want an orthonormal basis ψ_N of $L^2(\mathbb{R}^3)$ so that $H\psi_N = E_N\psi_N$. We call ψ_N the **state of energy** E_N (note $E_N \in \mathbb{R}$ since ψ self-adjoint). Consider $\psi(x, y, z, 0) = \sum c_N\psi_N(x, y, z)$. Here one has $c_N = (\psi(0), \psi_N)$. Given this initial condition, we get the solution

$$\psi(x, y, z, t) = \sum c_N e^{itE_N}\psi_N(x, y, z).$$

Thus, we only need to fine the eigenvectors ψ_N satisfying the **stationary Schrödinger equation** $H\psi_N = E_N\psi_N$.

This is similar to the story of compact operators, but more complicated. H is not compact, and also not bounded. It's spectrum won't be discrete. It'll have a discrete part (called 'bound states' if I heard correctly) as well as a continuous part (giving integrals instead of sums). At least, we can try to find the discrete spectrum.

[24] vectors of norm 1

[25] pavel is distinguishing these two and seemingly claiming self-adjoint is something complicated

Goal. Solve this equation ($H\psi = E\psi$ where E an eigenvalue), and figure out why we're talking about this in a Lie groups class.

Note everything is rotationally invariant, so we should utilize this symmetry. This amounts to passing to spherical coordinates. $1/r$ is already in spherical coordinates. The Laplacian splits into two pieces, $\Delta = \Delta_r + \frac{1}{r^2}\Delta_{sph}$, the *radial part* and the *spherical part*. These are

$$\Delta_r = \partial_r^2 + \frac{2}{r}\partial_r \text{ and } \Delta_{sph} = \frac{1}{\sin^2\varphi}\partial_\theta^2 + \frac{1}{\sin\varphi}\partial_\varphi \cdot \sin\varphi\partial_\varphi.$$

Above, our spherical coordinates are (r,θ,ψ) where r the radius, θ the angle in the horizontal plane, and ψ the angle in the vertical plane. Write $\vec{r} = r\vec{u}$ where $|\vec{u}| = 1$. We look for solutions of the form $\psi(r,\vec{u}) = f(r)\xi(\vec{u})$ ('separation of variables'[26]).

First note that if $\Delta_{sph}\xi + \lambda\xi = 0$, then f satisfies an ODE depending on λ. Second, we claim that λ will be positive. This is because

$$\int \Delta_{sph}\xi \cdot \xi = -\int (\nabla\xi)^2 \le 0 \implies \lambda \ge 0.$$

What will be the equation for f? It's a "calculus exercise" to compute that f satisfies the ODE

$$f'' + \frac{2}{r}f' + \left(\frac{2}{r} - \frac{\lambda}{r^2} + 2E\right)f = 0.$$

Here is where Lie groups start to come in. Δ_{sph} acts on $L^2(S^2)$ (really on some dense subspace) and is rotationally invariant (since Δ, Δ_r, and $1/r^2$ are; this is not obvious from its formula). Now, as SO(3)-reps, we have

$$L^2(S^2) = L_0 \oplus L_2 \oplus L_4 \oplus \dots \text{ with } \dim L_k = k+1$$

(apparently this was on some homework). Now, Δ_{sph} preserves each $L_{2\ell}$ and acts on it by a scalar. Once we compute these scalars, we'll know all the eigenvectors and eigenvalues on this operator. What are these scalars? There are a few ways to compute them. Here's one...

Let w_ℓ be the 0-weight vector in $L_{2\ell}$ (recall it has weights $2\ell, (2\ell-2), \dots, 0, \dots, (2-2\ell), -2\ell$). It turns out that $h \in \mathfrak{sl}_2$ acts by $-2i\partial_\theta$. Since w_ℓ is weight 0, $\partial_\theta w_\ell = 0$, so it depends only on φ. In fact, it is a degree ℓ polynomial in $\cos\varphi$, so write $w_\ell = P_\ell(\cos\varphi)$. Recall that the Jacobian in passing between sphereical and Euclidean coordinates is $J = r^2\sin\varphi$. Hence (matrix coefficients?),

$$\int_{-1}^{1} P_m(z)P_n(z)\mathrm{d}z = \int_0^\pi \sin\varphi \cdot P_m(\cos\varphi)\cdot P_n(\cos\varphi)\mathrm{d}\varphi = 0 \text{ if } m \ne n.$$

So P_n is a degree n polynomial and they are orthogonal under uniform measure; this makes them **Legendre polynomials**.

We can also calulate the action of Δ_{sph} on P_ℓ. Recall that $\Delta_{sph} = \frac{1}{\sin^2\varphi}\partial_\theta^2 + \frac{1}{\sin\varphi}\partial_\varphi \cdot \sin\varphi\partial_\varphi$ and note that $(\sin\varphi)^{-1}\partial_\varphi = \partial_z$. Using this (and independence from θ), one can show that

$$\Delta_{sph}P_\ell = \partial_z(1-z^2)\partial_z P_\ell = -\lambda P_\ell.$$

[26] Apparently we earlier separated time from space. Now we separate radius from angle operators

We want to compute λ. Write $P_\ell = Cz^\ell + \dots$. We compute the leading term of the LHS:

$$-\partial_z z^2 \partial_z z^\ell = -\ell(\ell+1)Cz^\ell + \dots.$$

Thus, $\lambda = \ell(\ell+1)$.

Proposition 13.4. Δ_{sph} *acts on each* $L_{2\ell}$ *by the scalar* $-\ell(\ell+1)$.

Here, we do we get a discrete spectrum even though the operator is unbounded.

Notation 13.5. Let y_ℓ^m denote the vector in $L_{2\ell}$ of weight $2m$, e.g. $y_\ell^0 = w_\ell$. This will be of the form

$$y_\ell^m = e^{im\theta} P_\ell^m(\varphi).$$

These functions are called **spherical harmonics**. These were known by quantum mechanics, but Laplace studied the Laplace operator on the sphere.

Note these spherical harmonics actually have some dependency on θ now. We ignored that $(\sin\varphi)^{-2}\partial_\theta^2$ before, but now this will acts on y_ℓ^m and generate a $-\frac{m^2}{1-z^2}$. We get

$$\partial_z(1-z^2)\partial_z P_\ell^m(z) - \frac{m^2}{1-z^2}P_\ell^m(z) + \ell(\ell+1)P_\ell^m(z) = 0.$$

This is called the **Legendre differential equation**. Note that $-\ell \le m \le \ell$ (in fact, it turns out these are the only values of the parameters for which this equation has a solution which is smooth near $x = 0$). This solution will be (almost?) a polynomial, unique up to scaling. One ends up with

$$P_\ell^m = \left(1-z^2\right)^{\frac{m}{2}} \partial_z^{\ell+m}(1-z^2)^\ell$$

which is a polynomial when m is even. These are called **associated Legendre polynomials**.

Remark 13.6. This P_ℓ^m is a matrix coefficient, so it's a trigonometric polynomial. You can write this as a polynomial of cos with some sin factor when the degree is odd (or something? I didn't quite catch what he was saying).

Let's go back to the radial equation. Recall it is

$$f'' + \frac{2}{r}f' + \left(\frac{2}{r} - \frac{\ell(\ell+1)}{r^2} + 2E\right) f = 0.$$

How do we deal with this? We start with the magic change of variables: write $f(r) = r^\ell e^{-\frac{r}{n}} h\left(\frac{2r}{n}\right)$. Letting $\rho = 2r/n$, h must satisfy

$$\rho h'' + (2\ell + 2 - \rho)\, h' + \left(n - \ell - 1 + \frac{1}{4}(1 + 2En^2)\rho\right) h = 0.$$

We should choose n so that the last term goes away, i.e. we take $n = \frac{1}{\sqrt{-2E}}$, i.e. $E = -\frac{1}{2n^2}$.[27] Thus, we have

$$\rho h'' + (2\ell + 2 - \rho)h' + (n - \ell - 1)h = 0, \tag{13.1}$$

[27] Since our potential is negative, one can show that $E < 0$ if you want a solution lying in L^2

called the **Laguerre equation**. Look at solutions near $\rho = 0$. They will have the form $h = \rho^s(1 + o(1))$ (for two values of s). The characteristic equation for s is (only first two terms relevant for this)

$$s(s-1) + s(2\ell + 2) = 0 \iff s(s + 2\ell + 1) = 0$$

with two solutions $s = 0$ and $s = -2\ell - 1$. We have a basis of two solutions, the first smooth and the second having a singularity. We claim the solution corresponding to $s = -2\ell - 1$ is not possible. Observe

Question: Why?

$$\int |\psi|^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int |f|^2 |\xi|^2 r^2 \sin\varphi \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi = \int |f|^2 r^2 \mathrm{d}r \cdot \underbrace{\int_{S^2} |\xi|^2}_{<\infty}$$

so our f should have the property that $\int |f|^2 r^2 \mathrm{d}r < \infty$ (since we want a solution in L^2). This is the case iff

$$\int \rho^{2\ell+2} e^{-\rho} |h(\rho)|^2 \, \mathrm{d}\rho < \infty.$$

If $h \sim \rho^{-2\ell-1}$ as $\rho \to 0$, then $\rho^{2\ell+2} |h(\rho)|^2 \sim \rho^{-2\ell}$ as $\rho \to 0$, so if $\ell > 0$, this is not integrable. Thus, $s = -2\ell - 1$ not possible when $\ell > 0$. Even when $\ell = 0$, this is not possible: $\psi(x, y, z) \sim \psi \sim r^{-1}$ near $r = 0$ and $h \sim \rho^{-1} \implies f \sim r^{-1}$ near $r = 0$. Then, $H\psi = -\frac{1}{2}\Delta\psi - \frac{1}{r}\psi = E\psi + \delta$ since $\Delta(1/r) \sim \delta_0(x, y, z)$. Thus, ψ won't satisfy Schrodinger at the origin (as a distribution), so $s = -2\ell - 1$ is impossible even when $\ell = 0$. Now allowing this behavior singles out a one dimensional span, the span of the solution corresponding to $s = 0$.

We see that h must be regular at $\rho = 0$. Use power series method: $h = \sum_{n \geq 0} a_n \rho^n$. We then must have

$$\sum \left[k(k-1)a_k\rho^{k-1} + (2\ell + 2 - \rho)ka_k\rho^{k-1} + (n - \ell - 1)a_k\rho^k\right]$$

We can shift

$$\sum (k+1)ka_{k+1}\rho^k + (2\ell+2)(k+1)a_{k+1}\rho^k - ka_k\rho^k + (n - \ell - 1)a_k\rho^k$$

to get a recursion

$$(k+1)(k + 2\ell + 2)a_{k+1} + (n - \ell - 1 - k)a_k = 0.$$

Starting with $a_0 = 1$, one can calculate

$$a_k = \frac{(1 + \ell - n)\dots(k + \ell - n)}{(2\ell + 2)\dots(2\ell + 1 + k) \cdot k!}.$$

Thus,

$$h(\rho) = \sum_{k \geq 0} \frac{(1 + \ell - n)\dots(k + \ell - n)}{(2\ell + 2)\dots(2\ell + 1 + k)} \frac{\rho^k}{k!}.$$

We see that this converges for all ρ (ratio behaves like a power of k and then it's divided by a factorial).

Exercise. $\frac{\log h(\rho)}{\rho} \to 1$ when $\rho \to +\infty$ except when the series terminates.

When does this series terminate? Well, when one of the factors in the numerator becomes 0, i.e. if $n - \ell - 1 \in \mathbb{Z}_{\geq 0}$. In which case you get a polynomial of this degree $n - \ell - 1$; it is denoted $L^{2\ell+1}_{n-\ell-1}(\rho)$

and called the **generalized Laguerre polynomial**. Recall, we need

$$\int \rho^{2\ell+2} e^{-\rho} |h(\rho)|^2 \,\mathrm{d}\rho < \infty$$

We looked at convergence near 0 before, but there's also convergence near ∞. This will fail unless $h(\rho)$ behaves like a polynomial (the alternative is it looks like e^ρ at infinity, so get something like $e^{-\rho}e^{2\rho}$ above).

Recall 13.7. The states with $E < 0$ are called **bound states**.

Theorem 13.8. *The bound states of the hydrogen atom are, up to normalization,*

$$\psi_{n,\ell,m}(r,\varphi,\theta) = r^{-\ell} e^{-\frac{r}{n}} L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{n}\right) y_\ell^m(\varphi,\theta)$$

where $n = 1, 2, 3, \dots,$ $\ell = 0, 1, \dots, n-1,$ *and* $-\ell \le m \le \ell$.

Definition 13.9. We call n above the **principal quantum number**, ℓ the **azimuthal quantum number**, and m the **magnetic quantum number**.

Remark 13.10. The energy can only take values $E = -\frac{1}{2n^2}$. When an electron jumps between energy levels, it emits a photon with energy/wavelength proportional to the difference $\frac{1}{2n^2} - \frac{1}{2n'^2}$.

We still have not achieved what we wanted yet. These eigenfunctions do not form a base in the Hilbert space. This functions $\psi_{n,\ell,m}$ span a space $L^2_0(\mathbb{R}^3) \subsetneq L^2(\mathbb{R}^3)$. For example, note that $(H\psi,\psi) \le 0$ for $\psi \in L^2_0(\mathbb{R}^3)$. This is not the case for all $\psi \in L^2(\mathbb{R}^3)$. Recall $H = -\frac{1}{2}\Delta - \frac{1}{r}$, so in general

$$(H\psi,\psi) = \frac{1}{2}\int \|\nabla\psi\|^2 - \int \frac{1}{r}|\psi|^2$$

Can cook up a ψ so this is positive. In addition to the discrete spectrum/bound states we found, there's also a continuous spectrum consisting of the whole positive real linear $\{r \ge 0\}$, but we will not discuss this. Pavel said more about this, but I didn't follow.

Remark 13.11. For each n, there are n choices of ℓ values, and each (n,ℓ) has 2ℓ choices of m values. Hence $\dim W_n = n^2$ is the dimension of the space of energy levels of n. In chemistry though, one observes a $2n^2$, so we're missing something. That something is spin. The real Hilbert space is $L^2(\mathbb{R}^3) \times C_2$.

There's more to the story that we will talk about next time.

14 Lecture 14 (4/8): Quantum stuff continued

Last time we studied the equation $H\psi = E\psi$ where $H = -\frac{1}{2}\Delta - \frac{1}{r}$. We saw that we hand 'bound states,' L^2-eigenfunctions with corresponding energy $E = -\frac{1}{2n^2}$ for $n = 1, 2, 3, \dots$. These eigenfunctions looked like

$$\psi_{n,\ell,m}(r,\varphi,\theta) = r^{\ell} e^{-\frac{r}{n}} L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{n}\right) Y_\ell^m(\varphi,\theta)$$

for $n \ge 1$, $0 \le \ell \le n-1$, and $-\ell \le m \le \ell$.

Recall 14.1. ℓ above is the *azimuthal quantum number* and m is the *magnetic quantum number.*

There's a geometric SO(3)-symmetry so $\mathfrak{so}(3) = \operatorname{Lie}\mathrm{SO}(3)$ acts by vector fields L_x, L_y, L_z. Set $\vec{\mathbb{L}} = (L_x, L_y, L_z) = \vec{r} \times \vec{p}$ where $\vec{r} = (x, y, z)$ and $\vec{p} = (p_x, p_y, p_z)$ with $p_x = -i\partial_x$, etc. This $\vec{L} = \vec{r} \times \vec{p}$ is called the **angular momentum operator**. Note that

$$L_x = -i\left(y\partial_z - z\partial_y\right).$$

These act on H-eigenspaces. Let $W_n = \left\{\psi : H\psi = -\frac{1}{2n^2}\psi\right\} = \operatorname{span}\{\psi_{n,\ell,m} : \text{any } \ell, m\}$. From our earlier restrictions on ℓ, m, we see that

$$\dim W_n = \sum_{\ell=0}^{n-1}(2\ell+1) = n^2.$$

We know

$$W_n = L_0 \oplus L_2 \oplus \cdots \oplus L_{2n-2}$$

as $\mathfrak{so}(3)$-reps.

Apparently, we studied the case where there's one electron 'orbiting' a nucleus of charge $+1$, but this also applies when there's a larger nucleus. If the nucleus is to big, things aren't too precise since there are many electrons interacting with each (and that's not taken into account here), but early in the periodic table this is good enough.

Isn't this supposed to be a math class?

Note 7. I'm finding it pretty hard to pay attention.

Because of chemistry stuff, our n^2 seems like it should really be a $2n^2$. We lost a factor of 2 in the physics. There's a thing called spin ('internal angular momentum') that we did not take into account in our model. This spin can be $\pm\frac{1}{2}$.

On the side of mathematics, this means that the Hilbert space for the theorem should not be $L^2(\mathbb{R}^3)$, but should be $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ where this $\mathbb{C}^2$ is the 2-dimensional rep of $\mathfrak{so}(3)$. On $\mathbb{C}^2$, we have the operator

$$S = \frac{1}{2}h = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

whose eigenvalues are $\pm\frac{1}{2}$. The **total spin** is $m + s \in \{m + 1/2, m - 1/2\}$. So the action of $\mathfrak{so}(3)$ is diagonal; the eigenvalues of h are $2m+1$ (or $2m-1$), odd numbers ('odd highest weight' or 'half-integer spin'); get a direct sum of representations L_{2k+1}. But Hamiltonian is the same, so instead of $\psi_{n,\ell,m}$, we have

$$\psi_{n,\ell,m,+} = \psi_{n,\ell,m} \otimes v_+ \ \text{ and } \ \psi_{n,\ell,m,-} = \psi_{n,\ell,m} \otimes v_-$$

where

$$v_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \text{ and } \ v_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now, $V_n := \left\{\psi : H\psi = E\psi \ \text{ with } \ E = -\frac{1}{2n^2}\right\} = W_n \otimes \mathbb{C}^2$, so

$$V_n = (L_0 \oplus \cdots \oplus L_{2n-2}) \otimes \underbrace{\mathbb{C}^2}_{L_1} \underset{\text{Clebsch-Gordan}}{=} L_1 + L_1 + L_3 + \cdots + L_{2n-3} + L_{2n-1} = 2L_1 \oplus 2L_3 \oplus \cdots \oplus 2L_{2n-3} \oplus L_{2n-1}$$

and $\dim V_n = 2n^2$.

This does not lift to a representation of SO(3), only to one of SU(2). The matrix $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SU}(2)$ acts by -1. This is called an '*anomaly*'. The point is that quantum states are elements up to phase factors, and this -1 is a phase factor, so we do have an SO(3)-action on the states; we just don't have one on vectors.

Say we have k electrons of the same energy $E = -\frac{1}{2n^2}$. In quantum mechanics, if you have a particle with state space V and another one with state space W, then the two together have state space $V \otimes W$. If particulars are indistinguishable from each, then you should mod out by permutation action. If elections where labelled, we'd have state space $V_n^{\otimes k}$. This they are in fact indistinguishable, we need to mod out by permutations. Hence, we would expect the state space to be $V = S^k V_n$; however, this is wrong. The correct answer is $V = V^{(k)} = \bigwedge^k V_n$ since electrons are ferminons, not bosons (for bosons, do get symmetric power).[28]

Remark 14.2 (**Pauli exclusion principle**). When $k > 2n^2$, we see $V^{(k)} = 0$.

Remark 14.3. A generic operator will have eigenspaces of dimension ≤ 1, but here we have large dimensions $\dim V_n = 2n^2$. This comes from symmetries grouping these eigenvalues into representations (apparently, we've seen two $\mathfrak{so}(3)$-symmetries and there's a third hidden one we'll see now).

14.1 Explanation for degeneracy of energy levels

There is another $\mathfrak{so}(3)$ related to the Laplace-Runge-Lenz vector (homework).

Note 8. Got distracted and missed some of what we said.

Total symmetry is $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and $V_n = L_{n-1} \boxtimes L_{n-1} \boxtimes L_1$. The first $\mathfrak{so}(3)$ comes from the Laplace-Runger-Lenz stuff, the second is the 'geometric' $\mathfrak{so}(3)$ ($\mathrm{SO}(3) \curvearrowright \mathbb{R}^3$ so on $L^2(\mathbb{R}^3)$), and the third is the 'spin' $\mathfrak{so}(3)$. Forgetting about spin, we have $W_n = L_{n-1} \boxtimes L_{n-1}$. Restricting to the diagonal gives

$$W_n|_{\mathrm{diag}} = L_0 \oplus L_2 \oplus \cdots \oplus L_{2n-2}.$$

Question: Why did he use $\boxtimes$ instead of $\otimes$?

Answer: It's an external tensor product, not an internal one

Remark 14.4. $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3) = \mathfrak{so}(4) \oplus \mathfrak{so}(3) = \mathfrak{so}(4) \oplus \mathfrak{su}(2)$.

There's apparently also another symmetric which doesn't commute with Hamiltonian, but which is sometimes useful to consider.

14.2 Back to math: automorphisms of semisimple Lie algebras

14.2.1 Summary of last semester

Let $\mathfrak{g}$ be a semisimple complex Lie algebra. We saw that $\mathrm{Aut}(\mathfrak{g})$ is a complex Lie group with $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g}) = \mathfrak{g}$ (I think in general $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$). In particular, this means there is a connected Lie group $\mathrm{Aut}(\mathfrak{g})^0$ with Lie algebra $\mathfrak{g}$. Furthermore, we showed last semester that $\mathrm{Aut}^0(\mathfrak{g})$ acts transitively on the Cartan subalgebras of $\mathfrak{g}$.

Definition 14.5. The **adjoint group of** $\mathfrak{g}$ is $G_{\mathrm{ad}} := \mathrm{Aut}^0(\mathfrak{g})$.

[28] Particles with half-integer spin are ferminions, while those with integer spin are bosons.

14.2.2 Maximal Tori

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Let $H \subset G_{\text{ad}}$ be the corresponding connected Lie subgroup. Elements $h \in H$ act on $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ as follows: $h|_{\mathfrak{h}} = 1$ and $h|_{\mathfrak{g}_{\alpha_j}} = \lambda_j \cdot \text{id} = e^{b_j} \cdot \text{id}$. Note that $h|_{\mathfrak{g}_{-\alpha_j}} = \lambda_j^{-1} = e^{-b_j}$. Furthermore, if $\alpha = \sum m_i \alpha_i$, then (by compatibility with conjugation)

$$h|_{\mathfrak{g}_\alpha} = \prod \lambda_i^{m_i}.$$

So if $x \in \mathfrak{h}$ s.t. $\alpha_i(x) = b_i$ (so $\lambda_j = e^{\alpha_j(x)}$), then $h|_{\mathfrak{g}_\alpha} = e^{\alpha(x)}$ so we see we have

$$H \cong \frac{\mathfrak{h}}{2\pi i P^\vee},$$

i.e. $x \mapsto e^{2\pi i x}$ defines an isomorphism $\mathfrak{h}/P^\vee \xrightarrow{\sim} H$ (recall: $P^\vee$ is the coweight lattice). Note that $H \cong (\mathbb{C}^\times)^{\text{rank}(\mathfrak{g})}$ is a complex torus; we call it the **maximal torus** corresponding to $\mathfrak{h} \subset \mathfrak{g}$.

We want to study its normalizer

$$N(H) = \left\{g \in G_{\text{ad}} : gHg^{-1} = H\right\}.$$

Proposition 14.6. *$N(H)$ = stabilizer of $\mathfrak{h} \subset \mathfrak{g}$, and contains H as a normal subgroup with quotient $N(H)/H \cong W$ isomorphic to the Weyl group.*

Proof. Recall $(\mathfrak{sl}_2)_i \subset \mathfrak{g}$ attached to simple roots. These give maps $\eta_i : \text{SL}_2(\mathbb{C}) \to \text{ad}G$ by fundamental theorems of Lie theory. Set

$$S_i = \eta_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \eta_i(e - f) \in G_{\text{ad}}.$$

This has the property that $\text{Ad}(S_i)|_{\mathfrak{h}} = s_i$ (with s_i the simple reflection). Note that $S_i^2 = \eta_i(-1) \neq 1$ in general, so we *do not* have a homomorphism $W \to \text{ad}G$, just some set-theoretic lift of W. For $w \in W$, write $w = s_{i_1} \dots s_{i_m}$ and define $\widetilde{w} = S_{i_1} \dots S_{i_m} \in G_{\text{ad}}$, so $\widetilde{w}$ acts on $\mathfrak{h}$ by w. Furthermore, if $w = w_1 w_2$, then $\widetilde{w} = \widetilde{w}_1 \widetilde{w}_2 h$ for some $h \in H$ s.t. h acts trivially on $\mathfrak{h}$. This implies that $\langle H, \widetilde{w} : w \in W \rangle$ generates a subgroup N of G_{ad} such that $N \supset H$ (with H normal) and $N/H = W$.

This was a homework problem once upon a time

By definition, $N \subset N(H)$, so we only need to show equality. Consider some $x \in N(H)$. Write $x(\alpha_i) = \alpha_i'$. Note that these α_i''s give another system of simple roots. Since the Weyl group acts transitively on systems of simple roots, there must be some $w \in W$ such that $w(\alpha_i') = \alpha_{p(i)}$ where p is some permutation of simple roots. Now consider $\widetilde{w}^{-1} x \in G_{\text{ad}}$. By construction, we have $\widetilde{w}^{-1} x(\alpha_i) = \alpha_{p(i)}$. Note that G_{ad} preserves all irreducible representations $\mathfrak{g}$ (since it acts by inner automorphisms), so $p = \text{id}$. Hence, $\widetilde{w}^{-1} x|_{\mathfrak{h}} = 1$, so $\widetilde{w}^{-1} x \in H$, so $x \in \widetilde{w} H \subset N$, and we win. ■

Warning 14.7. In general, the exact sequence

$$0 \longrightarrow H \longrightarrow N \longrightarrow W \longrightarrow 0$$

is not split, i.e. N is not a semi-direct product.

We've seen $\text{Aut}(\mathfrak{g}) \supset G_{\text{ad}}$. Another obvious subgroup is $\text{Aut}(D) \subset \text{Aut}(\mathfrak{g})$ where D is the Dynkin diagram. Moreover, $\mathrm{Aut}(D) \curvearrowright G_{\mathrm{ad}}$, so we get a homomorphism

$$\xi : \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}} \longrightarrow \mathrm{Aut}(\mathfrak{g}).$$

This is in fact injective; $\xi|_{G_{\mathrm{ad}}} = \mathrm{id}$ and a nontrivial Dynkin diagram automorphism can't act trivially on $\mathfrak{g}$ (something like this).

Theorem 14.8. *ξ is an isomorphism.*

Proof. We need to show that ξ is surjective. Fix some $a \in \mathrm{Aut}(\mathfrak{g})$. There exists a $g \in G_{\mathrm{ad}}$ such that $ga(\mathfrak{h}) = \mathfrak{h}$. We may replace a by ga, so assume WLOG that $a(\mathfrak{h}) = \mathfrak{h}$. By modifying a by an element of $N(H) \cdot \mathrm{Aut}(D)$,[29] can assume $a = 1$ (acts trivially on $\mathfrak{h}$ and each $\mathfrak{g}_{\alpha_i}$), so we win. ■

14.2.3 Forms of semisimple Lie algebras

We have classified semisimple Lie algebras over $\mathbb{C}$. What about their classification over other fields, in particular over $\mathbb{R}$?

Recall 14.9. A presentation of $\mathfrak{g}$ by generators and relations e_i, f_i, h_i contains only integers, so makes sense over any ring.

For any field K (say, $\mathrm{char}\, K = 0$), we have a Lie algebra $\mathfrak{g}_K$ defined by the same generators and relations; we call this **split semisimple Lie algebra**. Over an algebraically closed field, every semisimple Lie algebra is split, but this is not the case in general.

Let $\mathfrak{g}$ be a s.s. LA over K which splits over some finite Galois extension L/K (e.g. $K = \mathbb{R}$ and $L = \mathbb{C}$), i.e. $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$ is a split s.s. Lie alg. Can we classify such $\mathfrak{g}$? Let $\Gamma = \mathrm{Gal}(L/K)$, so $\mathfrak{g} = \mathfrak{g}_L^\Gamma$. Therefore, $\mathfrak{g}$ is determined by the action of Γ on $\mathfrak{g}_L$. This action is **twisted-linear**:

$$\gamma(\lambda x) = \gamma(\lambda)\gamma(x) \text{ for } \lambda \in L \text{ and } x \in \mathfrak{g}_L.$$

Example. $\rho_0(g \in \Gamma)$: preserves all generators and acts as Γ on scalars. This action gives rise to the split s.s. Lie algebra over K, $\mathfrak{g}_L^\Gamma = \mathfrak{g}_K$.

Other actions will be of the form $\rho(g) = \eta(g)\rho_0(g)$ with $\eta : \Gamma \to \mathrm{Aut}(\mathfrak{g}_L)$ not a homomorphism. Instead,

$$\eta(gh)\rho_0(g)\rho_0(h) = \rho(gh) = \rho(g)\rho(h) = \eta(g)\rho_0(\mathfrak{g})\eta(h)\rho_0(h),$$

from which we see

$$\eta(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h)^{-1} = \eta(g) \cdot g(\eta(h)),$$

so it's almost a homomorphism but twisted by the Γ-action on $\mathrm{Aut}(\mathfrak{g}_L)$. This is what's called a **1-cocycle** (or **twisted homomorphism**). Thus, any form of $\mathfrak{g}_K$ split over L is given by a 1-cocycle η; we call the corresponding form $\mathfrak{g}_\eta$.

Question 14.10. *When is $\mathfrak{g}_{\eta_1} \cong \mathfrak{g}_{\eta_2}$?*

[29] a sends simple roots to a different system of simple roots. Can use a (lift of an) element of the Weyl group to make it preserve the system of simple roots. Then use an automorphism of D to ensure $a(\alpha_i) = \alpha_i$. Then, $a|_{\mathfrak{g}_{\alpha_i}}$ acts by some scalar. Can use an element of H to make all these scalars 1.

Need some $a \in \mathrm{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$ which translates to

$$\eta_1(g) = a\eta_2(g)g(a)^{-1}$$

(**twisted conjugation**).

Definition 14.11. Equivalence classes of 1-cocycles, up to twisted conjugation, form the (pointed) set $\mathrm{H}^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$ called the **1st Galois cohomology**.

Proposition 14.12. *Forms of $\mathfrak{g}_L$ over K are labelled by elements of $\mathrm{H}^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$.*

15 Lecture 15 (4/13)

15.1 Forms of a semisimple Lie algebra, continued

Let $\mathfrak{g}$ be a s.s. Lie algebra over a field K of characteristic 0. Say there is a finite Galois extension $L \supset K$ such that $\mathfrak{g} \otimes_K L$ splits, i.e. is isomorphic to the standard semisimple Lie algebra $\mathfrak{g}_L$ given by the Serre relations. We showed last time that such forms of $\mathfrak{g}_L$ over K are classified by the cohomology set $\mathrm{H}^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$.

Today we specialize to the case of main interest to us, i.e. $K = \mathbb{R}$ and $L - \mathbb{C}$. That is, we wish to classify real forms of complex semisimple Lie algebras.

Remark 15.1. There's a parallel theory of forms for reductive Lie algebras.

In this case $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. We computed last time that

$$\mathrm{Aut}(\mathfrak{g}_L) = \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}} \text{ where } G_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g}_L)^\circ.$$

Consider a 1-cocycle $\eta : \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathfrak{g}_L)$. This must satisfy

$$\eta(xy) = \eta(x) \cdot x(\eta(y)).$$

Hence, $\eta(1) = \eta(1)\eta(1) \implies \eta(1) = 1$, so η is determined by the element

$$s := \eta(-1) \in \mathrm{Aut}(\mathfrak{g}_L) = \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}.$$

Not just any s will work. We require

$$1 = \eta(1) = \eta((-1)(-1)) = \eta(-1) \cdot (-1) \circ (\eta(-1)) = s\overline{s}.$$

Above, $\bar{\cdot}$ denotes complex conjugation, $\mathfrak{g}$ is the complexification of its split real form. s defined earlier is well-defined up to twisted conjugation: $s \mapsto as\overline{a}^{-1}$ (for $a \in \mathrm{Aut}(\mathfrak{g}_L)$). Putting this all together, we have...

Theorem 15.2. *Real semisimple Lie algebras with complexification isomorphic to $\mathfrak{g}$ (i.e. **real forms** of $\mathfrak{g}$) are classified by $s \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$ s.t. $s\overline{s} = 1$ modulo the equivalence relations $s \sim as\overline{a}^{-1}$ (for $a \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$; note $\bar{\cdot}$ acts trivially on $\mathrm{Aut}(D)$).*

The bijection in the theorem is given by

$$s \mapsto \mathfrak{g}_s := \{x \in \mathfrak{g} : \overline{x} = s(x)\}.$$

Example. $\mathfrak{g}_1 = \mathfrak{g}_{\mathbb{R}} = \{x \in \mathfrak{g} : \overline{x} = x\}$.

Note that we can compute s and $\bar{\cdot}$ to get the antilinear involution $\sigma_s(x) = \overline{s(x)}$ (note $\sigma_s^2(x) = \overline{s(\overline{s(x)})} = \overline{\overline{s}(s(x))} = \overline{s}(x) = x$). Hence, we can encode the real form $\mathfrak{g}_s$ using σ_s instead of s. In particular, note that s gives rise to an element $s_0 \in \operatorname{Aut}(D) = \operatorname{Out}(\mathfrak{g}) = \operatorname{Aut}(\mathfrak{g})/\operatorname{Inn}(\mathfrak{g})$ ($\operatorname{Inn}(\mathfrak{g}) = G_{\mathrm{ad}}$). Note that this satisfies $s_0^2 \in 1$, and that its conjugacy class is invariant under equivalences. This s_0 permutes connected components of the Dynkin diagram D (preserves some and matches others in pairs[30]). Hence, it's enough to consider to kinds of pictures.

(1) D connected with $s_0 : D \xrightarrow{\sim} D$.

(2) $D' = D \sqcup D$ and s_0 exchanges them.

Proposition 15.3. *If $\mathfrak{g}_{\mathbb{R}}$ is semisimple, then it is a direct sum of simple Lie algebras, and the simple Lie algebras are classified by such pictures.*

(1) In the first case, $\mathfrak{g}_{\mathbb{R}}$ is simple, and $\mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is also simple.

To be analyzed later...

(2) In the second, $\mathfrak{g}_{\mathbb{R}}$ is simple, but $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ has two summands (so is only semisimple).

Say $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$ with $\mathfrak{a}$ a simple complex Lie algebra. Write $s = (g, h)s_0$ with $g, h \in \operatorname{Aut}(\mathfrak{a})$. We know s switches the summands and that $s\overline{s} = 1$. This gives

$$(g\overline{h}, h\overline{g}) = (g, h)s_0(\overline{g}, \overline{h})s_0 = 1$$

so $h = \overline{g}^{-1}$. Thus,

$$s = (g, \overline{g}^{-1})s_0 = (g, 1)s_0\overline{(g, 1)}^{-1} \sim s_0,$$

so there is only one real form with such s_0. It is

$$\mathfrak{g}_{s_0} = \left\{(x, y) \in \mathfrak{a} \oplus \mathfrak{a} : \overline{(x, y)} = (y, x)\right\} = \{(x, \overline{x}) : x \in \mathfrak{a}\} \cong \mathfrak{a}$$

with the last iso an iso of real Lie algebras.

This just leaves the case when D is connected. We start with some new definitions.

Definition 15.4. We say $\mathfrak{g}_s$ is **inner to** $\mathfrak{g}_{s'}$ if $s' = g \circ s$ for some $g \in G_{\mathrm{ad}}$ (i.e. for some inner automorphism) $\iff s_0' \sim s_0$. The **inner class** of s is the set of s' which are inner to s. An **inner real form** is a member of the inner class of the split form (i.e. $s \in G_{\mathrm{ad}}$).

Definition 15.5. We say $\mathfrak{g}_s$ is **quasi-split** if $s = s_0 \in \operatorname{Aut}(D)$.

[30] since $s_0^2 = 1$

Note that any form is inner to a unique quasi-split form. The only quasi-split inner form is the split form.

There is one (non-split) distinguished form.

Definition 15.6. The **compact real form** is the one corresponding to the automorphism determined by

$$s(h_i) = -h_i, \;\; s(e_i) = -f_i, \;\text{ and }\; s(f_i) = -e_i.$$

The corresponding Lie group is denoted by $\mathfrak{g}^c$.

Proposition 15.7. *The Killing form of $\mathfrak{g}^c$ is negative definite.*

Proof. ($\mathfrak{g} = \mathfrak{sl}_2$) In this case, we have $s(h) = -h$, $s(e) = -f$, and $s(f) = -e$. We have

$$\mathfrak{g}^c = \{x \in \mathfrak{g} : \overline{x} = s(x)\} = \langle ih, e - f, i(e+f)\rangle .$$

We see the basis is given by the **Pauli matrices**

$$X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \;\; Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \;\text{ and }\; Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

These span $\mathfrak{su}(2) = \mathfrak{g}^c$ (recall $\mathfrak{su}(2) = \{x \in \mathfrak{gl}_2(\mathbb{C}) : x^t = -x \;\text{ and }\; \operatorname{tr} x = 0\}$).

The Killing form on $\mathfrak{su}(2)$ is (a scalar multiple of?) the trace form. We compute

$$\operatorname{tr} X^2 = -2 \;\text{ and }\; \operatorname{tr} Y^2 = -2 \;\text{ and }\; \operatorname{tr} Z^2 = -2,$$

so the Killing form on $\mathfrak{su}(2) = \mathfrak{sl}_2^c$ is negative definite.

(Same is true for any f.d irrep. The trace-form associated to any f.dim irrep will be negative-definite)

($\mathfrak{g}$ general) Consider the matrix

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2).$$

This preserves $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$ so $S_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SU}(2)_i$ (for ith simple root) preserves $\mathfrak{g}_c \supset \mathfrak{su}(2)_i$. Hence, for every $w \in W$ (i.e. $w = s_{i_1} \dots s_{i_r}$), the corresponding $\widetilde{w}$ ($= S_{i_1} \dots S_{i_r}$) also preserves $\mathfrak{g}_c$. For any root α, pick $w \in W$ s.t. $w(\alpha) = \alpha_i$ is a simple root. Then, $\widetilde{w}((_2)_\alpha) = (\mathfrak{sl}_2)_\mathrm{i}$. Thus, $(\mathfrak{sl}_2)_\alpha \cap \mathfrak{g}_C = \mathfrak{su}(2)$ so the Killing form is negative definite on $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$; also it is negative definite on $\mathfrak{h}_c = \mathfrak{h} \cap \mathfrak{g}_c$ since this is spanned by $\{ih_j : j = 1, \dots, r\}$ since positive on $\mathbb{R}$-span $\mathbb{R}\langle h_j\rangle$ (multiplying by i makes it negative). We have

$$\mathfrak{g}_c = \mathfrak{h}_c \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})_c.$$

∎

Remark 15.8. Above we used that the compact real form restricted to any simple root $(\mathfrak{sl}_2)_i$ is the corresponding compact real form.

Consider $\mathrm{Aut}(\mathfrak{g}^c)$ Killing is negative definite, so $\mathrm{Aut}(\mathfrak{g}^c) \subset O(\mathfrak{g}^c)$ is a closed subgroup in an orthogonal group, and hence compact.[31] Furthermore, $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g}^c) = \mathfrak{g}^c$ (not hard to show). Thus,

Corollary 15.9. *Let $G^c_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g}^c)^0$. Then, G^c_{ad} is a connected, compact Lie group with Lie algebra $\mathfrak{g}^c$.*

Remark 15.10. This gives a new proof that reps of complex semisimple Lie algebras are completely reducible.

Exercise (Homework). For $\mathfrak{g} = \mathfrak{sl}_n$, show $G^c_{\mathrm{ad}} = \mathrm{PSU}(n) = \mathrm{SU}(n)/\mu_n$ (where μ_n the nth roots of unity).

For $\mathfrak{g} = \mathfrak{so}_n$, show

$$G^c_{\mathrm{ad}} = \begin{cases} \mathrm{SO}(n) & \text{if } n \text{ odd} \\ \mathrm{SO}(n)/\{\pm 1\} & \text{if } n \text{ even.} \end{cases}$$

For $\mathfrak{g} = \mathfrak{sp}_{2n}$, show

$$G^c_{\mathrm{ad}} = U(n, \mathbb{H})/\{\pm 1\} \text{ where } U(n,\mathbb{H}) = \mathrm{Sp}_{2n}(\mathbb{C}) \cap U(2n).$$

Exercise. s_0 for the compact form is the involution corresponding to $-w_0$ (dualiziation of representations).

Exercise. The compact form is never quasisplit.

Question 15.11. *What kind of real forms do we know about?*

(1) A_{n-1} so $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

- split: $\mathfrak{sl}_n(\mathbb{R})$
- compact: $\mathfrak{su}(n)$
- When $n > 2$, the diagram A_{n-1} has the flip automorphism exchanging $e_i \leftrightarrow e_{n+1-i}$ ($e_i = E_{i,i+1}$). This corresponds to the involution $s(A) = -JA^tJ^{-1}$ where

$$J = \begin{pmatrix} & & & -1 \\ & & 1 & \\ & -1 & & \\ \iddots & & & \end{pmatrix}.$$

Thus, $\mathfrak{g}_s$ is the Lie algebra of traceless matrices A s.t. $A = -J\overline{A}^tJ^{-1}$ (i.e. $AJ + J\overline{A}^T = 0$). Thus, A preserves the (skew)hermitian form defined by J.[32] What is the signature of J? For even n, we have

$$J = \pm \sum \left(z_i\overline{z}_{n+1-i} \pm z_{n+1-i}\overline{z}_i\right),$$

while for odd J we have

$$J = \pm \sum \left(z_i\overline{z}_{n+1-i} \pm z_{n+1-i}\overline{z}_i\right) \pm z_{\frac{n+1}{2}}\overline{z}_{\frac{n+1}{2}}.$$

[31] Why is $O(n)$ compact? $A^tA = 1$ means $\sum_j a_{ij}^2 = 1$ so $O(n) \overset{\text{closed}}{\subset} (S^n)^n$

[32] If you take a Hermitian form and multiply by i, you get a skew-hermitian form (and vice versa), so the two types are not so different

When $n = 2p$, J has signature (p,p). When $n = 2p+1$, it has signature $(p, p+1)$ or $(p+1, p)$. The upshot of all of this is that the quasi-split form is $\mathfrak{su}(p,p)$ if $n = 2p$ and $\mathfrak{su}(p+1,p)$ if $n = 2p+1$.

- There are other forms: $\mathfrak{su}(p, n-p)$. These are neither compact nor quasi-split.

(2) Type B_n ($\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$)

- split: $\mathfrak{so}(n+1, n)$
- compact: $\mathfrak{so}(2n+1)$
- no quasi-split forms since the corresponding Dynkin diagram has no symmetrices

(3) Type C_n ($\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$):

- split: $\mathfrak{sp}_{2n}(\mathbb{R})$
- compact: $\mathfrak{u}(n, \mathbb{H})$

(4) D_n ($\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$)

- split: $\mathfrak{so}(n,n)$
- compact: $\mathfrak{so}(2n)$
- When $n > 4$, $\mathrm{Aut}(D_n) = \mathbb{Z}/2\mathbb{Z}$. When $n = 4$, $\mathrm{Aut}(D_4) = S_3$ (claw graph). However, we only care about conjugacy classes of involutions, and in either case, there's only one nontrivial such class: the one exchanging $\alpha_n = e_{n-1} + e_n$ and $\alpha_{n-1} = e_{n-1} - e_n$.

  Note that the split form consists of matrices A satisfying $A = -JA^tJ^{-1}$ where

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}$$

  To get the quasi-split form, we should use a matrix of the same structure, except it's the 2×2 identity I_2 in the center block (diagonal instead of antidiagonal at that point). Call this matrix $\widetilde{J}$. Then the quasi-split forms consists of matrices satisfying $A = -\widetilde{J}A^t\widetilde{J}^{-1}$, i.e. its the Lie algebra of skew-symmetric matrices under $\widetilde{J}$. This has signature $(n+1, n-1)$, so the quasi-split form is $\mathfrak{so}(n+1, n-1)$. ($n \geq 2$)

  TODO: Add rendition of matrix

(5) G_2 has a split form G_2^s and compact form G_2^c. No Dynkin diagram automorphisms.

(6) F_4 has a split form F_4^s and compact form F_4^c. No Dynkin diagram automorphisms.

(7) E_6 has a split form E_6^s and compact form E_6^c. There is a Dynkin diagram automorphisms, so also a quasi-split E_6^{qs}

(8) E_7 has a split form E_7^s and compact form E_7^c. No Dynkin diagram automorphisms.

(9) E_8 has a split form E_8^s and compact form E_8^c. No Dynkin diagram automorphisms.

Remark 15.12. There are some coincidences. For example,

- $B_1 = A_1$ gives $\mathfrak{so}(2,1) = \mathfrak{su}(1,1)$ and $\mathfrak{so}(3) = \mathfrak{su}(2)$
- $C_2 = B_2$ gives $\mathfrak{so}(5) = \mathfrak{u}(2,\mathbb{H})$ and $\mathfrak{so}(3,2) = \mathfrak{sp}_4(\mathbb{R})$
- $D_2 = A_1 \times A_1$ is two unconnected points. Comparing points of view shows

  $$\mathfrak{su}(1,1) \times \mathfrak{su}(1,1) = \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R}) = \mathfrak{so}(2,2) \text{ and } \mathfrak{su}(2) \times \mathfrak{su}(2) = \mathfrak{so}(4) \text{ and } \mathfrak{sl}_2(\mathbb{C})_{\mathbb{R}} = \mathfrak{so}(3,1).$$

  Above, $\mathfrak{so}(3,1)$ is called the **Lorentz Lie algebra**.
- $D_3 = A_3$ gives $\mathfrak{sl}_4(\mathbb{R}) = \mathfrak{so}(3,3)$, $\mathfrak{su}_4 = \mathfrak{so}(6)$, and $\mathfrak{su}(3,2) = \mathfrak{so}(4,2)$.

Note we still have not classifies all real forms. We've just looked at the compact, and quasi-split forms. There are still more.

16 Lecture 16 (4/15)

Last time we considered real forms of semisimple Lie algebras, and singled out a few particular forms of note.

In particular, we defined the compact form of a semisimple Lie algebra. This had corresponding involution $\omega : \mathfrak{g} \to \mathfrak{g}$ determined by $\omega(h_i) = -h_i$, $\omega(e_i) = -f_i$, and $\omega(f_i) = -e_i$. The corresponding (real) Lie algebra was $\mathfrak{g}^c = \{x \in \mathfrak{g} : \omega(x) = x\}$.

16.1 Twists of the compact form

In trying to classify forms of $\mathfrak{g}$ (a complex (semi)simple Lie algebra), we started with the split form, and then looked at the other versions of it. It turns out that it is actually more convenient to start with the compact form instead.

Write $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}^c + i\mathfrak{g}^c$. Hence we can write $z = x + iy$ and this has the natural involution $\omega(z) = \overline{z} = x - iy$. What are the other real structures on $\mathfrak{g}$?

Consider $\sigma : \mathfrak{g} \to \mathfrak{g}$ another antilinear involution. Then, $\sigma = \omega \circ g$ for some $\mathbb{C}$-linear $g \in \operatorname{Aut}(\mathfrak{g})$. We need

$$1 = \sigma^2 = \omega \circ g \circ \omega \circ g = \omega g \omega^{-1} \omega^2 g = \omega g \omega^{-1} \cdot g,$$

i.e. $\omega(g)g = 1$ where $\omega(g) := \omega g \omega^{-1}$. This is our old friend the cocycle condition. What's different? $\mathfrak{g}^c$ has a negative definite Killing form, so $\mathfrak{g} = \mathfrak{g}^c \otimes_{\mathbb{R}} \mathbb{C}$ naturally has a positive Hermitian form (complexification of $-$Killing).[33] Fix some $x \in \mathfrak{g}$. Then,

$$\mathrm{ad}\omega(x) = -(\mathrm{ad}x)^\dagger$$

is the Hermitian adjoint (negated). Hence, $\mathfrak{g}^c$ acts by skew-Hermitian operators, i.e. if $x \in \mathfrak{g}^c$, then $\mathrm{ad}x = -(\mathrm{ad}x)^\dagger$. Therefore, when acting on group elements, we sill have $\omega(g) = (g^\dagger)^{-1}$.

Now we see that the cocycle condition $\omega(g)g = 1$ is equivalent to saying that $g^\dagger = g$, so the condition on g is that it is a Hermitian operator on $\mathfrak{g}$.

[33] Any orthonormal basis for $\mathfrak{g}^c$ is also an orthogonal basis for $\mathfrak{g}$

Fact. Any Hermitian operator on a space with positive Hermitian form is diagonalizable with real eigenvalues.

For $g = g^\dagger$, we can write $\mathfrak{g} = \bigoplus_{\gamma \in \mathbb{R}} \mathfrak{g}(\gamma)$ as a sum of eigenspaces; moreover, this is a grading, i.e. $[\mathfrak{g}(\gamma), \mathfrak{g}(\beta)] \subset \mathfrak{g}(\beta\gamma)$. Since g is Hermitian, we can take its absolute value $|g| : \mathfrak{g} \to \mathfrak{g}$. This acts on $\mathfrak{g}(\gamma)$ by $|\gamma|$. Define $\theta := f\,|g|^{-1} : \mathfrak{g} \to \mathfrak{g}$ so $\theta|_{\mathfrak{g}(\gamma)} = \operatorname{sign}(\gamma)$. This is an automorphism satisfying

$$\theta^2 = 1 \text{ and } \theta\omega = \omega\theta$$

(second one since $\omega(\theta) = \left(\theta^\dagger\right)^{-1} = \theta$).

Claim 16.1. *θ and g define the same real structure.*

Proof. Note that θ, g define the same real structure $\iff \theta = ag\overline{a}^{-1} = ag\omega(a)^{-1}$ for some $a \in G_{\mathrm{ad}}$. We have take $a = |g|^{-1/2}$ (acts by $|\gamma|^{-1/2}$ of $\mathfrak{g}_\gamma$). Then,

$$|g|^{-1/2} g\omega\left(|g|^{1/2}\right) = |g|^{-1/2} g\,|g|^{-1/2} = g\,|g|^{-1} = \theta,$$

so we win. ■

Corollary 16.2. *WLOG we may assume $g = \theta$, i.e. $\sigma = \omega \circ \theta$ where $\omega\theta = \theta\omega$ and $\theta^2 = 1$.*

This replaces the mysterious equation $\omega(g)g = 1$ with the simpler equation $\theta^2 = 1$.

Any real form is determined by a conjugacy class (conjugate by $\mathfrak{g}^c$) of such θ. Conversely, if two such θ's define the same real structure, then they will be conjugate under $\mathfrak{g}^c$.

Claim 16.3. *θ, ξ as above define the same real form $\iff$ they are conjugate by* $\operatorname{Aut}(\mathfrak{g}^c)$*.*

Proof. ($\to$) We have $\xi = x\theta\omega\left(x\right)^{-1}$ for some $x \in \operatorname{Aut}(\mathfrak{g})$. Since $\omega(\xi) = \xi$, we see that $x\theta\omega\left(x\right)^{-1} = \omega(x)\theta x^{-1}$. Set $z := \omega\left(x\right)^{-1} x$, so we have $\theta z = z^{-1}\theta$ and $\omega(z) = z^{-1}$. Note that $z = x^\dagger x$ is a positive operator, so we can extract a square root and set $y := xz^{-1/2}$. Then, $\omega(y) = \omega(x)z^{1/2} = xz^{-1/2}$ (since $\omega(x) = xz^{-1}$), so $y \in \operatorname{Aut}(\mathfrak{g}^c)$. At the same time (use $\theta z = z^{-1}\theta$),

$$\xi = x\theta\omega(x)^{-1} = x\theta z x^{-1} = xz^{-1/2}\theta z^{1/2}x^{-1} = y\theta y^{-1},$$

so we win. ■

At this point, we have obtained the following theorem.

Theorem 16.4. *Real forms of $\mathfrak{g}$ are in bijection with conjugacy classes of involutions $\theta \in \operatorname{Aut}(\mathfrak{g}^c)$ (a compact Lie group), via $\theta \mapsto \sigma_\theta : \omega \circ \theta$.*

Corollary 16.5. *Have a canonical (up to* Aut *of $\mathfrak{g}^c$) decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the 1-eigenspace of θ, and $\mathfrak{p}$ is the -1-eigenspace. In particular, $\mathfrak{k}$ is a Lie subalgebra, and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ (so $\mathfrak{p}$ is a $\mathfrak{k}$-module). Furthermore, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ and $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$ ($\mathfrak{k}^c = \mathfrak{k} \cap \mathfrak{g}^c$ and $\mathfrak{p}^c = \mathfrak{g}^c \cap \mathfrak{p}$). Finally,*

$$\mathfrak{g}_\sigma = \mathfrak{k}^c + i\mathfrak{p}^c.$$

Example. Say $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, and let $\mathfrak{g}_\sigma = \mathfrak{sl}_2(\mathbb{R})$ be the split form. In this case, $\mathfrak{k} = \mathbb{C}(e - f)$. Compute $\mathfrak{p}$ as an exercise. Then, $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$ and $\mathfrak{g}_\sigma = \mathfrak{k}^c + i\mathfrak{p}^c$.

Figure 9: An example Vogan diagram. White vertices have sign + and black vertices have sign −.

Exercise. Show that $\mathfrak{k}$ is a reductive Lie algebra.

We would like to simplify our task even further. Classifying involutions in a Lie group is not so easy.

Proposition 16.6. *There exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ invariant under θ.*

Proof. Consider a generic $x \in \mathfrak{k}^c$. Note that all elements of $\mathfrak{g}^c$ are semisimple (act as skew-Hermitian operators so are diagonalizable). Hence, x is regular semisimple.[34] Let $\mathfrak{h}_+^C \subset \mathfrak{k}^c$ be the centralizer. Its complexification $\mathfrak{h}_+ = \mathfrak{h}_+^c \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ (and still the centralizer of x). Let $\mathfrak{h}_-^c \subset \mathfrak{p}^c$ be the maximal subspace s.t. $\mathfrak{h}^c = \mathfrak{h}_+^c \oplus \mathfrak{h}_-^c$ is a commutative subalgebra of $\mathfrak{g}^c$.

We claim that $\mathfrak{h} := \mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C} \subset \mathfrak{g}$ is a Cartan subalgebra. It consists of semisimple elements by construction (acts by normal operators on $\mathfrak{g}$). Suppose $z \in \mathfrak{g}$, $[z, \mathfrak{h}] = 0$. Write $z = z_+ + z_-$ where $z_+ \in \mathfrak{k}$ and $z_- \in \mathfrak{p}$. Note that

$$[z, \mathfrak{h}] = 0 \iff [z, \mathfrak{h}_+] = 0 \text{ and } [z, \mathfrak{h}_-] = 0 \iff [z_+, \mathfrak{h}] = 0 \text{ and } [z_-, \mathfrak{h}] = 0.$$

Since $\mathfrak{h}_+$ Cartan in $\mathfrak{k}$, we conclude that $z_+ \in \mathfrak{h}_+$. Write $z_- = x_- + iy_-$ with $x_-, y_- \in \mathfrak{p}^c$. Then, $[z_-, \mathfrak{h}] = 0 \implies [x_-, \mathfrak{h}^c] = 0$, so $x_-, y_- \in \mathfrak{h}_-^c$ (by maximality). Thus, $z \in \mathfrak{h}$, so $\mathfrak{h} \subset \mathfrak{g}$ is Cartian. It is θ-stable since $\theta|_{\mathfrak{h}_\pm} = \pm 1$. ■

Lemma 16.7. *$\mathfrak{h}_-$ does not contain any coroots of $\mathfrak{g}$.*

Proof. Suppose otherwise, so $\alpha^\vee \in \mathfrak{h}_-$. Then, $\theta(\alpha^\vee) = -\alpha^\vee$, so $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. Then, $\sigma(\mathfrak{g}_\alpha) = \omega \circ \theta(\mathfrak{g}_\alpha)$... (do this next time) ■

Corollary 16.8. *For generic $t \in \mathfrak{g}_+$ (regular semisimple) s.t. $\mathrm{Re}(t, \alpha^\vee) \neq 0$ for any coroot $\alpha^\vee$ (possible since no coroots in $\mathfrak{h}^-$), consider the polarization*

$$R_+ = \{\alpha \in R : \mathrm{Re}(t, \alpha^\vee) > 0\}.$$

Then, $\theta(R_+) = R_+$ (since $\theta(t) = t$).

With a polarization as above, the simple roots get permuted, so $\theta(\alpha_i) = \alpha_{\theta(i)}$ where $\theta(i)$ gives the action of θ on the Dynkin diagram of $\mathfrak{g}$. If $\theta(i) = i$, then $\theta(e_i) = \pm e_i$, $\theta(h_i) = h_i$, and $\theta(f_i) = \pm f_i$. If $\theta(i) \neq i$, we can normalize the generators so that $\theta(e_i) = e_{\theta(i)}$, $\theta(f_i) = f_{\theta(i)}$, and $\theta(h_i) = h_{\theta(i)}$.

We can encode this info in the Dynkin diagram, to produce a **Vogan diagram**. Any Vogan diagram gives rise to a real form, and any real form comes from some Vogan diagram. However, different diagrams can give rise to the same form (diagram depends on the choice of R_+ with $\theta(R_+) = R_+$).

TODO: Add picture

Exercise (Homework). Compute the signature of the Killing form for $\mathfrak{g}_\sigma$. It should be $(\dim \mathfrak{p}, \dim \mathfrak{k})$.

Deduce that for split form, $\dim \mathfrak{k} = |R_+|$.

If $\mathfrak{g}_\sigma$ is in compact inner class, then $\mathrm{rank}(\mathfrak{k}) = \mathrm{rank}(\mathfrak{g})$, so they will share a Cartan subalgebra.

[34] its centralizer is a Cartan algebra

16.2 Real forms of classical groups

16.2.1 Type A_{n-1}

The Dynkin diagram has two automorphisms (identity and flip), so there are two inner classes.

- We start with the compact inner class (i.e. θ an inner automorphism, conjugation by some element of order ≤ 2 in $\mathrm{PSU}(n)$).

  Such an element can be lifted to $g \in U(n)$ s.t. $g^2 = 1$. Then, $\theta(x) = gxg^{-1}$. We know what g can look like:

  $$g = \operatorname{diag}\left(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}\right) \quad \text{where } p + q = n,$$

  and we may assume $p \geq q$. The corresponding real form will be $\mathfrak{su}(p, q)$.

  The compact form was $\mathfrak{su}(n) : A = -\overline{A}^t$ (and $\operatorname{Tr} A = 0$). For the form attached to g, we need $A = -g\overline{A}^t g^{-1}$ (and $\operatorname{Tr} A = 0$). This is just the requirement that A be skew-Hermitian for the form defined by g.

  When $n = 2$, A_1 has no automorphisms, so all forms are inner to the compact form. In this case, there are only two forms: $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1) = \mathfrak{sl}(2, \mathbb{R})$.

- There's also the split inner class (assume $n > 2$)

  The Vogan has at most one fixed vertex (only exists when n even, so there are an odd number of vertices). Hence, there's no choice when n odd so only get the split form $\mathfrak{sl}_n(\mathbb{R})$. Here, $\mathfrak{k}^c = \mathfrak{so}_n(\mathbb{R})$ is skew-symmetric matrices while $\mathfrak{p}^c$ consists of symmetric matrices.

  When $n = 2k$ is even, there are two options. The single fixed vertex can be colored black or white. Exercise: if the vertex is white (positive sign), then $\mathfrak{k} = \mathfrak{sp}_{2k}$ while if it is black, then $\mathfrak{k} = \mathfrak{so}_{2k}$ (this is the split form $\mathfrak{sl}_{2k}(\mathbb{R})$). What is the Lie algebra corresponding to the white case? It is $\mathfrak{sl}(k, \mathbb{H})$, the Lie algebra of traceless quaternionic $k \times k$ matrices. There's a **trace map**

  $$\begin{aligned} \operatorname{Tr} : \quad \mathfrak{gl}(k, \mathbb{H}) &\longrightarrow \mathbb{R} \\ A &\longmapsto \operatorname{Re}\left(\sum a_{ii}\right). \end{aligned}$$

  This is the same as $\frac{1}{2} \operatorname{Tr} A$ as an operator on $\mathbb{C}^{2k}$. And

  $$\mathfrak{sl}(k, \mathbb{H}) = \ker \operatorname{Tr} = \{A \in \mathfrak{gl}(k, \mathbb{H}) : \operatorname{Tr} A = 0\}$$

  which has dimension $4k^2 - 1$.

16.3 Type B

There are no Dynkin diagram automorphisms, so all forms are inner. Furthermore $\mathrm{SO}(2n+1)$ has trivial center, so $\theta \in \mathrm{SO}(2n+1)$ of order 2. We know what all these elements look like (up to conjugation); we'll have $\theta = (-\operatorname{Id})_{2p} \oplus \operatorname{Id}_{2q+1}$ with $p + q = n$. The corresponding real form is $\mathfrak{so}(2p, 2q+1)$, $p = 0, \ldots, n$ (all distinct).

Holiday on Tuesday. Lecture on Thursday at MIT.

17 Lecture 17 (4/22)

Today we finish the classification of real forms.

17.1 Last time

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be an anti-involution giving rise to the real form $\mathfrak{g}^\sigma$. We single out the compact form $\mathfrak{g}^c$ obtained when $\sigma = \omega$ is the **Cartan involution**

$$\omega(e_i) = -f_i \text{ and } \omega(h_i) = -h_i \text{ and } \omega(f_i) = -e_i.$$

Characterizing σ in terms of how much it differs from the compact form led us to characterizing real forms in terms of an involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Given, θ, we write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $\mathfrak{k}$ the $(+1)$-eigenspace and $\mathfrak{p}$ the (-1)-eigenspace of θ. We intersect with $\mathfrak{g}^c$ to write $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$, and then $\mathfrak{g}^\sigma = \mathfrak{k}^c + i\mathfrak{p}^c$. Elements are $\mathfrak{k}$ are skew-Hermitian so expoentiate to unitary operators so we call $\mathfrak{k}^c$ the **compact directions**, while $i\mathfrak{p}^c$ has hermitian elements expoentiating to hermitian operators so we call these the **noncompact directions** (maybe typos in this sentence).

We also found a θ-stable Cartan subalgebra. While doing this, we had a lemma which we did not prove.

Recall 17.1. We chose $\mathfrak{h}_+^c \subset \mathfrak{k}^c$ and $\mathfrak{h}_-^c \subset \mathfrak{p}^c$. Then formed $\mathfrak{h}^c = \mathfrak{h}_+^c \oplus \mathfrak{h}_-^c$ and extended $\mathbb{C}$-linearly to get $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$.

Lemma 17.2. *$\mathfrak{h}_-$ does not contain any coroots of $\mathfrak{g}$ (w.r.t. $\mathfrak{h}$)*

Proof. If $\alpha^\vee$ is a coroot in $\mathfrak{h}_-$, then $\theta(\alpha^\vee) = -\alpha^\vee$, so $\theta(e_\alpha) = e_{-\alpha}$ and vice versa.[35] Therefore, $e_\alpha + e_{-\alpha} \in \mathfrak{k}$. Furthermore, $[\mathfrak{h}_+, e_\alpha + e_{-\alpha}] = 0$ since $\alpha|_{\mathfrak{h}_+} = 0$ (since $\alpha \in \mathfrak{h}_-^*$). We also know $(e_\alpha, \mathfrak{h}_+) = 0 = (e_{-\alpha}, \mathfrak{h}_+)$ so $e_\alpha + e_{-\alpha} \in \mathfrak{h}_+$, a contradiction (since $\mathfrak{h}_+$ maximal commutative subalgebra of $\mathfrak{k}$). ■

Question: Why?

17.2 Classification of real forms

- A_{n-1}
  - Compact inner class
    These are all $\mathfrak{su}(p,q)$ with $p+q=n$ and $p \geq q$.
  - Split inner class
    These are $\mathfrak{sl}_n(\mathbb{R})$ and $\mathfrak{sl}_n\left(\frac{n}{2}, \mathbb{H}\right)$ (when n even)
- B_n
  - Compact inner class (only one since Dynkin diagram has no nontrivial auto)
    $\mathfrak{so}(2p+1, 2q)$ where $p+q=n$.

This is where we stopped last time, so let's continue.

[35] normalize things so the coefficient is 1

- C_n

  Dynkin diagram has no nontrivial auto, so only one inner class. θ will be inner, so $\theta \in \mathrm{Sp}_{2n}(\mathbb{C})/\pm 1$ and $\theta^2 = 1$. Thus, $\theta(x) = gxg^{-1}$ where $g \in \mathrm{Sp}_{2n}$ and $g^2 = \pm 1$.

  - $g^2 = 1$

    We may write $V = \mathbb{C}^{2n} = V(1) \oplus V(-1)$. These eigenspaces each carry a symplectic form, so they are even dimensional. Hence $\dim V(1) = 2p$ and $\dim V(-1) = 2q$ with $p + q = n$. May assume $p \geq q$ (change $g \rightsquigarrow -g$). In this case, one finds

    $$g^\sigma = \mathfrak{u}(p, q, \mathbb{H}),$$

    the **quaternionic unitary Lie algebra**, the Lie algebra of symmetries of a quaternionic Hermitian form of signature (p, q). Can calculate that in this case, $\mathfrak{k} = \mathfrak{sp}_{2p} \oplus \mathfrak{sp}_{2q}$.

  - $g^2 = -1$

    In this case, we write $V = \mathbb{C}^{2n} = V(i) \oplus V(-i)$ with each eigenspace isotropic. This forces $V(\pm i)$ to be Lagrangian, both of dimension n. In this case, $\mathfrak{k} = \mathfrak{gl}_n$, and in obtains the split form $\mathfrak{g}^\sigma = \mathfrak{sp}_{2n}(\mathbb{R})$.

- D_n

  Split form is $\mathfrak{so}(n, n)$ so could be in either class depending on parity of n

  - Compact inner class

    We have $\theta \in G_{\mathrm{ad}} = \mathrm{SO}(2n)/\pm 1$ with $\theta^2 = 1$. Thus, $\theta(x) = gxg^{-1}$ where $g \in \mathrm{SO}(2n)$ and $g^2 = \pm 1$.

    * $g^2 = 1$

      Again write $V = \mathbb{C}^{2n} = V(1) \oplus V(-1)$. We need $\det g = 1$, so $\dim V(-1) =$ even. Hence, $\dim V(1) = 2p$ and $\dim V(-1) = 2q$ with $p + q = n$. Again, may assume $p \geq q$. In this case, $\mathfrak{k} = \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2q}$ and $\mathfrak{g}^\sigma = \mathfrak{so}(2p, 2q)$.

    * $g^2 = -1$

      Again $\mathbb{C}^{2n} = V(i) \oplus V(-i)$. These are Lagrangian as before, so $\dim V(i) = n = \dim V(-i)$. Thus, in this case $k = \mathfrak{gl}_n$ and one gets $\mathfrak{g}^\sigma = \mathfrak{so}^*(2n)$, the Lie algebra of symmetries of a skew-Hermitian quaternionic form

  - Other inner class

    Same story except $\theta \in O(n)/\pm 1$ so $\theta(x) = gxg^{-1}$ with $\det(g) = -1$. Note that we cannot have $g^2 = -1$ since that would imply $V = V(i) \oplus V(-i)$ both Lagrangian so $\det g = 1$. Thus, we have $g^2 = 1$ so $V = V(1) \oplus V(-1)$ with $\dim V(1) = 2p + 1$ and $\dim V(-1) = 2q - 1$ (and $q \leq p + 1$). One gets $\mathfrak{k} = \mathfrak{so}(2p + 1) \oplus \mathfrak{so}(2q - 1)$ and $\mathfrak{g}^\sigma = \mathfrak{so}(2p + 1, 2q - 1)$.

Remark 17.3. For real numbers, have symmetric and skew-symmetric forms. Symmetric have signature, but all skew-symmetric are the same.

For complex numbers, have Hermitian and skew-Hermitian. These are the same (multiply by i), and they have signature.

For quaternions, Hermitian and skew-Hermitian are again different.

Class	Real forms
A_{n-1} compact inner class	$\mathfrak{su}(p,q)$ with $p \geq q$ and $p+q=n$
A_{n-1} split inner class	$\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}(n/2,\mathbb{H})$ if n even
B_n	$\mathfrak{so}(2p+1,2q)$ with $p+q=n$
C_n	$\mathfrak{u}(p,q,\mathbb{H})$ with $p+q=n$ and $p \geq q$, $\mathfrak{sp}_{2n}(\mathbb{R})$
D_n compact inner class	$\mathfrak{so}(2p,2q)$ with $p+q=n$ and $p \geq q$, $\mathfrak{so}^*(2n)$
D_n other inner class	$\mathfrak{so}(2p+1,2q-1)$ with $p+q=n$ and $q \leq p+1$
G_2	compact and split
F_4	compact, split ($\mathfrak{k} = \mathfrak{sp}(6) \oplus \mathfrak{sl}(2)$), and the other ($\mathfrak{k} = \mathfrak{so}(9)$)
E_6 split inner class	split ($\mathfrak{k} = \mathfrak{sp}(8)$) and other ($\mathfrak{k} = F_4$)

Table 1: Real forms of simple complex Lie algebras (except E_6, E_7, E_8)

Example ($D_3 = A_3$)**.** Here, we can match up the real forms

$$\begin{aligned} \mathfrak{so}(6) &= \mathfrak{su}(4) \\ \mathfrak{so}(4,2) &= \mathfrak{su}(2,2) \quad \text{quasi-split} \\ \mathfrak{so}^*(6) &= \mathfrak{sl}(2,\mathbb{H}) \\ \mathfrak{so}(3,3) &= \mathfrak{sl}_4(\mathbb{R}) \quad \text{split} \\ \mathfrak{so}(5,1) &= \mathfrak{su}(3,1) \end{aligned}$$

We should also talk about exceptional Lie algebras. When dealing with these, one should consider Vogan diagrams. Recall that these are formed by paring up vertices transposed by the involution, and coloring the fixed vertices black or white.

- For black vertices, we set $\theta(e_i) = -e_i$
- For white vertices, we set $\theta(e_i) = +e_i$

Every real form gives rise to such a diagram, but there are some redundancies/equivalence relations. Note that, of the exceptional diagrams, only E_6 has automorphisms, so for the rest of them, we are just coloring each vertex black or white.

Let's consider the case when the automorphism of the Dynkin diagram is trivial (i.e. the compact inner class), so the Vogan diagram is simply the Dynkin diagram + coloring.

What are the equivalences? First note that the compact form = all white vertices ($\theta = \text{id}$) and so no other diagram gives the compact form. Hence we consider only diagrams have ≥ 1 black vertex.

Say we have $\theta \in G_{\text{ad}}$ giving our real form. This fixes all $\mathfrak{g}_\alpha$'s (trivial aut on Dynkin), so $\theta \in H := \exp(\mathfrak{h})$. We color vertex i white if $\alpha_i(\theta) = 1$ and black if $\alpha_i(\theta) = -1$. Recall the Weyl group sits in an exact sequence

$$1 \longrightarrow H \longrightarrow N(H) \longrightarrow W \longrightarrow 1.$$

Hence, we may modify θ by action of W. What do simple reflections do? Note that[36] (since $\alpha_k(\theta) = \pm 1$)

$$\alpha_j(s_i(\theta)) = s_i(\alpha_j)(\theta) = (\alpha_j - a_{ij}\alpha_i)(\theta) = \alpha_j(\theta) \cdot \alpha_i(\theta)^{-a_{ij}} = \begin{cases} \alpha_j(\theta) & \text{if } a_{ij} \text{ even } (\pm 2) \\ \alpha_j(\theta)\alpha_i(\theta) & \text{if } a_{ij} \text{ odd.} \end{cases}$$

[36] $s_i(\theta) := \widetilde{s}_i \theta \widetilde{s}_i^{-1}$ for some $\widetilde{s}_i \in N(H)$ lifting $s_i \in W$

Thus we get the follow equivalence relation: if we have a black vertex, then we can change the signs of all its neighbors except $\bullet \leftleftarrows \circ$ or $\bullet \leftleftarrows \bullet$ (and the color of the vertex itself doesn't change).[37]

Example (G_2)**.** The Dynkin diagram G_2 has four configurations

$$(\bullet, \bullet), \ (\bullet, \circ), \ (\circ, \bullet), \ \text{and} \ (\circ, \circ).$$

The last one is the compact form. The other three are all equivalent so must correspond to the split form. One can show that $\mathfrak{k}$ is the span of the long roots and then that $\mathfrak{k} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (for the split form). It must have rank 2 (same as G_2) and dimension $6 = |R_+|$.

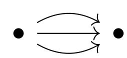


Figure 10: The Dynkin Diagram G_2

Example (F_4)**.** Now let's consider the F_4 case. Here we have the configurations (up to equivalence)

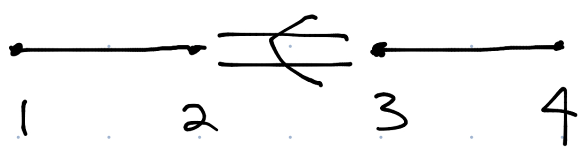


Figure 11: A Dynkin diagram of type F_4

$$(\circ, \circ, \circ, \circ), \ (\bullet, \circ, \circ, \circ), \ (\circ, \circ, \circ, \bullet), \ \text{and} \ (\circ, \bullet, \bullet, \bullet).$$

In fact, even two of these are equivalent. The last two are the same (since the right half can't be affect by the left half?). The first one is the compact form. What do the other two look like?

The roots of F_4 are $(\pm 1, 0, 0, 0)$ and all its permutations, $(\pm 1, \pm 1, 0, 0)$ and all its permutations, and $\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right)$ (for a total of $8+24+16 = 48$ roots). The $(\pm 1, 0, 0, 0)$ and $(\pm 1, \pm 1, 0, 0)$ roots generate an $\mathfrak{so}_9$ while the $(\pm 1/2, \dots, \pm 1/2)$ roots give the spinor representation S. Thus, $F_4 = \mathfrak{so}_9 \oplus S$.

In the second case $(\bullet, \circ, \circ, \circ)$, one can check that θ acts by 1 on $\mathfrak{so}(9)$ and by -1 on S, so $\mathfrak{k} = \mathfrak{so}(9)$. Hence, this will not give the split form sin $|R_+| = 24 \neq 32 = \dim \mathfrak{so}(9)$.

Thus, $(\circ, \circ, \circ, \bullet)$ gives the split form. Note that here you can observe an $\mathfrak{sp}(6)$ as the subdiagram using vertices $1, 2, 3$ (this is a copy of C_3).

This just leaves diagrams E_6, E_7, E_8.

17.2.1 Type E

We start with the E_6 split inner class. This corresponds to the nontrivial automorphism so only two

TODO: Add picture

[37]So change the colors of all neighbors of the black vertex except the neighbors with a double arrow coming into the black one.

vertices get colored. They can be colored

$$(\circ,\circ),\ \ (\circ,\bullet),\ \ (\bullet,\circ),\ \ \text{or}\ \ (\bullet,\bullet).$$

The last three colorings are equivalent, so there are only 2 real forms in this class (neither compact). In the first case $(\circ,\circ)$, you can check that $\mathfrak{k}$ gives a copy of F_4. This is not the split form (we call it E_6^1 instead) since $\dim F_4 = 52 \neq 36 = \#R_+$. Simple root generators of $\mathfrak{k}$ are $e_1+e_5, e_2+e_4, e_3, e_6$ (these all obviously satisfy $\theta(e)=e$ and one can check that they in fact generate $\mathfrak{k}$ as a Lie algebra); the Cartan algebra will be spanned by $h_1+h_5, h_2+h_4, h_3, h_6$.

18 Lecture 18 (4/27)

We were working on classifying real forms of Lie algebras last time. We filled out much of Table 1, but we still need to handle the exception Lie algebras of E type. We will complete the table this time, forming Table 2.

18.1 E type

We looked at the split inner class of E_6 last time. This is corresponding to the nontrivial automorphism, so there are only two colored vertices. In fact, there are only two equiv classes of colorings: $(+,+)$ and $\{(+,-),(-,+),(-,-)\}$ ($+$ is colored white). These correspond to E_6^1 with $\mathfrak{k}=F_4$ and E_6^{spl} with $\mathfrak{k}=\mathfrak{sp}(8)=C_4$.

This brings us to the compact inner class. Hence, the Vogan diagram is the Dynkin diagram with white and black vertices. We will be able to treat E_6, E_7, E_8 more-or-less simultaneously. If all vertices are white, we get the compact forms E_6^c, E_7^c, E_8^c. Hence, we may restrict ourselves to the case when we have at least 1 black vertex.

By applying equivalence transformations (i.e. change colors of neighbors of a black/- vertex), we can

Class	Real forms
A_{n-1} compact inner class	$\mathfrak{su}(p,q)$ with $p\geq q$ and $p+q=n$
A_{n-1} split inner class	$\mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{sl}(n/2,\mathbb{H})$ if n even
B_n	$\mathfrak{so}(2p+1,2q)$ with $p+q=n$
C_n	$\mathfrak{u}(p,q,\mathbb{H})$ with $p+q=n$ and $p\geq q$, $\mathfrak{sp}_{2n}(\mathbb{R})$
D_n compact inner class	$\mathfrak{so}(2p,2q)$ with $p+q=n$ and $p\geq q$, $\mathfrak{so}^*(2n)$
D_n other inner class	$\mathfrak{so}(2p+1,2q-1)$ with $p+q=n$ and $q\leq p+1$
G_2	compact and split ($\mathfrak{k}=\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$)
F_4	compact, split ($\mathfrak{k}=\mathfrak{sp}(6)\oplus\mathfrak{sl}(2)$), and the other ($\mathfrak{k}=\mathfrak{so}(9)$)
E_6 split inner class	split ($\mathfrak{k}=\mathfrak{sp}(8)$) and E_6^1 ($\mathfrak{k}=F_4$)
E_6 compact inner class	compact, E_6^2 ($\mathfrak{k}=\mathfrak{so}(10)\times\mathfrak{so}(2)$), and E_6^3 ($\mathfrak{k}=\mathfrak{sl}(6)\times\mathfrak{sl}(2)$)
E_7	compact, split ($\mathfrak{k}=\mathfrak{sl}(8)$), E_7^1 ($\mathfrak{k}=E_6\oplus\mathfrak{so}(2)$), E_7^2 ($\mathfrak{k}=\mathfrak{so}(12)\oplus\mathfrak{sl}(2)$)
E_8	compact, split ($\mathfrak{k}=\mathfrak{so}(16)$), E_8^1 ($\mathfrak{k}=E_7\times\mathfrak{sl}(2)$)

Table 2: Real forms of all simple complex Lie algebras

achieve

$$\begin{array}{ccccccccccc} (?) & \text{———} & (?) & \text{———} & (-) & \text{———} & (?) & \text{———} & \cdots & \text{———} & (?) \\ & & & & | & & & & & & \\ & & & & (?) & & & & & & \end{array}$$

i.e. force the vertex above the branch to be a minus. Similarly, we can then achieve

$$\begin{array}{ccccccccccc} (?) & \text{———} & (?) & \text{———} & (-) & \text{———} & (?) & \text{———} & \cdots & \text{———} & (?) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

(now can change color of nodal vertex whenever we want).

Now, focus on the 'right leg' (strictly to the right of the $(-)$ above).

- For E_6, this is $(?) - (?)$ so has colorings $++$ or $\{+-, -+, --\}$. This means we get 2 classes on the right. Now, the nodal transformation actually turns $++$ into $-+$, so the two classes on the right are one in the same (?).

  The 'node' is the valence 3 vertex

  Sounds like one ends up with the real forms $\mathfrak{su}(3)$ and $\mathfrak{su}(2,1)$ for this right leg.

- For E_7, the right leg is $(?) - (?) - (?)$. You end up with the forms $\mathfrak{su}(4), \mathfrak{su}(3,1)$, and $\mathfrak{su}(2,2)$. To get $\mathfrak{su}(4)$ you use $+++$. For $\mathfrak{su}(3,1)$, you use $\{-++, --+, +--, ++-\}$, and for $\mathfrak{su}(2,2)$ you can use $\{-+-, ---, +-+\}$.

  Example. Consider $++-$. These are the values of an involution on simple roots. Something liek these signs correspond to ratios of adjacent values so $++-$ gives

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

  which gives $\mathfrak{su}(3,1)$ while e.g. $+-+$ gives

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

  which gives $\mathfrak{su}(2,2)$.

- For E_8, end up with $\mathfrak{su}(5), \mathfrak{su}(4,1), \mathfrak{su}(3,2)$. One has

  ($\mathfrak{su}(5)$) $++++$

  ($\mathfrak{su}(4,1)$) $-+++$, $--++$, etc.

  ($\mathfrak{su}(3,2)$) $+-++$, etc.

Note that the nodal transformation take us between these classes, so they are again all equivalent (?)

Once the above is understood, the upshot is that we can arrange

$$\begin{array}{ccccccccccc} (?) & \text{——} & (?) & \text{——} & (-) & \text{——} & (-) & \text{——} & \cdots & \text{——} & (-) \\ & & & & | & & & & & & \\ & & & & (?) & & & & & & \end{array}$$

i.e. the right leg (+ the node) are all $-$'s (though the bottom vertex may become a + when applying the nodal transformation).

If we really want the bottom vertex to be minus, we may arrange

$$\begin{array}{ccccccccccccc} (?) & \text{——} & (?) & \text{——} & (-) & \text{——} & (-) & \text{——} & (-) & \text{——} & \cdots & \text{——} & (-) \\ & & & & | & & & & & & & & \\ & & & & (-) & & & & & & & & \end{array}$$

or

$$\begin{array}{ccccccccccccc} (?) & \text{——} & (?) & \text{——} & (-) & \text{——} & (+) & \text{——} & (-) & \text{——} & \cdots & \text{——} & (-) \\ & & & & | & & & & & & & & \\ & & & & (-) & & & & & & & & \end{array}$$

(i.e. a single +).

One knows that $+- \sim --$ and $+---- \sim ----$ so these two cases are actually the same for E_6 and E_8! However, they are inequivalent for E_7. This simplifies things a lot from when we began.

(E_6) We may arrange

$$\begin{array}{ccccccccc} (?) & \text{——} & (?) & \text{——} & (-) & \text{——} & (-) & \text{——} & (-) \\ & & & & | & & & & \\ & & & & (-) & & & & \end{array}$$

so the only two classes are $++$ on the left leg and $+-$ on the left leg. Hence, there are at most 2 non-compact real forms in the compact inner class. To finish the classification, we just produce 2 these two forms.

– Could consider

$$\begin{array}{ccccccccc} (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (-) \\ & & & & | & & & & \\ & & & & (+) & & & & \end{array}$$

Looking at all the $+$'s, we see that $\mathfrak{so}(10) \subset \mathfrak{k}$. Note that the root α_1 (the sole $-$) is miniscule. Hence, any positive root either does not contain α_1 or contains it with coefficient 1. It it does not, we are in the D_5 (the $\mathfrak{so}(10)$). We see that $\mathfrak{k} = \mathfrak{so}(10) \times \mathfrak{gl}(1) = \mathfrak{so}(10) \times \mathfrak{so}(2)$. We will call this form E_6^2.

Question: Why?

– Another possibility is

$$\begin{array}{ccccccccc} (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) \\ & & & & | & & & & \\ & & & & (-) & & & & \end{array}$$

We see that $\mathfrak{k}$ contains A_5, i.e. $\mathfrak{k} \supset \mathfrak{sl}(6)$. The remaining weight is not miniscule, so there will be other toos in the Lie algebra. One can show that $\mathfrak{k} = \mathfrak{sl}(6) \times \mathfrak{sl}(2)$ (homework). We will call this form E_6^3.

(E_8) We may arrange

$$\begin{array}{ccccccccccccc} (?) & \text{——} & (?) & \text{——} & (-) & \text{——} & (-) & \text{——} & (-) & \text{——} & (-) & \text{——} & (-) \\ & & & & | & & & & & & & & \\ & & & & (-) & & & & & & & & \end{array}$$

so only two classes for left leg, $++$ and $+-$. Hence, again at most 2 remaining real forms, so enough to product them.

– Consider

$$\begin{array}{ccccccccccccc} (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (-) \\ & & & & | & & & & & & & & \\ & & & & (+) & & & & & & & & \end{array}$$

We have $E_7 \subset \mathfrak{k}$, so this is not the split form ($\dim E_7 = 133$, $\dim E_8 = 248$, and $\dim \mathfrak{k}_{\text{spl}} = 120$). Note that E_8 has no miniscule weights. One can show (homework) that in this case $\mathfrak{k} = E_7 \times \mathfrak{sl}_2$. We call this form E_8^1.

– Second option is

$$\begin{array}{ccccccccccccc} (-) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) & \text{——} & (+) \\ & & & & | & & & & & & & & \\ & & & & (+) & & & & & & & & \end{array}$$

We see $D_7 \subset \mathfrak{k}$. Can show $\mathfrak{k} = D_8 = \mathfrak{so}(16)$, and that this is the split form. Sanity check: $\dim \mathfrak{k} = \binom{16}{2} = 120$.

Remark 18.1. $E_8 = \mathfrak{so}(16) \oplus R$ where $R = \mathfrak{p}$ is a 128-dimensional representation of $\mathfrak{k}$, the spinor representation S_+ (or S_-)

Thus, there are 3 real forms of E_8.

This just leaves E_7. There are a priori 4 variants, but two of them will be equivalent. Specifically,

$$\begin{array}{ccccccccccc} (+) & \text{---} & (-) & \text{---} & (-) & \text{---} & (+) & \text{---} & (-) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

is equivalent to (apply transformation to first $-$ from the left and then to leftmost vertex)

$$\begin{array}{ccccccccccc} (-) & \text{---} & (+) & \text{---} & (+) & \text{---} & (+) & \text{---} & (-) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

which is equiv to (apply transformation to bottom vertex)

$$\begin{array}{ccccccccccc} (-) & \text{---} & (+) & \text{---} & (-) & \text{---} & (+) & \text{---} & (-) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

which is equiv to (apply to nodal vertex)

$$\begin{array}{ccccccccccc} (-) & \text{---} & (-) & \text{---} & (-) & \text{---} & (-) & \text{---} & (-) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (+) & & & & & & \end{array}$$

which is equiv to (second from left)

$$\begin{array}{ccccccccccc} (+) & \text{---} & (-) & \text{---} & (+) & \text{---} & (-) & \text{---} & (-) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (+) & & & & & & \end{array}$$

which is equiv to (right of node)

$$\begin{array}{ccccccccccc} (+) & \text{---} & (-) & \text{---} & (-) & \text{---} & (-) & \text{---} & (+) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (+) & & & & & & \end{array}$$

which is equiv to (node)

$$\begin{array}{ccccccccccc} (+) & \text{---} & (+) & \text{---} & (-) & \text{---} & (+) & \text{---} & (+) & \text{---} & (-) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

which is equiv to (rightmost vertex)

$$\begin{array}{ccccccccccc} (+) & — & (+) & — & (-) & — & (+) & — & (+) & — & (-) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

The upshot is that when we have a + on the right leg, all configurations of the left leg are equivalent. Thus, there are only ≤ 3 possible non-compact real forms of E_7. These will all be different:

- First consider

  $$\begin{array}{ccccccccccc} (+) & — & (+) & — & (+) & — & (+) & — & (+) & — & (-) \\ & & & & | & & & & & & \\ & & & & (+) & & & & & & \end{array}$$

  We have $E_6 \subset \mathfrak{k}$. This is *not* split since $\dim E_6 = 78$ but $\dim \mathfrak{k}_{spl} = \#R_+ = \frac{1}{2}(\dim E_7 - 7) = 63$. The $-$ root above is miniscule. One gets that $\mathfrak{k} = E_6 \oplus \mathfrak{so}(2)$. We call this E_7^1. It is the "most compact" of the non-compact real forms ($\dim \mathfrak{k}$ maximal).

- Now consider

  $$\begin{array}{ccccccccccc} (-) & — & (+) & — & (+) & — & (+) & — & (+) & — & (+) \\ & & & & | & & & & & & \\ & & & & (+) & & & & & & \end{array}$$

  Here we have $D_6 = \mathfrak{so}(12) \subset \mathfrak{k}$ so $\dim \mathfrak{k} \geq \dim D_6 = \binom{12}{2} = 66 > 63$, so this is still not split. The $-$ root is not miniscule. One can show that $\mathfrak{k} = \mathfrak{so}(12) \oplus \mathfrak{sl}(2)$. We denote this by E_7^2.

- Finally, there is the split form

  $$\begin{array}{ccccccccccc} (+) & — & (+) & — & (+) & — & (+) & — & (+) & — & (+) \\ & & & & | & & & & & & \\ & & & & (-) & & & & & & \end{array}$$

  In this case, we have $\mathfrak{sl}(7) \subset \mathfrak{k}$, but in fact $\mathfrak{k} = \mathfrak{sl}(8)$ of dimension $8^2 - 1 = 63$ as it should be.

Thus, there are 4 real forms of E_7.

We now know all real semisimple Lie algebras. They're listed in Table 2.

18.2 Classification of connected compact Lie groups

Proposition 18.2 (Homework)**.** *If K is a compact Lie group, then $\mathfrak{k} := \mathrm{Lie}\, K$ is reductive, i.e. $\mathfrak{k} = \mathfrak{k}_{ss} \oplus \mathfrak{k}_{ab}$ is a sum of a semisimple Lie algebra plus an abelian Lie algebra.*

18.2.1 Classification of semisimple compact Lie groups

Definition 18.3. We say G is **semisimple** if $\mathrm{Lie}\, G$ is semisimple.

Lemma 18.4. *Let X be a compact manifold. Then, $\pi_1(X)$ is finitely generated.*

Proof idea. Cover X by (finitely many) small balls. Connect the centers of all these balls (use straight lines in local coordinates or choose a Riemannian metric and then use geodesics); this gives a finite graph Γ. Then, $\pi_1(\Gamma)$ is finitely generated (with # generates at most number of loops in Γ), and $\pi_1(\Gamma) \twoheadrightarrow \pi_1(M)$. This is because any closed path from a vertex x_0 to itself can be deformed to a graph walk. ■

Theorem 18.5. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and let G^c_{ad} be the compact adjoint group. Then, $\pi_1(G^c_{\mathrm{ad}}) = P^\vee/Q^\vee$ is a finite group of order* $\det(\mathit{Cartan})$. *In particular, the universal cover $\widetilde{G^c_{\mathrm{ad}}}$ is also a compact Lie group.*

Proof. Let $K \twoheadrightarrow G^c_{\mathrm{ad}}$ be a finite cover, so K a compact connected Lie group. Let $Z = \ker(K \to G^c_{\mathrm{ad}})$. K is compact, and its f.d. irreps are a subset of the f.d. irreps of $\mathrm{Lie}\, K_{\mathbb{C}} = \mathfrak{g}$ (since K connected by fundamental theorems), i.e. they are L_λ for $\lambda \in S$ where $P_+ \cap Q \subset S \subset P_+$. Note that the representations of G^c_{ad} are in bijection with $P_+ \cap Q$.

Question: Why?

Answer: See beginning of tomorrow's lecture

Z acts by scalar χ_λ on each L_λ. Since $L_{\lambda+\mu} \subset L_\lambda \otimes L_\mu$, we see that $\chi_{\lambda+\mu} = \chi_\lambda \chi_\mu$. Also $\chi_\lambda = 1$ for $\lambda \in Q$ (reps of G^c_{ad}). This implies that χ_λ depends only on λ mod Q, so get $\chi : P/Q \to Z^\vee$ (with $Z^\vee$ the character group). Now, Peter-Weyl says χ is surjective (all characters of Z must occur in $L^2(K) = \bigoplus_{\lambda \in S} L_\lambda \otimes L_\lambda^*$). The dual map gives an embedding $Z \hookrightarrow (P/Q)^\vee = P^\vee/Q^\vee$. Thus, you cannot have covers of big degree.

We next show $\pi_1(\mathrm{ad} G^c)$ is finite. We know it is finitely generated and abelian, so it is of the form $\mathbb{Z}^r \oplus F$ for F some finite group. Let $\Gamma \leq \pi_1(G^c_{\mathrm{ad}})$ be a subgroup of index N. This gives an N-sheeted covering $K \twoheadrightarrow G^c_{\mathrm{ad}}$ with kernel $Z = \pi_1/\Gamma$, so $|Z| = N$ and $Z \hookrightarrow P^\vee/Q^\vee$, so $N \leq |P^\vee/Q^\vee|$. Thus, we must have $r = 0$, so $\pi_1(\mathrm{ad} G^c) = F$.

Hence, $K = \widetilde{G^c_{\mathrm{ad}}}$ is compact and $\mathrm{Rep}\widetilde{G^c_{\mathrm{ad}}} = \mathrm{Rep}\mathfrak{g} = \langle L_\lambda : \lambda \in P \rangle$, so we must have $Z = P^\vee/Q^\vee = (P/Q)^\vee$. ■

Corollary 18.6.

(1) *If $\mathfrak{g}$ is a simple complex Lie algebra, then the simply connected group G^c with $\mathrm{Lie}\, G^c = \mathfrak{g}^c$ is compact with center $P^\vee/Q^\vee$.*

(2) *Let $\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i$ be a semisimple complex Lie algebra. Let G^c_i be the corresponding simply connected compact Lie groups, and let $Z_i = P_i^\vee/Q_i^\vee$. Then, any connected Lie group with Lie algebra $\mathfrak{g}^c$ is compact, and of the form*

$$\frac{\prod_{i=1}^n G^c_i}{Z} \quad \text{with } Z \subset Z_1 \times \ldots \times Z_n.$$

Hence, any semisimple connected compact Lie group is of this form.

Definition 18.7. A Lie group G is **simple** if $\mathrm{Lie}\, G$ is simple.

Example. $\mathrm{SU}(2)$ is a simple Lie group even though it has the nontrivial normal subgroup $\mathbb{Z}/2\mathbb{Z}$.

Remark 18.8. Abelian connected compact Lie groups are simply tori $(S^1)^n$. Their universal cover is $\mathbb{R}^n$ so $G = \mathbb{R}^n/L$ with L discrete, so $G = (S^1)^m \times \mathbb{R}^{n-m}$ and compactness forces $n = m$.

Corollary 18.9. *Any connected compact Lie group is the quotient of $T \times K$ by a finite central subgroup, where T is a torus and K is semisimple and simply connected.*

Proof. Let L be a connected, compact Lie group, and set $\mathfrak{l} = \operatorname{Lie} L$. We write $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{k}$ with $\mathfrak{t}$ abelian and $\mathfrak{k}$ semisimple. Let $T = \exp(\mathfrak{t}) \subset L$, a Lie subgroup. Note that $\operatorname{Lie} \overline{T} \subset \mathfrak{z}(\mathfrak{l}) = \mathfrak{t}$ so $T = \overline{T}$ is closed, so compact, so a torus. Similarly define $K = \exp(\mathfrak{k})$ which is also closed (K compact by previous theorem). The natural map $T \times K \to L$ is a surjective submersion, so a finite covering, so $Z = \ker(T \times K \to L)$ is finite central and $L = (T \times K)/Z$. ■

19 Lecture 19 (4/29)

19.1 Filling in a gap

We start by filling in a gap in the proof at the end of last time. We need to explain why representations of G^c_{ad} are related to dominant weights in the root lattice.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra, and let G be a connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\pi : G \to G_{\text{ad}}$ be the natural covering map, and let $Z = \ker \pi$. Hence, $Z = Z(G) \cong \pi_1(G_{\text{ad}})$ is the center of G (and fundamental group of G_{ad}).

Recall 19.1. f.dim reps of G are in bijection with f.d. representations of $\mathfrak{g}$ (since G simply connected). In particular, irreducible ones are the L_λ with $\lambda \in P_+$.

The center Z will act on L_λ by scalars, i.e. via a character $\chi_\lambda : Z \to \mathbb{C}^\times$. Since $L_{\lambda+\mu} \subset L_\lambda \otimes L_\mu$, we see that $\chi_{\lambda+\mu} = \chi_\lambda \chi_\mu$. Thus, more generally,

$$\chi_{\sum k_i \omega_i} = \prod_i \chi_{\omega_i}^{k_i}.$$

Thus, χ extends to a group homomorphism $\chi : P \to \operatorname{Hom}(Z, \mathbb{C}^\times)$, $\lambda \mapsto \chi_\lambda$.

Let θ be a maximal root, so $L_\theta = \mathfrak{g}$ is the adjoint rep (by definition of θ). Here, Z acts trivially, so $\chi_\theta = 1$.

Recall 19.2 (Exercise 31.10 in the notes)**.** If $\lambda(h_i)$ are large enough, then for all roots $\alpha \in R$, $L_{\lambda+\alpha} \subset L_\lambda \otimes \mathfrak{g}$.

(More specifically, this follows from $\operatorname{Hom}(L_\mu, L_\lambda \otimes V) = \left\{ v \in V[\mu - \lambda] : e_i^{\lambda(h_i)+1} v = 0 \right\}$)

In our case, $V = \mathfrak{g}$ and $\mu - \lambda = \alpha$, so $V[\mu - \lambda] = \mathfrak{g}_\alpha$. Thus, $\chi_\lambda \chi_\alpha = \chi_{\lambda+\alpha} = \chi_\lambda$, so $\chi_\alpha = 1$ for all roots $\alpha \in R$. Thus, $\chi|_Q = 1$, so χ really defines a map

$$\chi : P/Q \longrightarrow \operatorname{Hom}(Z, \mathbb{C}^\times),$$

i.e. it gives a pairing $\chi : P/Q \times Z \to \mathbb{C}^\times$. This is what we used in the proof (gives $Z \to \operatorname{Hom}(P/Q, \mathbb{C}^\times) = P^\vee/Q^\vee$).

Remark 19.3. In particular, $\chi|_Q = 1$ tells us that L_λ lifts to a rep of G_{ad} when $\lambda \in Q$.

19.2 Polar decomposition

Recall 19.4 (Linear algebra)**.** Let A be a complex invertible matrix. Then, it can be uniquely written in the form

$$A = UR \text{ where } U \text{ unitary \ and } R > 0,$$

i.e. R positive Hermitian.

Example. For $a \in \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$, this is $a = re^{i\theta}$.

Remark 19.5. Also, every matrix is sum of a Hermitian matrix with a skew-Hermitian one. Take real part + i(imaginary part). Real part Hermitian and i(imaginary part) skew-Hermitian.

Proof of Recall. Take $R = (A^\dagger A)^{1/2}$ (note $A^\dagger A$ positive Hermitian, so can take square root). Then, $U = A(A^\dagger A)^{-1/2}$. This gives existence. For uniqueness, say $U_1 R_1 = U_2 R_2$. Then, $U_2^{-1} U_1 R_1 = R_2$. Let $U = U_2^{-1} U_1$, so $UR_1 = R_2$. Take adjoint to see $R_1 U^{-1} = R_2$ and from this conclude that $U = \mathrm{Id}$. ■

We want to generalize this to any real semisimple group. Let $\mathfrak{g}^\sigma \subset \mathfrak{g}$ be a real form of $\mathfrak{g}$ with corresponding Lie group $G^\sigma \subset G_{\mathrm{ad}}$. Note this is a closed subgroup (if not, closure has a larger Lie algebra, but every element of it still fixed by σ). Recall the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \;\; \mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c, \;\text{ and }\; \mathfrak{g}^\sigma = \mathfrak{k}^c \oplus i\mathfrak{p}^c.$$

Let $K^c \subset G^c_{\mathrm{ad}}$ be the (closed) subgroup with $\operatorname{Lie} K^c = \mathfrak{k}^c$. Define

$$P^\sigma = \exp(i\mathfrak{p}^c) \subset G^\sigma.$$

Warning 19.6. This is not a group in general, e.g. since $\mathfrak{p}^c$ is not a Lie algebra but a module over $\mathfrak{k}^c$. Alternatively, $\mathfrak{p}^c$ acts by Hermitian matrices, so P^σ does as well, but products of Hermitian matrices need not be Hermitian.

Proposition 19.7. *The exponential map* $\exp : i\mathfrak{p}^c \to P^\sigma$ *is a diffeomorphism.*

Proof. We know that

$$\exp : i\mathfrak{u}(n) \xrightarrow{\sim} \mathrm{Herm}_{>0}(n)$$

onto positive Hermitian matrices is a diffeomorphism. Why? Take log of the eigenvalues to get inverse $\log : \mathrm{Herm}_{>0}(n) \xrightarrow{\sim} i\mathfrak{u}(n)$. The map in the statement is a restriction of this one. ■

Corollary 19.8. $P^\sigma \cong \mathbb{R}^N$ *where* $N = \dim \mathfrak{p}$.

Note that K^σ acts on P^σ by conjugation. Let

$$\mu : K^\sigma \times P^\sigma \longrightarrow G^\sigma$$

be the multiplication map.

Theorem 19.9. μ *is a diffeomorphism.*

Proof. Consider some $g \in G^\sigma \subset \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$. We also have $g^\dagger \in G^\sigma$, so $g^\dagger g$ is a positive definite automorphism. Hence, we can form $R_g := (g^\dagger g)^{1/2} \in \mathrm{Aut}(\mathfrak{g})$.[38] Set $U_g := g(g^\dagger g)^{-1/2}$, so $g = U_g R_g$. We see that $R_g \in P^\sigma$ and $U_g \in K^\sigma$. this gives inverse (bijective since polar decomp unique) to multiplication map $g \mapsto (U_g, R_g)$. This is smooth, so we win. ■

[38] $g^\dagger g$ is diagonalizable so splits Lie algebra into eigenspaces. Simply take square roots of those eigenvalues. That is, under $g^\dagger g$, $\mathfrak{g} = \bigoplus_{\Lambda > 0} \mathfrak{g}_\Lambda$ satisfying $[\mathfrak{g}_{\Lambda_1}, \mathfrak{g}_{\Lambda_2}] = \mathfrak{g}_{\Lambda_1 \Lambda_2}$. We set $R_g|_{\mathfrak{g}_\Lambda} = \sqrt{\Lambda}$ which makes sense since $\sqrt{\Lambda_1 \Lambda_2} = \sqrt{\Lambda_1}\sqrt{\Lambda_2}$

Corollary 19.10. *$G^\sigma \cong K^\sigma \times \mathbb{R}^{\dim \mathfrak{p}} \sim K^\sigma$ with $\cong$ denoting diffeomorphism and $\sim$ denoting homotopy equivalence here.*

Hence topology of semisimple Lie groups largely reduces to topology of compact Lie groups ($K^\sigma \overset{\text{closed}}{\subset} G^c_{\text{ad}}$).

Corollary 19.11. *$G_{\text{ad}} = G^c_{\text{ad}} \times P$ with $P \subset G_{\text{ad}}$ acting on $\mathfrak{g}$ by Hermitian positive operators. Hence,*

$$\pi_1(G_{\text{ad}}) = \pi_1(G^c_{\text{ad}}) = P^\vee/Q^\vee$$

(P here the weight lattice).

Question: Get this corollary by regarding G_{ad} as a real Lie group?

Question: and $P \cong \mathbb{R}^{\dim \mathfrak{p}}$?

Corollary 19.12. *Say G is a semisimple complex Lie group with center $Z = Z(G)$. Then, $Z \subset G^c$, so coincides with the center of G^c.*

In particular, the restriction of f.dim reps from G to G^c is an equivalence.

This generalizes straightforwardly to any complex semisimple Lie group G instead of G_{ad}, i.e. $G = G^c \times P$ and $\text{Rep}G = \text{Rep}G^c$.

Warning 19.13. G and G^c have the same topology, but G and G^σ do not. G^σ's topology is related to that of K^σ. In particular, it can happen that G is simply connected but G^σ is not.

Example. Say $G = \text{SL}_2(\mathbb{C})$ and $G^\sigma = \text{SL}_2(\mathbb{R})$ is its split form. Note that $\text{SL}_2(\mathbb{R}) \supset \text{SO}(2) \cong S^1$, and in fact we have a polar decomposition

$$\text{SL}_2(\mathbb{R}) = \text{SO}(2) \times P$$

with $P \cong \mathbb{R}^2$ consisting of positive symmetric matrices of determinant 1. Thus, $\text{SL}_2(\mathbb{R}) \cong S^1 \times \mathbb{R}^2$ (interior of a bagel). This has universal cover $\widetilde{\text{SL}_2(\mathbb{R})} \cong \mathbb{R}^3$.

Example. Take $G^\sigma = \text{SL}_n(\mathbb{C})$ (regarded as a real Lie group). Then, $K^\sigma = \text{SU}(n)$ and $P^\sigma =$ positive Hermitian matrices of determinant 1. Then, we recover the usual polar decomposition.

Example. If $G^\sigma = \text{SL}_n(\mathbb{R})$, then $K^\sigma = \text{SO}_n$ and P^σ is positive symmetric matrices of det 1. This gives the usual real polar decomposition.

19.3 Linear groups

Let G be a connected real or complex Lie group.

Definition 19.14. We say G is **linear** if it admits a faithful f.dim representation, i.e. it can be realized as a subgroup of GL_n.

Example. Every semisimple complex Lie group is linear. Let $P_G \subset G$ be the **weight lattice of** G (so $\lambda \in P_G \iff L_\lambda|\pi_1(G) = 1$). If P_G/Q is cyclic, we can take λ a generator, and then L_λ will be faithful.

P/Q is cyclic for all reduced irreducible root systems except D_{2n}, where it's $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. For $\mathfrak{so}(4n)$, take λ_1, λ_2 to generate P_G/Q, and then $L = L_{\lambda_1} \oplus L_{\lambda_2}$ is faithful.

We can characterize real linear semisimple Lie groups as well. Say $\mathfrak{g}^\sigma \subset \mathfrak{g}$ is a real form with corresponding Lie group $G^\sigma \subset G$. Then, G^σ is linear since G is, and all semisimple linear real groups are of this form.

Example. Let $G^\sigma = \mathrm{Sp}_{2n}(\mathbb{R})$ so $K^\sigma = U(n)$. Note $G^\sigma \subset \mathrm{Sp}_{2n}(\mathbb{C})$ which is simply connected and $\pi_1(U(n)) = \mathbb{Z}$. For every integer $m \geq 2$, $\mathrm{Sp}_{2n}(\mathbb{R})$ has an m-sheeted cover $\mathrm{Sp}_{2n}^{(m)}(\mathbb{R})$ with no f.dim faithful representations (in fact, all its f.dim reps will factor through $\mathrm{Sp}_{2n}(\mathbb{R})$).

Exercise (Homework). Classify simply connected real semisimple linear Lie groups.[39]

19.4 Connected complex reductive groups

Definition 19.15. A connected complex Lie group G is **reductive** if it is of the form $((\mathbb{C}^\times)^r \times G_0)/Z$ where G_0 semisimple, and $Z \subset (\mathbb{C}^\times)^r \times G_0$ a finite central subgroup. More generally, a complex Lie group G is **reductive** if G^0 is reductive, and G/G^0 is finite.

Fact. Connected G is reductive $\iff$ Rep$G$ are completely reducible.

Example. $\mathrm{GL}_n(\mathbb{C})$ is reductive, e.g. because

$$\mathrm{GL}_n(\mathbb{C}) = \frac{\mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})}{\mu_n}.$$

Remark 19.16. Let Z be the center of connected, reductive G. Then,

$$Z \subset (S^1)^r \times G_0^c \subset (\mathbb{C}^\times)^r \times G_0.$$

Hence, we get a compact subgroup

$$K := \frac{(S^1)^r \times G_0^c}{Z} \subset G.$$

Restriction of f.dim reps gives an equivalence

$$\mathrm{Rep}G \cong \mathrm{Rep}K,$$

so Rep$G$ is semisimple (i.e. reps of G are completely reducible).

How do we parametrize irreps of G as above? Looking at the construction of K, they parametrized by tuples $(n_1, \ldots, n_r, \lambda)$ with $n_i \in \mathbb{Z}$, $\lambda \in P_+$ subject to the global condition that they give a trivial character of Z.

19.5 Maximal tori

We talked about Cartan subalgebras last semester.

Recall 19.17. Cartan subalgebras of $\mathfrak{g}$ are conjugate, even when equipped with system of simple roots (use Weyl group acts (simply) transitively on systems of simple roots).

Definition 19.18. A **Cartan subalgebra** of $\mathfrak{g}^c$ is a maximal commutative subalgebra $\mathfrak{h}^c \subset \mathfrak{g}^c$ (note this automatically consists of semisimple elements). Equivalently, $\mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

Lemma 19.19. *All Cartan subalgebras (with systems of simple roots) of $\mathfrak{g}^c$ are conjugate.*

[39] Something something find those where $\mathfrak{k}$ is semisimple (not just reductive)

Proof. Let $(\mathfrak{h}_1^c, \Pi_1)$ and $(\mathfrak{g}_2^c, \Pi_2)$ be two such things. Then, there exists some $g \in G$ so that $g(\mathfrak{h}_1^c, \Pi_1)g^{-1} = (\mathfrak{h}_2^c, \Pi_2)$. Also, $\overline{g}(\mathfrak{h}_1^c, \Pi_1)\overline{g}^{-1} = (\mathfrak{h}_2^c, \Pi_2)$ (where $\overline{g} = \omega(g)$). Thus,

$$\overline{g}^{-1}g(\mathfrak{h}_1^c, \Pi_1)\left(\overline{g}^{-1}g\right)^{-1} = (\mathfrak{h}_1^c, \Pi_1),$$

so $\overline{g}^{-1}g =: h \in H = \exp(\mathfrak{h} := \mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C})$. Now we write the polar decomposition $g = kp$ so $\overline{g} = kp^{-1}$. Hence, $g = \overline{g}h$, $kp = kp^{-1}h$, $p = p^{-1}h$, and $p^2 = h$. Since $p \in P$ is positive, we see $p = \sqrt{h} = h^{1/2}$ as an operator on $\mathfrak{g}$. In particular, $p \in H$. Thus, conjugation by $g = kp$ is the same as conjugation by p, so $k(\mathfrak{h}_1^c, \Pi_1)k^{-1} = g(\mathfrak{h}_1^c, \Pi_1)g^{-1} = (\mathfrak{h}_2^c, \Pi_2)$, and we win (since $k \in K = G^c$). ■

Given a Cartan subalgebra $\mathfrak{h}^c \subset \mathfrak{g}^c$, its exponential $H^c = \exp(\mathfrak{h}^c) \subset G^c$ is a torus (connected, compact[40], abelian) $(S^1)^r$. In fact, H^c is a maximal torus (any larger torus would have a larger Lie algebra, but $\mathfrak{h}^c$ maximal).

Conversely, given a maximal torus H^c, $\operatorname{Lie} H^c$ is a commutative subalgebra, and maximality of H^c forces it to be a maximal commutative subalgebra. Thus, we have a bijection

$$\left\{\begin{matrix}\text{Cartan subalgebrs} \\ \text{in } \mathfrak{g}^c\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Maximal tori} \\ \text{in } G^c\end{matrix}\right\}.$$

Remark 19.20. Also Cartan subalgebras in $\mathfrak{g}$ are in bijection with maximal tori in G.

Corollary 19.21. *Any two maximal tori in G^c or G are conjugate.*

Theorem 19.22 (to be proved next time)**.** *Every element of G^c is contained in a maximal torus.*

Warning 19.23. This is false for complex groups (e.g. there exists non-semisimple elements like a matrix with nontrivial Jordan block).

Lecture at MIT on Tuesday.

20 Lecture 20 (5/4)

Let K be a compact connected Lie group. We proved last time that all maximal tori in K are conjugate (even with a choice of positive root system). The point was that maximal tori $T \subset K$ are in bijection with Cartan subalgebras $\mathfrak{t} \subset \mathfrak{k} = \operatorname{Lie} K$.

Today, we would like to prove the following theorem.

Theorem 20.1. *Every element of a connected compact Lie group K is contained in a maximal torus.*

(A generic element will be contained in the unique maximal torus which is its center, but a special element may be contained in many, e.g. a central element is contained in all)

Proof. The complexification $K_{\mathbb{C}} =: G$ will be a reductive connected group with $\mathfrak{k} = \mathfrak{g}^c$ ($\mathfrak{g} = \operatorname{Lie} C$). We may assume WLOG that K is semisimple (reductive groups are products of semisimple groups with torii,

[40] Since $\mathfrak{h}^c$ maximal $\implies H^c$ closed

up to finite quotient). Let $K' \subset K$ be the subset of elements contained in a maximal torus. Also, fix some maximal torus $T \subset K$. Consider the map

$$\begin{array}{rccc} f: & K \times T & \longrightarrow & K \\ & (k,t) & \longmapsto & ktk^{-1}. \end{array}$$

Note that $K' = \operatorname{im}(f)$, so K' is compact (so closed in K). Hence, $K \setminus K'$ is open. Now, say $x \in K$ is **regular** if the centralizer $\mathfrak{z}_x \subset \mathfrak{k}$ of x in the Lie algebra has dimension $\leq r := \operatorname{rank} K$. The set of such elements $K_{\text{reg}} \subset K$ is open (rank is lower semicontinuous) and nonempty (many regular elements in $\mathfrak{g}^c$ and exponentials of small regular elements will also be regular). On the other hand, any regular element x is contained in $\exp(\mathfrak{z}_x)$ which is a maximal torus. Therefore, $K_{\text{reg}} \subset K'$ so $K \setminus K' \subset K \setminus K_{\text{reg}}$. The set of non-regular elements is defined by polynomial equations[41]. Polynomials cannot vanish on an open set unless they vanish identically; these polynomials don't vanish identically (regular elements exist), so $K \setminus K'$ is empty. ■

Question: Why?

Corollary 20.2. *The exponential map* $\exp : \operatorname{Lie} K \to K$ *is surjective.*

Proof. If $T \subset K$ is a maximal torus, then $\exp : \operatorname{Lie} T \to T$ is surjective (since T commutative so exp a homomorphism with image containing an open neighborhood of identity). Applying this for all maximal tori gives the result. ■

Non-example. In $G = \mathrm{SL}_2(\mathbb{C}), \mathrm{SL}_2(\mathbb{R})$, $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$ is not in the image of the exponential map. It is the exponential of a matrix

$$\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix} = \exp \begin{pmatrix} \pi i & ? \\ & \pi i \end{pmatrix},$$

but it's not the exponential of a traceless matrix.

20.1 Semisimple and unipotent elements

We talked about semisimple and nilpotent elements in the Lie algebra last term. Now let's see the analogous notions for groups.

Let G be a connected complex reductive group.

Definition 20.3. We say that $g \in G$ is **semisimple** (resp. **unipotent**) if for every f.dim rep $\rho : G \to \mathrm{GL}(V)$, the operator $\rho(g)$ is semisimple[42] (resp. unipotent[43]).

Remark 20.4. For Lie algebras, we defined an element to be semisimple iff $\operatorname{ad} x$ was a semisimple operator, but this is the same as $\rho(x)$ being semisimple for any rep $\rho : \mathfrak{g} \to \operatorname{End}(V)$ since $x \in \mathfrak{g}$ semisimple iff it's contained in a Cartan subalgebra.

Similarly, $g \in G$ will be semisimple iff it's contained in a maximal torus.

We won't delve into this theory here, but developing it is done in a series of homework exercises.

[41] ranker smaller than expected, so certain minors have to vanish
[42] diagonalizable since V a $\mathbb{C}$-rep. In general, 'semisimple' means diagonalizable over algebraic closure
[43] only eigenvalue is 1

Exercise. Let Y be a faithful f.dim rep of G. Then, $g \in G$ is semisimple (resp. unipotent) iff $\rho_Y(g)$ is semisimple (resp. unipotent).

Exercise. The exponential map $\exp : \mathcal{N}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(G)$ gives a homeomorphism from the nilpotent elements of g to the unipotent elements of G.

Exercise. Let $Z = Z(G) \subset G$ be the center of G, and let $\pi : G \to G/Z =: G_{\mathrm{ad}}$ be the natural projection.

Note $\dim G_{\mathrm{ad}}$ may be less than $\dim G$, e.g. if Z contains a torus

(1) $\mathcal{U}(G) \xrightarrow{\sim} \mathcal{U}(G/Z)$ is a homeomorphism.

Example. If G is a torus, then $G/Z = 1$, so tori have no nontrivial unipotent elements.

(2) $SS(G) = \pi^{-1}(SS(G/Z))$ where $SS(\cdot)$ denotes semisimple elements.

Exercise (**Jordan Decomposition**). Any $g \in G$ can be uniquely written as a product $g = g_s g_u$ where g_s semisimple, g_u unipotent, and $g_s g_u = g_u g_s$.

Remark 20.5 (to be proved later). $g \in G$ is semisimple $\iff$ g is contained in some (complex) maximal torus (i.e. copy of $(\mathbb{C}^\times)^r$)

20.2 Cartan Decomposition

Let G be a complex connected *semisimple* group (actually, what we'll say extends to reductive groups) with Lie algebra $\mathfrak{g} := \mathrm{Lie}\, G$. Let $\mathfrak{g}^c \subset \mathfrak{g}$ be the compact form. Pick some Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ defining a real form; let $\sigma = \theta \circ \omega$ so $\theta \rightsquigarrow \mathfrak{g}_\sigma$ (ω is the antilinear involution defining the compact form). Recall

$$\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c, \quad \mathfrak{g}_\sigma = \mathfrak{k}^c \oplus i\mathfrak{p}^c, \quad \text{and} \quad G_\sigma = K + \sigma \cdot \underbrace{\exp(i\mathfrak{p}^c)}_{P_\sigma)}$$

where $\theta|_{\mathfrak{k}^c} = 1$ and $\theta|_{\mathfrak{p}^c} = -1$.

As a manifold, we have $G_\sigma = K_\sigma \times P_\sigma$ and P_σ is some Euclidean space. This P_σ is in general not a group. Recall we have a Cartan subalgebra compatible with θ, $\mathfrak{h}^c_+ \oplus \mathfrak{h}^c_- = \mathfrak{h}^c \subset \mathfrak{g}^c$ (with $\mathfrak{h}^c_+ \subset \mathfrak{k}^c$ and $\mathfrak{h}^c_- \subset \mathfrak{p}^c$). Define $A := \exp(i\mathfrak{h}^c_-) \subset P_\sigma$. This is an abelian group (since $\mathfrak{h}^c_-$ abelian).

Theorem 20.6 (**Cartan Decomposition**).

$$G_\sigma = K_\sigma A K_\sigma,$$

i.e. any $g \in G_\sigma$ can be written as $g = k_1 a k_2$ with $k_1, k_2 \in K_\sigma$ and $a \in A$.

Warning 20.7. This decomposition is *not* unique. In particular, $K_\sigma \times A \times K_\sigma \to G_\sigma$ is not injective.

Remark 20.8. This extends to reductive groups e.g. by forming the decomposition separately for the torus and semisimple factors.

Example. Let $G_\sigma = \mathrm{GL}_n(\mathbb{C})$. Then, $K_\sigma = U(n)$ and $A = \{\text{positive diagonal matrices}\}$. This says any invertible matrix g over $\mathbb{C}$ is of the form $u_1 a u_2$ where u_1, u_2 are unitary and a is diagonal matrix with positive entries.

Example. Let $G_\sigma = \mathrm{GL}_n^+(\mathbb{R})$, invertible real matrices with positive determinant. Any $g \in G_\sigma$ can be written as $g = O_1 a O_2$ with O_i orthogonal with determinant 1 and a positive diagonal.

Easy to go from this to $\mathrm{GL}_n(\mathbb{R}) = O(n) A\, \mathrm{SO}(n)$ with A consisting of diagonal matrices with pos. entries.

These decompositions are well-known classically. For example.

Proof of first example. Write $g = UR$ the polar decomposition, so U unitary and R positive hermitian. We may diagonalize $R = U' a\, (U')^{-1}$ with U' unitary. Then, $g = UU' a\, (U')^{-1}$. ■

How does one prove Theorem 20.6?

Lemma 20.9 (Homework)**.** *$\mathfrak{h}_-^c$ is a maximal abelian subalgebra of $\mathfrak{p}^c$, and all such subalgebras are conjugate under K_σ.*

Proof of Theorem 20.6. We know $G_\sigma = K_\sigma P_\sigma$. Hence, it is enough to show that every element $p \in P_\sigma$ is conjugate to an element of a by action of K_σ. This follows from the Lemma. Consider $\mathfrak{h}_{p-}^c$ a maximal abelian subalgebra of $\mathfrak{p}^c$ containing $i \log p$. Then, by the lemma, there exists $g \in K^c$ such that $\mathrm{Ad}(g)(\mathfrak{h}_{p-}^c) = \mathfrak{h}_-^c$. Thus, $\mathrm{Ad}(g)(i \log p) \in \mathfrak{h}_-^c$, so $\mathrm{Ad}(g)(\log p) \in i\mathfrak{h}_-^c$, so $gpg^{-1} \in \exp(i\mathfrak{h}_-^c) = A$. ■

20.3 Integral form of character orthogonality

Let K be a connected compact Lie group with maximal torus $T \subset K$. We know that characters of irreducible representations of K are orthonormal under the inner product

$$(f, g) := \int_K f(k)\overline{g(k)}\mathrm{d}k$$

on $C(K)^{K_{\mathrm{ad}}}$, continuous function invariant under adjoint action.[44] But every $f \in C(K)^{K_{\mathrm{ad}}}$ is determined by its values on T (since every element conjugate to an element of T), so we should be able to write this inner product just in terms of T. That is, we should have

$$(f, g) = \int_T f(t)\overline{g(t)}w(t)\mathrm{d}t$$

for some weight function $w(t)$. All our functions are Weyl group invariant, this weight should be W-invariant as well.

What is $w(t)$? You can compute it directly by doing a computation in differential geometry. However, we will not have to do this, because we secretly know what it is from the Weyl character formula.

Theorem 20.10. *For any $f \in C(K)^{K_{\mathrm{ad}}}$,*

$$\int_K f(k)\mathrm{d}k = \int_T f(t)w(t)\mathrm{d}t,$$

where $w(t) = \frac{1}{\#W}\left|\Delta(t)\right|^2$ and $\Delta(t)$ is the ***Weyl denominator***

$$\Delta(t) = \prod_{\alpha \in R_+}(1 - \alpha(t)).$$

[44] If you wanted, could have taken $L^2(K)^{K_{\mathrm{ad}}}$ instead; it doesn't matter

Example. Take $K = U(n)$ with

$$T = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} : z_j \in \mathbb{C} \text{ with } |z_j| = 1 \right\} \cong (S^1)^n.$$

The (positive) roots are $\alpha = \alpha_{jm} = e_j - e_m$, i.e. $\alpha(t) = z_j/z_m$. We see that

$$\int_{U(n)} f(k)\mathrm{d}k = \frac{1}{n!} \int_T f(z_1, \dots, z_n) \left(\prod_{j<m} \left| 1 - \frac{z_j}{z_m} \right|^2 \right) \frac{\mathrm{d}\theta_1 \dots \mathrm{d}\theta_n}{(2\pi)^n} \text{ where } z_j = e^{i\theta_j}.$$

This can be simplified. Write

$$\left| 1 - \frac{z_j}{z_m} \right| = \left| 1 - e^{i(\theta_j - \theta_m)} \right| = \left| e^{i(\theta_j - \theta_m)/2} - e^{-i(\theta_j - \theta_m)/2} \right| = \left| 2\sin\left(\frac{\theta_j - \theta_m}{2} \right) \right|,$$

and so see that

$$\int_{U(n)} f(k)\mathrm{d}k = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \left(\prod_{j<m} \left(2\sin\left(\frac{\theta_j - \theta_m}{2} \right) \right)^2 \right) f \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n.$$

Proof of theorem 20.10. We know the characters $\chi_\lambda(k)$ are dense in $C(K)^{K_{\text{ad}}}$, so it's enough to check this equality for $f = \chi_\lambda$, the character of L_λ. Characters are orthogonal, so

$$\int_K \chi_\lambda(k)\mathrm{d}k = (\chi_\lambda, 1) = (\chi_\lambda, \chi_0) = \delta_{0,\lambda}.$$

Compare this with (use Weyl character formula for first equality[45] and Weyl denominator formula for the second)

Possibly a typo below

$$\begin{aligned} \frac{1}{\#W} \int_T \chi_\lambda(t) \left| \prod_{\alpha \in R_+} (1 - \alpha(t)) \right|^2 \mathrm{d}t &= \frac{1}{\#W} \frac{\int_T \sum_{w \in W} \operatorname{sign}(w)(w(\lambda + \rho))(t)}{\rho(t) \prod_{\alpha \in R_+} (1 - \alpha(t)^{-1})} \prod_{\alpha \in R_+} (1 - \alpha(t)) \prod_{\alpha \in R_+} (1 - \alpha(t)^{-1})\mathrm{d}t \\ &= \frac{1}{\#W} \int_T \left(\sum_{w \in W} \operatorname{sign}(w) w(\lambda + \rho)(t) \right) \left(\sum_w \operatorname{sign}(w) w(\rho)^{-1}(t) \right) \mathrm{d}t \end{aligned}$$

Now, if $\lambda \neq 0$, then this is 0.[46] If $\lambda = 0$, then the above becomes

$$\frac{1}{\#W} \sum_{w \in W} 1 = 1.$$

This completes the proof. ■

You can reverse this. If you do the differential geometry calculation giving the integral formula, then

[45] Also use $\alpha(t) \in S^1$ so $\overline{\alpha(t)} = \alpha(t)^{-1}$

[46] Think, $\int_{S^1} e^{im\theta} \cdot e^{-in\theta} = 0$ when $n \neq m$

you can use it to obtain the Weyl character formula instead. This is what Weyl did.

20.4 Topology of Lie Groups

This will be the subject of the next few lectures.

We want to understand the (co)homology/homotopy groups of Lie groups. There are many cohomology theories computing the same thing; for Lie groups, it will be convenient to use de Rham cohomology.

Let M be a manifold. Recall the space $\Omega^i(M)$ of complex differential i-forms as well as the exterior derivative $\mathrm{d} : \Omega^i(M) \to \Omega^{i+1}(M)$ which satisfies $\mathrm{d}^2 = 0$.

Definition 20.11. The ith **de Rham cohomology group** of M is

$$\mathrm{H}^i(M) = \mathrm{H}^i(M, \mathbb{C}) = \frac{\ker \mathrm{d}|_{\Omega^i}}{\operatorname{im} \mathrm{d}|_{\Omega^{i-1}}}.$$

Forms in $\ker \mathrm{d}$ are called **closed forms** while those in $\operatorname{im} \mathrm{d}$ are called **exact forms**.

What input from differential geometry will we need to use?

Let X be a vector field on M. Then one can form $L_X : \Omega^r(M) \to \Omega^r(M)$. First note that for $X = \sum_i a_i \frac{\partial}{\partial x_i}$, we have $X(x_i) = a_i$. This action can be extended to the contraction map

$$\iota_X : \Omega^j \to \Omega^{j-1}.$$

In particular, $\iota_X \omega(X_1, \ldots, X_{n-1}) = \omega(X, X_1, \ldots, X_{n-1})$. The map L_X is locally given by

$$L_X\left(f \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r}\right) = (L_X f)\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_r} + \sum_{j=1}^{n} f \cdot \mathrm{d}x_{i_1} \wedge \cdots \wedge a_{i_j} \wedge \cdots \wedge \mathrm{d}x_{i_r} \text{ where } X = \sum_i a_i \frac{\partial}{\partial x_i}.$$

Theorem 20.12 (Cartan's magic formula). *On differential forms,* $L_X = \iota_X \mathrm{d} + \mathrm{d}\iota_X$.

Friday is a holiday, so homework due date moved to Monday. There will be one more homework after the current one, due on Monday of the last week.

21 Lecture 21 (5/6): Cohomology of Lie Groups

At the end of last time we switched topics to 'cohomology of Lie groups.' Let's pick up where we left off.

Recall 21.1. Let M be a manifold. Its cohomology $\mathrm{H}^i(M, \mathbb{C})$ can be computed using the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^n(M) \to 0,$$

where $n = \dim M$. Here, $\Omega^i(M)$ is the space of (smooth, $\mathbb{C}$-valued) differential i-forms, and d is the de Rham differential determined by

$$\mathrm{d}(f \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m) = \mathrm{d}f \wedge \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m.$$

This satisfies d^2, and the cohomology of this complex

$$\mathrm{H}^i(M,\mathbb{C}) = \frac{\ker \mathrm{d}}{\operatorname{im}\mathrm{d}}$$

is the cohomology of M.

Fact. If M is compact, then $\dim \mathrm{H}^i(M) < \infty$.

Definition 21.2. The **Betti numbers** of M are $b_i(M) = \dim \mathrm{H}^i(M)$.

Example. $b_0(M) = \#$connected components, so M connected $\iff b_0 = 1$.

We would like to compute these b_i for compact Lie groups.

Recall 21.3. There is a product structure on cohomology. If $\omega \in \Omega^i$ and $\xi \in \Omega^j$, can get an $(i+j)$-form $\omega \wedge \xi \in \Omega^{i+j}$. Moreover,

$$\mathrm{d}(\omega \wedge \xi) = \mathrm{d}\omega \wedge \xi + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\xi$$

(Above, you can think of the sign as coming from commuting d past ω). The **Leibniz rule** above tells us that $\wedge$ descnds to

I don't know if this reasoning works in general to always get the right sign in these graded situations

$$\mathrm{H}^*(M) = \bigoplus_{i=0}^{n} \mathrm{H}^n(M)$$

giving it the structure of an associative **graded commutative** algebra. Graded commutative means

$$ab = (-1)^{\deg(a)\deg(b)} ba.$$

Remark 21.4. Let $f : M \to N$ be a differential (i.e. smooth) map. Then, we get a pullback $f^* : \Omega^i(N) \to \Omega^i(M)$ which commutes with d and preserves $\wedge$. Hence, it induces a graded algebra homomorphism

$$f^* : \mathrm{H}^*(M) \to \mathrm{H}^*(N).$$

Exercise. Say $f_t : M \to N$ is a smooth family of maps for $t \in (0,1)$ (i.e. $f : (0,1) \times M \to N$ smooth). Then, $f_t^* : \mathrm{H}^*(N) \to \mathrm{H}^*(M)$ is independent of t. Hint: show that if $\mathrm{d}\omega = 0$, then $\frac{\partial}{\partial t} f_t^* \omega$ is exact.

(f^* does not change under deformations of f).

Before turning to Lie groups, we recall Cartan's magic formulas. Let v be a vector field on M. Then we get Lie derivative $L_v : \Omega^i \to \Omega^i$ as well as a contraction operator $\iota_v : \Omega^i \to \Omega^{i-1}$. This latter operator is defined by

$$\iota_v \left(g \mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_r\right) = \operatorname{Alt}\left(g \cdot \iota_v f_1 \mathrm{d} f_2 \wedge \cdots \wedge \mathrm{d} f_r\right),$$

average the application of L_v over all permutations (or something like this). One can check

$$\iota_v(\omega \wedge \xi) = \iota_v \omega \wedge \xi + (-1)^{\deg \omega} \omega \wedge \iota_v \xi$$

and $L_v(\omega \wedge \xi) = L_v \omega \wedge \xi + \omega \wedge L_v \xi$.

L_v degree 0 operator so no sign

Lemma 21.5 (Cartan's magic formula). $L_v = \iota_v \mathrm{d} + \mathrm{d}\iota_v$.

ι_v is a chain homotopy from L_v to the zero map

Note that $\iota_v \circ \mathrm{d} + \mathrm{d} \circ \iota_v = [\iota_v, \mathrm{d}]$ is a (graded) commutator.

Proof. We showed last semester that the commutator of two derivations is a derivation. The same holds true for graded commutators, so $[\iota_v, \mathrm{d}]$ is a derivation of degree 0 (exercise). Hence, we can check this equality on generators in a local chart.

That is, we may assume $\omega = f$ or $\omega = \mathrm{d}f$ (everything else is a wedge/product of these). We see

$$L_v f = \mathrm{d}f(v) = \iota_v \mathrm{d}f = (\iota_v \mathrm{d} + \mathrm{d}\iota_v) f$$

since $\iota_v f = 0$. Similarly,

$$L_v \mathrm{d}f = \mathrm{d}L_v f = \mathrm{d}\iota_v \mathrm{d}f = (\iota_v \mathrm{d} + \mathrm{d}\iota_v)\mathrm{d}f$$

since $\mathrm{d}^2 f = 0$. ■

Corollary 21.6. *L_v maps closed forms to exact forms.*

Proof. If $\mathrm{d}\omega = 0$, then $L_v\omega = \mathrm{d}\iota_v\omega$. ■

Corollary 21.7. *L_v defines the zero map on $\mathrm{H}^*(M)$.*

Corollary 21.8. *If a connected Lie group G acts on M, then it acts trivially on $\mathrm{H}^*(M)$*

(A path in G gives a homotopy of actions of its elements, so anything in the path component of 1 acts via the identity).

Theorem 21.9. *Suppose of compact connected Lie group G acts on M. Then $\mathrm{H}^*(M)$ is computed by the complex $\Omega^*(M)^G \subset \Omega^*(M)$ of G-invariant forms.*

Proof. Let $P : \Omega^*(M) \to \Omega^*(M)$ be averaging over G, i.e.

$$P\omega = \int_G g^*\omega \mathrm{d}g.$$

Then $P^2 = P$ and we have

$$\Omega^*(M) = \Omega^*(M)_1 \oplus \Omega^*(M)_0 = \Omega^*(M)^G \oplus \ker P.$$

This decomposition is respected by d, so cohomology of M is a sum of the cohomology of these two subcomplexes. Suppose $\omega \in \Omega^*(M)_0$ is a closed form, $\mathrm{d}\omega = 0$. Then, $[\omega] = [g\omega]$ for any $g \in G$ (G acts trivially on cohomology). Thus we can take the average

$$[\omega] = [g\omega] = \int_G [g\omega]\mathrm{d}g = \left[\int_G g\omega \mathrm{d}g\right] = [P\omega] = 0.$$

Thus, $\omega = \mathrm{d}\eta$ for some $\eta = \eta_1 + \eta_0 \in \Omega^{i-1}(M)$. Thus, $\omega = \mathrm{d}\eta_1 + \mathrm{d}\eta_0 \implies \mathrm{d}\eta_1 = 0$ so $\omega = \mathrm{d}\eta_0$ which means that $\Omega^*(M)_0$ is exact (it has zero cohomology). ■

Corollary 21.10. *Let G be a compact Lie group. Then $\mathrm{H}^*(G)$ is computed by $\Omega^*(G)^G$, the complex of left-invariant differential forms.*

Recall that the space of left-invariant vector fields is isomorphic to the Lie algebra $\mathrm{Lie}\, G$. By the same reasoning, one shows that

$$\Omega^i(G)^G \cong \bigwedge^i \mathfrak{g}^* \text{ where } \mathfrak{g} = (\mathrm{Lie}\, G)_{\mathbb{C}}.$$

That is, cohomology of a compact Lie group is computed using a complex of the form

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{g}^* \longrightarrow \bigwedge^2 \mathfrak{g}^* \longrightarrow \ldots \longrightarrow \bigwedge^n \mathfrak{g}^* \longrightarrow 0$$

(this gives a way to see cohomology of a compact Lie group is finite dimensional).

Question 21.11. *What is the different* d*?*

Before proving a description entirely in terms of the Lie algebra, we need another lemma from differential geometry.

Lemma 21.12 (Cartan's differentiation formula). *Let $\omega \in \Omega^m(M)$, and let $v_0, \ldots, v_m$ be vector fields on M. Then,*

$$\mathrm{d}\omega(v_0, \ldots, v_m) = \sum_{i=0}^{m} (-1)^i L_{v_i}\left(\omega(v_0, \ldots, \widehat{v_i}, \ldots, v_m)\right) + \sum_{0 \leq i < j \leq m} (-1)^{i+j} \omega\left([v_i, v_j], v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_m\right)$$

Proof Sketch. **(1)** $RHS(fv_0, v_1, \ldots, v_m) = f \cdot RHS(v_0, \ldots, v_m)$ so the RHS is linear over functions (in each variable $v_0, v_1, \ldots, v_m$).

(2) Now it's enough to check this when $v_i = \frac{\partial}{\partial x_{k_i}}$. Say $\omega = f \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_m}$. This it's a "straightforward" calculation to verify this equality. ■

Corollary 21.13. *If $\omega \in \Omega^*(G)^G$ is left-invariant and $v_0, v_1, \ldots, v_m$ are left-invariant vector fields, then*

$$\mathrm{d}\omega(v_0, \ldots, v_m) = \sum_{0 \leq i < j \leq m} (-1)^{i+j} \omega\left([v_i, v_j], v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_j}, \ldots, v_m\right). \tag{21.1}$$

Proof. $\omega(v_0, \ldots, \widehat{v_i}, \ldots, v_m)$ is locally constant. ■

Corollary 21.14. *(21.1) defines the differential in the complex $\Omega^*(G)^G$ computed the cohomology of a compact, connected Lie group.*

Note that this complex

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathfrak{g}^* \longrightarrow \bigwedge^2 \mathfrak{g}^* \longrightarrow \ldots \longrightarrow \bigwedge^n \mathfrak{g}^* \longrightarrow 0$$

makes sense for any Lie algebra $\mathfrak{g}$ (now that we've defined the differential just in terms of the Lie bracket).

Definition 21.15. This complex is called the **standard complex** (or **Chevally-Eilenberg complex**) of $\mathfrak{g}$, denoted $\mathrm{CE}^*(\mathfrak{g})$. Its cohomology is called **Lie algebra** cohomology of $\mathfrak{g}$, and is denoted by $\mathrm{H}^*(\mathfrak{g})$.

This makes sense for any Lie algebra over any field. One has $\mathrm{d}^2 = 0$ because of the Jacobi identity.

Proposition 21.16. $\mathrm{H}^*(G) \cong \mathrm{H}^*(\mathfrak{g})$ *when G compact connected.*

Remark 21.17. There is an algebra structure on $CE^*(\mathfrak{g})$ induced by $\wedge$ which descends to $\mathrm{H}^*(\mathfrak{g})$, making it an associative graded commutative algebra. This isomorphism of the previous prop is one of graded algebras.

Note we need G compact to compute its cohomology using its Lie algebra.

Example. Say $\mathfrak{g}$ is abelian. Then $\mathrm{d} = 0$ since all Lie brackets vanish. Thus $\mathrm{H}^*(\mathfrak{g}) = \bigwedge^* \mathfrak{g}^*$ is the exterior algebra of the dual of $\mathfrak{g}$.

Example. If $G = (S^1)^n$, then $\mathfrak{g} = \mathbb{C}^n$ and one has

$$\mathrm{H}^*(G) = \mathrm{H}^*(\mathfrak{g}) = \bigwedge^* (\xi_1, \ldots, \xi_n) \quad \text{where} \quad \deg \xi_i = 1.$$

In particular, $\mathrm{H}^*(S^1) = \bigwedge(\xi) = \langle 1, \xi \rangle$ with 1 generating H^0 and ξ generating H^1.

Non-example. If you replace the circle by its universal cover, you get $\mathbb{R}$ and $\mathrm{H}^*(R) \neq \mathrm{H}^*(S^1) = \mathrm{H}^*((\mathrm{Lie}\,\mathbb{R})_{\mathbb{C}})$.

Corollary 21.18. *Finite covers of compact Lie groups induce an isomorphism in* $\mathrm{H}^*(-;\mathbb{C})$*.*

This is not true with $\mathbb{Z}$-coefficients.

Non-example. $S^1 \to S^1, z \mapsto z^2$ induces multiplication by 2 in cohomology. This is an iso on $\mathrm{H}^*(-;\mathbb{C})$, but not on $\mathrm{H}^*(-;\mathbb{Z})$ and certainly not on $\mathrm{H}^*(-;\mathbb{F}_2)$.

Non-example. $\mathrm{SU}(2) \cong S^3$ is a double cover of $\mathrm{SO}(3) \cong \mathbb{RP}^3$. The have different integral cohomology.

21.1 Künneth formula

Say M, N are manifolds. Then $\Omega^i(M) \otimes \Omega^j(N) \to \Omega^{i+j}(M, N)$ gives a map $\mathrm{H}^i(M) \otimes \mathrm{H}^j(N) \to \mathrm{H}^{i+j}(M \times N)$. These induce a graded iso.

Theorem 21.19. *This induces an isomorphism*

$$\mathrm{H}^*(M) \otimes \mathrm{H}^*(N) \xrightarrow{\sim} \mathrm{H}^*(M \times N)$$

as graded algebras.

Remark 21.20. This is a graded tensor product above, so e.g. we have

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')}(aa' \otimes bb').$$

Remark 21.21. The map

$$\bigoplus_{i+j=k} \Omega^i(M) \otimes \Omega^j(N) \hookrightarrow \Omega^k(M \times N)$$

is an injection, but is not an isomorphism in general. What is true is that the image is dense w.r.t an appropriate topology. This makes proving Künnth a bit subtle.

However, for Lie groups, Künnth formula comes for free:

$$\Omega^*(G \times K)^{G \times K} = \Omega^*(G)^G \otimes \Omega^*(K)^K.$$

21.2 Main Theorem

Theorem 21.22. *If G is a connected compact Lie group, then the cohomology of G is*

$$\mathrm{H}^*(G) \cong \left(\bigwedge^* \mathfrak{g}^*\right)^{\mathfrak{g}}$$

as a graded algebra, where we're taking $\mathfrak{g}$-invariants under the adjoint action.

Proof. G has an action of $G \times G$, so cohomology of G is computed by

$$\Omega^*(G)^{G\times G} = \left(\bigwedge^* \mathfrak{g}^*\right)^{\mathfrak{g}}.$$

Hence, we only need to show that $\mathrm{d} = 0$ on this space. This is easy to see from invariance, e.g.

$$\omega\left([v_0, v_1], v_2, \dots, v_m\right) + \omega(v_1, [v_0, v_2], \dots, v_m) + \cdots + \omega(v_1, v_2, \dots, [v_0, v_m]) = 0.$$

Similarly with v_i replacing v_0 above. Equation (21.1) tells us that the alternating sum of these (which are all 0) is $2\mathrm{d}\omega(v_0, v_1, \dots)$, so $\mathrm{d} = 0$. ■

Example. Say $\omega \in \bigwedge^2 \mathfrak{g}^*$. Then,

$$\mathrm{d}\omega(x, y, z) = \omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y).$$

If ω is ad-invariant, then

$$\omega([x, y], z) + \omega(y, [x, z]) = 0 = \omega([y, x], z) + \omega(x, [y, z]) = 0 = \omega([z, w], y) + \omega(x, [z, y]).$$

Adding these up with alternating signs shows that

$$2\omega([x, y], z) + 2\omega([z, x], y) + 2\omega([y, z], x) = 0.$$

This says $2\mathrm{d}\omega(x, y, z) = 0$.

To understand this answer a bit better, first note

$$\dim \mathrm{H}^*(G) = \sum_i \dim \mathrm{H}^i(G) = \sum_i \dim \left(\bigwedge^i \mathfrak{g}^*\right)^{\mathfrak{g}}.$$

Use the Weyl character formula. We have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$, so $\bigwedge^* \mathfrak{g}^* = \bigwedge^* \mathfrak{h}^* \oplus \bigoplus_{\alpha\in R} \bigwedge \mathfrak{g}_\alpha^*$. Hence, letting $r = \mathrm{rank}(\mathfrak{g})$,

$$\left(\mathrm{ch} \bigwedge^* \mathfrak{g}^*\right)(t) = 2^r \prod_{\alpha\in R} (1 + \alpha(t))$$

($\mathbb{C} \oplus \mathfrak{g}_{-\alpha}^*$ contributes $1 + e^\alpha$) where $t \in T \subset G$. Thus,

$$\begin{aligned}
\dim \left(\bigwedge^* \mathfrak{g}^*\right)^{\mathfrak{g}} &= \left(\mathrm{ch} \bigwedge^* \mathfrak{g}^*, \mathrm{ch}\, \mathbb{C}\right) \\
&= \frac{1}{\#W} \int_T 2^r \prod_{\alpha\in R} (1 + \alpha(t)) \prod_{\alpha\in R} (1 - \alpha(t)) \mathrm{d}t \\
&= \frac{2^r}{\#W} \int_T \prod_{\alpha\in R} (1 - \alpha(t^2)) \mathrm{d}t \\
&= \frac{2^r}{\#W} \int_T w(t^2) \mathrm{d}t.
\end{aligned}$$

We change variables $t \mapsto t^2$ to see that this is equal to

$$\frac{2^r}{\#W}\int_T w(t)\mathrm{d}t = 2^r \text{ since } \frac{1}{\#W}\int_T w(t)\mathrm{d}t = (\mathrm{ch}\,\mathbb{C}, \mathrm{ch}\,\mathbb{C}) = 1.$$

Why did we get a power of 2? This is related to the fact that the cohomology of a Lie group is a graded Hopf algebra. Let $m : G \times G \to G$ be the multiplication map. This induces a coproduct

$$\Delta : \mathrm{H}^*(G) \to \mathrm{H}^*(G \times G) \cong \mathrm{H}^*(G) \otimes \mathrm{H}^*(G)$$

map. This is coassociative in the sense that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ and is an algebra homomorphism. This makes $\mathrm{H}^*(G)$ a graded bialgebra.

Exercise. Deduce from this that $\mathrm{H}^*(G)$ is a free (graded commutative) algebra. Hence, all generators are odd.[47]

Corollary 21.23.

$$\mathrm{H}^*(G) = \left(\bigwedge^* \mathfrak{g}^*\right)^{\mathfrak{g}} \cong \bigwedge(\xi_1, \ldots, \xi_k) \quad \textit{where} \quad \deg \xi_i = 2m_i + 1.$$

Thus, $\dim \mathrm{H}^i(G) = 2^k$.

Corollary 21.24.

$$\mathrm{H}^*(G) = \bigwedge^* (\xi_1, \ldots, \xi_r) \quad \textit{and} \quad \deg \xi_i = 2m_i + 1$$

where $m_1 \leq m_2 \leq \cdots \leq m_r$ *are integers.*

We will discuss what these numbers are next time. They turn out to be the exponents of G (See section 9.1).

22 Lecture 22 (5/11)

Last time we discussed the (complex) cohomology of Lie groups. In the end, we saw that the cohomology of a compact Lie group is a free graded algebra with generators in odd degrees, computed as the invariants of the exterior algebra on the dual of the Lie algebra.

Recall 22.1. For G a compact Lie group of rank r,

$$\mathrm{H}^*(G) = \bigwedge^* (\xi_1, \ldots, \xi_r) \quad \text{and} \quad \deg \xi_i = 2m_i + 1$$

where $m_1 \leq m_2 \leq \cdots \leq m_r$ are integers.

What do we know about these numbers $m_1, \ldots, m_r$?

- We know $r + 2\sum m_i = \sum_{i=1}^r (2m_i + 1) = \dim \mathfrak{g}$ so $\sum m_i = \frac{\dim \mathfrak{g} - r}{2} = \#R_+$.

  Exercise. $\left(\bigwedge^3 \mathfrak{g}^*\right)^{\mathfrak{g}} = \mathbb{C}$ spanned by the triple product $([x, y], z)$ (a linear functional on $\mathfrak{g}^{\otimes 3}$.

  From this it follows that $m_1 = 1$.

Question: Why do these degrees add to to $\dim \mathfrak{g}$?

[47] An even generator would give nontrivial cohomology in arbitrarily high degree

Example ($\mathfrak{g}$ simple of rank 2)**.** We get $m_2 = 2$ for A_2, $m_2 = 3$ for $B_2 = C_2$, $m_2 = 5$ for G_2, etc. This is because $m_2 = \#R_+ - m_1 = \#R_+ - 1$ for these cases.

In fact, we have the following general theorem, not to be proven here

Theorem 22.2. *The numbers m_i are the exponents of $\mathfrak{g}$ defined in Section 9.1. In other words, the degrees $2m_i + 1$ of generators of the cohomology ring are the dimensions of simple modules occurring in the decomposition of $\mathfrak{g}$ over its principal $\mathfrak{sl}_2$-subalgebra.*

Definition 22.3. For a space X, it's **Poincaré series** (sometimes polynomial) is

$$P_X(z) = \sum_{n \geq 0} (\dim_{\mathbb{C}} \mathrm{H}^n(X; \mathbb{C})) z^n.$$

Remark 22.4. The Poincaré polynomial $P(z)$ of $\left(\bigwedge^* \mathfrak{g}^*\right)^{\mathfrak{g}}$ is given by the formula

$$P(z) = \frac{(1+z)^r}{\#W} \int_T \prod_{\alpha \in R} (1 + z\alpha(t))(1 - \alpha(t)).$$

Hence, the above theorem is equivalent to the statement that this integral equals $\prod_i (1_t^{2m_i+1})$.

We will prove this for the case of type A.

Corollary 22.5. *For $\mathfrak{g} = \mathfrak{sl}_n$, we have $m_i = i$. Equivalently, the same is true for $\mathfrak{g} = \mathfrak{gl}_n$ if we add $m_0 = 0$.*

How do we prove this (w/o using the theorem)?

Proof. Let $\mathfrak{g} = \mathfrak{gl}_n$, $V = \mathbb{C}^n$. We need to compute the Poincaré polynomial of $\bigwedge^* (V \otimes V^*)^{\mathfrak{g}}$. To this end, we employ the skew-Howe duality.

Exercise (**skew-Howe duality**). Show that $\bigwedge^*(V \otimes V^*) = \bigoplus S^\lambda V \otimes S^{\lambda^t} V^*$ where λ^t is the conjugate partition to λ (i.e. transpose the Young diagram).

Remark 22.6. Taking exterior power is (something) like taking tensor power and then taking antiinvariants of the symmetric group (homomorphisms from the sign representation, I think).

We need to take ad-invariants of $\bigoplus S^\lambda V \otimes S^{\lambda^t} V^*$. These invariants will only exist if $\lambda = \lambda^t$ (need the irreps to be the same). Thus,

$$P(z) = \sum_{\lambda = \lambda^t} z^{|\lambda|}$$

with sum taken over λ with $\leq n$ parts. There are exactly 2^n such symmetric partitions λ; they consist of a sequence of hooks $(k, 1^{k-1})$ with decreasing values of k. The degree of such a hook is $2k - 1$, and so we see that

$$P(z) = (1+z)(1+z^3)\dots(1+z^{2n-1}).$$

■

Corollary 22.7.

$$\mathrm{H}^*(U(n)) = \bigwedge^*(\xi_1, \xi_3, \dots, \xi_{2n-1})$$

with subscripts denoting the degrees, and

$$\mathrm{H}^*(\mathrm{SU}(n)) = \bigwedge^*(\xi_3, \ldots, \xi_{2n-1}).$$

Remark 22.8. For $\mathfrak{gl}_n$, one gets the same cohomology even integrally. This is not true for other Lie algebras.

22.1 Cohomology of homogeneous spaces

Let G be a compact connected Lie group with complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$, and let $K \subset G$ be a closed subgroup with $\mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$. Consider the homogeneous space G/K.

Question 22.9. *How can we compute the cohomology* $\mathrm{H}^*(G/K)$*?*

(recall we implicitly use $\mathbb{C}$-coefficients)

Since the group G acts on G/K, this cohomology is computed by the complex $\Omega^*(G/K)^G = \left(\bigwedge^*(\mathfrak{g}/\mathfrak{k})^*\right)^K$ (for equality, use translation by G to see that an invariant differential form is determined by its value at the identity). We denote this complex by $\mathrm{CE}^*(\mathfrak{g}, K)$ and call it the **relative Chevalley-Eilenberg complex**.

For example, if $K = \Gamma$ is finite, this is just the Γ-invariant part of the Chevalley-Eilenberg complex. We now Γ acts trivially on the cohomology (since G connected), so we get $\mathrm{H}^*(G/\Gamma) = \mathrm{H}^*(G)$ (as noted before).

What happens if $\dim K > 0$? Can we reduce to a purely algebraic problem as we did for $K = 1$?

Notation 22.10. For $\mathfrak{k} \subset \mathfrak{g}$ a pair of Lie algebras (over any field, of any dimension), let

$$\mathrm{CE}^i(\mathfrak{g}, \mathfrak{k}) := \left(\bigwedge^* \left(\frac{\mathfrak{g}}{\mathfrak{k}}\right)^*\right)^{\mathfrak{k}}.$$

Exercise. $\mathrm{CE}^\bullet(\mathfrak{g}, \mathfrak{k})$ is a subcomplex of $\mathrm{CE}^\bullet(\mathfrak{g})$.

Definition 22.11. The complex $\mathrm{CE}^\bullet(\mathfrak{g}, \mathfrak{k})$ is called the **relative Chevalley-Eilenberg complex**, and its cohomology is called the **relative Lie algebra cohomology**, denoted $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{k})$.

Going back to compact Lie groups, we have $\mathrm{CE}^\bullet(\mathfrak{g}, K) = \mathrm{CE}^\bullet(\mathfrak{g}, \mathfrak{k})^{K/K^\circ}$, so

Corollary 22.12.

$$\mathrm{H}^*(G/K) \cong \mathrm{H}^*(\mathfrak{g}, \mathfrak{k})^{K/K^\circ}$$

as algebras.

Thus computation of the cohomology of G/K reduced to the computation of relative Lie algebra cohomology, which is again purely algebraic.

Warning 22.13. The differentials won't always be trivial in this case.

Corollary 22.14. *Suppose $z \in K$ is an element acting by -1 on $\mathfrak{g}/\mathfrak{k}$. Then,*

$$\left(\bigwedge^i(\mathfrak{g}/\mathfrak{k})^*\right)^K = 0 \ \textit{for} \ \ \textit{odd } i.$$

Hence, the differential $\mathrm{CE}^\bullet(\mathfrak{g}, K)$ *is* 0 *and thus*

$$\mathrm{H}^*(G/K) \cong \left(\bigwedge^\bullet (\mathfrak{g}/\mathfrak{k})^*\right)^K,$$

with cohomology present only in even degrees.

Example (Grassmannians)**.** Let $G = U(n+m)$ and $K = U(n) \times U(m)$, so that G/K is the **Grassmannian** $G_{n+m,n}(\mathbb{C}) \cong G_{n+m,m}(\mathbb{C})$ (the manifold of $m-$ (or $n-$)dimensional subspaces of $\mathbb{C}^{m+n}$). The element $z = I_n \oplus (-I_m)$ acts by -1 on $\mathfrak{g}/\mathfrak{k} = V \otimes W^* \oplus W \otimes V^*$, where V, W are the tautological representations of $U(n)$ and $U(m)$. So we get that the Grassmannian has cohomology only in even degrees, and that

$$\mathrm{H}^{2i}(G_{m+n,m}(\mathbb{C})) = \bigwedge^{2i} (V \otimes W^* \oplus W \otimes V^*)^{U(n)\times U(m)}.$$

We can therefore use skew Howe duality to see that $\dim \mathrm{Hom}^{2i}(G_{m+n,m}(\mathbb{C})) = N_i(n,m)$, where $N_i(n,m)$ is the number of partitions λ whose Young diagram has i boxes and fits into the $m \times n$ rectangle.

We have an exterior power of a tensor product of dual spaces

Has at most n parts with transpose having at most m parts

To compute $N_i(m,n)$, consider the generating function $f_{n,m}(q) = \sum_i N_i(n,m)q^i$. Denote by p_i the jumps of our partition, so

$$\sum_{n\geq 0} f_{n,m}(q) z^n = \sum_{p_0,p_1,\ldots,p_m \geq 0} z^{p_0+\cdots+p_m} q^{p_1+2p_2+\cdots+mp_m} = \prod_{j=0}^m \frac{1}{1-q^j z}.$$

Hence the Betti numbers of Grassmannians are the coefficients of this series, e.g. if $m=1$, we see that

$$\sum_{n\geq 0} f_{n,1}(q) z^n = \frac{1}{(1-z)(1-qz)} = \sum_n (1+q+\cdots+q^n)\, z^n$$

which recovers the Poincaré polynomial $1+q+\cdots+q^n$ of the complex projective space $\mathbb{CP}^n = G_{n+1,1}$.

The polynomials $f_{n,m}(q)$ are called the **Gaussian binomial coefficients**, and they can be computed explitictly:

$$f_{m,n}(q) = \binom{n+m}{n}_q = \frac{[m+n]_q!}{[m_q]![n_q]!} \text{ where } [m]_q := \frac{q^m-1}{q-1} \text{ and } [m_q]! := [1]_q[2]_q \ldots [m]_q.$$

In other words, we have the q**-binomial theorem**

$$\sum_{n\geq 0} \binom{n+m}{n}_q z^n = \prod_{j=0}^m \frac{1}{1-q^j z}.$$

Note that setting $q=1$ recovers the familar identity

$$\sum_{n\geq 0} \binom{n+m}{m} z^n = \frac{1}{(1-z)^{m+1}}.$$

Exercise. Compute the Betti number of $G_{N,2}(\mathbb{C})$.

Exercise. Prove the q-binomial theorem.

There is a more geometric way to compute the Betti numbers of Grassmannians by working with Schubert cells. Let $F_i \subset \mathbb{C}^{n+m}$ be spanned by the last i basis vectors $e_{m+n-i+1}, \ldots, e_{m+n}$. Thus, we have a complete flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n+m} = \mathbb{C}^{m+n}.$$

Given an m-dimensional subspace $V \subset \mathbb{C}^{m+n}$, let ℓ_j be the smallest integer for which $\dim(F_{\ell_j} \cap V) = j$. Then,

$$1 \le \ell_1 < \ell_2 < \cdots < \ell_m \le m+n,$$

which defines a partition with parts $\lambda_1 = \ell_m - m$, $\lambda_2 = \ell_{m-1} - (m-1)$, ..., $\lambda_m = \ell_1 - 1$ fitting in the $m \times n$ box. We let $S_\lambda \subset G_{n+m,m}(\mathbb{C})$ be the set of V giving such numbers λ_i.

Exercise. Show that S_λ is an embedded (non-closed) complex submanifold of the Grassmannian isomorphic to the affine space $\mathbb{C}^r$ of dimension $r := |\lambda| = \sum_i \lambda_i$.

Definition 22.15. The subset S_λ of the Grassmannian is called the **Schubert cell** corresponding to λ.

We see that $G_{m+n,m}(\mathbb{C})$ has a cell decomposition into a disjoint union of Schubert cells. This allows one to rederive the same formula for the Poincaré polynomial of the Grassmannian from the following fact from algebraic topology:

Proposition 22.16. *If X is a connected cell complex which only has even-dimensional cells, then the cohomology of X vanishes in odd degrees, and the groups $\mathrm{H}^{2i}(X;\mathbb{Z})$ are free abelian groups of ranks $b_{2i}(X)$, where the Betti number $b_{2i}(X)$ is just the number of cells in X of dimension i. Moreover, X is simply connected.*

(use cellular chain complex)

Corollary 22.17. *$\mathrm{H}^{2i}(G_{n+m,n}(\mathbb{C}),\mathbb{Z})$ is a free abelian group of rank $\binom{n+m}{m}_q$, and the odd cohomology groups are zero. Moreover, Grassmannians are simply connected.*

In particular, this gives Betti numbers of any field.

22.1.1 Flag manifolds

Definition 22.18. The **flag manifold** $\mathcal{F}_n(\mathbb{C})$ is the space of all flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \text{ with } \dim V_i = i.$$

It is a homogeneous space since $\mathcal{F}_n = G/T$, where $G = U(n)$ and $T = U(1)^n$ is a maximal torus in G.

We have fibrations $\pi : \mathcal{F}_n(\mathbb{C}) \to \mathbb{CP}^{n-1}$ sending $(V_1, \dots, V_{n-1})$ to V_{n-1}, whose fiber is the space of flags in V_{n-1}, i.e. is $\mathcal{F}_{n-1}(\mathbb{C})$. By induction[48], one argues that the flag manifolds can be decomposed into even-dimensional cells isomorphic to $\mathbb{C}^r$ (also called **Schubert cells**). Thus, the Betti numbers of $\mathcal{F}_n$ vanish in odd degrees, and in even degrees they are given by the generating function

As a vector space, the cohomology of $\mathcal{F}_n(\mathbb{C})$ will be tensor product of cohomology of $\mathbb{CP}^k$ for $k = 1, \dots, n-1$

$$\sum b_{2i}(\mathcal{F}_n) q^n = [n]_q! = (1+q)(1+q+q^2)\dots(1+q+\cdots+q^{n-1}).$$

Remark 22.19. There is also a map $\pi_m : \mathcal{F}_{m+n}(\mathbb{C}) \to G_{m+n,m}(\mathbb{C})$ sending $(V_1, \dots, V_{n+m-1}) \mapsto V_m$. This is a fibration with fiber $\mathcal{F}_m(\mathbb{C}) \times \mathcal{F}_n(\mathbb{C})$. From this one gets another proof of the formula for Betti numbers of the Grassmannian.

[48] The fiber bundle will become trivial over the cells?

We can also define the **partial flag manifold** $\mathcal{F}_S(\mathbb{C})$ for $S \subset [1, n-1]$, i.e. it is the space of partial flags $(V_s : s \in S)$ with $V_s \subset \mathbb{C}^n$, $\dim V_s = s$, and $V_s \subset V_t$ if $s < t$. These include both (complete) flag manifolds and Grassmannians.

Exercise. Let $S = \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{k-1}\}$ and let $n_k = n - n_1 - \cdots - n_{k-1}$. Show that the even Betti numbers of the partial flag manifold are the coefficients of the polynomials

$$P_S(q) = \frac{[n]_q!}{[n_1]_q! \ldots [n_k]_q!},$$

while the odd Betti numbers vanish. Also, show the partial flag manifold is simply connected.

23 Lecture 23 (5/13): Lie algebra cohomology

We've talked recently about cohomology of Lie algebras. We can generalize our definitions to talk about cohomology with non-trivial coefficients, i.e. with coefficients in a representation of the Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra (over any field, of any dimension), and let V be a $\mathfrak{g}$-module. We can define the **Chevalley-Eilenberg complex**

$$\mathrm{CE}^\bullet(\mathfrak{g}, V) := \mathrm{Hom}_k(\bigwedge\nolimits^\bullet \mathfrak{g}, V),$$

so it looks like

$$0 \to V \to \mathrm{Hom}(\mathfrak{g}, V) \to \mathrm{Hom}(\bigwedge\nolimits^2 \mathfrak{g}, V) \to \cdots.$$

The differential is given by the Cartan differentiation formula

$$\mathrm{d}\omega(a_0, \ldots, a_m) = \sum_{i=0}^{m} (-1)^i a_i \omega(a_0, \ldots, \widehat{a_i}, \ldots, a_m) + \sum_{0 \le i < j \le m} (-1)^{i+j} \omega([a_i, a_j], a_0, \ldots, \widehat{a_i}, \ldots, \widehat{a_j}, \ldots, a_m)$$

for $\omega \in \mathrm{CE}^m$.

Example. If G is a Lie group, $\mathfrak{g} = \mathrm{Lie}\, G$, and V f.dim, then

$$\mathrm{CE}^*(\mathfrak{g}, V) = (\Omega^*(G) \otimes V)^{\mathfrak{g}}$$

with $\mathfrak{g}$ acting diagonally, and the differential is the de Rham differential.

The cohomology of this complex $\mathrm{CE}^\bullet(\mathfrak{g}, V)$ is called the **cohomology of $\mathfrak{g}$ with coefficients in** V and is denoted $\mathrm{H}^*(\mathfrak{g}, V)$. The cohomology we studied before is simply $\mathrm{H}^i(\mathfrak{g}) = \mathrm{H}^i(\mathfrak{g}, \mathbb{C})$.

Proposition 23.1. *If G is compact and V is a f.dim nontrivial irrep, then $\mathrm{H}^i(\mathfrak{g}, V) = 0$ for all $i > 0$.*

(I missed the explanation, but this follows from what we did before. Something about cohomology being computed using invariant forms so all nontrivial irreps drop out or something, who knows).

Remark 23.2. In general, $\mathrm{H}^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$ is $\mathfrak{g}$-invariants.

Proposition 23.3 (Whitehead's lemma). *If $\mathfrak{g}$ is semisimple, then $\mathrm{H}^1(\mathfrak{g}, V) = \mathrm{H}^2(\mathfrak{g}, V) = 0$ for any f.dim V.*

Proof. Can assume V irreducible. If $V \neq \mathbb{C}$, this follows from previous prop, so say $V = \mathbb{C}$. The standard complex starts

$$0 \to \mathbb{C} \xrightarrow{0} \mathfrak{g}^* \xrightarrow{\mathrm{d}} \bigwedge^2 \mathfrak{g}^* \to \cdots .$$

Above, $\mathrm{d}(f) = f([x,y])$. Hence, $\mathrm{H}^1(\mathfrak{g}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathbb{C}) = 0$. Similarly, $\mathrm{H}^2(\mathfrak{g}, \mathbb{C}) = \left(\bigwedge^2 \mathfrak{g}^*\right)^{\mathfrak{g}}$. Why is this 0? Can assume $\mathfrak{g}$ simple. This is the space of $\mathfrak{g}$-invariant skew-symmetric homomorphisms $A : \mathfrak{g} \to \mathfrak{g}^*$ ($A^* = -A$). Note that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}^*) = \mathbb{C}K$ is 1-dimensional, spanned by the killing form. The Killing form is symmetric, not skew-symmetric, so there are no skew-symmetric invariant forms. ■

Remark 23.4. If you look at cohomology of non-semisimple Lie algebras or with coefficients in an infinite-dimensional rep, then things are more complicated.

23.1 Interpretations of $\mathrm{H}^i(\mathfrak{g}, V)$ for small i

23.1.1 $i = 0$

We start with $\mathrm{H}^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$.

23.1.2 $i = 1$

Onto $\mathrm{H}^1(\mathfrak{g}, V) = Z^1(\mathfrak{g}, V)/B^1(\mathfrak{g}, V)$. Here, we have **1-cocycles**

$$Z^1(\mathfrak{g}, V) = \{\omega : \mathfrak{g} \to V \mid \omega([x,y]) = [x, \omega(y)] - [y, \omega(x)]\}$$

and **1-coboundaries** $B^1(\mathfrak{g}, V) = \{\omega = \mathrm{d}v : v \in V\}$ (i.e. $\omega(x) = xv$ which satisfies $[x,y]v = x(yv) - y(xv)$).

Proposition 23.5. *Say V, W are representations of $\mathfrak{g}$. Then,*

$$\mathrm{H}^1(\mathfrak{g}, \mathrm{Hom}_k(V, W)) = \mathrm{Ext}^1(V, W)$$

classifies extensions $0 \to W \to Y \to V \to 0$ *of V by W.*

Proof. Given an extension $0 \to W \to Y \to V \to 0$, we choose a complement of W in Y to write $Y = W \oplus V$ as vector spaces. Under this decomposition, $\mathfrak{g}$ acts by block upper triangular matrices

$$\rho_Y(x) = \begin{pmatrix} \rho_W(x) & \omega(x) \\ 0 & \rho_V(x) \end{pmatrix}.$$

For this to be a representation, we need $\rho_Y([x,y]) = [\rho_Y(x), \rho_Y(y)]$. Note that

$$\rho_Y(x)\rho_Y(y) = \begin{pmatrix} \rho_W(x)\rho_W(y) & \rho_W(x)\omega(y) + \omega(x)\rho_V(y) \\ 0 & \rho_V(w)\rho_V(y) \end{pmatrix}.$$

The condition we get is that (ρ's omitted for brevity)

$$\omega([x,y]) = x\omega(y) - y\omega(x).$$

Hence, ρ_Y is a representation $\iff \omega \in Z^1(\mathfrak{g}, \mathrm{Hom}_k(V,W))$.

Exercise. If Y_1, Y_2 are two such representations, then $Y_1 \cong Y_2$ as extensions iff $\omega_1 - \omega_2 \in B^1(\mathfrak{g}, \mathrm{Hom}_k(V,W))$.

■

Note 9. I have been pretty distracted most of this lecture, so I keep missing small things.

We're talking about semidirect products now.

Definition 23.6. $\mathfrak{g} \ltimes V$ is $V \oplus \mathfrak{g}$ with Lie bracket

$$[(v_1, x_1), (v_2, x_2)] := (x_1 v_2 - x_2 v_1, [x_1, x_2]) .$$

This comes with a natural surjection $\mathfrak{g} \ltimes V \twoheadrightarrow \mathfrak{g}$. What are the splittings $x \mapsto (\omega(x), x)$ of this map? The condition for ω is precisely the 1-cocycle condition: $\omega([x,y]) = x\omega(y) - y\omega(x)$, so we need $\omega \in Z^1(\mathfrak{g}, V)$. Note that V acts on $\mathfrak{g} \ltimes V$ by automorphisms: $w \cdot (v, x) = (v + xw, x)$. We call this 'conjugation by v.'

Exercise. Sections s_1, s_2 are conjugate $\iff \omega_1 - \omega_2 \in B^1$ differ by a coboundary.

Corollary 23.7. *Splittings of $\mathfrak{g} \ltimes V \twoheadrightarrow \mathfrak{g}$, up to conjugation, are classified by $\mathrm{H}^1(\mathfrak{g}, V)$.*

Remark 23.8. By previous interpretation, we also know

$$\mathrm{H}^1(\mathfrak{g}, V) = \mathrm{Ext}^1(\mathbb{C}, V).$$

Let's see yet another interpretation. Consider $V = \mathfrak{g}$ the adjoint representation. Consider $\omega : \mathfrak{g} \to \mathfrak{g}$ with $\omega \in Z^1(\mathfrak{g}, \mathfrak{g})$. Then,

$$\omega([x,y]) = [x, \omega(y)] - [y, \omega(x)] = [x, \omega(y)] + [\omega(x), y],$$

so $\omega \in \mathrm{Der}(\mathfrak{g})$, i.e. ω is a cocycle if it's a derivation. The coboundaries $\omega \in B^1(\mathfrak{g}, \mathfrak{g})$ are the **inner derivations**, $\omega(x) = [\mathrm{d}, x]$ for some $\mathrm{d} \in \mathfrak{g}$. Thus,

$$\mathrm{H}^1(\mathfrak{g}, \mathfrak{g}) = \frac{\mathrm{Der}(\mathfrak{g})}{\mathrm{Inn}(\mathfrak{g})} = \mathrm{Out}(\mathfrak{g})$$

is the space of **outer derivations**.

23.1.3 $i = 2$

For H^2, we'll need to talk about abelian extensions.

Definition 23.9. An **abelian extension** is a Lie algebra $\widetilde{\mathfrak{g}}$ sitting in a short exact sequence

$$0 \longrightarrow V \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

with V an abelian ideal.

Example. $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes V$ is a split abelian extension (of $\mathfrak{g}$ by V)

Example. The Heisenberg algebra $\mathcal{H}$ has basis x, y, c with c central ($[x, c] = [y, c] = 0$) and $[x, y] = c$. Let $V = \langle c \rangle$, a 1-dimensional abelian ideal. We have $\mathcal{H} = k^2 = \langle x, y \rangle$ (abelian quotient). This gives an exact sequence

$$0 \to k \to \mathcal{H} \to k^2 \to 0$$

which is *not* split.

How can we classify abelian extensions of $\mathfrak{g}$ by V?

Note that exact sequences of Lie algebras always split as vector spaces, so we can write $\widetilde{\mathfrak{g}} = V \oplus \mathfrak{g}$ as vector spaces. We then get a commutator

$$[(v, x), (w, y)] = (xw - yv + \omega(x, y), [x, y]) \text{ with } \omega : \bigwedge^2 \mathfrak{g} \to V.$$

What is the condition of ω for this to be a Lie algebra structure? The condition is given by the Jacobi identity (this is already skew-symmetric by definition). One checks that this satisfies Jacobi $\iff \omega \in Z^2(\mathfrak{g}, V)$ (exercise). Furthermore, $\widetilde{\mathfrak{g}}_1 \cong \widetilde{\mathfrak{g}}_2$ (as extensions) iff $\omega_1 - \omega_2 = \mathrm{d}\eta \in B^2(\mathfrak{g}, V)$.

Proposition 23.10. *Up to equivalence, abelian extensions of* $\mathfrak{g}$ *by* V *are classified by* $\mathrm{H}^2(\mathfrak{g}, V)$.

Example. Say $\mathfrak{g} = k^2 = \langle x, y \rangle$ and let $V = k$ be the trivial rep. Then,

$$\mathrm{H}^*(\mathfrak{g}, k) = \mathrm{H}^*(\mathfrak{g}) = \mathrm{H}^*(S^1 \times S^1) = \begin{cases} k & \text{if } * = 0, 2 \\ k^2 & \text{if } * = 1 \\ 0 & \text{if } * \geq 3. \end{cases}$$

In particular, $\mathrm{H}^2(\mathfrak{g}) = k$ with nonzero element corresponding to the Hiesenberg algebra (up to some scaling)

Corollary 23.11. *If* $\mathfrak{g}$ *is semisimple and* $\widetilde{\mathfrak{g}}$ *is an abelian extension of* $\mathfrak{g}$ *by a f.dim rep* V*, then* $\widetilde{\mathfrak{g}} = \mathfrak{g} \ltimes V$.

Whitehead says $\mathrm{H}^2(\mathfrak{g}, V) = 0$.

There's another interpretation in terms of deformations of the Lie algebra. Say $\mathfrak{g}$ over k with Lie bracket $[\,,\,] : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. Can we deform it with parameter t? Something like

$$[x, y]_t = [x, y] + tc_1(x, y) + t^2 c_2(x, y) + \ldots,$$

a formal power series. These coefficients will be maps $c_i : \bigwedge^2 \mathfrak{g} \to \mathfrak{g}$. We want the above to be a Lie bracket (i.e. satisfy Jacobi) for all t; that is, it should give a Lie algebra structure on $\mathfrak{g}[\![t]\!]$ so that, mod t, you recover the original one.

We'd like to understand/analyze things term-by-term. We start with first order analysis, work mod t^2. That is, we work with the ring $\mathfrak{g}[t]/t^2\mathfrak{t}[t] = \mathfrak{g} \oplus t\mathfrak{g}$. Note we have an exact sequence

$$0 \to t\mathfrak{g} \to \mathfrak{g} \oplus t\mathfrak{g} \to \mathfrak{g} \to 0$$

with $t\mathfrak{g}$ abelian, so we have an abelian extension of $\mathfrak{g}$ by itself with zero commutator. Hence, the condition on c_1 is that it should be a 2-cocycle: $c_1 \in Z^2(\mathfrak{g}, \mathfrak{g})$. Up to isomorphisms: $a = 1 + ta_1 + t^2 a_2 + \ldots$ with

$a_i \in \operatorname{End}_k \mathfrak{g}$. Possible first order deformations c_1 are classified up to isomorphism by $\mathrm{H}^2(\mathfrak{g},\mathfrak{g})$. In particular if $\mathrm{H}^2(\mathfrak{g},\mathfrak{g}) = 0$, then all deformations of $\mathfrak{g}$ are in fact trivial (isom to $c_1 = c_2 = c_3 = \cdots = 0$).

Question: What is a?

(1) kill c_1 by applying $a^{(1)} = 1 + ta_1 + \dots$. This gives

$$[x,y]_t = [x,y] + t^2 c_2(x,y) + \dots.$$

(2) Now one discovers that we have $[c_2] \in Z^2(\mathfrak{g},\mathfrak{g}) = B^2(\mathfrak{g},\mathfrak{g})$ so we can kill it as well by $a^{(2)} = 1 + t^2 a_2 + \dots$. This gives $[x,y]_t = [x,y] + t^3 c_3(x,y) + \dots$.

(3) Now one continues. Use the composition $\cdots \circ a^{(3)} \circ a^{(2)} \circ a^{(1)} =: a$ (this makes sense since only finitely many degrees involved in each step). This transforms the original deformation to the trivial one with $[x,y]_t = [x,y]$,

Corollary 23.12. *If $\mathfrak{g}$ is a semisimple Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, then it is* ***deformationally rigid*** *in the sense that all its deformations are trivial.*

Example. Say $\mathfrak{g} = k^2 = \langle x, y\rangle$ ($[x,y] = 0$). Then, $\mathrm{H}^2(\mathfrak{g},\mathfrak{g}) = k^2$ so have 2-parameter deformation. Can take $[x,y] = tx + sy$ with $t, s \in \mathbb{C}$. If $(t,s) \neq 0$, all are isomorphic as Lie algebras (though not as deformations) by action of $\mathrm{GL}_2(\mathbb{C})$. Can always bring it to the form $[x,y] = y$, i.e. to the Lie algebra

$$\mathrm{aff}_1 := \mathrm{Lie}\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in \mathbb{C} \right\} \subset \mathfrak{gl}_2.$$

Suppose $\mathfrak{g}$ a Lie algebra with $c_1 \in Z^2(\mathfrak{g},\mathfrak{g})$ and $[c_1] \neq 0 \in \mathrm{H}^2(\mathfrak{g},\mathfrak{g})$. Can we lift our deformation mod t^3? Can we find c_2 so that

$$[x,y]_t = [x,y] + tc_1(x,y) = t^2 c_2(x,y)$$

satisfies Jacobi? If $c_1 = 0$, the condition would be $\mathrm{d}c_2 = 0$ (that it is a cocycle). In general, the condition is

$$\mathrm{d}c_2 = \frac{1}{2}[c_1, c_1] \in \operatorname{Hom}_k(\textstyle\bigwedge^3 \mathfrak{g}, \mathfrak{g})$$

where $[-,-]$ is the Schouten bracket (this is some explicit quadratic expression we won't write down).

Exercise. $[c_1, c_1]$ is a cocycle.

Hence we get a cohomology class $\Big[[c_1, c_1]\Big] \in \mathrm{H}^3(\mathfrak{g},\mathfrak{g})$. To get a lifting (i.e. to solve $\mathrm{d}c_2 = [c_1, c_1]$), this needs to be a coboundary, i.e. the **obstruction class** $[[c_1, c_1]]$ needs to vanish.

Remark 23.13. Solving these extension problems depends on the choices you make along the way (i.e. whether or not you can find c_2 depends on what you choose for c_1), so things can get hairy fast.

One can also consider deformations of modules. So you have $\mathfrak{g}$ and a module V, and you want to deform to a module $V[\![t]\!]$. Say $\rho = \rho_V : \mathfrak{g} \to \operatorname{End} V$. We now want

$$\rho_t = \rho + t\rho_1 + t^2 \rho_2 + \dots \quad \text{with} \quad \rho_i : \mathfrak{g} \to \operatorname{End} V.$$

We start again with first order analysis (i.e. with working mod t^2): $\rho_t = \rho + t\rho_1 + O(t^2)$. Note $V[t]/t^2 V[t] = V \oplus tV$ so we get an extension

$$0 \longrightarrow tV \to V \oplus tV \longrightarrow V \longrightarrow 0$$

of modules. We see that first order deformations of V, up to isomorphism, are classified by $\mathrm{H}^1(\mathfrak{g}, \mathrm{End}_k V) = \mathrm{Ext}^1(V, V)$. Deformations are a non-linear problem, so we are not done yet.

Can we lift $\rho_t = \rho + t\rho_1 + O(t^2)$ modulo t^3? Again, one gets that they need $\mathrm{d}\rho_2 = [\rho_1, \rho_1]$. Thus, you have an obstruction class $[[\rho_1, \rho_1]] \in \mathrm{H}^2(\mathfrak{g}, \mathrm{End}_k V)$ and you can lift iff it vanishes.

23.2 Levi decomposition theorem

Recall 23.14. The **radical** of $\mathfrak{g}$ is its maximal solvable ideal, denoted $\mathrm{rad}(\mathfrak{g})$. The quotient $\mathfrak{g}_{ss} := \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is the largest semisimple quotient of $\mathfrak{g}$.

Theorem 23.15 (**Levi decomposition**)**.** *Let $\mathfrak{g}$ be a f.dim Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. The exact sequence*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{ss} \to 0$$

splits. In particular, $\mathfrak{g}_{ss}$ acts on the radical $\mathrm{rad}(\mathfrak{g})$*.*

Warning 23.16. $\mathrm{rad}(\mathfrak{g})$ is *not* abelian in general.

Once we establish this, we'll use it to prove the 3rd fundamental theorem (that every f.dim Lie algebra is the Lie algebra of some simply connected Lie group).

Tuesday's lecture will be prerecorded and posted online at the usual time. No zoom meeting/real-time class meeting on Tuesday.

24 Lecture 24 (5/18)

Last time we introduced the Levi decomposition theorem.

Recall 24.1 (Levi decomposition)**.** Let $\mathfrak{g}$ be a f.dim Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. The exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{ss} \to 0$$

splits. In particular, $\mathfrak{g} \cong \mathfrak{g}_{ss} \ltimes \mathrm{rad}(\mathfrak{g})$, and $\mathfrak{g}_{ss}$ acts on the radical $\mathrm{rad}(\mathfrak{g})$.

Above, recall $({}_{\mathrm{red}}\mathfrak{g})$ is the sum of all solvable ideals in $\mathfrak{g}$.

We stated this last semester and said the proof will be given later. It's later.

Proof of Levi decomposition. Choose a splitting of vector spaces $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$. It's commutator will be of the form

$$[(a, x), (b, y)] = ([x, b] - [y, a] + [a, b] + \omega(x, y), [x, y]) \text{ for some } \omega : \bigwedge^2 \mathfrak{g}_{ss} \to \mathrm{rad}(\mathfrak{g}),$$

where $a, b \in \mathrm{rad}(\mathfrak{g})$ and $x, y \in \mathfrak{g}_{ss}$. Since $\mathrm{rad}(\mathfrak{g})$ is solvable, it has the filtration

$$\mathrm{rad}(\mathfrak{g}) = D^0 \supset D^1 \supset \cdots \supset D^n \supset D^{n+1} = 0 \text{ with } D^{i+1} = [D^i, D^i]$$

(we suppose $D^n \neq 0$). We can replace $\mathfrak{g}$ by $\mathfrak{g}/D^n$ and then use induction in $\dim \mathfrak{g}$ to assume that $\omega = 0$ mod D^n, i.e. $\omega : \bigwedge^2 \mathfrak{g}_{ss} \to D^n$. Now, D^n is an *abelian* ideal in $\mathfrak{g}$. Hence, since D^n is abelian, our commutator satisfies Jacobi iff ω is a 2-cocycle. Now, Whitehead's lemma says that $\mathrm{H}^2(\mathfrak{g}_{ss}, D^n) = 0$, so $\omega = \mathrm{d}\eta$ is a coboundary. Now, we can use η to modify the splitting, so that ω becomes 0. ∎

We would like to prove the 3rd Lie theorem (any f.dim Lie algebra is the Lie algebra of a Lie group) and also Ado's theorem (any f.dim Lie algebra has a faithful rep). Doint so will require some more technology, which brings us to....

24.1 The nilradical

We will consider nilradicals of *solvable* Lie algebras.

Say $\mathfrak{a}$ is a solvable Lie algebra. By Lie's theorem, in some basis of the adjoint representation (or any rep), the matrices of $\mathrm{ad}(x)$ $(x \in \mathfrak{a})$ are upper triangular:

$$\mathrm{ad}(x) = \begin{pmatrix} \lambda_1(a) & & * \\ & \ddots & \\ & & \lambda_n(a) \end{pmatrix}.$$

Definition 24.2. The **nilradical** of $\mathfrak{a}$ is the subset $\mathfrak{n}$ of nilpotent elements (i.e. $a \in \mathfrak{a}$ s.t. $\mathrm{ad}(a)$ is nilpotent).

Using this upper triangular basis, one can write this as

$$\mathfrak{n} = \{a \in \mathfrak{a} : \mathrm{ad}(a) \text{ is strictly upper triangular}\}.$$

This is an ideal containing $[\mathfrak{a}, \mathfrak{a}]$ (commutator of two triangular matrices is strictly upper triangular), so $\mathfrak{a}/\mathfrak{n}$ is abelian.

The characters $\lambda_1, \ldots, \lambda \in (\mathfrak{a}/\mathfrak{n})^*$ are a spanning set. If not, there is an element of $\mathfrak{a}$, not in $\mathfrak{n}$, but whose adjoint matrix is nilpotent. Note that some λ_i may be zero (e.g. if $\mathfrak{a}$ is nilpotent, they are all 0).

Lemma 24.3. *If* $\mathrm{d} : \mathfrak{a} \to \mathfrak{a}$ *is a derivation, then* $\mathrm{d}(\mathfrak{a}) \subset \mathfrak{n}$.

Proof. Look at 1-parameter group of automorphisms $e^{td} : \mathfrak{a} \to \mathfrak{a}$. If $b \in \mathfrak{a}$ with $[a, b] = \lambda(a)b$ $(\lambda \in (\mathfrak{a}/\mathfrak{n})^*)$, then

$$[e^{td}(a), e^{td}(b)] = \lambda(a)e^{td}(b),$$

i.e. $[\widetilde{a}, e^{td}(b)] = \lambda(e^{-td}\widetilde{a})e^{td}(b)$. Hence, if $\lambda(a)$ occurs in $\mathfrak{a}$, then so does $\lambda_t(a) = \lambda(e^{-td}(a))$. By 'occurs' we mean shows up as a Jordan-Hölder factor. Only finitely many characters can occur, so this 1-parameter family must be constant. Thus, $e^{td}\lambda_i = \lambda_i$ for all i. Therefore, e^{td} acts trivially on $(\mathfrak{a}/\mathfrak{n})^*$ so it acts trivially on $\mathfrak{a}/\mathfrak{n}$, i.e. $\mathrm{d}|_{\mathfrak{a}/\mathfrak{n}} = 0$ which exactly says $\mathrm{d}(\mathfrak{a}) \subset \mathfrak{n}$. ∎

Corollary 24.4. *If* $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$*, then* $\mathfrak{g}$ *acts trivially on* $\mathfrak{a}/\mathfrak{n}$.

24.2 Exponentiating nilpotent and solvable Lie algebras, and 3rd Lie theorem

Say $\mathfrak{g}$ is a f.dim *solvable* Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theorem 24.5. *There exists a simply connected Lie group G with Lie algebra $\mathfrak{g}$ with the exponential map $\exp : \mathfrak{g} \to G$ being a diffeomorphism. Moreover, if $\mathfrak{g}$ is nilpotent and we identify $\exp : \mathfrak{g} \xrightarrow{\sim} G$, then multiplication is given by a polynomial*

$$p : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

Example. Say $\mathfrak{g}$ is the **Heisenberg Lie algebra** $\mathfrak{H} = \langle x, y, c\rangle$ with $[x, y] = c$ and $[x, c] = 0 = [y, c]$. Equivalently,

$$\mathcal{H} = \left\{ \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

is the Lie algebra of 3×3 upper triangular matrices. Can check that

$$\exp \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \alpha & \gamma + \frac{\alpha\beta}{2} \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix}.$$

In these coordinates, multiplication looks like

$$(\alpha, \beta, \gamma) * (\alpha', \beta', \gamma') = \left(\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \frac{1}{2}(\alpha\beta' - \beta\alpha')\right).$$

Proof of Theorem 24.5. Induct on $\dim \mathfrak{g}$. Suppose known for all Lie algebras of dimension $< \dim \mathfrak{g}$. Fix $\chi : \mathfrak{g} \to \mathbb{K}$ a nontrivial character (exists since $\mathfrak{g}$ solvable). Let $\mathfrak{g}_0 := \ker \chi$, an ideal of codimension 1 in $\mathfrak{g}$. Hence, $\mathfrak{g} = \mathbb{K}\mathrm{d} \oplus \mathfrak{g}_0$ is a semidirect products (d acts as a derivative of $\mathfrak{g}_0$). We know by inductive assumption that $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$ for some G_0 with

$$\exp : \mathfrak{g}_0 \to G_0$$

a diffeomorphism, and $P : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$ a regular multiplication map, polynomial if $\mathfrak{g}_0$ nilpotent.

Consider the 1-parameter group of automorphisms $e^{td} : \mathfrak{g}_0 \to \mathfrak{g}_0$. Can now introduce group law on $\mathfrak{g} = \mathbb{K}\mathrm{d} \oplus \mathfrak{g}_0$:[49]

$$(x, t) * (y, s) = \big(P(x, e^{td}(y)), t + s\big)$$

where $x, y \in \mathfrak{g}_0$ and $t, s \in \mathbb{K}$. One can check (exercise)

(1) this is a group law, defining a Lie group G with $\mathrm{Lie}(G) = \mathfrak{g}$ and $\exp : \mathfrak{g} \to G$ a diffeomorphism. More precisely,

$$\exp(td + x) = (t, x_t) \text{ where } x_t = \sum_{n \geq 1} \frac{t^{n-1}\mathrm{d}^{n-1}(x)}{n!} = \left.\frac{e^u - 1}{u}\right|_{u=td}(x).$$

Example.

$$\exp \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^\alpha & \frac{e^\alpha - 1}{\alpha} \cdot \beta \\ 0 & 1 \end{pmatrix}$$

(2) If $\mathfrak{g}$ is nilpotent, the multiplication law is polynomial

[49] We want a semidirect product

■

Definition 24.6. If $\mathfrak{g}$ is nilpotent, the corresponding simply connected group G is called **unipotent** (it acts by unipotent operators in the adjoint representation).

Theorem 24.7 (Third Lie Theorem). *Every f.dim Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ is the Lie algebra of some Lie group.*

Proof. By Levi decomposition, we have $\mathfrak{g} = \mathfrak{g}_{ss} \ltimes \mathfrak{a}$ where $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$ is solvable. By previous theorem, $\mathfrak{a} = \mathrm{Lie}\, A$ with A simply connected. Furthermore, since $\mathfrak{g}_{ss}$ is semisimple, we can write $\mathfrak{g}_{ss} = \mathrm{Lie}\, G_{ss}$ with G_{ss} simply connected. Furthermore, G_{ss} acts on $\mathfrak{a}$ by automorphisms, so it acts on A by automorphisms. We can now form $G = G_{ss} \ltimes A$ and by construction $\mathrm{Lie}\, G = \mathfrak{g}$. ■

Corollary 24.8. *A simply connected complex Lie group G has homotopy type of its semisimple part G_{ss}, and hence of G_{ss}^c. Specifically,*

$$G \cong G_{ss}^c \times \mathbb{R}^m$$

as a manifold.

Remark 24.9. Almost the same thing holds for real group. If G is a simply connected real Lie group, we also have $G \sim G_{ss}$ (homotopy equivalent) and $G_{ss} \sim K_{ss}$, the simply connected compact group corresponding to $\mathfrak{k}_{ss} \subset \mathfrak{k}$, the semisimple part of $\mathfrak{k} = \mathfrak{g}_{ss}^{\sigma}$.

The upshot is that any Lie group has the homotopy type of some compact Lie group (its maximal compact subgroup).

24.3 Algebraic Lie algebras

We want to show every Lie algebra has a faithful representation. We will show this over $\mathbb{R}, \mathbb{C}$, but it is in fact true over any characteristic.

Definition 24.10. A f.dim Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ is **algebraic** if it is the Lie algebra of an algebraic group, i.e. $\mathfrak{g} = \mathrm{Lie}(G)$ where $G = K \ltimes N$ with K reductive and $\mathrm{Lie}\, N$ nilpotent (i.e. N unipotent).

Non-example. Consider $\mathfrak{g}_1 = \langle \mathrm{d}, x, y \rangle$ with $[x, y] = 0$, $[\mathrm{d}, x] = x$ and $[\mathrm{d}, y] = \sqrt{2} y$.

Exercise. This is not an algebraic Lie algebra, ultimately because $\sqrt{2}$ is irrational.

Non-example. Consider $\mathfrak{g}_2 = \langle \mathrm{d}, x, y \rangle$ with $[x, y] = 0$, $[\mathrm{d}, x] = x$, and $[\mathrm{d}, y] = y + x$.

Exercise. This is not algebraic, ultimately because d is a nontrivial Jordan block.

On the other hand

Proposition 24.11. *Every f.dim Lie algebra over $\mathbb{C}$ is a Lie subalgebra of an algebraic Lie algebra.*

Proof. We first make a definition. Say $\mathfrak{g}$ is n**-algebraic** if $\mathfrak{g} = \mathrm{Lie}\, G$ and $G = K \ltimes A$, where K is reductive and $\mathfrak{a} = \mathrm{Lie}\, A$ is solvable with $\dim(\mathfrak{a}/\mathfrak{n}) \leq n$. Note 0-algebraic = algebraic.

Example. $\mathfrak{g}_1, \mathfrak{g}_2$ are 1-algebraic.

Now on to the proof. Any $\mathfrak{g}$ is of the form $\mathfrak{g} = \mathfrak{g}_{ss} \ltimes \mathfrak{a}$ with $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$ solvable. Thus, any f.dim Lie algebra is n-algebraic for some n. Hence, it suffices to show that an n-algebraic Lie algebra can be embedded into an $(n-1)$-algebraic Lie algebra (when $n \geq 1$).

Suppose $\mathfrak{g}$ is n-algebraic, so $\mathfrak{g} = \mathrm{Lie}\, G$ and $G = K \ltimes A$ with K reductive and A simply connected with $\mathfrak{a} = \mathrm{Lie}(A)$ solvable satisfying $\dim(\mathfrak{a}/\mathfrak{n}) = n$. Pick $\mathrm{d} \in \mathfrak{a}$ but not in $\mathfrak{n}$ s.t d is K-invariant (exists since K acts trivially on $\mathfrak{a}/\mathfrak{n}$ and since reps of K are completely reducible). Thus, $\mathrm{ad(d)}$ is a derivation of $\mathfrak{a}$, so we can write

$$\mathfrak{a} = \bigoplus_{i=1}^{r} \mathfrak{a}(\beta_i)$$

as a sum of generalized eigenspaces for ad(d). This decomposition is K-stable (K commutes with d). Pick character $\chi : \mathfrak{a} \to \mathbb{C}$ so that $\chi(\mathrm{d}) = 1$.

Consider subgroup $\Gamma \subset \mathbb{C}$ generated by β_i, $\Gamma \cong \mathbb{Z}^m$. Let $\alpha_1, \ldots, \alpha_m$ be a basis and write

$$\beta_i = \sum_{j=1}^{m} b_{ij}\alpha_j, \ \ b_{ij} \in \mathbb{Z}.$$

Let $T = (\mathbb{C}^\times)^m$, and m-dimensional torus. We make T act on $\mathfrak{a}$ via $z = (z_1, \ldots, z_m) \in T$ satisfies

$$z|_{\mathfrak{a}[\beta_i]} = \prod_{j=1}^{m} z_j^{b_{ij}}.$$

This T commutes with K, so T acts on $G = K \ltimes A$. Form

$$\widetilde{G} = T \ltimes G = (K \times T) \ltimes A.$$

Define $\mathfrak{a}' \subset \mathrm{Lie}(T) \ltimes \mathfrak{a} \subset \mathrm{Lie}(\widetilde{G})$ by

$$\mathfrak{a}' = \langle \ker \chi, \mathrm{d} - \alpha \rangle \ \text{ where } \ \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathrm{Lie}\, T = \mathbb{C}^m.$$

Note that $\alpha|_{\mathfrak{a}[\beta_i]} = \sum b_{ij}\alpha_j = \beta_i$. Thus, d, α have the same eigenvalues (α semisimple, d possibly not), so $\mathrm{d} - \alpha$ is nilpotent. Thus, $\mathfrak{n}' \subset \mathfrak{a}'$, the nilradical, is bigger: $\mathfrak{n}' = \langle \mathfrak{n}, \mathrm{d} - \alpha \rangle$ and $\dim(\mathfrak{a}'/\mathfrak{n}') = n - 1$. We now let A' be the simply connected Lie group corresponding to $\mathfrak{a}'$, so $A' \subset (K \times T) \ltimes A$. Note that

$$\widetilde{G} = (K \times T) \ltimes A = (K \times T) \ltimes A'$$

so $\widetilde{\mathfrak{g}} = \mathrm{Lie}\, \widetilde{G}$ is $(n-1)$-algebraic and contains $\mathfrak{g} = \mathrm{Lie}(K \ltimes A)$. ■

Example. Recall $\mathfrak{g}_1 = \langle d, x, y \rangle$ with $[d, x] = x$, $[d, y] = \sqrt{2}y$. Add new generator δ so that $[\delta, y] = y$ and $[\delta, x] = 0 = [\delta, \mathrm{d}]$. Call the result algebra $\widetilde{\mathfrak{g}}_1$. Note that it is algebraic as $\widetilde{\mathfrak{g}}_1 = \langle \delta, y \rangle \oplus \langle \mathrm{d} - \sqrt{2}\delta, x \rangle$. We see that $\widetilde{\mathfrak{g}}_1 = \mathfrak{b} \oplus \mathfrak{b}$ with $\mathfrak{b}$ a 2 dimensional noncommutative Lie algebras, $= \mathrm{Lie}\, \mathrm{Aff}(1)$, the Lie algebra of the group of affine transformations of a line.

Example. Recall $\mathfrak{g}_2 = \langle d, x, y \rangle$ with $[d, x] = x$, $[d, y] = x + y$. Adjoint δ satisfying $[\delta, x] = 0 = [\delta, d]$ and $[\delta, y] = x$. Let $\widetilde{\mathfrak{g}}_2 = \langle \delta, d, x, y \rangle$. This is $\mathbb{C}(d - \delta) \ltimes \mathfrak{H}$ with $\mathcal{H} = \langle \delta, x, y \rangle$ the Heisenberg Lie algebra.

24.4 Faithful representations of nilpotent Lie algebras

Let $\mathfrak{n}$ be a f.d nilpotent Lie algebra. We now $\mathfrak{n} = \operatorname{Lie} N$ with N unipotent, and $\exp : \mathfrak{n} \xrightarrow{\sim} N$ an isomorphism. Furthermore, the induced group law on $\mathfrak{n}$ is polynomial $P : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$.

Proposition 24.12. *Let $\mathscr{O}(N)$ be the space of polynomial functions on $N \cong \mathfrak{n}$, $\mathscr{O}(N) = \mathbb{C}[\mathfrak{n}] = S\mathfrak{n}^*$. Then, $\mathscr{O}(N)$ is invariant under the action of $\mathfrak{n}$ by left-invariant vector fields. Moreover, we have a canonical filtration*

$$\mathscr{O}(N) = \bigcup_{n \geq 1} V_n$$

where $V_n \subset \mathscr{O}(N)$ are f.dim subspaces, $V_1 \subset V_2 \subset \dots$, and $nV_i \subset V_{i-1}$.

Prove this next time.

25 Lecture 25 (5/20): Last Class

25.1 Ado's Theorem

We're working towards proving Ado's theorem. We will first prove it in the case of nilpotent Lie algebras.

Recall 25.1. Say $\mathfrak{n}$ is a nilpotent Lie algebra, then we can write $\mathfrak{n} = \operatorname{Lie} N$ with N a unipotent group and $\exp : \mathfrak{n} \xrightarrow{\sim} N$ an isomorphism. The induced group law on $\mathfrak{n}$, a deformation of addition, is a polynomial $P : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$.

We ended last time with the statement of the following proposition.

Proposition 25.2. *Let $\mathscr{O}(N)$ be the space of polynomial functions on $N \cong \mathfrak{n}$, $\mathscr{O}(N) = \mathbb{C}[\mathfrak{n}] = S\mathfrak{n}^*$. Then, $\mathscr{O}(N)$ is invariant under the action of $\mathfrak{n}$ by left-invariant vector fields. Moreover, we have a canonical filtration*

$$\mathscr{O}(N) = \bigcup_{n \geq 1} V_n$$

where $V_n \subset \mathscr{O}(N)$ are f.dim subspaces, $V_1 \subset V_2 \subset \dots$, and $nV_i \subset V_{i-1}$.

The existence of such a filtration make $\mathscr{O}(N)$ a 'locally finite module' (I think this is the terminology)

Proof. Say $x \in \mathfrak{n}$, so it has an associated left-invariant vector field L_x. For $f \in \mathscr{O}(N) = S\mathfrak{n}^*$, we have, by definition,

$$(L_x f)(y) = \left.\frac{\partial}{\partial t}\right|_{t=0} f\left(y * tx\right) = \left.\frac{\partial}{\partial t}\right|_{t=0} f\left(P(y, tx)\right).$$

Since f, P are both polynomials, we see that $L_x f$ is a polynomial in y, so $L_x f \in \mathscr{O}(N)$. Thus, $\mathscr{O}(N)$ is invariant under the action of $\mathfrak{n}$ by left-invariant vector fields.

Now we given the filtration. Recall that $\mathfrak{n}$ has a lower central series filtration

$$\mathfrak{n} = D_0(\mathfrak{n}) \supset D_1(\mathfrak{n}) \supset \dots \supset D_m(\mathfrak{n}) = 0 \text{ where } D_i(\mathfrak{n}) = [\mathfrak{n}, D_{i-1}(\mathfrak{n})].$$

We can take orthogonal complements of these spaces to get

$$0 = D_0(\mathfrak{n})^\perp \subset D_1(\mathfrak{n})^\perp \subset \dots \subset D_m(\mathfrak{n})^\perp = \mathfrak{n}.$$

Now pick a suff. large positive integer d, and give $D_i(\mathfrak{n})^\perp$ degree d^i. This gives an increasing filtration $F^\bullet$ on the symmetric algebra $S\mathfrak{n}^* = \mathscr{O}(N)$. Write

$$P(x,y) = x + y + \sum_{i \geq 1} Q_i(x,y) \text{ where } Q_i : \mathfrak{n} \times \mathfrak{n} \to [\mathfrak{n}, \mathfrak{n}]$$

and Q_i is degree i in y.[50] Now note that[51]

$$(L_x f)(y) = \partial_x f(y) + \left(\partial_{Q_1(x,y)} f\right)(y).$$

Note that $f(y) \mapsto \partial_x f(y)$ lowers the degree in $F^\bullet$.

Exercise. If $d \gg 0$ big enough, then also the second term $f \mapsto \partial_{Q_1(x,y)f}$ lowers the degree.

Hence, we may take $V_n = F^n(S\mathfrak{n}^*)$ and will have $L_x : V_n \to V_{n-1}$. ■

Example. Consider the Heisenberg Lie algebra $\mathfrak{H} = \langle x, y, c\rangle$ with $[x,y] = c$ while $[x,c] = 0 = [y,c]$ (i.e. c central). Then,

$$e^{tx}e^{sy} = e^{tx+sy+\frac{1}{2}tsc}$$

(all higher commutators vanish). If $u = px + qy + rc$ with $p, q, r \in \mathbb{C}$ then multiplication in these coordinates is given by

$$(p_1, q_1, r_1) * (p_2, q_2, r_2) = (p_1 + p_2, q_1 + q_2, r_1 + r_2 + \frac{1}{2}(p_1 q_2 - p_2 q_1)).$$

You can alternative describe things using upper triangular nilpotent matrices.

$$\begin{pmatrix} 1 & p_1 & r_1 \\ & 1 & q_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & p_2 & r_2 \\ & 1 & q_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & p_1 + p_2 & r_1 + r_2 + p_1 q_2 \\ & 1 & q_1 + q_2 \\ & & 1 \end{pmatrix}$$

which is a slightly different, but isomorphic, group law. One can check that

$$L_c = \partial_r, \;\; L_x = \partial_p - \frac{1}{2} q \partial_r, \;\text{ and }\; L_y = \partial_q + \frac{1}{2} p \partial_r.$$

Setting $\deg p = \deg q = d$ and $\deg r = d^2$, these operators lower degree if $d > 1$.

Corollary 25.3. *Every f.dim nilpotent Lie algebra over $\mathbb{C}$ has a faithful f.dim representation, and therefore is isomorphic to a Lie subalgebra of the Lie algebra of strictly upper triangular matrices.*

Proof. By definition, $\mathscr{O}(N)$ is a faithful $\mathfrak{n}$-module. Hence, for some n, the space V_n is also faithful. ■

We now prove Ado's theorem.

Theorem 25.4 (Ado's Theorem). *Every f.dim Lie algebra over $\mathbb{C}$ has a faithful f.dim representation, i.e. is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.*

[50] Apply Campbell-Hausdorff expansion to $P(x,y) = \log(e^x e^y)$

[51] Keep in mind $P(y, tx) = y + tx + \sum Q_i(y, tx)$

Proof. We know from last time that $\mathfrak{g}$ can be embedded into an algebraic Lie algebra, so we may assume $\mathfrak{g}$ is itself algebraic, i.e. that $\mathfrak{g} = \operatorname{Lie} G$ where $G = K \ltimes N$ with K reductive and N unipotent (i.e. $\operatorname{Lie} N = \mathfrak{n}$) with action of K. Thus $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n}$ with $\mathfrak{k} = \operatorname{Lie} K$ and $\mathfrak{n} = \operatorname{Lie} N$. Let $\mathfrak{z} \subset \mathfrak{k}$ be the centralizer of $\mathfrak{n}$. Since $\mathfrak{k}$ is reductive and $\mathfrak{z}$ is an ideal, there is a complementary ideal $\mathfrak{k}'$ s.t. $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{z}$. Hence,

$$\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n} = \mathfrak{k}' \ltimes \mathfrak{n} \oplus \mathfrak{z}.$$

Now, note that if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_i$ has a faithful rep V_i, then $\mathfrak{g}$ has faithful rep $V_1 \oplus V_2$. Therefore, we may assume $\mathfrak{g}$ is indecomposable, so assume that $\mathfrak{z} = 0$.

Now, $\mathfrak{k}$ acts faithfully on $\mathfrak{n}$. $\mathfrak{g} = \mathfrak{k} \ltimes \mathfrak{n}$ acts on $\mathscr{O}(N)$ where $x \in \mathfrak{n}$ acts by L_x, and $y \in \mathfrak{k}$ acts by $L_y - R_y$ (adjoint action). Thus, it also acts on the spaces V_n. Fix n so that the action of $\mathfrak{n}$ on V_n is faithful. We claim all of $\mathfrak{g}$ acts faithfully on V_n. Suppose that nonzero $y \in \mathfrak{g}$ acts by 0 on V_n. Write $y = (y_1, y_2)$ ($y_1 \in \mathfrak{k}$ and $y_2 \in \mathfrak{n}$), and pick $z \in \mathfrak{n}$ so that $a = [y, z] \neq 0$ (possible since $\mathfrak{z} = 0$). Then, $a \in \mathfrak{n}$ acts by 0 on V_n, a contradiction. ∎

25.2 Last topic: Borel subalgebras and flag manifold

Note 10. Got distracted for a few minutes and missed some stuff that looks important... whoops

TODO: Find what you missed and fill it in here

Definition 25.5. A **Borel subalgebra** is a Lie subalgebra conjugate to $\mathfrak{b}_+$. A **Borel subgroup** of G is a Lie subgroup conjugate to B_+. A **parabolic subalgebra** is a Lie subalgebra containing a Borel subalgebra. A **parabolic subgroup** is a Lie subgroup containing a Borel subgroup.

Since all pairs $(\mathfrak{h}, \Pi)$ are conjugate, this definition does not depend on the choice of $(\mathfrak{h}, \Pi)$.

Lemma 25.6. *B_+ is its own normalizer in G.*

Proof. Take $\gamma \in G$ such that $\gamma B_+ \gamma^{-1} = B_+$. Let $H' = \gamma B_+ \gamma^{-1} \subset B_+$ ($H \subset B_+$ a maximal torus). It is easy to see that we can conjugate H' back to H inside B_+. Therefore, we may assume that $H' = H$. Thus, $\gamma \in N(H)$ which we recall fits in an exact sequence

Something about Levi decomposition and having a vanishing H^1

$$1 \longrightarrow H \longrightarrow N(H) \longrightarrow W \longrightarrow 1.$$

We also remark that γ preserves positive roots, so preserves the set Π of simple roots. The only element of the Weyl group which preserves the set Π is the identity, so actually $\gamma \in H \subset B_+$, and we win. ∎

Corollary 25.7. *The set of Borel subgroups (subalgebras) is G/B_+, a homogeneous space and complex manifold. We call this the **flag manifold** of G.*

Note that this manifold is canonically attached to G, and depends only on $\mathfrak{g}_{ss} \subset \mathfrak{g} = \operatorname{Lie} G$.

Remark 25.8.

$$\dim G/B_+ = |R_+| = \frac{1}{2}\left(\dim \mathfrak{g} - \dim \mathfrak{h}\right).$$

Example. If $G = \mathrm{GL}_n$ (or SL_n), then we can take B_+ to be the upper triangular matrices, and $G/B_+ = F_n$ is the set of complete flags in $\mathbb{C}^n$. For example, if $G = \mathrm{SL}_2$, then $G/B_+ = \mathbb{CP}^1$ is the Riemann sphere.

Note that G/B_+ is compact in the above example. This is in fact true in general.

Let $G^c \subset G$ be the compact form.

Remark 25.9. $\mathfrak{g}^c + \mathfrak{b}_+ = \mathfrak{g}$. Note $\mathfrak{g}^c$ contains things like $e_\alpha \pm e_{-\alpha}$ and $i(e_\alpha \mp e_{-\alpha})$ while $\mathfrak{b}_+ \ni ie_\alpha, e_\alpha$ ($\alpha > 0$ throughout this sentence). Hence, their sum contains all the e_α's and the Cartan subalgebra.

As a consequence, the orbit $G^c \cdot 1 \subset G/B_+$ contains a neighborhood of $1 \in G/B_+$ (since $\mathfrak{g}^c \twoheadrightarrow \mathfrak{g}/\mathfrak{b}_+$). By translation, we see $G^c \cdot 1$ contains a neighborhood of all its elements, so it is open. It is also closed since it is compact. Since G/B_+ is connected, we conclude that $G^c \cdot 1 \twoheadrightarrow G/B_+$ is surjective, so G/B_+ is compact.

The above shows that G^c acts transitively on G/B_+. It's stabilizer is $\text{Stab}(1) = G^c \cap B_+$.

Note 11. Distracted and missed more stuff

Sounds like the stabilizer is $H^c = (S^1)^r$, a maximal torus in G^c.

Corollary 25.10. *$G/B_+ \cong G^c/H^c$.*

Corollary 25.11 (Iwasawa Decomposition). *The usual notation is $K = G^c$, $N = N_+$, and $A = \exp(i\mathfrak{h}^c) \subset H$, the non-compact part H. The multiplication map*

$$K \times A \times N \to G$$

is a diffeomorphism, so $G = KAN$.

(Compare e.g. with Polar decomposition)

Proof. As shown above, the map $\varphi : G^c \times B_+ \to G$ is surjective. Further,

$$\varphi(g_1, b_1) = \varphi(g_2, b_2) \iff g_1 = g_2 s, b_1 = s^{-1} b_2 \text{ for some } s \in H^c.$$

Let $B_+^0 = AN_+$ so $B_+ = H^c B_+^0$. Hence, $G^c \times B_+^0 \xrightarrow{\sim} G$ is a diffeomorphism. Also, $A \times N_+ \xrightarrow{\sim} B_+^0$ is a diffeomorphism, so

$$G^c \times A \times N_+ \to G$$

is a diffeomorphism. ∎

Another realization of flag manifold One can construct the flag manifold alternatively as the orbit of a highest weight line in an irreducible representation with regular highest weight. Say $\lambda \in P_+$ dominant integral regular, where **regular** means $\lambda(h_i) \geq 1$ for all i (e.g. $\lambda = \rho$ so $\rho(h_i) = 1$). Let L_λ be the irrep with highest weight λ, and let v_λ be a highest weight vector, so $\mathbb{C}v_\lambda \in \mathbb{P}L_\lambda$. Let $\mathscr{O} := G \cdot \mathbb{C}v_\lambda \subset \mathbb{P}L_\lambda$ be the orbit of this line.

Claim 25.12.

$$\mathscr{O} \cong G/B_+.$$

Proof. What is the stabilizer S of $\mathbb{C}v_\lambda$? Clearly, $S \supset B_+$ since v_λ a highest weight vector. Also, for $\alpha \in R_+$, $e_{-\alpha} v_\lambda \neq 0$ as $\lambda(h_i) \geq 1$.[52] We see from this that the stabilizer of $\mathbb{C}v_\lambda$ in $\mathfrak{g}$ is $\mathfrak{b}_+$. Hence, S normalizes $\mathfrak{b}_+$, so $S \subset B_+$, so $S = B_+$. This shows that $\mathscr{O}$ is a closed orbit in $\mathbb{P}L_\lambda$, so G/B_+ is a complex (smooth) projective variety. ∎

Remark 25.13. Partial flag manifolds are also complex projective varieties. Can prove similarly using non-regular weights.

[52] also wrote $h_\alpha v_\lambda = m v_\lambda$ with $m > 0$, but I don't see why this is relevant

Borel fixed point theorem Only 8 minutes left, so let's end with a bang.

Theorem 25.14. *Let $\mathfrak{a}$ be a solvable Lie algebra over $\mathbb{C}$, and let V be a f.dim $\mathfrak{a}$-module. Let $X \subset \mathbb{P}V$ be a closed subset preserved by A. Then, there exists $x \in X$ such that $Ax = x$.*

Non-example. $\mathrm{SL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1$ without fixed points.

Proof. Induct in $n = \dim \mathfrak{a}$. The base $n = 0$ is trivial. Since $\mathfrak{a}$ is solvable, it has an ideal $\mathfrak{a}'$ of codimension 1. By induction, $Y = X^{\mathfrak{a}'} \neq \emptyset$. Furthermore, $\mathfrak{a}/\mathfrak{a}'$ acts on Y, so we only need to prove the theorem when $\dim \mathfrak{a} = 1$.

Say $\mathfrak{a} = \langle a \rangle$ with a acting by a linear operator $a : V \to V$. We can scale a by complex numbers. In particular, by rotating, we may assume that the real parts of all its eigenvalues are different. Pick $x_0 \in X$, and consider $e^{ta} \cdot x_0$. If we send $t \to \infty$, the eigenvalue with largest real part will 'dominate' resulting in the existence of a limit $x \in X$ (no particular vector has a limit, but the whole line does). This limit is fixed by a, so we win. ■

Question: Use compactness of X?

Corollary 25.15. *Any solvable subalgebra of $\mathfrak{g}$ is contained in a Borel. Thus, Borels are simply maximal solvable subalgebras.*

Proof. Say $\mathfrak{a} \subset \mathfrak{g}$ solvable. Then, it has a fixed when acting on $G/B_+ \subset \mathbb{P}L_\lambda$. This fixed point is a Borel subalgebra $\mathfrak{b}$, so $\exp(\mathfrak{a})$ normalizes $\mathfrak{b}$, so $\mathfrak{a} \subset \mathfrak{b}$. ■

Corollary 25.16. *Any element of $\mathfrak{g}$ is contained in some Borel subalgebra.*

Example. When $\mathfrak{g} = \mathfrak{gl}_n$, this says any matrix can be upper triangularized in some basis.

We don't have time to give the proof (it's in the notes), but similarly...

Proposition 25.17. *Any nilpotent subalgebra of $\mathfrak{g}$ (consisting of nilpotent elements) is contained in a conjugate to $\mathfrak{n}_+$. Hence, conjugates of $\mathfrak{n}_+$ are the same thing as maximal nilpotent subalgebras.*

One can show that the normalizer of $\mathfrak{n}_+$ is B_+, so any maximal nilpotent subalgebra $\mathfrak{n}$ is contained in a unique Borel $\mathfrak{n}$, and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. Therefore, maximal nilpotent subalgebras are also parameterized by the flag manifold G/B_+.

26 List of Marginal Comments

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